EXOTIC CHARACTERISTIC CLASSES OF CERTAIN $\Gamma$-FOLIATIONS

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§ 0. Introduction

In this paper we study characteristic classes and exotic characteristic classes [B] of foliations. We deal with a certain sort of $\Gamma$-foliations ($\langle G, \iota \rangle$-foliations) which is a generalization of Riemannian, projective and conformal foliations [NS, P]. The main purpose of this paper is to prove vanishing theorems for certain exotic characteristic classes of such $\Gamma$-foliations. As a step toward this, we obtain results which are relevant to strong vanishing theorems for characteristic classes of the $\Gamma$-foliations. These generalize the result of Nishikawa and Sato [NS]. In order to obtain these results, we use neither normal Cartan connections nor classifying spaces, but a product formula for secondary invariants [CS] and a technique used by Kobayashi and Ochiai in [KO].

Throughout this paper, all manifolds and mappings are assumed to be smooth ($C^\infty$). In § 1, Chern-Weil theory of characteristic classes are reviewed, and the technique used in [KO] is slightly improved so that it may be applied to the case of foliations. In § 2, certain automorphisms of $G$-structures are specialized to $\iota$-automorphisms, which are generalized notions of affine, projective and conformal transformations, and then $(G, \iota)$-foliations are defined. In § 3, so-called strong vanishing theorems for characteristic classes of $(G, \iota)$-foliations are proved, where the results prepared in § 1 are applied. In § 4, we review some notions about exotic characteristic classes [B, H] such as cochain complexes $WO_q$ and $W_q$, generalized characteristic homomorphisms for foliations and Vey-basis. In § 5, the vanishing theorems for certain exotic characteristic classes of $(G, \iota)$-foliations are proved. In § 6, more detailed results are obtained in the case of projective and conformal foliations. Especially we find that all of the rigid exotic characteristic classes of conformal foliations vanish and the rest of the exotic characteristic classes coincide with the Godbillon-Vey invariant up to scalar multiples. In § 7, we prove the product formula and a derivative formula [H] for secondary invariants in a simple and unified manner.

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§ 1. Characteristic homomorphism and \((I, \theta)\)-equivalence

Let \(G\) be a Lie group, \(\mathfrak{g}\) its Lie algebra and \(I^r(G)\) the space of all \(\text{Ad}(G)\)-invariant symmetric multilinear functions on \(\mathfrak{g} \times \cdots \times \mathfrak{g}\) \((r\ times)\). We will use the convention that if \(\varphi \in I^r(G)\) contains less than \(r\) arguments, the last one is repeated a number of times to make \(\varphi\) a function of \(r\) arguments. Let \(M\) be a manifold and \(P\) a principal \(G\)-bundle over \(M\). The space of all \(\mathfrak{g}\)-valued \(\text{resp. real valued}\) \(s\)-forms on \(P\) \(\text{resp. } M\) is denoted by \(A^s(P, \mathfrak{g})\) \(\text{resp. } A^s(M)\).

Let \(\omega^0, \omega^1 \in A^r(P, \mathfrak{g})\) be connection forms.

**DEFINITION.** Homomorphisms of modules

\[ \lambda(\omega^0) : I^r(G) \rightarrow A^{2r}(M) \]

and

\[ \lambda(\omega^0, \omega^1) : I^r(G) \rightarrow A^{2r-1}(M) \]

are defined by

\[ \lambda(\omega^0) \varphi := \varphi(\Omega^1) \]

and

\[ \lambda(\omega^0, \omega^1) \varphi := r \int_0^1 \varphi(\omega^1 - \omega^0, \Omega^1) dt \quad \text{for } \varphi \in I^r(G), \]

where \(\Omega^1\) is the curvature form of the connection \(\omega^1 = t \omega^0 + (1-t) \omega^0\).

We obtain the following (see § 7):

**Fact 1.1.**

\[ d(\lambda(\omega^0) \varphi) = 0 \]

and

\[ d(\lambda(\omega^0, \omega^1) \varphi) = \lambda(\omega^0) \varphi - \lambda(\omega^0) \varphi. \]

In other words, the closed form \(\lambda(\omega^0) \varphi\) represents a de Rham cohomology class \([\lambda(\omega^0)] \in H^{2r}(M)\), and the induced homomorphism

\[ \lambda(\omega^0)^* : I^r(G) \rightarrow H^{2r}(M) \]

does not depend on the choice of connections on \(P\). We call this homomorphism the characteristic homomorphism of \(P\) and denote by \(\lambda_p\).

In § 7, we prove the following product formula.

**Proposition 1.1.** For \(\varphi, \psi \in I(G)\),

\[ \lambda(\omega^0, \omega^1)(\varphi \cdot \psi) \sim \lambda(\omega^0, \omega^0) \varphi \wedge \lambda(\omega^0) \psi + \lambda(\omega^0)^* \psi \lambda(\omega^0) \varphi, \]

where \(\sim\) means “cohomologous to”.

Moreover we have the derivative formula:

**Fact 1.2 (Heitsch [H]).** For \(\varphi \in I^r(G)\), and a smooth family of connections \(\omega_s \in \mathcal{C}(P)\) \((s \in R)\),

\[
\frac{1}{r} \frac{\partial}{\partial s} \lambda(\omega^0, \omega^1) \varphi = \varphi \left( \frac{\partial}{\partial s} \omega^1, \Omega^1 \right) + (r-1) \int_0^1 t \varphi \left( \frac{\partial}{\partial s} \omega^1, \omega^1 - \omega^0, \Omega^1 \right) dt,
\]
where $\Omega^t_2$ is the curvature form of $\omega^t := t\omega^1 + (1-t)\omega^0$.

Integrating both sides of this formula, we get

Fact 1.3. For $\varphi \in \Gamma^r(G)$,

$$\lambda(\omega^a, \omega^t)\varphi - \lambda(\omega^a, \omega^0)\varphi \sim r \int_0^1 \varphi \left( \frac{\partial}{\partial s} \omega^t_s, \Omega^t_2 \right) ds.$$ 

We shall give simple proofs of Facts 1.1 and 1.2 in §7.

Let $V$ be a vector space and $GL(V)$ the general linear group. Hereafter we assume that $G \subset GL(V)$. Let $I$ be a Lie algebra which contains the Lie algebra $g + V$ of the semidirect product $G \cdot V$ as a Lie subalgebra. Denote the space of all $I$-valued (resp. $V$-valued) $s$-forms on $P$ by $A^s(P, I)$ (resp. $A^s(P, V)$). Let $\theta \in A^s(P, V)$, $C_{\theta}(P) := \{ \omega \in C(P) | d\theta + [\omega, \theta] = 0 \}$, where $C(P)$ denotes the set of all $G$-connections on $P$. By the definition and the Jacobi identity in $g + V$, we get

**Lemma 1.1.** $[\theta, \Omega] = 0$ for $\omega \in C_{\theta}(P)$.

**Definition.** We say that $\omega^0$, $\omega^1 \in C(P)$ are $(I, \theta)$-equivalent if there exists $\rho \in A^r(P, I)$ such that $\omega^1 - \omega^0 = [\theta, \rho]$.

From the Jacobi identity in $I$ and the fact that $[V, V] = 0$, we see the following lemma is true.

**Lemma 1.2.** If $\omega^0 \in C(P)$ and $\omega^1 \in C_{\theta}(P)$ are $(I, \theta)$-equivalent, then $\omega^0 \in C_{\theta}(P)$.

We denote the set $\{ X \in I | [X, Y] \in g + V \}$ for any $Y \in V$ by $I$. It is obvious that $I$ is an $ad(V)$-invariant subspace of $I$ and contains $g + V$. From the definition of $I$, we get

**Lemma 1.3.** If $\theta_\rho : T_y P \to V$ is surjective for each $\rho \in P$, $[\theta, \rho] = \omega^r - \omega^0$ for $\omega^r$, $\omega^0 \in C(P)$ and $\rho \in A^r(P, I)$, then $\rho \in A^r(P, I)$.

