N. ABE
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EXOTIC CHARACTERISTIC CLASSES OF CERTAIN Γ-FOLIATIONS

BY NAOTO ABE

§ 0. Introduction

In this paper we study characteristic classes and exotic characteristic classes [B] of foliations. We deal with a certain sort of Γ-foliations ($(G, \mathfrak{l})$-foliations) which is a generalization of Riemannian, projective and conformal foliations [NS, P]. The main purpose of this paper is to prove vanishing theorems for certain exotic characteristic classes of such Γ-foliations. As a step toward this, we obtain results which are relevant to strong vanishing theorems for characteristic classes of the Γ-foliations. These generalize the result of Nishikawa and Sato [NS]. In order to obtain these results, we use neither normal Cartan connections nor classifying spaces, but a product formula for secondary invariants [CS] and a technique used by Kobayashi and Ochiai in [KO].

Throughout this paper, all manifolds and mappings are assumed to be smooth (C^∞). In § 1, Chern-Weil theory of characteristic classes are reviewed, and the technique used in [KO] is slightly improved so that it may be applied to the case of foliations. In § 2, certain automorphisms of G-structures are specialized to l-automorphisms, which are generalized notions of affine, projective and conformal transformations, and then $(G, \mathfrak{l})$-foliations are defined. In § 3, so-called strong vanishing theorems for characteristic classes of $(G, \mathfrak{l})$-foliations are proved, where the results prepared in § 1 are applied. In § 4, we review some notions about exotic characteristic classes [B, H] such as cochain complexes WO_q and W_q, generalized characteristic homomorphisms for foliations and Vey-basis. In § 5, the vanishing theorems for certain exotic characteristic classes of $(G, \mathfrak{l})$-foliations are proved. In § 6, more detailed results are obtained in the case of projective and conformal foliations. Especially we find that all of the rigid exotic characteristic classes of conformal foliations vanish and the rest of the exotic characteristic classes coincide with the Godbillon-Vey invariant up to scalar multiples. In § 7, we prove the product formula and a derivative formula [H] for secondary invariants in a simple and unified manner.

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§ 1. Characteristic homomorphism and $(\mathfrak{I}, \theta)$-equivalence

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $I^r(G)$ the space of all $\mathrm{Ad}(G)$-invariant symmetric multilinear functions on $\mathfrak{g}\times\cdots\times\mathfrak{g}$ (r times). We will use the convention that if $\varphi\in I^r(G)$ contains less than r arguments, the last one is repeated a number of times to make φ a function of r arguments. Let M be a manifold and P a principal G-bundle over M. The space of all $\mathfrak{g}$-valued (resp. real valued) s-forms on P (resp. M) is denoted by $A^s(P, \mathfrak{g})$ (resp. $A^s(M)$).

Let $\omega^0, \omega^1\in A^1(P, \mathfrak{g})$ be connection forms.

DEFINITION. Homomorphisms of modules

$$\lambda(\omega^1):\ I^r(G)\longrightarrow A^{2r}(M)$$

and

$$\lambda(\omega^0, \omega^1):\ I^r(G)\longrightarrow A^{2r-1}(M)$$

are defined by

$$\lambda(\omega^1)\varphi := \varphi(\Omega^1)$$

and

$$\lambda(\omega^0, \omega^1)\varphi := r\int_0^1\varphi(\omega^1-\omega^0, \Omega^t)dt \qquad \text{for} \quad \varphi\in I^r(G)\,,$$

where Ω^t is the curvature form of the connection $\omega^t=t\omega^1+(1-t)\omega^0$.

We obtain the following (see § 7):

Fact 1.1.

$$d(\lambda(\omega^1)\varphi)=0$$

and

$$d(\lambda(\omega^0, \omega^1)\varphi)=\lambda(\omega^1)\varphi-\lambda(\omega^0)\varphi\,.$$

In other words, the closed form $\lambda(\omega^1)\varphi$ represents a de Rham cohomology class $[\lambda(\omega^1)]\in H^{2r}(M)$, and the induced homomorphism

$$\lambda(\omega^1)^*:\ I^r(G)\longrightarrow H^{2r}(M)$$

does not depend on the choice of connections on P. We call this homomorphism the characteristic homomorphism of P and denote by λ_P.

In § 7, we prove the following product formula.

PROPOSITION 1.1. *For $\varphi, \psi\in I(G)$,*

$$\lambda(\omega^0, \omega^1)(\varphi\cdot\psi)\sim\lambda(\omega^0, \omega^1)\varphi\wedge\lambda(\omega^1)\psi+\lambda(\omega^0)\varphi\wedge\lambda(\omega^0, \omega^1)\psi\,,$$

where "$\sim$" means "cohomologous to".

Moreover we have the derivative formula:

Fact 1.2 (Heitsch [H]). For $\varphi\in I^r(G)$, and a smooth family of connections $\omega^1_s\in C(P)$ $(s\in R)$,

$$\frac{1}{r}\frac{\partial}{\partial s}\lambda(\omega^0, \omega^1_s)\varphi=\varphi\Big(\frac{\partial}{\partial s}\omega^1_s, \Omega^1_s\Big)+(r-1)d\int_0^1 t\varphi\Big(\frac{\partial}{\partial s}\omega^1_s, \omega^1_s-\omega^0, \Omega^t_s\Big)dt\,,$$

where Ω_s^t is the curvature form of $\omega_s^t := t\omega_s^1 + (1-t)\omega^0$.

Integrating the both sides of this formula, we get

Fact 1.3. For $\varphi \in I^r(G)$,

$$\lambda(\omega^0, \omega_1^1)\varphi - \lambda(\omega^0, \omega_0^1)\varphi \sim r\int_0^1 \varphi\Big(\frac{\partial}{\partial s}\omega_s^1,\ \Omega_s^1\Big) ds\,.$$

We shall give simple proofs of Facts 1.1 and 1.2 in §7.

Let V be a vector space and $GL(V)$ the general linear group. Hereafter we assume that $G \subset GL(V)$. Let $\mathfrak{l}$ be a Lie algebra which contains the Lie algebra $\mathfrak{g}+V$ of the semidirect product $G\cdot V$ as a Lie subalgebra. Denote the space of all $\mathfrak{l}$-valued (resp. V-valued) s-forms on P by $A^s(P, \mathfrak{l})$ (resp. $A^s(P, V)$). Let $\theta \in A^1(P, V)$, $C_\theta(P) := \{\omega \in C(P) \mid d\theta + [\omega, \theta] = 0\}$, where $C(P)$ denotes the set of all G-connections on P. By the definition and the Jacobi identity in $\mathfrak{g}+V$, we get

LEMMA 1.1. $[\theta, \Omega] = 0$ *for* $\omega \in C_\theta(P)$.

DEFINITION. We say that $\omega^0, \omega^1 \in C(P)$ are $(\mathfrak{l}, \theta)$-equivalent if there exists $\rho \in A^0(P, \mathfrak{l})$ such that $\omega^1 - \omega^0 = [\theta, \rho]$.

From the Jacobi identity in $\mathfrak{l}$ and the fact that $[V, V] = 0$, we see the following lemma is true.

LEMMA 1.2. *If* $\omega^0 \in C(P)$ *and* $\omega^1 \in C_\theta(P)$ *are* $(\mathfrak{l}, \theta)$-*equivalent, then* $\omega^0 \in C_\theta(P)$.

We denote the set $\{X \in \mathfrak{l} \mid [X, Y] \in \mathfrak{g}+V$ for any $Y \in V\}$ by $\hat{\mathfrak{l}}$. It is obvious that $\hat{\mathfrak{l}}$ is an $ad(V)$-invariant subspace of $\mathfrak{l}$ and contains $\mathfrak{g}+V$. From the definition of $\hat{\mathfrak{l}}$, we get

LEMMA 1.3. *If* $\theta_p ; T_pP \to V$ *is surjective for each* $p \in P$, $[\theta, \rho] = \omega^1 - \omega^0$ *for* $\omega^1, \omega^0 \in C(P)$ *and* $\rho \in A^0(P, \mathfrak{l})$, *then* $\rho \in A^0(P, \hat{\mathfrak{l}})$.

