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ON THE ZERO-ONE SET OF AN ENTIRE FUNCTION, II

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1. Introduction. Let $\{a_n\}$ and $\{b_n\}$ be two disjoint sequences with no finite limit points. If it is possible to construct an entire function f whose zeros are exactly $\{a_n\}$ and whose d-points are exactly $\{b_n\}$, the given pair $(\{a_n\}, \{b_n\})$ is called the zero-d set of f. Here of course $d\neq 0$. If further there exists only one entire function f, whose zero-d set is just the given pair $(\{a_n\}, \{b_n\})$, then the pair is called unique. It is well-known that unicity in this sense does not hold in general.

In this paper we shall prove the following

THEOREM. *Let $(\{a_n\}, \{b_n\})$ and $(\{a_n\}, \{c_n\})$ be the zero-one set and the zero-d set of an entire function N, where $d\neq 0, 1$. Then at least one of two given pairs is unique, unless N is an arbitrary entire function of the following form: e^L+A, where A is an arbitrary constant and L is an entire function.*

As a corollary we have the following fact.

COROLLARY. *Let N be an entire function with no finite lacunary value. Then every zero-d set of N excepting at most one is unique.*

Our proof depends on the impossibility of Borel's identity [1]. One of its form is the following

LEMMA. *Let $\{\alpha_j\}$ be a set of non-zero constant and $\{g_j\}$ a set of entire functions satisfying*

$$\sum_{j=1}^{p}\alpha_j g_j=1 .$$

Then

$$\sum_{j=1}^{p}\delta(0, g_j)\leqq p-1 ,$$

where $\delta(0, g_j)$ denotes the Nevanlinna deficiency.

This form was stated in [2]. In our present case g_j is e^{L_j} and hence $\delta(0, g_j)=1$. Hence Lemma gives evidently a contradiction.

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In our previous paper [3] we proved the following fact: The non-unicity of the given zero-one set $(\{a_n\}, \{b_n\}_{n=1}^{\infty})$ implies that $(\{a_n\}, \{b_n\}_{n\geqq n_0})$ $(n_0\geqq 2)$ is not a zero-one set of any entire function. We shall prove a corresponding fact in this paper.

2. Proof of Theorem. The emptyness of $\{a_n\}$ implies $N=e^L$, which is an exceptional entire function. The same holds for $\{b_n\}$ and for $\{c_n\}$, Hence we may assume that the three sets $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are not empty. Assume that there are entire functions f and g such that

$$f=Ne^{\alpha}, \qquad f-1=(N-1)e^{\beta},$$

$$g=Ne^{\gamma}, \qquad g-d=(N-d)e^{\delta}$$

with entire functions α, β, γ and δ. Suppose that α is a constant. Then $f=CN$, $C=e^{\alpha}$. By $f(b_n)=N(b_n)=1$, $f=N$. This is just the desired unicity of the given zero-one set. Hence we may assume that α is not a constant. Similarly we may assume that β, γ and δ are not constants. Suppose that $\alpha-\beta$ and $\gamma-\delta$ are constants c and a, respectively. Then

$$f=e^{c}Ne^{\beta}, \qquad f-1=(N-1)e^{\beta}.$$

Hence

$$f=\frac{e^{c}N}{(e^{c}-1)N+1}.$$

$e^{c}=1$ implies $f=N$. Hence we may assume that $e^{c}\neq 1$. Since f is entire,

$$(e^{c}-1)N+1=e^{L}$$

with entire L. Then

$$N=\frac{e^{L}-1}{e^{c}-1},$$

which is just the exceptional entire function.

If $e^{a}=1$, then $g=N$, which is the desired unicity of the zero-d set of N. Hence we may assume that $e^{a}\neq 1$. Then we have

$$g=\frac{de^{a}N}{(e^{a}-1)N+d}.$$

Since g is entire,

$$N=\frac{d(e^{L'}-1)}{e^{a}-1}.$$

This must coincide with the one already mentioned. Hence

$$d=\frac{e^{a}-1}{e^{c}-1}. \qquad L'=L.$$

Suppose that $\alpha-\beta$ and $\gamma-\delta$ are not constants. Then

$$\frac{e^{-\beta}-1}{e^{\alpha-\beta}-1}=N=\frac{d(e^{-\delta}-1)}{e^{\gamma-\delta}-1},$$

that is,

$$e^{\gamma-\delta-\beta}-e^{\gamma-\delta}-e^{-\beta}-de^{\alpha-\beta-\delta}+de^{\alpha-\beta}+de^{-\delta}=d-1.$$

Lemma implies that $\gamma-\delta-\beta$ is a constant, unless $\alpha-\beta-\delta$ is.