Hereafter we assume that $\theta \in A^r(P, V)$ satisfies the condition described in Lemma 1.3. Note that the canonical form of the tangent frame bundle of a manifold will do. Let $S^r(I^*_*)$ be the space of all multilinear functions on $I^* \times \cdots \times I^*$ (r times). For $X \in V$, let the linear map $ad(X)^* : S^r(I^*_*) \to S^r(I^*_*)$ be defined by, for $\phi \in S^r(I^*_*)$, $X_i \in I$,

$$(ad(X)^* \phi)(X_1, \ldots, X_r) := \sum_{i=1}^r \phi(X_{-i}, \ldots, [X, X_i], \ldots, X_r).$$

Set $S^r(I^*_*)^* := \{ \phi \in S^r(I^*_*) | ad(V)^* \phi = 0 \}$, $I^r_{\theta}(G) := (S^r(I^*_*)^* | g) \cap I(G)$.

**Proposition 1.2.** If $\omega^a$, $\omega^1 \in C_{\theta}(P)$ are $(I, \theta)$-equivalent, then

$$\lambda(\omega^a, \omega^1) I^r_{\theta}(G) = 0 \quad \text{for} \quad r \geq 1.$$

**Proof.** Let $\rho \in A^r(P, I)$ satisfy $\omega^1 - \omega^0 = [\theta, \rho]$. Since $\omega^i = t\omega^1 + (1-t)\omega^0 = \omega^0 + [\theta, t\rho]$, $\omega^0$ and $\omega^i$ are $(I, \theta)$-equivalent. From Lemmas 1.1 and 1.2, we get $[\theta, \Omega^t_2] = 0$. For $\phi \in S^r(I^*_*)^*$, $X \in V$ and $X_i \in I$, we see that
\[ \sum_{i=1}^{r} \phi(X_i, \ldots, [X, X_i], \ldots, X_r) = 0. \]

Hence \( \psi([\theta, \rho], \Omega^1) + (r-1)\phi(\rho, [\theta, \Omega^1], \Omega^1) = 0 \), so that, we get

\[ \phi(\omega^1 - \omega^0, \Omega^1) = \psi([\theta, \rho], \Omega^1) = 0. \quad Q. E. D. \]

This proposition and Fact 1.1 imply

**Corollary 1.3.** If \( \omega^0, \omega^1 \in \mathfrak{C}_\theta(P) \) are \((1, \theta)\)-equivalent, then

\[ \lambda(\omega^0) \varphi = \lambda(\omega^1) \varphi \quad \text{for} \quad \varphi \in I_{(\theta)}(G). \]

**Remark 1.1.** Let \( S^r(\Omega^*)_{L} := \{ \phi \in S^r(\Omega^*)|_{L'} \phi = 0 \} \), where \( L \) is the Lie algebra of \( L \). Denote the image of the restriction map \( I(L) \to I(G) \) by \( I_{L}(G) \). Since \( I_{L}(G) \subseteq I_{(\theta)}(G) \), our results in this section generalize the corresponding results in [KO].

**Remark 1.2.** Let \( L \) be a Lie group which contains \( G \) as a Lie subgroup compatible with the inclusion \( \mathfrak{g} + V \subseteq \mathfrak{l} \), where \( \mathfrak{l} \) is the Lie algebra of \( L \). Denote the image of the restriction map \( I(L) \to I(G) \) by \( I_{L}(G) \). Since \( I_{L}(G) \subseteq I_{(\theta)}(G) \), our results in this section generalize the corresponding results in [KO].

**Remark 1.3.** For a \( G \)-bundle \( P \) with \( \theta \in A^1(P, V) \) which satisfy the condition in Lemma 1.3, we can find a Lie algebra \( \mathfrak{l} \) such that any \( \omega^0, \omega^1 \in \mathfrak{C}_\theta(P) \) are \((1, \theta)\)-equivalent. Let \( \mathfrak{g}^(p) \) be the \( p \)-th prolongation of \( \mathfrak{g} \), i.e., the vector space \( \{ t \in S^p+(V^*) \otimes V \} \) the linear map \( V \ni X \to t(X, X_1, \ldots, X_p) \in V \) belongs to \( \mathfrak{g} \subseteq \mathfrak{g}(V) \) \( = V^* \otimes V \) for any \( X_1, \ldots, X_p \in V \) where \( S^r(V^*) \otimes V \) is the space of all symmetric multilinear functions on \( V \times \cdots \times V \) (\( r \) times) with values in \( V \). Note that \( \mathfrak{g} \) is a Lie algebra. By the definition of the bracket product, it is clear that \( \mathfrak{g} \) becomes a Lie algebra.

If is well-known that \( \mathfrak{g} := \sum_{p=1}^{\infty} \mathfrak{g}^{(p)} \) becomes a Lie algebra. By the definition of the bracket product, it is clear that \( \mathfrak{g} \) contains the Lie algebra \( \mathfrak{g} + V \) of the semidirect product \( G \cdot V \) as a Lie subalgebra. Let \( \omega^0, \omega^1 \in \mathfrak{C}_\theta(P) \), then \( [\omega^1 - \omega^0, \theta] = 0 \). Fix a basis of \( V \), and denote the components of \( \theta \) (resp. \( \omega^1 - \omega^0 \)) by \( \theta^i \) (resp. \( \tau^i_j \)). Then \( \tau^i_j \land \theta = 0 \) for each \( i \). Here we employ the Einstein summation convention. Since the form \( \theta \) satisfies the condition in Lemma 1.3, \( \theta^i \) are linearly independent at each point. Then there exist functions \( \rho^i_\theta \) such that \( \tau^i_j = \ldots \)
Clearly \( \rho_j \) define an element \( \rho \) of \( g \) satisfying \( \omega_1 - \omega_0 = [\rho, \theta] \). Thus any \( \omega_0, \omega_1 \in C_0(P) \) are \((g^{(0)}, \theta)\)-equivalent.

Note that \( \mathfrak{l} \subset (\mathfrak{g}^{(-1)} + \mathfrak{g}^{(+0)} + \mathfrak{g}^{(+0)}) \cap \mathfrak{l} \) if \( \mathfrak{l} \) is a Lie subalgebra of \( \mathfrak{g}^{(0)} \) and contains \( V + \mathfrak{g} = g^{(-1)} + g^{(0)} \).

\[ \mathfrak{g}^{(-1)}, \mathfrak{g}^{(+0)} \]

\section{2. \((G, \mathfrak{l})\)-foliation}

Let \( M \) be an \( n \)-dimensional manifold, \( TM \) the tangent bundle of \( M \). An integrable subbundle \( E \) of \( TM \) is called a foliation of \( M \). We call the quotient bundle \( TM/E \) the normal bundle, and the frame bundle of \( TM/F \) the normal frame bundle of \( E \). We will take the following point of view toward \( \Gamma \)-foliations (with coordinates) (cf. \([BH]\), \([NS]\) and \([P]\)):

A \( \Gamma \)-foliation \( \mathcal{F} \) of codimension \( q \) \((\leq n)\) on \( M \) is given by the following data:

1. An open covering \( \{U_\lambda\} \) of \( M \).
2. An auxiliary \( q \)-dimensional manifold \( N \) with a pseudogroup \( \Gamma \) of local transformations on \( TV \).
3. A system of maps \( h_\lambda : U_\lambda \rightarrow N \) of rank \( q \).
4. A system of elements \( \gamma_{\mu \lambda} \in \overline{\Gamma} \) which satisfy \( h_\mu = \gamma_{\mu \lambda} h_\lambda \) on \( U_\mu \cap U_\lambda \) for each \( \lambda, \mu \), where \( U_{\mu \lambda} := U_\mu \cap U_\lambda \).

Then the kernels of differentials \( \langle h_\lambda \rangle \) constitute an integrable subbundle \( F \) of \( TM \). Thus \( \mathcal{F} \) defines a usual foliation as an underlying structure. The quotient bundle \( TM/F \) is called the normal bundle of \( \mathcal{F} \). Of course, the fibre dimension of \( TM/F \) is the codimension of \( \mathcal{F} \). Conversely Frobenius theorem shows that an integrable subbundle of \( TM \) defines a \( \Gamma \)-foliation with \( N = \mathbb{R}^q \) and \( \Gamma = \{ \text{local diffeomorphisms on } \mathbb{R}^q \} \).