Hereafter we assume that $\theta \in A^1(P, V)$ satisfies the condition described in Lemma 1.3. Note that the canonical form of the tangent frame bundle of a manifold will do. Let $S^r(\hat{\mathfrak{l}}^*)$ be the space of all multilinear functions on $\hat{\mathfrak{l}} \times \cdots \times \hat{\mathfrak{l}}$ (r times). For $X \in V$, let the linear map $ad(X)^* : S^r(\hat{\mathfrak{l}}^*) \to S^r(\hat{\mathfrak{l}}^*)$ be defined by, for $\psi \in S^r(\hat{\mathfrak{l}}^*)$, $X_i \in \hat{\mathfrak{l}}$,

$$(ad(X)^*\psi)(X_1, \cdots, X_r) := \sum_{i=1}^r \psi(X_1, \cdots, [X, X_i], \cdots, X_r)$$

Set $S^r(\hat{\mathfrak{l}}^*)^V := \{\psi \in S^r(\hat{\mathfrak{l}}^*) \mid ad(V)^*\psi = 0\}$, $I^r{}_{(\mathfrak{l})}(G) := (S^r(\hat{\mathfrak{l}}^*)^V \mid \mathfrak{g}) \cap I(G)$.

PROPOSITION 1.2. *If* $\omega^0, \omega^1 \in C_\theta(P)$ *are* $(\mathfrak{l}, \theta)$-*equivalent, then*

$$\lambda(\omega^0, \omega^1) I^r{}_{(\mathfrak{l})}(G) = 0 \qquad \textit{for} \quad r \geqq 1\,.$$

Proof. Let $\rho \in A^0(P, \hat{\mathfrak{l}})$ satisfy $\omega^1 - \omega^0 = [\theta, \rho]$. Since $\omega^t = t\omega^1 + (1-t)\omega^0 = \omega^0 + [\theta, t\rho]$, ω^t and ω^0 are $(\mathfrak{l}, \theta)$-equivalent. From Lemmas 1.1 and 1.2, we get $[\theta, \Omega^t] = 0$. For $\psi \in S^r(\hat{\mathfrak{l}}^*)^V$, $X \in V$ and $X_i \in \mathfrak{l}$, we see that

$$\sum_{i=1}^{r} \psi(X_1, \cdots, [X, X_i], \cdots, X_r) = 0 .$$

Hence $\psi([\theta, \rho], \Omega^t)+(r-1)\psi(\rho, [\theta, \Omega^t], \Omega^t)=0$, so that, we get

$$\psi(\omega^1-\omega^0, \Omega^t) = \psi([\theta, \rho], \Omega^t) = 0 . \qquad \text{Q. E. D.}$$

This proposition and Fact 1.1 imply

COROLLARY 1.3. *If ω^0, $\omega^1 \in C_\theta(P)$ are $(\mathfrak{l}, \theta)$-equivalent, then*

$$\lambda(\omega^0)\varphi = \lambda(\omega^1)\varphi \qquad for \quad \varphi \in I_{(\mathfrak{l})}(G) .$$

Remark 1.1. Let $S^r(\mathfrak{l}^*)_V := \{\psi \in S^r(\mathfrak{l}^*)^V \mid \iota_V \psi = 0\}$, where $\iota_X : S^r(\mathfrak{l}^*) \to S^{r-1}(\mathfrak{l}^*)$ is the inner product operator for $X \in V$. Then Proposition 1.2 and Corollary 1.3 hold for ω^0, $\omega^1 \in C(P)$ and $\varphi \in (S^r(\mathfrak{l}^*)_V \mid \mathfrak{g}) \cap I(G)$ also.

Remark 1.2. Let L be a Lie group which contains G as a Lie subgroup compatible with the inclusion $\mathfrak{g}+V \subset \mathfrak{l}$, where $\mathfrak{l}$ is the Lie algebra of L. Denote the image of the restriction map $I(L) \to I(G)$ by $I_L(G)$. Since $I_L(G) \subset I_{(\mathfrak{l})}(G)$, our results in this section generalize the corresponding results in [KO].

Remark 1.3. For a G-bundle P with $\theta \in A^1(P, V)$ which satisfy the condition in Lemma 1.3, we can find a Lie algebra $\mathfrak{l}$ such that any ω^0, $\omega^1 \in C_\theta(P)$ are $(\mathfrak{l}, \theta)$-equivalent. Let $\mathfrak{g}^{(p)}$ be the p-th prolongation of $\mathfrak{g}$, i.e., the vector space $\{t \in S^{p+1}(V^*) \otimes V \mid$ the linear map $V \ni X \mapsto t(X, X_1, \cdots, X_p) \in V$ belongs to $\mathfrak{g} \subset \mathfrak{gl}(V) = V^* \otimes V$ for any $X_1, \cdots, X_p \in V\}$ where $S^r(V^*) \otimes V$ is the space of all symmetric multilinear functions on $V \times \cdots \times V$ (r times) with values in V. Note that $\mathfrak{g}^{(-1)} = V$ and $\mathfrak{g}^{(0)} = \mathfrak{g}$. For $t \in \mathfrak{g}^{(r)}$ and $t' \in \mathfrak{g}^{(s)}$, we define $[t, t'] \in \mathfrak{g}^{(r+s)}$ by, for $X_i \in V$,

$$[t, t'](X_0, \cdots, X_{r+s}) := \frac{1}{r!(s+1)!} \sum t(t'(X_{i_0}, \cdots, X_{i_s}), X_{i_{s+1}}, \cdots, X_{i_{s+r}})$$
$$- \frac{1}{s!(r+1)!} \sum t'(t(X_{j_0}, \cdots, X_{j_r}), X_{j_{r+1}}, \cdots X_{j_{r+s}}) ,$$

where the summations are taken over all permutations of $(0, \cdots, r+s)$. We explicitly set $[\mathfrak{g}^{(-1)}, \mathfrak{g}^{(-1)}]=0$. In particular, if $t \in \mathfrak{g}^{(p)}$, $p \geqq 0$, and $X \in V$, then

$$[t, X](X_1, \cdots, X_p) = t(X, X_1, \cdots, X_p) .$$

If is well-known that $\mathfrak{g}^{(*)} := \sum_{p \geqq -1} \mathfrak{g}^{(p)}$ becomes a Lie algebra. By the definition of the bracket product, it is clear that $\mathfrak{g}^{(*)}$ contains the Lie algebra $\mathfrak{g}+V$ of the semidirect product $G \cdot V$ as a Lie subalgebra. Let ω^0, $\omega^1 \in C_\theta(P)$, then $[\omega^1 - \omega^0, \theta]=0$. Fix a basis of V, and denote the components of θ (resp. $\omega^1-\omega^0$) by θ^i (resp. τ^i_j). Then $\tau^i_j \wedge \theta^j = 0$ for each i. Here we employ the Einstein summation convention. Since the form θ satisfies the condition in Lemma 1.3, θ^i are linearly independent at each point. Then there exist functions ρ^i_{jk} such that $\tau^i_j =$

$\rho^i_{jk}\cdot\theta^k$ and $\rho^i_{jk}=\rho^i_{kj}$. Clearly ρ^i_{jk} define an element ρ of $\mathfrak{g}^{(1)}$ satisfying $\omega^1-\omega^0=[\rho, \theta]$. Thus any $\omega^0, \omega^1\in C_\theta(P)$ are $(\mathfrak{g}^{(*)}, \theta)$-equivalent.

Note that $\hat{\mathfrak{l}}\subset(\mathfrak{g}^{(-1)}+\mathfrak{g}^{(0)}+\mathfrak{g}^{(1)})\cap\mathfrak{l}$ if $\mathfrak{l}$ is a Lie subalgebra of $\mathfrak{g}^{(*)}$ and contains $V+\mathfrak{g}=\mathfrak{g}^{(-1)}+\mathfrak{g}^{(0)}$.

§ 2. $(G, \mathfrak{l})$-foliation

Let M be an n-dimensional manifold, TM the tangent bundle of M. An integrable subbundle E of TM is called a foliation of M. We call the quotient bundle TM/E the normal bundle, and the frame bundle of TM/F the normal frame bundle of E. We will take the following point of view toward Γ-foliations (with coordinates) (cf. [BH], [NS] and [P]):

A Γ-foliation $\mathcal{F}$ of codimension q $(\leqq n)$ on M is given by the following data:

(1) An open covering $\{U_\lambda\}$ of M.

(2) An auxiliary q-dimensional manifold N with a pseudogroup Γ of local transformations on N.

(3) A system of maps $h_\lambda: U_\lambda\to N$ of rank q.

(4) A system of elements $\gamma_{\mu\lambda}\in\Gamma$ which satisfy $h_\mu=\gamma_{\mu\lambda}\circ h_\lambda$ on $U_{\mu\lambda}\neq\phi$ for each λ, μ, where $U_{\mu\lambda}:=U_\mu\cap U_\lambda$.