If $\gamma-\delta-\beta=x$ is a constant but $\alpha-\beta-\delta$ is not, then

$$-e^{\gamma-\delta}-e^{x}e^{\delta-\gamma}-de^{x}e^{\alpha-\gamma}+de^{x}e^{\alpha+\delta-\gamma}+de^{-\delta}=d-1-e^{x}.$$

Hence $d-1=e^{x}$ by Lemma. Thus we have

$$-e^{2(\gamma-\delta)}-d(d-1)e^{\alpha-\delta}+d(d+1)e^{\alpha}+de^{-2\delta+\gamma}=d-1.$$

Lemma again implies that $\alpha-\delta$ is a constant, unless $\gamma-2\delta$ is. If $\gamma-2\delta=y$ is a constant but $\alpha-\delta$ is not, then

$$-e^{2y}e^{2\delta}-d(d-1)e^{\alpha-\delta}+d(d-1)e^{\alpha}=d-1-de^{y}.$$

This shows that $de^{y}=d-1$ and

$$(d-1)e^{2\delta}+d^{3}e^{\alpha-\delta}-d^{3}e^{\alpha}=0.$$

Hence

$$(d-1)e^{2\delta-\alpha}+d^{3}e^{-\delta}=d^{3},$$

which is impossible. If $\alpha-\delta=y$ is a constant but $\gamma-2\delta$ is not, then

$$-e^{2(\gamma-\delta)}+d(d-1)e^{y}e^{\delta}+de^{-2\delta+\gamma}=d-1+d(d-1)e^{y}.$$

Hence $e^{y}=-1/d$ and

$$de^{\gamma-3\delta}-e^{2\gamma-3\delta}=d-1,$$

which is impossible. If both of $\alpha-\delta=z$ and $\gamma-2\delta=y$ are constants, then

$$y-x=\gamma-2\delta-\gamma+\delta+\beta=\beta-\delta$$

and

$$\alpha-\beta=\alpha-\delta-\beta+\delta=z-y+x.$$

This is absurd, since $\alpha-\beta$ is not a constant. Hence the case that $\gamma-\delta-\beta$ is a constant but $\alpha-\beta-\delta$ is not is now rejected. If $\alpha-\beta-\delta=x$ is a constant but $\gamma-\delta-\beta$ is not, then

$$e^{\gamma-\delta-\beta}-e^{\gamma-\delta}-e^{-\beta}+e^{x}e^{\delta}+de^{-\delta}=d-1+de^{x}.$$

Hence $de^{x}=1-d$ and

$$de^{\gamma-\beta}-de^{\gamma}-de^{\delta-\beta}-(d-1)e^{2\delta}=-d^{2}.$$

This implies that $\gamma-\beta$ is a constant, unless $\delta-\beta$ is. If $\gamma-\beta=y$ is a constant but $\delta-\beta$ is not, then

$$de^{\gamma}+de^{y}e^{\delta-\gamma}+(d-1)e^{2\delta}=d^{2}+de^{y}.$$

Hence $e^y=-d$ and

$$(d-1)e^{2\delta-\gamma}-d^2e^{\delta-2\gamma}=-d\,.$$

which is impossible. If $\delta-\beta=y$ is a constant but $\gamma-\beta$ is not, then

$$de^ye^{\gamma-\delta}-de^{\gamma}-(d-1)e^{2\delta}=de^y-d^2\,.$$

Hence $e^y=d$ and

$$d^2e^{-\delta}-(d-1)e^{2\delta-\gamma}=d\,,$$

which is absurd. If $\gamma-\beta$ and $\delta-\beta$ are constants, then $\gamma-\delta$ reduces to a constant. This is impossible. Hence the case that $\alpha-\beta-\delta$ is a constant but $\gamma-\delta-\beta$ is not is now rejected. If $\alpha-\beta-\delta=x$ and $\gamma-\beta-\delta=y$ are constants, then

$$-e^{\gamma-\delta}-e^{-\beta}+de^{\alpha-\beta}+de^{-\delta}=d-1-e^y+de^x\,.$$

Hence $d-1=e^y-de^x$ and

$$-e^ye^{\beta+\delta}-e^{\delta-\beta}+de^xe^{2\delta}=-d\,.$$

Since $\beta+\delta$ is not a constant, $\delta-\beta$ should be a constant. Let us put $z=\delta-\beta$. Then

$$-e^{y-z}e^{2\delta}+de^xe^{2\delta}=e^z-d\,.$$

Hence $e^z=d$ and $e^y=d^2e^x$. In this case we have

$$d^2f=g \quad \text{and} \quad \frac{f-1}{g-d}=\frac{N-1}{dN-d^2}$$

Thus

$$g-d^2=-\frac{d^2(N-1)}{N}$$

Hence N should be of the form e^L and the given set $\{a_n\}$ should be empty. This is a contradiction.