\textbf{Example 2.1.} If \( N \) is a complex analytic manifold and \( \Gamma = \{ \text{complex analytic local diffeomorphisms on } N \} \), \( \mathcal{F} \) is called a complex analytic foliation. If \( M \) is a complex analytic manifold, then a holomorphic integrable subbundle of the holomorphic tangent bundle of \( M \) defines a complex analytic foliation (see \([B]\), for example).

\textbf{Example 2.2.} If \( N \) is a manifold with an affine connection \( \omega \) and \( \Gamma = \{ \text{local affine transformations on } N \} \), \( \mathcal{F} \) is called an affine foliation.

\textbf{Example 2.3.} If \( N \) is a Riemannian manifold and \( \Gamma = \{ \text{local isometries of } N \} \), \( \mathcal{F} \) is called a Riemannian foliation \([P]\). If \( M \) admits a bundle-like metric \([R]\) with respect to a foliation on \( M \), this foliation with the metric defines a Riemannian foliation.

\textbf{Example 2.4.} If \( N \) is a Riemannian manifold and \( \Gamma = \{ \text{local projective transformations on } N \} \), \( \mathcal{F} \) is called a projective foliation \([NS]\). A foliation of codimension one defines a projective foliation. We will also call \( \mathcal{F} \) a projective foliation when \( N \) is a manifold with an affine connection \( \omega \) and \( \Gamma = \{ \text{local projective transformations on } N \} \). This foliation is a generalization of an affine foliation.
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EXAMPLE 2.5. If \( N \) is a Riemannian manifold and \( \Gamma = \{ \text{local conformal transformations on } N \} \), \( \mathcal{F} \) is called a conformal foliation \([\text{NS}]\). Both a foliation of codimension one and a complex analytic foliation of complex codimension one define conformal foliations.

We see that Example 2.2, 2.4 and 2.5 are generalizations of Example 2.3. In each example, \( \Gamma \) consists of local automorphisms of a certain \( G \)-structure, that is, integrable \( GL(q/2, \mathbb{C}) \)-structure (\( q \): even) in Example 2.1, \( GL(q, R) \)-structure with an affine connection in Example 2.2, \( O(q) \)-structure in Example 2.3, \( GL(q, R) \)-structure with a torsion free connection in Example 2.4 and \( CO(q) \)-structure in Example 2.5, where \( CO(q) := \{ aA | a \in R - \{0\}, A \in O(q) \} \). Thus we will restrict ourselves to \( \Gamma \)-foliations which are defined by \( G \)-structures on \( N \) with connection.

Let \( B_N \) be the tangent frame bundle of \( N \), \( GL_q := GL(q, R) \) its structure group and \( \Theta_N \) the canonical form. Let \( G \) be a subgroup of \( GL_q \) and \( P_N \) a \( G \)-subbundle of \( B_N \) (a \( G \)-structure on \( N \)). The restriction of \( \Theta_N \) to \( P_N \) is also denoted by \( \Theta_N \).

For a local diffeomorphism \( \gamma \) on \( N \), the induced local bundle isomorphism of \( B_N \) is denoted by \( f \). Let \( \Gamma(P_N) \) be the pseudogroup of all local diffeomorphisms on \( N \) which preserve \( P_N \). Let \( \iota \) be an auxiliary Lie algebra which contains the Lie algebra \( \mathfrak{g} + R^q \) of the semidirect product \( G \cdot R^q \) as a Lie subalgebra. Let \( \omega \) be a \( G \)-connection on \( P_N \).

**DEFINITION.** \( \gamma \in \Gamma(P_N) \) is said to be a (local) \( \iota \)-automorphism of \( (P_N, \omega) \) if \( f^* \omega \) is \( (\iota, \Theta_N) \)-equivalent to \( \omega \).

**Remark 2.1.** If \( \mathfrak{l} \) is a Lie subalgebra of a Lie algebra \( \mathfrak{g} \) and \( \gamma \in \Gamma(P_N) \) is a \( \mathfrak{l} \)-automorphism of \( (P_N, \omega) \), then \( \gamma \) is a \( \mathfrak{l} \)-automorphism. For a torsion free connection \( \omega \in C(P_N) \), any \( \gamma \in \Gamma(P_N) \) is a \( \mathfrak{l}(\omega) \)-automorphism, where \( \mathfrak{l}(\omega) \) is the Lie algebra which was defined in Remark 1.3.

**EXAMPLE 2.2'** (Affine case). Let \( N \) be a manifold with an affine connection \( \omega \in C(B_N) \). Put \( G = GL_q \), \( P_N = B_N \) and \( \mathfrak{l} = \mathfrak{gl}_{q+1} \). For \( X + v \in \mathfrak{gl}_{q+1} \), the corresponding element is given by \( \begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix} \). If \( \gamma \) is an affine transformation on \( N \), then \( \tilde{\gamma}^* \omega - \omega = 0 \) or \( \tilde{\gamma}^* \omega - \omega = [\theta_N, 0] \) in \( \mathfrak{l} \).

**EXAMPLE 2.4'** (Projective case \([\text{KO}]\)). Let \( N \) be a Riemannian manifold, \( \omega \) the Levi-Civita connection. Put \( G = GL_q \), \( P_N = B_N \) and \( \mathfrak{l} = \mathfrak{sl}(q+1, R) \subset \mathfrak{gl}_{q+1} \), where \( \mathfrak{sl}(q+1, R) \) is the Lie algebra of the special linear group \( SL(q+1, R) \). For \( X + v \in \mathfrak{sl} + R^q \), the corresponding element is given by \( \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \), where \( a := - \frac{1}{q+1} \text{ Tr } X \), and \( A := X + aI_q \). If \( \gamma \) is a projective transformation on \( N \), then there exists a \( \mathfrak{l}'(R^q) \)-valued function \( \rho \) on \( P_N \) such that \( \tilde{\gamma}^* \omega - \omega = \theta_N \rho + ([\theta_N, \rho])_q \) or \( \tilde{\gamma}^* \omega - \omega = [\theta_N, \rho] \) in \( \mathfrak{l} \), where \( \mathfrak{l}'(R^q) \) is contained in \( \mathfrak{l}' \) in the form \( \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \) for \( u \in \mathfrak{l}'(R^q) \).
EXAMPLE 2.5’ (Conformal case [KO]). Let $N$ be a Riemannian manifold, $\omega$ the Levi-Civita connection. Put $G=\mathcal{O}(q)$, $P_N$ the $\mathcal{O}(q)$-extension of the orthonormal frame bundle, and $1=\mathcal{O}(q+1, 1)\subset \mathfrak{so}_{q+2}$, where $\mathcal{O}(q+1, 1)$ is the Lie algebra of $\mathcal{O}(q+1, 1)$. The Lie group $O(q+1, 1)$ is defined to be $\{X\in GL_{q+2}|^{\mathfrak{t}}X\mathfrak{S}X=S\}$, where $S=\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 1 \end{pmatrix}$. For $X+\nu\in \mathfrak{g}+\mathfrak{r}$, the corresponding element is given by $\begin{pmatrix} -a & \nu & 0 \\ 0 & A & \nu \\ 0 & 0 & a \end{pmatrix}$, where $a=-\frac{1}{q} \text{Tr} X$ and $A:=X+\alpha I_q$. If $\gamma$ is a conformal transformation on $N$, then there exists a $\mathfrak{t}(\mathfrak{r})$-valued function $\rho$ on $P_N$ such that $\gamma^*\omega-\omega=\theta N$, where $\gamma^*\omega-\omega=[\theta, \rho]$ in $I$, where $\mathfrak{t}(\mathfrak{r})$ is contained in $I$ in the form $u_0 u_0 0$ for $u_0 \in \mathfrak{r}$. Now we get the following generalization of all the above foliations.

DEFINITION. If $\Gamma$ consists of $\mathfrak{l}$-automorphisms of $(P_N, \omega)$ for a $G$-structure $P_N$ and $\omega\subset C(P_N)$, then the $\Gamma$-foliation $\mathcal{F}$ is called a $(G, \mathfrak{l})$-foliation.