Then the kernels of differentials $(h_\lambda)_*$ constitute an integrable subbundle F of TM. Thus $\mathcal{F}$ defines a usual foliation as an underlying structure. The quotient bundle TM/F is called the normal bundle of $\mathcal{F}$. Of course, the fibre dimension of TM/F is the codimension of $\mathcal{F}$. Conversely Frobenius theorem shows that an integrable subbundle of TM defines a Γ-foliation with $N=R^q$ and $\Gamma=$ {local diffeomorphisms on R^q}.

EXAMPLE 2.1. If N is a complex analytic manifold and $\Gamma=$ {complex analytic local diffeomorphisms on N}, $\mathcal{F}$ is called a complex analytic foliation. If M is a complex analytic manifold, then a holomorphic integrable subbundle of the holomorphic tangent bundle of M defines a complex analytic foliation (see [B], for example).

EXAMPLE 2.2. If N is a manifold with an affine connection ω and $\Gamma=$ {local affine transformations on N}, $\mathcal{F}$ is called an affine foliation.

EXAMPLE 2.3. If N is a Riemannian manifold and $\Gamma=$ {local isometries of N}, $\mathcal{F}$ is called a Riemannian foliation [P]. If M admits a bundle-like metric [R] with respect to a foliation on M, this foliation with the metric defines a Riemannian foliation.

EXAMPLE 2.4. If N is a Riemannian manifold and $\Gamma=$ {local projective transformations on N}, $\mathcal{F}$ is called a projective foliation [NS]. A foliation of codimension one defines a projective foliation. We will also call $\mathcal{F}$ a projective foliation when N is a manifold with an affine connection ω and $\Gamma=$ {local projective transformations on N}. This foliation is a generalization of an affine foliation.

EXAMPLE 2.5. If N is a Riemannian manifold and $\Gamma=\{$local conformal transformations on $N\}$, $\mathscr{F}$ is called a conformal foliation [NS]. Both a foliation of codimension one and a complex analytic foliation of complex codimension one define conformal foliations.

We see that Example 2.2, 2.4 and 2.5 are generalizations of Example 2.3. In each example, Γ consists of local automorphisms of a certain G-structure, that is, integrable $GL(q/2, C)$-structure (q: even) in Example 2.1, $GL(q, R)$-structure with an affine connection in Example 2.2, $O(q)$-structure in Example 2.3, $GL(q, R)$-structure with a torsion free connection in Example 2.4 and $CO(q)$-structure in Example 2.5, where $CO(q):=\{aA \mid a\in R-\{0\}, A\in O(q)\}$. Thus we will restrict ourselves to Γ-foliations which are defined by G-structures on N with connection.

Let B_N be the tangent frame bundle of N, $GL_q:=GL(q, R)$ its structure group and θ_N the canonical form. Let G be a subgroup of GL_q and P_N a G-subbundle of B_N (a G-structure on N). The restriction of θ_N to P_N is also denoted by θ_N. For a local diffeomorphism γ on N, the induced local bundle isomorphism of B_N is denoted by $\tilde{\gamma}$. Let $\Gamma(P_N)$ be the pseudogroup of all local diffeomorphisms on N which preserve P_N. Let $\mathfrak{l}$ be an auxiliary Lie algebra which contains the Lie algebra $\mathfrak{g}+R^q$ of the semidirect product $G\cdot R^q$ as a Lie subalgebra. Let ω be a G-connection on P_N.

DEFINITION. $\gamma\in\Gamma(P_N)$ is said to be a (local) $\mathfrak{l}$-automorphism of (P_N, ω) if $\tilde{\gamma}^*\omega$ is $(\mathfrak{l}, \theta_N)$-equivalent to ω.

Remark 2.1. If $\mathfrak{l}$ is a Lie subalgebra of a Lie algebra $\mathfrak{l}'$ and $\gamma\in\Gamma(P_N)$ is a $\mathfrak{l}$-automorphism of (P_N, ω), then γ is a $\mathfrak{l}'$-automorphism. For a torsion free connection $\omega\in C(P_N)$, any $\gamma\in\Gamma(P_N)$ is a $\mathfrak{g}^{(*)}$-automorphism, where $\mathfrak{g}^{(*)}$ is the Lie algebra which was defined in Remark 1.3.

EXAMPLE 2.2′ (Affine case). Let N be a manifold with an affine connection $\omega\in C(B_N)$. Put $G=GL_q$, $P_N=B_N$ and $\mathfrak{l}=\mathfrak{gl}_{q+1}$. For $X+v\in\mathfrak{g}+R^q$, the corresponding element is given by $\begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix}$. If γ is an affine transformation on N, then $\tilde{\gamma}^*\omega-\omega=0$ or $\tilde{\gamma}^*\omega-\omega=[\theta_N, 0]$ in $\mathfrak{l}$.

EXAMPLE 2.4′ (Projective case [KO]). Let N be a Riemannian manifold, ω the Levi-Civita connection. Put $G=GL_q$, $P_N=B_N$ and $\mathfrak{l}=\mathfrak{sl}(q+1, R)\subset\mathfrak{gl}_{q+1}$, where $\mathfrak{sl}(q+1, R)$ is the Lie algebra of the special linear group $SL(q+1, R)$. For $X+v\in\mathfrak{g}+R^q$, the corresponding element is given by $\begin{pmatrix} A & v \\ 0 & a \end{pmatrix}$, where $a:=-\frac{1}{q+1}\operatorname{Tr} X$ and $A:=X+aI_q$. If γ is a projective transformation on N, then there exists a ${}^t(R^q)$-valued function ρ on P_N such that $\tilde{\gamma}^*\omega-\omega=\theta_N\rho+(\rho\theta_N)I_q$ or $\tilde{\gamma}^*\omega-\omega=[\theta_N, \rho]$ in $\mathfrak{l}$, where ${}^t(R^q)$ is contained in $\mathfrak{l}'$ in the form $\begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}$ for $u\in{}^t(R^q)$.

EXAMPLE 2.5′ (Conformal case [KO]). Let N be a Riemannian manifold, ω the Levi-Civita connection. Put $G=CO(q)$, $P_N=$ the $CO(q)$-extension of the orthonormal frame bundle, and $\mathfrak{l}=O(q+1,1)\subset\mathfrak{gl}_{q+2}$, where $O(q+1,1)$ is the Lie algebra of $O(q+1,1)$. The Lie group $O(q+1,1)$ is defined to be $\{X\in GL_{q+2}\,|\,{}^tXSX=S\}$, where $S=\begin{pmatrix}0 & 0 & -1\\ 0 & I_q & 0\\ -1 & 0 & 0\end{pmatrix}\in GL_{q+2}$. For $X+v\in\mathfrak{g}+R^q$, the corresponding element is given by $\begin{pmatrix}-a & {}^tv & 0\\ 0 & A & v\\ 0 & 0 & a\end{pmatrix}$, where $a:=-\dfrac{1}{q}\operatorname{Tr}X$ and $A:=X+aI_q$. If γ is a conformal transformation on N, then there exists a ${}^t(R^q)$-valued function ρ on P_N such that $\tilde\gamma^*\omega-\omega=\theta_N\rho+(\rho\theta_N)I_q-{}^t\rho{}^t\theta_N$ or $\tilde\gamma^*\omega-\omega=[\theta_N,\rho]$ in $\mathfrak{l}$, where ${}^t(R^q)$ is contained in $\mathfrak{l}$ in the form $\begin{pmatrix}0 & 0 & 0\\ {}^tu & 0 & 0\\ 0 & u & 0\end{pmatrix}$ for $u\in{}^t(R^q)$.

Now we get the following generalization of all the above foliations.

DEFINITION. If Γ consists of $\mathfrak{l}$-automorphisms of (P_N,ω) for a G-structure P_N and $\omega\in C(P_N)$, then the Γ-foliation $\mathcal{F}$ is called a $(G,\mathfrak{l})$-foliation.

Remark 2.2. According to Remark 2.1, if the G-structure admits a torsion free connection $\omega\in C(P_N)$ and $\Gamma\subset\Gamma(P_N)$, then the Γ-foliation with (P_N,ω) is a $(G,\mathfrak{g}^{(*)})$-foliation. Note that a $(G,\mathfrak{l})$-foliation $\mathcal{F}$ is a $(G,\mathfrak{l}')$-foliation if $\mathfrak{l}$ is a Lie subalgebra of $\mathfrak{l}'$. In order to study characteristic homomorphisms for $(G,\mathfrak{l})$-foliations, we may take a larger Lie algebra $\mathfrak{l}'$.