We shall consider the case that $\alpha-\beta=c$ is a constant but $\gamma-\delta$ is not. Then

$$e^{\gamma-\delta-\beta}-e^{\gamma-\delta}-e^{-\beta}-d(e^c-1)e^{-\delta}=d-1-de^c\,.$$

If $\gamma-\delta-\beta$ is not a constant, then $de^c=d-1$. Then

$$e^{\gamma-\beta}-e^{\gamma}-e^{\delta-\beta}=-1\,.$$

This implies that $\gamma-\beta$ is a constant, unless $\delta-\beta$ is. If $\gamma-\beta=x$ is a constant and $\delta-\beta$ is not, then

$$e^{\gamma}+e^xe^{\delta-\gamma}=1-e^x\,.$$

Therefore $e^x=1$ and $2\gamma-\delta$ is a constant and

$$e^x+e^z=0\,, \qquad z=2\gamma-\delta.$$

Hence

$$\frac{f}{g}=e^{\alpha-\gamma}=e^c=\frac{d-1}{d}$$

and

$$\frac{g}{f-1}=\frac{N}{N-1}e^{\gamma-\beta}=\frac{N}{N-1}e^{x}=\frac{N}{N-1}.$$

By these relations

$$-\frac{d-1}{d}g=\frac{(d-1)N}{N-d}.$$

Hence $N-d$ has no zero, that is $\{c_n\}$ is empty, which is absurd. If $\delta-\beta=x$ is a constant but $\gamma-\beta$ is not, then

$$e^{\gamma-\beta}-e^{\gamma}=e^{x}-1,$$

which easily gives a contradiction. If $\delta-\beta$ and $\gamma-\beta$ are constants, then $\gamma-\delta$ is so. This is impossible. Hence $\gamma-\delta-\beta=a$ reduces to a constant. In this case

$$e^{a}e^{\beta}+e^{-\beta}+d(e^{c}-1)e^{-\delta}=e^{a}+de^{c}-d+1,$$

from which $e^{a}=d-1-de^{c}\neq 0$ and

$$(d-1-de^{c})e^{2\beta}+d(e^{c}-1)e^{-\delta+\beta}=-1.$$

This is absurd.

We can similarly consider the remaining case that $\gamma-\delta$ is a constant and $\alpha-\beta$ is not. And finally we arrive at a contradiction.

3. Examples. Let N be e^{z}. Then all the zero-d sets of N are not unique. This has been implicitly shown in our theorem. Explicitly

$$g=d^{2}e^{-z}$$

satisfies

$$g=Nd^{2}e^{-2z}, \qquad g-d=-(N-d)de^{-z}.$$

Let N be $e^{z}(1-e^{z})$. Then $f=e^{-z}(1-e^{-z})$ satisfies

$$f=-Ne^{-3z}, \qquad f-1=(N-1)e^{-2z}.$$

All the zero-d sets of N excepting the zero-one set are unique.

Let N be an entire function of finite non-integral order. Then all the zero-d sets of N are unique.

4. We can prove the following fact: Let N be an entire function whose zero-one set is not unique but all other zero-d sets $(\{a_n\}, \{c_n\}_{n\geq 1})$ are unique. Then $(\{a_n\}, \{c_n\}_{n\geq n_0})$ $(n_0\geqq 2)$ is not a zero-d set of any entire function.

In this case we have

$$f=Ne^{\alpha}, \qquad f-1=(N-1)e^{\beta},$$

$$g=Ne^{\gamma}, \qquad (g-d)P=(N-d)e^{\delta}$$

with entire $\alpha, \beta, \gamma, \delta$ and a non-constant polynomial $P=c(z-c_1)\cdots(z-c_{n_0-1})$.

This gives a contradiction, although we need a similar discussion as in §2. Now the existence of g is excluded.

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