Remark 2.2. According to Remark 2.1, if the $G$-structure admits a torsionfree connection $\omega\subset C(P_N)$ and $\Gamma\subset \Gamma(P_N)$, then the $\Gamma$-foliation with $(P_N, \omega)$ is a $(G, \mathfrak{g}^{\mathfrak{l}})$-foliation. Note that a $(G, \mathfrak{l})$-foliation $\mathcal{F}$ is a $(G, \mathfrak{l})$-foliation if $\mathfrak{l}$ is a Lie subalgebra of $\mathfrak{g}^{\mathfrak{l}}$. In order to study characteristic homomorphisms for $(G, \mathfrak{l})$-foliations, we may take a larger Lie algebra $\mathfrak{g}$. §3. Characteristic classes of $(G, \mathfrak{l})$-foliations

Let $\mathcal{F}$ be a $\Gamma$-foliation on $M$ as in §2, $B$ the normal frame bundle. Each $h_\lambda: U_\lambda \rightarrow N$ is covered by the canonical bundle map $\tilde{h}_\lambda: B|U_\lambda \rightarrow B_N$ which satisfies $\tilde{h}_\mu \circ h_\lambda = \tilde{h}_\mu \circ \tilde{h}_\lambda$. Let $P_N$ be a $G$-reduction of $B_N$, $\Gamma\subset \Gamma(P_N)$ at first. Then we have

LEMMA 3.1. $\tilde{h}_\lambda$ and $P_N$ defines a canonical $G$-subbundle $P\subset B$ on which $\theta\in A^1(P, \mathfrak{r})$ satisfying $\tilde{h}_\lambda^*\theta_N=\theta|P_\lambda$, exists, where $P_\lambda$ is the restriction $P|U_\lambda$.

Such a reduction is a special case of $\mathcal{F}$-reductions defined in [A]. We assume that the covering $\{U_\lambda\}$ in the definition of $\mathcal{F}$ is locally finite and admits a partition of unity $\{f_\lambda\}$. Let $\pi: P \rightarrow M$ be the projection, $\tilde{f}_\lambda := \pi^* f_\lambda$. Let $\omega$ be a connection form on $P_N$. Define $\omega^1 \subset C(P)$ by $\omega^1 := \sum f_\lambda \circ \tilde{h}_\lambda^* (\tilde{h}_\lambda^* \omega)$. We also denote the $GL_{q^2}$-extension of $\omega^1$ to $B$ by $\omega^1$. Clearly this connection is a basic connection defined by Bott [B].

Let $F$ be the corresponding integrable subbundle of $TM$, $F^\mathfrak{g}:=\{\alpha \in A^1(M)|_{\Gamma F} \alpha = 0 \text{ for any } X \in \Gamma F\}$ and $\mathfrak{r} A := (\wedge^r F^\mathfrak{g}) \wedge A(M)$ an ideal of $A(M)$, where $\Gamma F$ is the space of all cross sections of $F$. Clearly $\mathfrak{r} A=0$ for $r>q$. The following facts
were proved by Bott, and are called the vanishing theorems for characteristic classes of foliations.

**Fact 3.1.**
\[ \lambda(\omega^I) \supset A \cap A^{2r}(M), \]
and hence
\[ \lambda(\omega^I) = 0 \quad \text{for} \quad r > q. \]

For the $GL_q$-extension of $\omega^I$ to $B$, we get

**Fact 3.2.**
\[ \lambda(\omega^I) \supset A \cap A^q(M), \]
and hence
\[ \lambda(\omega^I) = 0 \quad \text{for} \quad r > q. \]

In the rest of this section, we assume that $\mathcal{F}$ is a $(G, I)$-foliation. Then we can find a basic connection which is convenient to study characteristic homomorphisms.

**Proposition 3.1.** Let $\omega = \sum \hat{f}_\mu \times (\hat{h}_\mu \omega)$ ($\in C(P)$), then $\omega|_{P_{\frac{1}{2}}}$ and $\hat{h}_\mu \omega$ are $(1, \theta|P_{\frac{1}{2}})$-equivalent.

**Proof.** The restriction is $\omega|_{P_{\frac{1}{2}}} = \sum (\hat{f}_\mu| P_{\frac{1}{2}} \times (\hat{h}_\mu \omega| P_{\frac{1}{2}})$, where $P_{\frac{1}{2}} := P|_{U_{\frac{1}{2}}}$. On the other hand, there is $\rho_{\frac{1}{2}} \in A(\hat{h}_\mu(P_{\frac{1}{2}}), 1)$ such that
\[ \hat{g}_{\frac{1}{2}}(\omega| P_{\frac{1}{2}}) = \omega| P_{\frac{1}{2}} + [\theta| P_{\frac{1}{2}}, \hat{h}_\mu \rho_{\frac{1}{2}}], \]
so that
\[ \hat{g}_{\frac{1}{2}}(\omega| P_{\frac{1}{2}}) = (\hat{h}_\mu \omega| P_{\frac{1}{2}} + [\theta| P_{\frac{1}{2}}, \hat{h}_\mu \rho_{\frac{1}{2}}]. \]
Then we obtain
\[ \omega|_{P_{\frac{1}{2}}} = \sum (\hat{f}_\mu| P_{\frac{1}{2}} \times (\hat{h}_\mu \omega| P_{\frac{1}{2}}) + [\theta| P_{\frac{1}{2}}], \sum (\hat{f}_\mu| P_{\frac{1}{2}} \times (\hat{h}_\mu \rho_{\frac{1}{2}})] \]
\[ = \hat{h}_\mu \omega + [\theta| P_{\frac{1}{2}}, \sum (\hat{f}_\mu| P_{\frac{1}{2}} \times (\hat{h}_\mu \rho_{\frac{1}{2}}). \quad \text{Q. E. D.} \]

This proposition generalizes the converse version of the holonomy theorem in [A] in the case of normal bundles of foliations.

Furthermore, we assume that $\omega \in C(P_N)$ is torsion free, then $\omega \in C_0(P)$ from Lemma 1.2. We get the following generalizations of strong vanishing theorems in [NS] and [P].

**Theorem 3.2.**
\[ \lambda(\omega^I) \supset A \cap A^{2r}(M), \]
and hence
\[ \lambda(\omega^I) = 0 \quad \text{for} \quad r > q/2. \]

**Proof.** It is clear that the form $\theta$ in Lemma 3.1 satisfies the condition in Lemma 1.3. By Corollary 1.3 and Proposition 3.1, we get
\[ \lambda(\omega^I) \gamma(U_{\frac{1}{2}} = \hat{h}_\mu \lambda(\omega) \gamma \quad \text{for} \quad \psi \in I(\gamma)(G). \quad \text{Q. E. D.} \]

**Remark 3.1.** According to Remark 2.2, we see that Theorem 3.2 is applicable to any $\Gamma$-foliation with $\Gamma \subset I(P_N)$ for a $G$-structure $P_N$ on $N$ with a torsion
free connection. Studying the relation between $I_{GL_q}(G)$ and $I_{(\mathfrak{g})}(G)$, we can generalize vanishing theorems for $I(GL_q)$ as follows.

**Corollary 3.3.** If $I(G)$ is generated by $I_{(\mathfrak{g})}(G)$ and odd degree elements of $I(G)$, then

$$\lambda_p(I'(G)) = 0 \quad \text{for } r > q/2.$$  

**Corollary 3.4.** If $I_{GL_q}(G)$ is generated by $I_{(\mathfrak{g})}(G)$ and odd degree elements of $I_{GL_q}(G)$, then

$$\lambda_h(I'(GL_q)) = 0 \quad \text{for } r > q/2,$$

where $I_{GL_q}(G):=I(GL_q)\cap\mathfrak{g}$.

For example, $I_{GL_q}(G)=I_{(\mathfrak{g})}(G)$ in the case of affine foliations. $I_{GL_q}(G)$ is generated by $I_{(\mathfrak{g})}(G)$ and the first Chern polynomial $c_1(\mathfrak{g})\in I_{GL_q}(G)$ in the case of projective or conformal foliations (see § 6).