§ 3. Characteristic classes of $(G,\mathfrak{l})$-foliations

Let $\mathcal{F}$ be a Γ-foliation on M as in § 2, B the normal frame bundle. Each $h_\lambda: U_\lambda\to N$ is covered by the canonical bundle map $\tilde h_\lambda: B|U_\lambda\to B_N$ which satisfies $\tilde h_\mu=\tilde\gamma_{\mu\lambda}\circ\tilde h_\lambda$. Let P_N be a G-reduction of B_N, $\Gamma\subset\Gamma(P_N)$ at first. Then we have

LEMMA 3.1. *$\tilde h_\lambda$ and P_N defines a canonical G-subbundle $P\subset B$ on which $\theta\in A^1(P,R^q)$ satisfying $\tilde h_\lambda^*\theta_N=\theta|P_\lambda$ exists, where P_λ is the restriction $P|U_\lambda$.*

Such a reduction is a special case of $\mathcal{F}$-reductions defined in [A]. We assume that the covering $\{U_\lambda\}$ in the definition of $\mathcal{F}$ is locally finite and admits a partition of unity $\{f_\lambda\}$. Let $\pi: P\to M$ be the projection, $\tilde f_\lambda:=\pi^*f_\lambda$. Let ω be a connection form on P_N. Define $\omega^1\in C(P)$ by $\omega^1:=\sum_\lambda\tilde f_\lambda\cdot(\tilde h_\lambda^*\omega)$. We also denote the GL_q-extension of ω^1 to B by ω^1. Clearly this connection is a basic connection defind by Bott [B].

Let F be the corresponding integrable subbundle of TM, $F^a:=\{\alpha\in A^1(M)\,|\,\iota_X\alpha=0$ for any $X\in\Gamma F\}$ and ${}^rA:=(\wedge^rF^a)\wedge A(M)$ an ideal of $A(M)$, where ΓF is the space of all cross sections of F. Clearly ${}^rA=0$ for $r>q$. The following facts were proved by Bott, and are called the vanishing theorems for characteristic classes of foliations.

Fact 3.1. $\lambda(\omega^1)I^r(G)\subset{}^rA\cap A^{2r}(M)$,

and hence $\lambda(\omega^1)I^r(G)=0$ for $r>q$.

For the GL_q-extension of ω^1 to B, we get

Fact 3.2. $\lambda(\omega^1)I^r(GL_q)\subset{}^rA\cap A^{2r}(M)$,

and hence $\lambda(\omega^1)I^r(GL_q)=0$ for $r>q$.

In the rest of this section, we assume that $\mathcal{F}$ is a $(G, \mathfrak{l})$-foliation. Then we can find a basic connection which is convenient to study characteristic homomorphisms.

PROPOSITION 3.1. *Let* $\omega^1:=\sum_\mu \tilde{f}_\mu\cdot(\tilde{h}_\mu^*\omega)$ $(\in C(P))$, *then* $\omega^1|P_\lambda$ *and* $\tilde{h}_\lambda^*\omega$ *are* $(\mathfrak{l}, \theta|P_\lambda)$*-equivalent.*

Proof. The restriction is $\omega^1|P_\lambda=\sum_\mu(\tilde{f}_\mu|P_\lambda)\cdot(\tilde{h}_\mu^*\omega|P_{\mu\lambda})$, where $P_{\mu\lambda}:=P|U_{\mu\lambda}$. On the other hand, there is $\rho_{\mu\lambda}\in A^0(\tilde{h}_\lambda(P_{\mu\lambda}), \mathfrak{l})$ such that

$$\tilde{\gamma}_{\mu\lambda}{}^*(\omega|\tilde{h}_\mu(P_{\mu\lambda}))=\omega|\tilde{h}_\lambda(P_{\mu\lambda})+[\theta_N|\tilde{h}_\lambda(P_{\mu\lambda}), \rho_{\mu\lambda}],$$

so that

$$\tilde{h}_\mu^*\omega|P_{\mu\lambda}=(\tilde{h}_\lambda^*\tilde{\gamma}_{\mu\lambda}^*\omega)|P_{\mu\lambda}=\tilde{h}_\lambda^*\omega|P_{\mu\lambda}+[\theta|P_{\mu\lambda}, \tilde{h}_\lambda^*\rho_{\mu\lambda}].$$

Then we obtain

$$\begin{aligned}\omega^1|P_\lambda&=\sum_\mu(\tilde{f}_\mu|P_\lambda)\cdot(\tilde{h}_\mu^*\omega|P_{\mu\lambda})+[\theta|P_\lambda, \sum_\mu(\tilde{f}_\mu|P_\lambda)\cdot(\tilde{h}_\lambda^*\rho_{\mu\lambda})]\\&=\tilde{h}_\lambda^*\omega+[\theta|P_\lambda, \sum_\mu(\tilde{f}_\mu|P_\lambda)\cdot(\tilde{h}_\lambda^*\rho_{\mu\lambda}).\end{aligned}$$

Q. E. D.

This proposition generalizes the converse version of the holonomy theorem in [A] in the case of normal bundles of foliations.

Furthermore, we assume that $\omega\in C(P_N)$ is torsion free, then $\omega^1\in C_\theta(P)$ from Lemma 1.2. We get the following generalizations of strong vanishing theorems in [NS] and [P].

THEOREM 3.2. $\lambda(\omega^1)I^r{}_{(\mathfrak{l})}(G)\subset{}^{2r}A\cap A^{2r}(M)$,

and hence $\lambda(\omega^1)I^r{}_{(\mathfrak{l})}(G)=0$ *for* $r>q/2$.

Proof. It is clear that the form θ in Lemma 3.1 satisfies the condition in Lemma 1.3. By Corollary 1.3 and Proposition 3.1, we get

$$\lambda(\omega^1)\varphi|U_\lambda=h_\lambda^*\lambda(\omega)\varphi \quad \text{for} \quad \varphi\in I^r{}_{(\mathfrak{l})}(G).$$

Q. E. D.

Remark 3.1. According to Remark 2.2, we see that Theorem 3.2 is applicable to any Γ-foliation with $\Gamma\subset\Gamma(P_N)$ for a G-structure P_N on N with a torsion free connection. Studying the relation between $I_{GL_q}(G)$ and $I_{(1)}(G)$, we can generalize vanishing theorems for $I(GL_q)$ as follows.

COROLLARY 3.3. *If $I(G)$ is generated by $I_{(1)}(G)$ and odd degree elements of $I(G)$, then*

$$\lambda_P(I^r(G)) = 0 \qquad \text{for} \quad r>q/2 .$$

COROLLARY 3.4. *If $I_{GL_q}(G)$ is generated by $I_{(1)}(G)$ and odd degree elements of $I_{GL_q}(G)$, then*

$$\lambda_B(I^r(GL_q)) = 0 \qquad \text{for} \quad r>q/2 ,$$

where $I_{GL_q}(G) := I(GL_q)|\mathfrak{g}$.

For example, $I_{GL_q}(G)=I_{(1)}(G)$ in the case of affine foliations. $I_{GL_q}(G)$ is generated by $I_{(1)}(G)$ and the first Chern polynomial $c_1|\mathfrak{g}\in I^1{}_{GL_q}(G)$ in the case of projective or conformal foliations (see § 6).

Remark 3.2. The subring $\lambda_B(I(GL_q))\subset H^*(M)$ is generated by the Pontryagin classes of TM/F. Corollary 3.4 was obtained by Nishikawa and Sato [NS] in the case of projective (resp. conformal) foliations provided that $q\geqq 2$ (resp.$q\geqq 3$). Since they used the normal Cartan connections in their paper, the restriction on the codimension was not avoidable. In a recent paper [NT], Nishikawa and Takeuchi generalized the theorem in [NS] to Γ-foliations which relate to flat homogeneous spaces of order two, using the normal Cartan connections.

§ 4. Exotic characteristic classes

In [BH], Bott and Haefliger constructed cochain complexes denoted by WO_q and W_q. We recall their construction [B]. Let $R[c_1, \cdots, c_q]$ be the polynomial ring over R in variables $c_1, \cdots, c_q$ with dimensions $\dim c_j=2j$ for $j=1, 2, \cdots, q$, I_{2q} the ideal generated by monomials whose dimensions are greater than $2q$. Denote the quotient ring $R[c_1, \cdots c_q]/I_{2q}$ by $R_q[c_1, \cdots, c_q]$. Let $E(h_1, \cdots)$ be the exterior algebra over R generated by indicated h's with $\dim h_i=2i-1$.

As a graded algebra,

$$WO_q := E(h_1, h_3, \cdots, h_{2[(q+1)/2]-1})\otimes R_q[c_1, \cdots c_q] .$$

On WO_q, a unique differential $d_W : WO_q \to WO_q$ is defined by requiring

$$d_w c_j=0 , \quad 1\leqq j\leqq q , \quad \text{and} \quad d_w h_i=c_i , \quad i=1, 3, \cdots 2[(q+1)/2]-1 .$$

The cochain complex W_q is defined similarly, that is,

$$W_q := E(h_1, h_2, \cdots, h_q)\otimes R_q[c_1, c_2, \cdots, c_q] .$$

with the differential $d_w : W_q \to W_q$ defined by requiring

$$d_w c_j=0 , \quad 1\leqq j\leqq q , \quad \text{and} \quad d_w h_i=c_i , \quad 1\leqq i\leqq q .$$

Clearly $d_w{}^2=0$. Denote the cohomology ring of the cochain complex WO_q

(resp. W_q) by $H^*(WO_q)$ (resp. $H^*(W_q)$). Let $\mathcal{F}$ be a Γ-foliation, B the normal frame bundle, and $\omega^1 \in C(B)$ be a basic connection [B]. From Fact 3.2, the following homomorphism defined by Bott is well-defined.

DEFINITION. A homomorphism of graded algebras $\lambda_{\mathcal{F}}: WO_q \to A(M)$ is defined by requiring

$$\lambda_{\mathcal{F}}(c_j) := \lambda(\omega^1)c_j, \qquad 1 \leqq j \leqq q,$$

and

$$\lambda_{\mathcal{F}}(h_i) := \lambda(\omega^0, \omega^1)c_i, \qquad i = 1, 3, \cdots, 2[(q+1)/2]-1,$$

where ω^0 is a fixed metric connection on B and c_j in the right hand side is the j-th Chern polynomial defined in the following fact.

Fact 4.1. $I(GL_q) = R[c_1, \cdots, c_q]$, where $c_i \in I^i(GL_q)$ are defined by

$$\sum t^i c_i(X) := \det\left(I_q - \frac{t}{2\pi} X\right) \qquad \text{for} \quad X \in \mathfrak{gl}_q.$$

From Fact 1.1, it is clear that $\lambda_{\mathcal{F}}$ is a cochain homomorphism, that is, $\lambda_{\mathcal{F}} d_w = d\lambda_{\mathcal{F}}$. Bott proved the following:

Fact 4.2. The induced homomorphism $\lambda_{\mathcal{F}}^*: H^*(WO_q) \to H^*(M)$ does not depend on the choice of basic connections and metric connections.

This homomorphism is called the generalized characteristic homomorphism for $\mathcal{F}$. The elements of $\lambda_{\mathcal{F}}^*(H^*(WO_q) - [R_q[c_1, \cdots, c_q]])$ are called exotic characteristic classes of $\mathcal{F}$, where [ ] denotes the cohomology class. J. Vey determined a basis for $H^*(WO_q)$.

Fact 4.3. A basis for exotic classes of $H^*(WO_q)$ is given by the classes of

$$h_I \otimes c_J = h_{i_1} \wedge \cdots \wedge h_{i_l} \otimes c_{j_1} \cdots c_{j_m},$$

where $I = (i_1, \cdots, i_l)$ and $J = (j_1, \cdots, j_m)$ satisfy

$$1 \leqq i_1 < \cdots < i_l \leqq q \qquad (l \geqq 1),$$

$$1 \leqq j_1 \leqq \cdots \leqq j_m \leqq q \qquad \text{with} \quad |J| := j_1 + \cdots + j_m \leqq q, \quad \text{and}$$

1) $i_1 + |J| \geqq q+1$ (condition to be cocycle),

2) $i_1 \leqq j^0$,

where j^0 is the smallest odd integer in J or $j^0 := \infty$.

For the proof, see [H]. The cohomology classes $[h_I \otimes c_J]$ are called Vey-basis for exotic classes.

When there exists a flat connection ω^0 on B, Bott defined the following homomorphism:

DEFINITION. A homomorphism of graded algebras $\lambda_{\mathcal{F},\omega^0}: W_q \to A(M)$ is defined by requiring

$$\lambda_{\mathcal{F},\omega^0}(c_j) := \lambda(\omega^1)c_j, \qquad 1 \leqq j \leqq q,$$

and

$$\lambda_{\mathcal{F},\omega^0}(h_i) := \lambda(\omega^0, \omega^1)c_i, \qquad 1 \leqq i \leqq q.$$

Remark 4.1. This homomorphism is originaly defind only for foliations with trivialized normal bundles [B].

From Fact 1.1, it is clear that $\lambda_{\mathcal{F},\omega^0}$ is a cochain homomorphism.

Fact 4.4. The induced homomorphism $\lambda^*_{\mathcal{F},\omega^0}: H^*(W_q) \to H^*(M)$ does not depend on the choice of basic connections.

Remark 4.2. In contrast to the case of WO_q, $\lambda^*_{\mathcal{F},\omega^0}$ depends also on the $\mathcal{F}$-equivalence class $[TM/F$ with $\omega^0]_{\mathcal{F}}$ which was defined in [A]. The flat connection ω^0 is not reducible to P in general for a G-subbundle P defined in §3.

Following to the method of Vey [H], we get

Fact 4.5. A basis for $H^*(W_q)$ is given by the classes of $h_I \otimes c_J$ as in Fact 4.3, provided that the odd integer restriction on $i_1, \cdots, i_l$ is deleted and $j^0 := j_1$.

§5. Exotic characteristic classes of $(G, \mathfrak{l})$-foliations

In this section, we study exotic characteristic classes, using Theorem 3.2. Let $\mathcal{F}$ be a Γ-foliation, B the normal frame bundle, $\omega^1 \in C(B)$ a basic connection. Let $K^r := \{\varphi \in I^r(GL_q) \mid \lambda(\omega^1)\varphi \subset {}^{2r}A\}$, and $K := \sum_{r \geqq 0} K^r$ is a subring of $I(GL_q)$. There exist integers $s_1, \cdots, s_k$ $(1 \leqq s_1 < \cdots < s_k \leqq q)$ satisfying $I(GL_q) = K[c_{s_1}, \cdots, c_{s_k}]$.

It is clear that

LEMMA 5.1. *If $s_1, \cdots, s_k$ are odd, then*

$$\lambda(\omega^1) I^r(GL_q) \sim 0 \qquad \text{for} \quad r > q/2 .$$

Hereafter in this paper we will deal with only the cocycles $h_I \otimes c_J$ as in Facts 4.3 and 4.5. Fixing a basic connection $\omega^1 \in C(B)$, we use the convention that $\hat{\varphi} := \lambda(\omega^1)\varphi$ for $\varphi \in I(GL_q)$ and $\hat{h}_I := \lambda_{\mathcal{F}} h_I$ (*or* $\lambda_{\mathcal{F},\omega^0} h_I$).

PROPOSITION 5.1. *If $s_1, \cdots, s_k$ are odd, $|J| > q/2$ and $i_1 + |J| \geqq q + s_k$ for $h_I \otimes c_J \in WO_q$, then*

$$\begin{aligned} \hat{h}_I \wedge \hat{c}_J &\sim b_{i_1, J} \hat{h}_{s_k} \wedge \hat{h}_{I_1} \wedge (\hat{c}_{s_k})^{q/s_k} \qquad \text{for} \quad i_1 + |J| = q + s_k, \quad s_k \mid q \quad \text{and} \quad s_k \mid i_1 , \\ &\sim 0 \qquad \text{otherwise}, \end{aligned}$$

where $I_1 := (i_2, \cdots, i_l)$ and $b_{i_1, J}$ is a real number.