**Remark 3.2.** The subring $\lambda_h(I(GL_q))\subset H^k(M)$ is generated by the Pontryagin classes of $TM/F$. Corollary 3.4 was obtained by Nishikawa and Sato [NS] in the case of projective (resp. conformal) foliations provided that $q \geq 2$ (resp. $q \geq 3$). Since they used the normal Cartan connections in their paper, the restriction on the codimension was not avoidable. In a recent paper [NT], Nishikawa and Takeuchi generalized the theorem in [NS] to $\Gamma$-foliations which relate to flat homogeneous spaces of order two, using the normal Cartan connections.

### § 4. Exotic characteristic classes

In [BH], Bott and Haefliger constructed cochain complexes denoted by $WO_q$ and $W_q$. We recall their construction [B]. Let $R[c_1, \ldots, c_q]$ be the polynomial ring over $R$ in variables $c_1, \ldots, c_q$ with dimensions $\dim c_j = 2j$ for $j=1, 2, \ldots, q$, $I_q$ the ideal generated by monomials whose dimensions are greater than $2q$. Denote the quotient ring $R[c_1, \ldots, c_q]/I_q$ by $R_q[c_1, \ldots, c_q]$. Let $E(h_1, \ldots)$ be the exterior algebra over $R$ generated by indicated $h_i$'s with $\dim h_i = 2i - 1$.

As a graded algebra,

$$WO_q := E(h_1, h_2, \ldots, h_{2[(q+1)/2]-1}) \otimes R_q[c_1, \ldots, c_q].$$

On $WO_q$, a unique differential $d_w : WO_q \rightarrow WO_q$ is defined by requiring

$$d_w c_j = 0, \quad 1 \leq j \leq q, \text{ and } d_w h_i = c_i, \quad i = 1, 3, \ldots, 2[(q+1)/2] - 1.$$  

The cochain complex $W_q$ is defined similarly, that is,

$$W_q := E(h_1, h_2, \ldots, h_{2[(q+1)/2]-1}) \otimes R_q[c_1, c_2, \ldots, c_q].$$

with the differential $d_w : W_q \rightarrow W_q$ defined by requiring

$$d_w c_j = 0, \quad 1 \leq j \leq q, \text{ and } d_w h_i = c_i, \quad 1 \leq i \leq q.$$  

Clearly $d_w^2 = 0$. Denote the cohomology ring of the cochain complex $WO_q$.
EXOTIC CHARACTERISTIC CLASSES

Let $\mathcal{F}$ be a $\Gamma$-foliation, $B$ the normal frame bundle, and $\omega \in \mathcal{C}(B)$ be a basic connection [B]. From Fact 3.2, the following homomorphism defined by Bott is well-defined.

**Definition.** A homomorphism of graded algebras $\lambda_{\mathcal{F}} : W_\omega \to A(M)$ is defined by requiring

$$
\lambda_{\mathcal{F}}(c_j) := \lambda(\omega^i)c_j, \quad 1 \leq j \leq q,
$$

and

$$
\lambda_{\mathcal{F}}(h_i) := \lambda(\omega^\phi, \omega^i)c_i, \quad i = 1, 3, \ldots, 2[(q+1)/2] - 1,
$$

where $\omega^\phi$ is a fixed metric connection on $B$ and $c_j$ in the right hand side is the $j$-th Chern polynomial defined in the following fact.

**Fact 4.1.** $I(GL_q) = R \xi_1, \ldots, \xi_q$, where $c_i \in I^q(GL_q)$ are defined by

$$
\Sigma t^ic_i(X) := \det(I_q - \frac{t}{2\pi}X) \quad \text{for} \quad X \in \mathfrak{gl}_q.
$$

From Fact 1.1, it is clear that $\lambda_{\mathcal{F}}$ is a cochain homomorphism, that is, $\lambda_{\mathcal{F}} \circ d = d\lambda_{\mathcal{F}}$. Bott proved the following:

**Fact 4.2.** The induced homomorphism $\lambda_{\mathcal{F}}^*: H^*(W_\omega) \to H^*(M)$ does not depend on the choice of basic connections and metric connections.

This homomorphism is called the generalized characteristic homomorphism for $\mathcal{F}$. The elements of $\lambda_{\mathcal{F}}^*(H^*(W_\omega) - [R_q[c_1, \ldots, c_q]])$ are called exotic characteristic classes of $\mathcal{F}$, where $[\cdot]$ denotes the cohomology class. J. Vey determined a basis for $H^*(W_\omega)$.

**Fact 4.3.** A basis for exotic classes of $H^*(W_\omega)$ is given by the classes of

$$
h_I \otimes c_J = h_{i_1} \wedge \cdots \wedge h_{i_\ell} \otimes c_{j_1} \cdots c_{j_m},
$$

where $I = (i_1, \ldots, i_\ell)$ and $J = (j_1, \ldots, j_m)$ satisfy

1) $1 \leq i_1 < \cdots < i_\ell \leq q$ (if $\ell \geq 1$),

2) $1 \leq j_1 \leq \cdots \leq j_m \leq q$ with $|J| := j_1 + \cdots + j_m \leq q$, and

1) $i_1 + |J| \geq q+1$ (condition to be cocycle),

2) $i_1 \leq j_0$,

where $j_0$ is the smallest odd integer in $J$ or $j_0 := \infty$.

For the proof, see [H]. The cohomology classes $[h_I \otimes c_J]$ are called Vey-basis for exotic classes.

When there exists a flat connection $\omega^\phi$ on $B$, Bott defined the following homomorphism:

**Definition.** A homomorphism of graded algebras $\lambda_{\mathcal{F},\omega^\phi} : W_\omega \to A(M)$ is defined by requiring

$$
\lambda_{\mathcal{F},\omega^\phi}(c_j) := \lambda(\omega^\phi)c_j, \quad 1 \leq j \leq q,
$$

and

$$
\lambda_{\mathcal{F},\omega^\phi}(h_i) := \lambda(\omega^\phi, \omega^i)c_i, \quad 1 \leq i \leq q.
$$
Remark 4.1. This homomorphism is originally defined only for foliations with trivialized normal bundles [B].

From Fact 1.1, it is clear that $\lambda_\omega$ is a cochain homomorphism.

Fact 4.4. The induced homomorphism $\lambda^\ast_\omega: H^\ast(W_q) \to H^\ast(M)$ does not depend on the choice of basic connections.

Remark 4.2. In contrast to the case of $WO_q$, $\lambda_\omega$ depends also on the equivalence class $[TM/F]$ with $\omega^0$ which was defined in [A]. The flat connection $\omega^0$ is not reducible to $P$ in general for a $G$-subbundle $P$ defined in §3.

Following to the method of Vey [H], we get

Fact 4.5. A basis for $H^\ast(W_q)$ is given by the classes of $h^C_j$ as in Fact 4.3, provided that the odd integer restriction on $i_1, \ldots, i_l$ is deleted and $j^0 := j_i$.

§5. Exotic characteristic classes of $(G, 1)$-foliations

In this section, we study exotic characteristic classes, using Theorem 3.2. Let $\mathcal{F}$ be a $\Gamma$-foliation, $B$ the normal frame bundle, $\omega^i \in C(B)$ a basic connection. Let $K^r := \{ \varphi \in I(GL_q) | |\varphi| = r \}$, and $K := \sum K^r$ is a subring of $I(GL_q)$. There exist integers $s_1, \ldots, s_k (1 \leq s_1 < \cdots < s_k \leq q)$ satisfying $I(GL_q) = K[c_1, \ldots, c_k]$.

It is clear that

Lemma 5.1. If $s_1, \ldots, s_k$ are odd, then $\lambda(\omega)I^r(GL_q) \sim 0$ for $r > q/2$.

Hereafter in this paper we will deal with only the cocycles $h_j \otimes c_j$ as in Facts 4.3 and 4.5. Fixing a basic connection $\omega^i \in C(B)$, we use the convention that $\varphi := \lambda(\omega)^0 \varphi$ for $\varphi \in I(GL_q)$ and $\hat{h}_1 := \lambda^g \hat{h}_1$ (or $\lambda^g \omega^g h_1$).

Proposition 5.1. If $s_1, \ldots, s_k$ are odd, $|J| > q/2$ and $i_1 + |J| \equiv q + s_k$ for $h_1 \otimes c_j \in WO_q$, then

$$\hat{h}_1 \wedge \hat{e}_j \sim b_{i_1, j} \hat{h}_{s_k} \wedge \hat{h}_1 \wedge (\hat{e}_{s_k})^{i_1}$$

for $i_1 + |J| = q + s_k$, $s_k | q$ and $s_k | i_1$, $\sim 0$ otherwise,

where $I_1 := (i_1, \ldots, i_l)$ and $b_{i_1, j}$ is a real number.

Proof. There exist $\varphi_j^A \in K^{[J] - [A]}$ such that

$$c_j = \sum_{0 \leq |J| \leq |J|} \varphi_j^A \cdot c_A,$$

where $A := (a_1, \ldots)$ with $\{a_1, \ldots, s_k \} \subseteq \{i_1, \ldots, s_k \}$ and $a_1 \leq a_2 \leq \cdots$. Since $|J| > q/2$ and $a_1$ is odd, we get

$$\hat{e}_j = \sum_{0 \leq |J| \leq |J|} \varphi_j^A \wedge \hat{e}_A = \sum (d\hat{h}_A) \wedge \hat{e}_A \wedge \varphi_j^A = \sum d(\hat{h}_A) \wedge \hat{e}_A \wedge \varphi_j^A),$$

where $A_1 := (a_1, \ldots)$. The condition $i_1 + |J| \equiv q + s_k$ and Fact 3.2 imply
**Exotic Characteristic Classes**  

It follows from the definition of $K$ and Fact 3.2 that

$$
\hat{h}_I \wedge \hat{\epsilon}_J \sim (-1)^{i+1} \sum (d\hat{h}_I) \wedge \hat{h}_{\alpha I} \wedge \hat{\epsilon}_{\alpha J} \wedge \hat{\phi}^A
$$

$$
= \sum \hat{h}_{\alpha I} \wedge \hat{h}_I \wedge \hat{\epsilon}_I \wedge \hat{\phi}^A.
$$

Since $i_1 + |J| \geq q + s_k$ and $s_k \geq a_1$,

$$
i_1 + |A| - a_1 + 2(|J| - |A|) \geq q + (s_k - a_1) + (|J| - |A|) \geq q,
$$

where the equality holds if and only if $i_1 + |J| = q + s_k$, $|A| = |J|$ and $a_1 = s_k$. As $a_1$ is the smallest in $A$, the condition $s_k | (q - 1)$ is necessary. If the equality holds, then

$$
\hat{h}_I \wedge \hat{\epsilon}_J \sim \sum_{i_1 \leq |J|} \hat{h}_{s_k} \wedge \hat{h}_I \wedge \hat{\epsilon}_{s_k} \wedge \hat{\phi}^A
$$

$$
= b_{i_1} \hat{h}_{s_k} \wedge \hat{h}_I \wedge \hat{\epsilon}_{s_k} \wedge \hat{\phi}^A.
$$

Apply the same method as above to the cocycle $h_{s_k} \wedge h_I \otimes c_{s_k} \cdot (c_{s_k})^{(q - 1)/s_k}$ instead of $h_I \otimes c_J$. Then we get

$$
\hat{h}_{s_k} \wedge \hat{h}_I \wedge \hat{\epsilon}_{s_k} \wedge \hat{\phi}^A \sim b_{i_1} \hat{h}_{s_k} \wedge \hat{h}_I \wedge \hat{\epsilon}_{s_k} \wedge \hat{\phi}^A
$$

for $s_k \mid q$,

$$
\sim 0 \quad \text{otherwise.} \quad \text{Q. E. D.}
$$

**Remark 5.1.** If $J^0 \neq \emptyset$, that is, $J$ contains an odd integer, then the condition $|J| > q/2$ is automatically satisfied.

In the case of a foliation with a flat connection, the assumption that $s_1, \cdots, s_k$ are odd and $|J| > q/2$ is avoidable, that is,

**Proposition 5.2.** Let $\mathcal{F}$ be a foliation with a flat connection, $s_k$ as above. If $i_1 + |J| \geq q + s_k$ for $h_I \otimes c_J \in W_q$, then

$$
\hat{h}_I \wedge \hat{\epsilon}_J \sim b_{i_1} \hat{h}_{s_k} \wedge \hat{h}_I \wedge \hat{\phi}^A
$$

for $i_1 + |J| = q + s_k$, $s_k \mid q$ and $s_k \mid i_1$,

$$
\sim 0 \quad \text{otherwise.} \quad \text{Q. E. D.}
$$

**Proof.** Since $i_1 + |J| \geq q + 1$ and $i_1 \leq j$, $2|J| \geq i_1 + |J| \geq q + 1$, that is, the assumption $|J| > q/2$ in Proposition 5.1 is satisfied. The same method as in the proof of Proposition 5.1 completes the proof.

Let $\mathcal{F}$ be a $(G, \omega)$-foliation, $\omega$ as in Theorem 3.2. Let $I_{(G, \omega)} := \{ \phi \in I(GL_q) \mid \phi|_g = \phi|_q \text{ for some } \phi \in I(G) \}$, then $I_{(G, \omega)} \subseteq K$ for this basic connection by Theorem 3.2. If $I(GL_q)$ is generated by $I_{(G, \omega)}$ and $c_{s_1}, \cdots, c_{s_k}$, then $I(GL_q) = K[c_{s_1}, \cdots, c_{s_k}]$. Thus Propositions 5.1 and 5.2 are applicable to $(G, \omega)$-foliations.

§ 6. Exotic characteristic classes of projective and conformal foliations

Let $\mathcal{F}$ be a projective (resp. conformal) foliation of codimension $q$ on $M$, $G = GL_q$ (resp. $CO(q)$) and $L = SL(q+1, R)$ (resp. $O(q+1, 1)$) as in Example 2.4 (resp.
2.5'. The subring \{\varphi \in I(GL_q) | \varphi \circ \varphi = \varphi \circ \varphi \text{ for some } \varphi \in I(L)\} in \(I(GL_q)\) is denoted by \(I_{(G,L)}\). Note that \(I_{(G,L)} \subset I_{(G,1)}\). We now study the relation between \(I(GL_q)\) and \(I_{(G,L)}\). Let \(c_f^L \in I^k(GL_q)\) be defined by

\[c_f^L := \sum_{j=0}^{k-1} B_{kj}(c_j) c_j + c_k,\]

where \(B_{kj} := \left(-\frac{1}{q+1}\right)^{k-j} \left(q+1-j\right) \left(\frac{1}{q-1}\right)^{k-j} \left(q+1-j\right) \left(q+1-j\right) \left(\frac{1}{q-1}\right)^{k-j} \left(q+1-j\right) \left(q+1-j\right) \left(\frac{1}{q-1}\right)^{k-j} \left(q+1-j\right) \left(q+1-j\right)\) and \(c_0 := 1\).

**Lemma 6.1.** \(c_f^L \in I^k_{(G,L)}\).

**Proof.** For \(X \in L\), let \(Y\) be the corresponding element in \(l\), see Example 2.4' (resp. 2.5'). A direct calculation shows

\[\det \left( I - \frac{t}{2\pi} Y \right) = \sum t^k c_f^L(X),\]

where \(I = I_{q+1}\) (resp. \(I_{q+2}\)). Q. E. D.

Obviously \(c_f^L = c_x = 1\) and \(c_f^L = 0\).

**Lemma 6.2.** In the conformal case, we have \(c_f^L | = 0\) for \(k \text{ odd}\).

Rearranging the expression in the definition, we obtain

**Lemma 6.3.**

\[c_f = \sum_{k=0}^{k-1} B_{kj}(c_j) c_j + c_k,\]

where \(B_{kj} := \left(-\frac{1}{q+1}\right)^{k-j} \left(q+1-j\right) \left(\frac{1}{q-1}\right)^{k-j} \left(q+1-j\right) \left(q+1-j\right) \left(\frac{1}{q-1}\right)^{k-j} \left(q+1-j\right) \left(q+1-j\right) \left(\frac{1}{q-1}\right)^{k-j} \left(q+1-j\right) \left(q+1-j\right)\).

Thus \(I(GL_q)\) is generated by \(I_{(G,L)}\) and \(c_f \in I^1(GL_q)\). From now on, we will consider only the cocycles described in Facts 4.3 and 4.5. We may assume, as is allowable from Facts 4.2 and 4.4, \(\omega^0\) is the basic connection defined in Proposition 3.1. We will use the same notations as in §5.

First we study the homomorphism \(\lambda_f : WO_q \to A(M)\). We may assume that \(\omega^0\) is a metric connection which is reducible to \(P\). Since \(I_{(G,L)} \subset I_{(G,1)} \subset K\) and \(I(GL_q) = K[c_f]\), Proposition 5.1 implies

**Proposition 6.1.** If \(|J| > q/2\) for \(h_1 \otimes c_j \in WO_q\), then

\[\hat{h}_1 \wedge \hat{c}_j \sim b_{1,J} \hat{h}_1 \wedge \hat{h}_1 \wedge (\hat{c}_j)^{\theta} \text{ for } \theta_{1 + |J| = q + 1},\]

\[\sim 0 \text{ for } \theta_{1 + |J| \geq q + 2}.\]

**Remark 6.1.** We find that \(b_{1,J} = B^{\theta_{1,J}} \prod_{i=1}^m B^{\theta_{i,J}}\), by chasing the proof of Proposition 5.1 with Lemma 6.3.
Especially when $\mathcal{F}$ is a conformal foliation, we obtain the following theorem in which the condition $|J| > q/2$ is avoidable by Lemma 6.2.

**Theorem 6.2.** For $h_1 \otimes c_j \in \mathcal{W}_q$,
\[
\hat{h}_1 \wedge \hat{c}_j \sim b_{i_1} \hat{h}_1 \wedge (\hat{c}_i)^g \quad \text{for} \quad I = (i_1) \quad \text{and} \quad i_1 + |J| = q + 1,
\]
\[
\sim 0 \quad \text{otherwise}.
\]

**Proof.** Lemma 6.3 and Proposition 1.1 imply
\[
\lambda(\omega^0, \omega^1)c_i \sim \lambda(\omega^0, \omega^1)c_i^g + (\lambda(\omega^0, \omega^1)c_i) \wedge \lambda(\omega^1)\phi_i
\]
for certain $\phi_i \in \mathcal{I}^t(GL_q)$, that is, $\hat{h}_1 \sim \hat{h}_1^i + \hat{h}_1 \wedge \phi_i$, where $h_i = \lambda(\omega^0)c_i$. It follows from Fact 3.2 that
\[
\hat{h}_1 \wedge \hat{c}_j \sim \hat{h}_1 \wedge \hat{c}_j + \hat{h}_1 \wedge \hat{h}_1 \wedge \phi_{i_1} \wedge \hat{c}_j.
\]
Since $\omega^0$ is reducible to $P$, Lemma 6.2 imply $\hat{h}_1 \wedge \hat{c}_j = 0$, and then $\hat{h}_1 \wedge \hat{c}_j = 0$ for $I_1 \neq \phi$ or $i_1 + |J| \geq q + 2$. In the same way as the proof of Proposition 5.1, we get $\phi_{i_1} \wedge \hat{c}_j = b_{i_1} \phi(\hat{c}_i)^g$ for $i_1 + |J| = q + 1$.

The classes $h_1 \otimes c_j \in \mathcal{W}_q$ with $i_1 + |J| \geq q + 2$ are said to be rigid [H]. The class of $h_1 \otimes (c_i)^g$ is called the Godbillon-Vey invariant of foliations of codimension $q$.

Next we study the homomorphism $\lambda^*: \mathcal{W}_q \to \mathcal{A}(M)$. Let $\mathcal{F}$ be a projective or conformal foliation with a flat connection $\omega^0$ on the normal frame bundle $B$.

**Proposition 6.3.** For $h_1 \otimes c_j \in \mathcal{W}_q$,
\[
\hat{h}_1 \wedge \hat{c}_j \sim b_{i_1} \hat{h}_1 \wedge \hat{h}_1 \wedge (\hat{c}_i)^g \quad \text{for} \quad i_1 + |J| = q + 1,
\]
\[
\sim 0 \quad \text{for} \quad i_1 + |J| \geq q + 2.
\]

Hereafter we assume that $\mathcal{F}$ is a conformal foliation.

**Theorem 6.4.** If the flat connection $\omega^0$ is reducible to $P$ and $I_1$ contains an odd integer, then $\hat{h}_1 \wedge \hat{c}_j = 0$.

The proof is almost the same as that of Theorem 6.2.

**Corollary 6.5.** If the flat connection $\omega^0$ is given by a trivialization of $B$ and $I_1$ contains an odd integer, then $\hat{h}_1 \wedge \hat{c}_j = 0$.

For the proof of this corollary, we need the following lemma which is derived from Fact 1.3.

**Lemma 6.4.** If $\{\omega^0_i : 0 \leq i \leq 1\}$ is a smooth family of flat connections on $B$, then
\[
\lambda(\omega^0_i, \omega^1)c_i \sim \lambda(\omega^0_i, \omega^1)c_i \quad \text{for} \quad i \geq 2.
\]

**Remark 6.2.** This lemma also shows exotic characteristic classes of $h_1 \otimes c_j$ depends on the $\mathcal{F}$-homotopy class $[A]$ of $[TM/F]$ if $i_1 > 1$. 

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Proof of Corollary 6.5. Since \( O(q) \subseteq CO(q) \), any trivialization of \( B \) is homotopic to one of \( P \). Then there is a smooth family \( \omega_t^i \) of flat connections such that \( \omega_t^i \) is reducible to \( P \). Lemma 6.4 completes the proof. Q. E. D.

Remark 6.3. In the case of projective or conformal foliations with trivialized normal bundles, Proposition 6.3 and Corollary 6.5 were proved by Morita [M] and Yamato [Y], where they used the normal Cartan connections as in [NS].

§ 7. Appendix

In this section, we apply the method of integration along fibre to the case of secondary invariants, that is, exotic characteristic classes [B] and Chern-Simons classes [CS]. By this method, certain formulae for secondary invariants (derivation formulae, especially) can be proved in a simple manner.

Let \( G \) be a Lie group, \( q \) and \( l^r(G) \) as in § 1. Let \( P \) be a smooth manifold. We denote the space of \( q \)-valued (resp. real-valued) \( r \)-forms on \( P \) by \( A^r(P, q) \) (resp. \( A^r(P) \)). Denote \( R \times P \) by \( \bar{P} \), where \( R \) is the real numbers field. Let \( \pi: \bar{P} \to P \) be the canonical projection, \( j_t: P \to \bar{P} \) the inclusion map defined by \( j_t(p) := (t, p) \in \bar{P} \). For the interval \( I := [0, 1] \subseteq R \), a linear map \( \pi^I: A^r(\bar{P}) \to A^{r-1}(P) \) is defined by

\[
\pi^I(\alpha) := \int_0^1 (t \alpha) dt \quad \text{for} \quad \alpha \in A^r(\bar{P}),
\]

where \( T \) is the vector field on \( \bar{P} \) which is the canonical extension of \( \partial/\partial t \) on \( R \). We can easily get the following:

**Lemma 7.1.**

\[ d\pi^I + \pi^I d = j_t^* - j_0^* . \]

For \( \omega \in A^r(P, q) \), define \( \Omega = \Omega(\omega) \in A^q(P, q) \) by \( \Omega(\omega) := d\omega + 1/2[\omega, \omega] \in A^q(P, q) \). When \( \omega \) is a connection form on a principal \( G \)-bundle \( P \), \( \Omega \) is the curvature form of \( \omega \). Let \( \phi \in l^r(G) \). From \( \text{Ad}(G) \)-invariancy of \( \phi \), we obtain

**Lemma 7.2.**

\[ d\phi(\Omega) = 0 . \]

For a smooth family of \( q \)-valued \( 1 \)-forms \( \omega^R := \{ \omega^t \in A^1(P, q) | t \in R \} \), \( \omega^R \in A^1(\bar{P}, q) \) is defined by \( \langle \omega^R, (t, p) \rangle := \pi^*(\omega^t)_p \) for \( (t, p) \in \bar{P} \).

**Definition.** A linear map

\[ \mu(\omega^R): l^r(G) \longrightarrow A^{r-1}(P) \]

is defined by \( \mu(\omega^R)\phi := \pi^I(\phi(\bar{\Omega}^R)) \) for \( \phi \in l^r(G) \), where \( \bar{\Omega}^R := \Omega(\omega^R) \).

**Lemma 7.3.**

\[ \mu(\omega^R)\phi = \int_0^1 \phi \left( \frac{\partial}{\partial t} \omega^t, \Omega^t \right) dt , \]

where \( \Omega^t := \Omega(\omega^t) \).
Note that \( \mu(\omega^R)\phi \) can be regarded as a form on \( M \) when each \( \omega^i \) is a connection form on a principal \( G \)-bundle \( P \) over \( M \). From Lemma 7.1 and 7.2, we get

**Proposition 7.1.**
\[
d(\mu(\omega^R)\phi) = \phi(\Omega^1) - \phi(\Omega^0)
\]

Especially when \( \omega^i = t\omega^i + (1-t)\omega^0 \), we denote \( \mu(\omega^R) \) by \( \lambda(\omega^0, \omega^i) \), and \( T(\omega) := \lambda(0, \omega) \).

**Corollary 7.2.**
\[
d(\lambda(\omega^0, \omega^i)\phi) = \phi(\Omega^1) - \phi(\Omega^0)
\]

**Corollary 7.3.**
\[
d(T(\omega)\phi) = \phi(\Omega)
\]

If each \( \omega^i \) is a connection form on a principal \( G \)-bundle \( P \) over \( M \), Proposition 7.1 shows that the characteristic homomorphism \( \Gamma(G) \to [\phi(\Omega)] \in H^0(M) \) is independent of the choice of connections (cf. [C]).

**Proposition 7.4** (Product formula). For \( \varphi, \psi \in I(G) \),
\[
\mu(\omega^R)(\varphi \cdot \psi) = (\mu(\omega^R)\varphi) \wedge \phi(\Omega^1) + \varphi(\Omega^0) \wedge (\lambda(\omega^0, \omega^i)\psi) - d\pi^I(\varphi(\Omega^R) \wedge (\lambda(\pi^*\omega^1, \omega^R)\psi)).
\]

**Proof.** It follows from Lemma 7.1 and Corollary 7.2 that
\[
d\pi^I(\varphi(\Omega^R) \wedge (\lambda(\pi^*\omega^1, \omega^R)\psi)) = -\pi^I(\varphi(\Omega^R) \wedge d(\lambda(\pi^*\omega^1, \omega^R)\psi)) + j^*(\varphi(\Omega^R) \wedge (\lambda(\pi^*\omega^1, \omega^R)\psi))) - \pi^I(\psi(\Omega^R) - \phi(\pi^*\varphi(\Omega^1))) - \phi(\Omega^0) \wedge \lambda(\omega^1, \omega^0)\psi
\]
\[
= -\mu(\omega^R)(\varphi \cdot \psi) - \mu(\omega^R)\varphi \wedge \phi(\Omega^1) + \varphi(\Omega^0) \wedge \lambda(\omega^0, \omega^i)\psi .
\]

Q. E. D.

**Corollary 7.5.**
\[
\lambda(\omega^0, \omega^i)(\varphi \cdot \psi) = (\lambda(\omega^0, \omega^i)\varphi) \wedge \phi(\Omega^1) + \varphi(\Omega^0) \wedge (\lambda(\omega^0, \omega^i)\psi) - d\pi^I(\varphi(\Omega^R) \wedge (\lambda(\pi^*\omega^1, \omega^R)\psi)).
\]

**Corollary 7.6** (Chern and Simons [CS]).
\[
T(\omega)(\varphi \cdot \psi) = (T(\omega)\varphi) \wedge (\phi(\Omega) + \text{exact}).
\]

We need a lemma to prove derivation formulae. Let \( \varphi \in I^*(G) \), \( \omega = A^1(P, \mathfrak{g}) \mid s \in R \) a smooth family.

**Lemma 7.4.**
\[
\frac{1}{r} \frac{\partial}{\partial s} \varphi(\Omega_s) = d\varphi(\frac{\partial}{\partial s} \omega_s, \Omega_s).
\]

**Proof.** For a family \( \omega(s)^R := \{\omega(s)^i := \omega_{s, s} \mid s \in R \} \) for fixed \( s \in R \), Proposition 7.1 implies
\[
d(\mu(\omega(s)^R)\varphi) = \varphi(\Omega(s)^1) - \varphi(\Omega(s)^0) = \varphi(\Omega_s) - \varphi(\Omega_s).
\]
where $\Omega(s)^t := \Omega(\omega(s)^t)$. On the other hand, we have

$$\mu(\omega(s)^R) = \int_0^1 \varphi \left( \frac{\partial}{\partial t} \omega(s)^t, \Omega(s)^t \right) dt = \int_0^1 \varphi \left( \frac{\partial}{\partial t} \omega_t, \Omega_t \right) dt,$$

so that, the proof is completed. Q. E. D.

Let $\{\omega_t \in \mathcal{A}(P, g) | t \in R, s \in R \}$ be a smooth two-parameters family on $R \times R \times P = R \times \tilde{P}$. Denote the smooth one-parameter family $\{\omega_t | t \in R \}$ on $\tilde{P}$ by $\omega_t^R$ for each $s \in R$.

**Proposition 7.7 (Generalized derivation formula).** For $\varphi \in \mathcal{I}(G)$,

$$\frac{1}{r} \frac{\partial}{\partial s} (\mu(\omega_t^R)) = -d \pi^t \left( \varphi \left( \frac{\partial}{\partial s} \omega_t^R, \tilde{\Omega}_t^R \right) \right) + \varphi \left( \frac{\partial}{\partial s} \omega_t^R, \Omega_t \right) - \varphi \left( \frac{\partial}{\partial s} \omega_t, \Omega_t \right),$$

where $\tilde{\Omega}_t^R := \Omega(\omega_t^R)$ and $\Omega_t^R := \Omega(\omega_t^R)$.

**Proof.** Applying Lemma 7.4 to the one-parameter family $\{\omega_t \in \mathcal{A}(\tilde{P}, g) | s \in R \}$, we get

$$\frac{1}{r} \frac{\partial}{\partial s} \pi^t(\varphi(\tilde{\Omega}_t^R)) = \pi^t \left( d \varphi \left( \frac{\partial}{\partial s} \omega_t^R, \tilde{\Omega}_t^R \right) \right) = -d \pi^t \left( \varphi \left( \frac{\partial}{\partial s} \omega_t^R, \tilde{\Omega}_t^R \right) \right) + \varphi \left( \frac{\partial}{\partial s} \omega_t^R, \Omega_t ^R \right) - \varphi \left( \frac{\partial}{\partial s} \omega_t, \Omega_t \right).$$

Q. E. D.

For $\omega^R \in \mathcal{A}(P, g)$ and a smooth family $\{\omega_t \in \mathcal{A}(P, g) | s \in R \}$, we get the following:

**Corollary 7.8 (Heitsch [H]).** For $\varphi \in \mathcal{I}(G)$,

$$\frac{1}{r} \frac{\partial}{\partial s} (\lambda(\omega^R, \omega_t^R)) = -d \pi^t \left( \varphi \left( \frac{\partial}{\partial s} \omega_t^R, \tilde{\Omega}_t^R \right) \right) + \varphi \left( \frac{\partial}{\partial s} \omega_t^R, \Omega_t^R \right).$$

Note that each term in this corollary can be regarded as a form on $M$ if $\omega^R$ and $\omega_t^R$ are connection forms on a principal $G$-bundle $P$ over $M$. Putting $\omega^R = 0$ in this corollary, we obtain

**Corollary 7.9.** (Chern and Simons [CS, KO]).

$$\frac{1}{r} \frac{\partial}{\partial s} (T(\omega_t^R)) = -d \pi^t \left( \varphi \left( \frac{\partial}{\partial s} \omega_t^R, \tilde{\Omega}_t^R \right) \right) + \varphi \left( \frac{\partial}{\partial s} \omega_t^R, \Omega_t^R \right).$$

**References**


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