Proof. There exist $\varphi^K_{JA} \in K^{|J|-|A|}$ such that

$$c_J = \sum_{0 \leqq |A| \leqq |J|} \varphi^K_{JA} \cdot c_A ,$$

where $A := (a_1, \cdots)$ with $\{a_1, \cdots\} \subset \{s_1, \cdots, s_k\}$ and $a_1 \leqq a_2 \leqq \cdots$. Since $|J| > q/2$ and a_1 is odd, we get

$$\hat{c}_J = \sum_{0 < |A| \leqq |J|} \hat{\varphi}^K_{JA} \wedge \hat{c}_A = \sum (d\hat{h}_{a_1}) \wedge \hat{c}_{A_1} \wedge \hat{\varphi}^K_{JA} = \sum d(\hat{h}_{a_1} \wedge \hat{c}_{A_1} \wedge \hat{\varphi}^K_{JA}) ,$$

where $A_1 := (a_2, \cdots)$. The condition $i_1 + |J| \geqq q + s_k$ and Fact 3.2 imply

$$\hat{h}_I \wedge \hat{c}_J \sim (-1)^{l-1} \sum (d\hat{h}_I) \wedge \hat{h}_{a_1} \wedge \hat{c}_{A_1} \wedge \hat{\varphi}^K_{JA}$$
$$= \sum \hat{h}_{a_1} \wedge \hat{h}_{I_1} \wedge \hat{c}_{i_1} \wedge \hat{c}_{A_1} \wedge \hat{\varphi}^K_{JA}.$$

It follows from the definition of K and Fact 3.2 that

$$\hat{c}_{i_1} \wedge \hat{c}_{A_1} \wedge \hat{\varphi}^K_{JA} \in {}^{i_1+(|A|-a_1)}A \cdot {}^{2(|J|-|A|)}A \subset {}^{i_1+|A|-a_1+2(|J|-|A|)}A.$$

Since $i_1+|J| \geqq q+s_k$ and $s_k \geqq a_1$,

$$i_1+|A|-a_1+2(|J|-|A|) \geqq q+(s_k-a_1)+(|J|-|A|) \geqq q,$$

where the equality holds if and only if $i_1+|J|=q+s_k$, $|A|=|J|$ and $a_1=s_k$. As a_1 is the smallest in A, the condition $s_k|(q-i_1)$ is necessary. If the equality holds, then

$$\hat{h}_I \wedge \hat{c}_J \sim \sum_{|A|=|J|} \hat{h}_{s_k} \wedge \hat{h}_{I_1} \wedge \hat{c}_{i_1} \wedge (\hat{c}_{s_k})^{(q-i_1)/s_k}$$
$$= b_J \hat{h}_{s_k} \wedge \hat{h}_{I_1} \wedge \hat{c}_{i_1} \wedge (\hat{c}_{s_k})^{(q-i_1)/s_k}.$$

Apply the same method as above to the cocycle $h_{s_k} \wedge h_{I_1} \otimes c_{i_1} \cdot (c_{s_k})^{(q-i_1)/s_k}$ instead of $h_I \otimes c_J$. Then we get

$$\hat{h}_{s_k} \wedge \hat{h}_{I_1} \wedge \hat{c}_{i_1} \wedge (\hat{c}_{s_k})^{(q-i_1)/s_k} \sim b_{i_1} \hat{h}_{s_k} \wedge \hat{h}_{I_1} \wedge (\hat{c}_{s_k})^{q/s_k} \quad \text{for} \quad s_k|q,$$
$$\sim 0 \quad \text{otherwise}.$$

Q. E. D.

Remark 5.1. If $j^0 \neq \infty$, that is, J contains an odd integer, then the condition $|J|>q/2$ is automatically satisfied.

In the case of a foliation with a flat connection, the assumption that $s_1, \cdots, s_k$ are odd and $|J|>q/2$ is avoidable, that is,

PROPOSITION 5.2. *Let $\mathcal{F}$ be a foliation with a flat connection, s_k as above. If $i_1+|J| \geqq q+s_k$ for $h_I \otimes c_J \in W_q$, then*

$$\hat{h}_I \wedge \hat{c}_J \sim b_{i_1,J} \hat{h}_{s_k} \wedge \hat{h}_{I_1} \wedge (\hat{c}_{s_k})^{q/s_k} \quad \textit{for} \quad i_1+|J|=q+s_k, \ s_k|q \ \textit{and} \ s_k|i_1,$$
$$\sim 0 \quad \textit{otherwise}.$$

Proof. Since $i_1+|J| \geqq q+1$ and $i_1 \leqq j_1$, $2|J| \geqq i_1+|J| \geqq q+1$, that is, the assumption $|J|>q/2$ in Proposition 5.1 is satisfied. The same method as in the proof of Proposition 5.1 completes the proof. Q. E. D.

Let $\mathcal{F}$ be a $(G, \mathfrak{l})$-foliation, ω^1 as in Theorem 3.2. Let $I_{(G,\mathfrak{l})} := \{\varphi \in I(GL_q) \mid \varphi|\mathfrak{g} = \psi|\mathfrak{g}$ for some $\psi \in I_{(\mathfrak{l})}(G)\}$, then $I_{(G,\mathfrak{l})} \subset K$ for this basic connection by Theorem 3.2. If $I(GL_q)$ is generated by $I_{(G,\mathfrak{l})}$ and $c_{s_1}, \cdots, c_{s_k}$, then $I(GL_q) = K[c_{s_1}, \cdots, c_{s_k}]$. Thus Propositions 5.1 and 5.2 are applicable to $(G, \mathfrak{l})$-foliations.

§ 6. Exotic characteristic classes of projective and conformal foliations

Let $\mathcal{F}$ be a projective (resp. conformal) foliation of codimension q on M, $G = GL_q$ (resp. $CO(q)$) and $L=SL(q+1, R)$ (resp. $O(q+1, 1)$) as in Example 2.4′ (resp.

2.5′). The subring $\{\varphi\in I(GL_q)\,|\,\varphi|\mathfrak{g}=\psi|\mathfrak{g}$ for some $\psi\in I(L)\}$ in $I(GL_q)$ is denoted by $I_{(G,L)}$. Note that $I_{(G,L)}\subset I_{(G,\mathfrak{l})}$. We now study the relation between $I(GL_q)$ and $I_{(G,L)}$. Let $c_k^L\in I^k(GL_q)$ be defined by

$$c_k^L := \sum_{j=0}^{k-1} B_{kj}(c_1)^{k-j}c_j + c_k\,,$$

where $B_{kj} := \left(-\frac{1}{q+1}\right)^{k-j}\binom{q+1-j}{k-j}$ $\left(\text{resp. } \left(-\frac{1}{q}\right)^{k-j}\left\{\binom{q+1-j}{k-j}-\binom{q+1-j}{k-j-1}\right\}\right)$ and $c_0 := 1$.

LEMMA 6.1. $$c_k^L \in I^k{}_{(G,L)}\,.$$

Proof. For $X\in\mathfrak{g}$, let Y be the corresponding element in l, see Example 2.4′ (resp. 2.5′). A direct calculation shows

$$\det\left(I-\frac{t}{2\pi}Y\right) = \sum t^k c_k^L(X)\,,$$

where $I=I_{q+1}$ (resp. I_{q+2}). Q. E. D.

Obviously $c_0^L = c_0 = 1$ and $c_1^L = 0$.

LEMMA 6.2. *In the conformal case, we have*

$$c_k^L|\mathfrak{g} = 0 \qquad \textit{for} \quad k: \textit{odd}\,.$$

Rearranging the expression in the definition, we obtain

LEMMA 6.3. $$c_i = \sum_{k=0}^{i-1} B^{ik}(c_1)^{i-k}c_k^L + c_i^L\,,$$

where $$B^{ik} := \left(\frac{1}{q+1}\right)^{i-k}\binom{q+1-k}{i-k}\left(\textit{resp. } \left(\frac{1}{q}\right)^{i-k}\sum_{s=k}^{i}\binom{q+2-k}{s-k}(-2)^{i-s}\right).$$

Thus $I(GL_q)$ is generated by $I_{(G,L)}$ and $c_1\in I^1(GL_q)$. From now on, we will consider only the cocycles described in Facts 4.3 and 4.5. We may assume, as is allowable from Facts 4.2 and 4.4, ω^1 is the basic connection defined in Proposition 3.1. We will use the same notations as in § 5.

First we study the homomorphism $\lambda_{\mathcal{F}}: WO_q \to A(M)$. We may assume that ω^0 is a metric connection which is reducible to P. Since $I_{(G,L)}\subset I_{(G,\mathfrak{l})}\subset K$ and $I(GL_q)=K[c_1]$, Proposition 5.1 implies

PROPOSITION 6.1 *If* $|J|>q/2$ *for* $h_I\otimes c_J\in WO_q$, *then*

$$\begin{aligned} \hat{h}_I\wedge\hat{c}_J &\sim b_{i_1,J}\hat{h}_1\wedge\hat{h}_{I_1}\wedge(\hat{c}_1)^q \qquad \textit{for} \quad i_1+|J|=q+1\,,\\ &\sim 0 \qquad\qquad\qquad\qquad\qquad \textit{for} \quad i_1+|J|\geqq q+2\,. \end{aligned}$$

Remark 6.1. We find that $b_{i_1,J}=B^{i_1,0}\prod_{s=1}^{m}B^{j_s,0}$, by chasing the proof of Proposition 5.1 with Lemma 6.3.

Especially when $\mathcal{F}$ is a conformal foliation, we obtain the following theorem in which the condition $|J|>q/2$ is avoidable by Lemma 6.2.

THEOREM 6.2. *For* $h_I\otimes c_J\in WO_q$,

$$\hat{h}_I\wedge\hat{c}_J\sim b_{i_1,J}\hat{h}_1\wedge(\hat{c}_1)^q \quad \textit{for} \quad I=(i_1) \quad \textit{and} \quad i_1+|J|=q+1\,,$$
$$\sim 0 \quad \textit{otherwise}.$$

Proof. Lemma 6.3 and Proposition 1.1 imply

$$\lambda(\omega^0,\omega^1)c_i\sim\lambda(\omega^0,\omega^1)c_i^L+(\lambda(\omega^0,\omega^1)c_1)\wedge\lambda(\omega^1)\varphi_i$$

for certain $\varphi_i\in I^{i-1}(GL_q)$, that is, $\hat{h}_i\sim\hat{h}_i^L+\hat{h}_1\wedge\hat{\varphi}_i$, where $h_i^L:=\lambda_{\mathcal{F}}c_i^L$. It follows from Fact 3.2 that

$$\hat{h}_I\wedge\hat{c}_J\sim\hat{h}_I^L\wedge\hat{c}_J+\hat{h}_1\wedge\hat{h}_{I_1}^L\wedge\hat{\varphi}_{i_1}\wedge\hat{c}_J\,.$$

Since ω^0 is reducible to P, Lemma 6.2 imply $\hat{h}_I^L\wedge\hat{c}_J=0$, and then $\hat{h}_I\wedge\hat{c}_J\sim 0$ for $I_1\neq\phi$ or $i_1+|J|\geqq q+2$. In the same way as the proof of Proposition 5.1, we get $\hat{\varphi}_{i_1}\wedge\hat{c}_J=b_{i_1,J}(\hat{c}_1)^q$ for $i_1+|J|=q+1$. Q.E.D.

The classes $h_I\otimes c_J\in WO_q$ with $i_1+|J|\geqq q+2$ are said to be rigid [H]. The class of $h_1\otimes(c_1)^q$ is called the Godbillon-Vey invariant of foliations of codimension q.

Next we study the homomorphism $\lambda^*_{\mathcal{F},\omega^0}: W_q\to A(M)$. Ler $\mathcal{F}$ be a projective or conformal foliation with a flat connection ω^0 on the normal frame bundle B. Proposition 5.2 implies

PROPOSITION 6.3. *For* $h_I\otimes c_J\in W_q$,

$$\hat{h}_I\wedge\hat{c}_J\sim b_{i_1,J}\hat{h}_1\wedge\hat{h}_{I_1}\wedge(\hat{c}_1)^q \quad \textit{for} \quad i_1+|J|=q+1\,,$$
$$\sim 0 \quad \textit{for} \quad i_1+|J|\geqq q+2\,.$$

Hereafter we assume that $\mathcal{F}$ is a conformal foliation.

THEOREM 6.4. If the flat connection ω^0 is reducible to P and I_1 contains an odd integer, then $\tilde{h}_I\wedge\hat{c}_J\sim 0$.

The proof is almost the same as that of Theorem 6.2.

COROLLARY 6.5. *If the flat connection ω^0 is given by a trivialization of B and I_1 contains an odd integer, then $\hat{h}_I\wedge\hat{c}_J\sim 0$.*

For the proof of this corollary, we need the following lemma which is derived from Fact 1.3.

LEMMA 6.4. *If $\{\omega_s^0\,|\,0\leqq s\leqq 1\}$ is a smooth family of flat connections on B, then*

$$\lambda(\omega_1^0,\omega^1)c_i\sim\lambda(\omega_0^0,\omega^1)c_i \quad \textit{for} \quad i\geqq 2\,.$$

Remark 6.2. This lemma also shows exotic characteristic classes of $h_I\otimes c_J$ depends on the $\mathcal{F}$-homotopy class [A] of [TM/F with $\omega^0]_{\mathcal{F}}$ if $i_1>1$.

Proof of Corollary 6.5. Since $O(q)\subset CO(q)$, any trivialization of B is homotopic to one of P. Then there is a smooth family ω_s^0 of flat connections such that ω_0^0 is reducible to P. Lemma 6.4 completes the proof. Q. E. D.

Remark 6.3. In the case of projective or conformal foliations with trivialized normal bundles, Proposition 6.3 and Corollary 6.5 were proved by Morita [M] and Yamato [Y], where they used the normal Cartan connections as in [NS].

§ 7. Appendix

In this section, we apply the method of integration along fibre to the case of secondary invariants, that is, exotic characteristic classes [B] and Chern-Simons classes [CS]. By this method, certain formulae for secondary invariants (derivation formulae, especially) can be proved in a simple manner.

Let G be a Lie group, $\mathfrak{g}$ and $I^r(G)$ as in § 1. Let P be a smooth manifold. We denote the space of $\mathfrak{g}$-valued (resp. real-valued) r-forms on P by $A^r(P, \mathfrak{g})$ (resp. $A^r(P)$). Denote $R\times P$ by $\tilde{P}$, where R is the real numbers field. Let π : $\tilde{P}\rightarrow P$ be the canonical projection, $j_t: P\rightarrow\tilde{P}$ the inclusion map defined by $j_t(p):=(t, p)\in\tilde{P}$. For the interval $I:=[0, 1]\subset R$, a linear map $\pi^I: A^r(\tilde{P})\rightarrow A^{r-1}(P)$ is defined by

$$\pi^I(\alpha):=\int_0^1(\iota_T\alpha)dt \qquad \text{for} \quad \alpha\in A^r(\tilde{P}),$$

where T is the vector field on $\tilde{P}$ whicn is the canonical extension of $\partial/\partial t$ on R. We can easily get the following:

LEMMA 7.1. $$d\pi^I+\pi^I d = j_1^*-j_0^*.$$

For $\omega\in A^1(P, \mathfrak{g})$, define $\Omega=\Omega(\omega)\in A^2(P, \mathfrak{g})$ by $\Omega(\omega):=d\omega+1/2[\omega, \omega]\in A^2(P, \mathfrak{g})$. When ω is a connection form on a principal G-bundle P, Ω is the curvature form of ω. Let $\varphi\in I^r(G)$. From $Ad(G)$-invariancy of φ, we obtain

LEMMA 7.2. $$d\varphi(\Omega)=0.$$

For a smooth family of $\mathfrak{g}$-valued 1-forms $\omega^R=\{\omega^t\in A^1(P, \mathfrak{g})\,|\,t\in R\}$, $\tilde{\omega}^R\in A^1(\tilde{P}, \mathfrak{g})$ is defined by $(\tilde{\omega}^R)_{(t,p)}:=\pi^*(\omega^t)_p$ for $(t, p)\in\tilde{P}$.

DEFINITION. A linear map

$$\mu(\omega^R): I^r(G)\longrightarrow A^{2r-1}(P)$$

is defined by $\mu(\omega^R)\varphi:=\pi^I(\varphi(\tilde{\Omega}^R))$ for $\varphi\in I^r(G)$, where $\tilde{\Omega}^R:=\Omega(\tilde{\omega}^R)$.

LEMMA 7.3. $$\mu(\omega^R)\varphi=r\int_0^1\varphi\Big(\frac{\partial}{\partial t}\omega^t,\ \Omega^t\Big)dt,$$

where $\Omega^t:=\Omega(\omega^t)$.

Note that $\mu(\omega^R)\varphi$ can be regarded as a form on M when each ω^t is a connection form on a principal G-bundle P over M. From Lemma 7.1 and 7.2, we get

PROPOSITION 7.1. $$d(\mu(\omega^R)\varphi)=\varphi(\Omega^1)-\varphi(\Omega^0).$$

Especially when $\omega^t=t\omega^1+(1-t)\omega^0$, we denote $\mu(\omega^R)$ by $\lambda(\omega^0, \omega^1)$, and $T(\omega):=\lambda(0, \omega)$.

COROLLARY 7.2. $$d(\lambda(\omega^0, \omega^1)\varphi)=\varphi(\Omega^1)-\varphi(\Omega^0).$$

COROLLARY 7.3. $$d(T(\omega)\varphi)=\varphi(\Omega).$$

If each ω^t is a connection form on a principal G-bundle P over M, Proposition 7.1 shows that the characteristic homomorphism $I^r(G)\ni\varphi\rightarrow[\varphi(\Omega)]\in H^{2r}(M)$ is independent of the choice of connections (cf. [C]).

PROPOSITION 7.4 (Product formula). *For* $\varphi, \psi\in I(G)$,

$$\mu(\omega^R)(\varphi\cdot\psi)=(\mu(\omega^R)\varphi)\wedge\psi(\Omega^1)+\varphi(\Omega^0)\wedge(\lambda(\omega^0, \omega^1)\psi)$$
$$-d\pi^I(\varphi(\tilde{\Omega}^R)\wedge(\lambda(\pi^*\omega^1, \tilde{\omega}^R)\psi)).$$

Proof. It follows from Lemma 7.1 and Corollary 7.2 that

$$\begin{aligned}
&d\pi^I(\varphi(\tilde{\Omega}^R)\wedge(\lambda(\pi^*\omega^1, \tilde{\omega}^R)\psi))\\
&=-\pi^I(\varphi(\tilde{\Omega}^R)\wedge d(\lambda(\pi^*\omega^1, \tilde{\omega}^R)\psi))\\
&\quad+j_1^*(\varphi(\tilde{\Omega}^R)\wedge\lambda(\pi^*\omega^1, \tilde{\omega}^R)\psi))-j_0^*(\varphi(\tilde{\Omega}^R)\wedge(\lambda(\pi^*\omega^1, \tilde{\omega}^R)\psi))\\
&=-\pi^I(\varphi(\tilde{\Omega}^R)\wedge(\psi(\tilde{\Omega}^R)-\psi(\pi^*\Omega^1)))-\varphi(\Omega^0)\wedge\lambda(\omega^1, \omega^0)\psi\\
&=-\mu(\omega^R)(\varphi\cdot\psi)+\mu(\omega^R)\varphi\wedge\psi(\Omega^1)+\varphi(\Omega^0)\wedge\lambda(\omega^0, \omega^1)\psi.
\end{aligned}$$
Q. E. D.

COROLLARY 7.5.

$$\lambda(\omega^0, \omega^1)(\varphi\cdot\psi)=(\lambda(\omega^0, \omega^1)\varphi)\wedge\psi(\Omega^1)+\varphi(\Omega^0)\wedge(\lambda(\omega^0, \omega^1)\psi)$$
$$-d\pi^I(\varphi(\tilde{\Omega}^R)\wedge(\lambda(\pi^*\omega^1, \tilde{\omega}^R)\psi)).$$

COROLLARY 7.6 (Chern and Simons [CS]).

$$T(\omega)(\varphi\cdot\psi)=(T(\omega)\varphi)\wedge\psi(\Omega)+exact.$$

We need a lemma to prove derivation formulae. Let $\varphi\in I^r(G)$, $\{\omega_s\in A^1(P, \mathfrak{g})\mid s\in R\}$ a smooth family.

LEMMA 7.4. $$\frac{1}{r}\frac{\partial}{\partial s}\varphi(\Omega_s)=d\varphi\Big(\frac{\partial}{\partial s}\omega_s, \Omega_s\Big).$$

Proof. For a family $\omega(s)^R:=\{\omega(s)^t:=\omega_{st}\mid s\in R\}$ for fixed $s\in R$, Proposition 7.1 implies

$$d(\mu(\omega(s)^R)\varphi)=\varphi(\Omega(s)^1)-\varphi(\Omega(s)^0)=\varphi(\Omega_s)-\varphi(\Omega_0),$$

where $\Omega(s)^t := \Omega(\omega(s)^t)$. On the other hand, we have

$$\mu(\omega(s)^R)\varphi = r\int_0^1 \varphi\Big(\frac{\partial}{\partial t}\omega(s)^t,\ \Omega(s)^t\Big)dt = r\int_0^s \varphi\Big(\frac{\partial}{\partial t}\omega_t,\ \Omega_t\Big)dt\,,$$

so that, the proof is completed. Q. E. D.

Let $\{\omega_s^t \in A^1(P, \mathfrak{g}) \mid t \in R,\ s \in R\}$ be a smooth two-parameters family on $R\times R \times P = R\times \tilde{P}$. Denote the smooth one-parameter family $\{\omega_s^t \mid t\in R\}$ on $\tilde{P}$ by ω_s^R for each $s\in R$.

PROPOSITION 7.7 (Generalized derivation formula). *For* $\varphi \in I^r(G)$,

$$\frac{1}{r}\frac{\partial}{\partial s}(\mu(\omega_s^R)\varphi) = -d\pi^I\Big(\varphi\Big(\frac{\partial}{\partial s}\tilde{\omega}_s^R,\ \tilde{\Omega}_s^R\Big)\Big) + \varphi\Big(\frac{\partial}{\partial s}\omega_s^1,\ \Omega_s^1\Big) - \varphi\Big(\frac{\partial}{\partial s}\omega_s^0,\ \Omega_s^0\Big),$$

where $\tilde{\Omega}_s^R := \Omega(\tilde{\omega}_s^R)$ *and* $\Omega_s^t := \Omega(\omega_s^t)$.

Proof. Applying Lemma 7.4 to the one-parameter family $\{\tilde{\omega}_s^R \in A^1(\tilde{P}, \mathfrak{g}) \mid s\in R\}$, we get

$$\frac{1}{r}\frac{\partial}{\partial s}\pi^I(\varphi(\tilde{\Omega}_s^R)) = \pi^I\Big(d\varphi\Big(\frac{\partial}{\partial s}\tilde{\omega}_s^R,\ \tilde{\Omega}_s^R\Big)\Big)$$

$$= -d\pi^I\Big(\varphi\Big(\frac{\partial}{\partial s}\tilde{\omega}_s^R,\ \tilde{\Omega}_s^R\Big)\Big) + j_1^*\varphi\Big(\frac{\partial}{\partial s}\omega_s^R,\ \Omega_s^R\Big) - j_0^*\varphi\Big(\frac{\partial}{\partial s}\omega_s^R,\ \Omega_s^R\Big).$$

Q. E. D.

For $\omega^0 \in A^1(P, \mathfrak{g})$ and a smooth family $\{\omega_s^1 \in A^1(P, \mathfrak{g}) \mid s\in R\}$, we get the following:

COROLLARY 7.8 (Heitsch [H]). *For* $\varphi\in I^r(G)$,

$$\frac{1}{r}\frac{\partial}{\partial s}(\lambda(\omega^0,\ \omega_s^1)\varphi) = -d\pi^I\Big(\varphi\Big(\frac{\partial}{\partial s}\tilde{\omega}_s^R,\ \tilde{\Omega}_s^R\Big)\Big) + \varphi\Big(\frac{\partial}{\partial s}\omega_s^1,\ \Omega_s^1\Big).$$

Note that each term in this corollary can be regarded as a form on M if ω^0 and ω_s^1 are connection forms on a principal G-bundle P over M. Putting $\omega^0=0$ in this corollary, we obtain

COROLLARY 7.9. (Chern and Simons [CS, KO]).

$$\frac{1}{r}\frac{\partial}{\partial s}(T(\omega_s^1)\varphi) = -d\pi^I\Big(\varphi\Big(\frac{\partial}{\partial s}\tilde{\omega}_s^R,\ \tilde{\Omega}_s^R\Big)\Big) + \varphi\Big(\frac{\partial}{\partial s}\omega_s^1,\ \Omega_s^1\Big).$$

REFERENCES

[A] N. ABE, On foliations and exotic characteristic classes, Kōdai Math. Sem. Rep. **28** (1977) 324-341.

[B] R. BOTT, Lectures on characteristic classes and foliations, Lecture Notes in Math. 279, Springer-Verlag. (1972) 1-80.

[BH] R. BOTT AND A. HAEFLIGER, On characteristic classes of Γ-foliations, Bull.

Amer. Math. Soc. **78** (1972), 1039-1044.

[C] S.S. CHERN, Complex manifolds without potential theory, van Nostrand (1967).

[CS] S.S. CHERN AND J. SIMONS, Characteristic forms and geometric invariants, Ann. of Math. **99** (1974) 48-69.

[H] J.L. HEITSCH, Deformations of secondary characteristic classes, Topology **12** (1973) 381-388.

[KO] S. KOBAYASHI AND T. OCHIAI, G-structures of order two and transgression operators, J. Diff. Geo. **6** (1971) 213-230.

[M] S. MORITA, Secondary characteristic classes for projective and conformal foliations, preprint.

[NS] S. NISHIKAWA AND H. SATO, On characteristic classes of Riemannian, conformal and projective foliations, J. Math. Soc. Japan **28** (1976) 223-241.

[NT] S. NISHIKAWA AND M. TAKEUCHI, Γ-foliations and semisimple flat homogeneous spaces, to appear.

[P] J. PASTERNACK, Foliations and compact Lie group actions, Comment. Math. Helv. **46** (1971) 467-477.

[R] B.L. REINHART, Foliated manifolds with boudle-like metrics, Ann. of Math. **69** (1959) 119-132.

[Y] K. YAMATO, On exotic characteristic classes of conformal and projective foliations, preprint.

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY