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REMARKS ON SQUARE-INTEGRABLE BASIC COHOMOLOGY SPACES ON A FOLIATED RIEMANNIAN MANIFOLD

Dedicated to Professor T. Otsuki on his 60th birthday

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B. L. Reinhart [7] showed that on compact foliated manifold with "bundle-like" metric, the cohomology of basic differential forms is isomorphic to the harmonic space of a certain semidefinite Laplacian. It is well known that the complex

$$d'' : o \longrightarrow \wedge^{0,0}(M) \longrightarrow \wedge^{0,1}(M) \longrightarrow \wedge^{0,2}(M) \longrightarrow \cdots$$

is not elliptic. We shall define the completion $L_2^{0,s}(M)$ of compactly supported basic $(0, s)$-forms and discuss the squareintegrable basic cohomology spaces in complete case analogous to K. Okamoto and H. Ozeki [2] on a Hermitian manifold.

§ 1. Definitions

Let M be an n dimensional C^∞ manifold which, topologically, is connected, orientable, paracompact Hausdorff space. We shall assume given on M a foliation E of codimension q, and we may find about each point a coordinate neighbourhood with coordinates $(x^1, \cdots, x^p, y^1, \cdots, y^q)$ $(n=p+q)$ such that

(i) $|x^i|\leqq 1$, $|y^\alpha|\leqq 1$,

(ii) The integral manifolds of E are given locally by $y^1=c^1, \cdots, y^q=c^q$ for constants c^α satisfying $|c^\alpha|\leqq 1$. (Here and hereafter, Latin indices run from 1 to p, and Greek indices from 1 to q.)

Such a coordinate neighbourhood will be called flat, while each of slices given by a set of equations $y^\alpha=c^\alpha$ will be called a plaque. If U is a flat neighbourhood, the quotient space of U by its plaques will be called the local base and be denoted by U_y.

We may assume that there exist in U differential forms w^i and vectors v_α such that

(i) $\{\partial/\partial x^i\}$ forms the base for the space of cross-sections of E in U at each point,

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(ii) $\{w^1, \cdots, w^p, dy^1, \cdots, dy^q\}$ and $\{\partial/\partial x^1, \cdots, \partial/\partial x^p, v_1, \cdots, v_q\}$ are dual bases for the cotangent and tangent spaces at each point of U respectively. Hence, $w^i = dx^i + \sum_\alpha a^i_\alpha dy^\alpha$ and $v_\alpha = \partial/\partial y^\alpha + \sum_i b^i_\alpha \partial/\partial x^i$.

Throughout this note, all local expressions for differential forms and vectors will be taken with respect to these bases.

§ 2. Square-integrable basic cohomology spaces

On a foliated manifold we may have the decomposition of differential forms into components in following way: Any $C^\infty - m$-form ϕ may be expressed locally as

$$\sum_{\substack{i_1 < \cdots < i_r \\ \alpha_1 < \cdots < \alpha_s}} \sum_{r+s=m} \phi_{i_1 \cdots i_r \alpha_1 \cdots \alpha_s}(x, y) w^{i_1} \wedge \cdots \wedge w^{i_r} \wedge dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_s}.$$

We then define $\Pi_{r,s}\phi$ to be the sum of all these terms with a fixed r and s. Since under change of flat coordinate system, $\{\{dy^\alpha\}\}$ goes into $\{\{dy^{*\alpha}\}\}$ and $\{\{w^i\}\}$ goes into $\{\{w^{*i}\}\}$, this operator $\Pi_{r,s}$ is independent of the choice of coordinate system. Here by $\{\{\cdot\}\}$ we mean the vector space generated by the set $\{\cdot\}$. $\Pi_{r,s}\phi$ is called the component of type (r, s) of ϕ. The type decomposition of forms induces a type decomposition of the exterior derivative d by the rule $(\Pi_{t,u} d)\phi = \sum_{r,s} \Pi_{r+t, s+u} d \Pi_{r,s}\phi$. Let $\Pi_{1,0} d = d'$ and $\Pi_{0,1} d = d''$. In general, there will be a component $\Pi_{-1,2} d$; since we are interested only in forms of type $(0, s)$, we shall not introduce a notion for this component.

PROPOSITION 2.1. (cf. [7]) *If ϕ is of type $(0, s)$, then $d\phi = d'\phi + d''\phi$. Moreover, $d'\phi = 0$ if and only if ϕ depends only upon y, in the sense that locally*

$$\phi = \sum \phi_{\alpha_1 \cdots \alpha_s}(y) dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_s}.$$

DEFINITION 2.1. A form of type $(0, s)$ which is annihilated by d' will be called a basic form.

DEFINITION 2.2. A riemannian metric is bundle-like if and only if it is representable in each flat neighbourhood by an expression of the form

$$ds^2 = \sum g_{ij}(x, y) w^i w^j + \sum g_{\alpha\beta}(y) dy^\alpha dy^\beta.$$

Hereafter, we assume that the riemannian metric on M is bundle-like and all leaves are compact.

Remark. In this paragraph, we may replace "all leaves are compact" by "the volume of M is finite", then the compact support of the basic form is replaced by the tranversally compact support, i.e. $\phi_{\alpha_1 \cdots \alpha_s}(y) dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_s}$ has compact support in variable y's in local expression, and, for example, we may consider the torus foliated by a family of irrational spirals.

Let $\wedge^{0,s}(M)$ be the space of all C^∞-basic form of type $(0, s)$ and $\wedge^{0,s}_o(M)$ the subspace of $\wedge^{0,s}(M)$ composed of forms with compact supports. Restricted to

$\wedge^{0,*}(M)=\sum_{s=0}^{\infty}\wedge^{0,s}(M)$, $d''^2=d^2=0$, so we may consider the cohomology of $\wedge^{0,*}(M)$ and d''. (This is called the base-like cohomology by B. L. Reinhart [7].)

B. L. Reinhart [7] introduces the $*''$-operation on $\wedge^{0,s}(M)$, and defined by

$$*''\phi=\sum_{\substack{\alpha_1<\cdots<\alpha_s \\ \beta_1<\cdots<\beta_{q-s}}}\operatorname{sgn}\begin{pmatrix}1\cdots\cdots\cdots\cdots\cdots\cdots\cdots q\\ \alpha_1\cdots\alpha_s\,\beta_1\cdots\beta_{q-s}\end{pmatrix}(\det(g_{\alpha\beta}))^{1/2}$$

$$g^{\alpha_1\nu_1}\cdots g^{\alpha_s\nu_s}\phi_{\nu_1\cdots\nu_s}dy^{\beta_1}\wedge\cdots\wedge dy^{\beta_{q-s}}.$$

According to B. L. Reinhart [7], we may define a riemannian metric on $\wedge^{0,s}(M)$ by

$$\langle\phi, \psi\rangle=\phi\wedge*''\psi\wedge dx^1\wedge\cdots\wedge dx^p,$$

and obtain a pre-Hilbertian metric on $\wedge_o^{0,s}(M)$ by

$$(\phi, \psi)=\int_M\langle\phi, \psi\rangle$$

$$=\int_M\phi\wedge*''\psi\wedge dx^1\wedge\cdots\wedge dx^p.$$

The differential operator d'' maps $\wedge^{0,s}(M)$ into $\wedge^{0,s+1}(M)$. We define δ'': $\wedge^{0,s}(M)\to\wedge^{0,s-1}(M)$ by

$$\delta''\phi=(-1)^{qs+q+1}*''d''*''\phi.$$

Then we have

$$(d''\phi, \psi)=(\phi, \delta''\psi)$$

for $\phi\in\wedge_o^{0,s}M)$, $\psi\in\wedge_o^{0,s+1}(M)$.

Let $L_2^{0,s}(M)$ be the completion of $\wedge_o^{0,s}(M)$ with respect to the inner product $(\,,)$. We will denote by ∂ the restriction of d'' to $\wedge_o^{0,s}(M)$ and by θ the restriction of δ'' to $\wedge_o^{0,s}(M)$. Define

$$\bar\partial=(\theta)^*\quad\text{and}\quad\bar\theta=(\partial)^*$$

where $(\ )^*$ denotes the adjoint operator of $(\ )$ with respect to the inner product $(\,,)$. Then $\bar\partial$ (resp. $\bar\theta$) is a closed, densely defined operator of $L_2^{0,s}(M)$ into $L_2^{0,s+1}(M)$ (resp. $L_2^{0,s-1}(M)$). Let $D_{\bar\partial}^{0,s}$ (resp. $D_{\bar\theta}^{0,s}$) be the domain of the operator $\bar\partial$ (resp. $\bar\theta$) in $L_2^{0,s}(M)$. We put

$$Z_{\bar\partial}^{0,s}(M)=\{\phi\in D_{\bar\partial}^{0,s}\,|\,\bar\partial\phi=0\}$$

$$Z_{\bar\theta}^{0,s}(M)=\{\phi\in D_{\bar\theta}^{0,s}\,|\,\bar\theta\phi=0\}.$$

Since $\bar\partial$ and $\bar\theta$ are closed operators, $Z_{\bar\partial}^{0,s}(M)$ and $Z_{\bar\theta}^{0,s}(M)$ are closed in $L_2^{0,s}(M)$. Let $B_{\bar\partial}^{0,s}(M)$ and $B_{\bar\theta}^{0,s}(M)$ be the closure of $\bar\partial(D_{\bar\partial}^{0,s-1})$ and $\bar\theta(D_{\bar\theta}^{0,s+1})$ respectively.

DEFINITION 2.3. $H_2^{0,s}(M)=Z_{\bar\partial}^{0,s}(M)\ominus B_{\bar\partial}^{0,s}(M)$ is the square-integrable basic cohomology spaces, where $\ominus$ denotes the orthogonal complement of $B_{\bar\partial}^{0,s}(M)$.

It is easy to see that

$$H_2^{0,s}(M)=Z_{\bar\partial}^{0,s}(M)\cap Z_{\bar\theta}^{0,s}(M).$$

Since $Z_{\bar\partial}^{0,s}(M)$ and $Z_{\bar\theta}^{0,s}(M)$ are closed in $L_2^{0,s}(M)$, $H_2^{0,s}(M)$ has canonically the structure of a Hilbert space.

THEOREM 2.1. (*The orthogonal decomposition theorem*)

$$L_2^{0,s}(M)=H_2^{0,s}(M)\oplus B_{\bar\partial}^{0,s}(M)\oplus B_{\bar\theta}^{0,s}(M).$$

Proof is analogous to L. Hörmander [3], in fact, we have only to notice that $B_{\bar\partial}^{0,s}(M)$ and $B_{\bar\theta}^{0,s}(M)$ are mutually orthgonal and $B_{\bar\partial}^{0,s}(M)^\perp\cap B_{\bar\theta}^{0,s}(M)^\perp=H_2^{0,s}(M)$, where $\perp$ denotes the orthogonal complement in $L_2^{0,s}(M)$.

Then we have the Dolbeault-Serre type theorem.

THEOREM 2.2. *If the bundle-like metric on M is complete, then,*

$$H_2^{0,s}(M)=H_2^{0,q-s}(M) \qquad (\textit{isomorphic as Hilbert space}).$$

In fact, we have only to notice that the following diagram is commutative.

$$\begin{array}{ccc} \wedge_o^{0,s}(M) & \xrightarrow{\;*''\;} & \wedge_o^{0,q-s}(M) \\ \bar\theta\downarrow\uparrow\bar\partial & (-1)^{s}*'' & \bar\partial\downarrow\uparrow\bar\theta \\ \wedge_o^{0,s-1}(M) & \longrightarrow & \wedge_o^{0,q-s+1}(M). \end{array}$$

COROLLARY 2.1. (cf. [7]) *If* $\dim H_2^{0,s}(M)$ *is finite, then* $\dim H_2^{0,s}(M)=\dim H_2^{0,q-s}(M)$.

§ 3. Harmonic forms in complete bundle-like metric

Hereafter, we assume that the bundle-like metric is complete and all leaves are compact.

PROPOSITION 3.1.

$$N_{d''}^{0,s}(M)\cap L_2^{0,s}(M)\subset Z_{\bar\partial}^{0,s}(M)$$

$$N_{\delta''}^{0,s}(M)\cap L_2^{0,s}(M)\subset Z_{\bar\theta}^{0,s}(M),$$

where $N_{d''}^{0,s}(M)=\{\phi\in\wedge^{0,s}(M)\mid d''\phi=0\}$ *and* $N_{\delta''}^{0,s}(M)=\{\phi\in\wedge^{0,s}(M)\mid\delta''\phi=0\}$.

In order to prove this proposition, we need some facts analogous to A. Andreotti and E. Vesentini [1].

We consider a differentiable function μ on $\boldsymbol{R}$ (the reals) satisfying

(i) $0\leqq\mu\leqq1$ on $\boldsymbol{R}$,

(ii) $\mu(t)=1$ for $t\leqq1$,

(iii) $\mu(t)=0$ for $t\geqq2$.

It is known that a geodesic orthogonal to a leaf is orthogonal to all leaves (cf. B. L. Reinhart [6]). We fix a point o in M, and for each point p in M, we denote by $\rho(p)$ the distance between leaves through o and p. Then we set

$$w_k(p)=\mu(\rho(p)/k) \quad \text{for} \quad k=1, 2, 3, \cdots .$$

LEMMA 3.1. *Under the above notations, there exists a positive number A, depending only on μ, such that*

(i) $\|d''w_k\wedge\phi\|^2<\dfrac{nA^2}{k^2}\|\phi\|^2$

(ii) $\|d''w_k\wedge *''\phi\|^2<\dfrac{nA^2}{k^2}\|\phi\|^2$

for all $\phi\in\wedge^{0,s}(M)$, where $\|\phi\|^2=(\phi, \phi)$.

In order to prove this lemma, we have to notice that the function $\rho(p)$ is a locally Lipschitz function and, at points where the derivatives exist, it holds

$$\sum g^{\alpha\beta}v_\alpha(\rho)v_\beta(\rho)<n .$$

Then we have

$$|d''w_k|^2=\sum g^{\alpha\beta}v_\alpha(w_k)v_\beta(w_k)<\frac{nA^2}{k^2}$$

where A is a positive number depending only on $\sup\left|\dfrac{d\mu}{dt}\right|$.

We remark that $d'w_k=0$ and $w_k\phi$ has compact support for each $\phi\in\wedge^{0,s}(M)$. Then we have that $w_k\phi\in D_{\bar\partial}^{0,s}\cap D_{\bar\theta}^{0,s}$ for $\phi\in\wedge^{0,s}(M)$, and that

$$\bar\partial(w_k\phi)=d''(w_k\phi)$$

$$\bar\theta(w_k\phi)=\delta''(w_k\phi) .$$

Now we prove Proposition 3.1. Let ϕ be in $N_{d''}^{0,s}\cap L_2^{0,s}(M)$. By the above remarks

$$\begin{aligned}\bar\partial(w_k\phi)&=d''(w_k\phi)\\&=d''w_k\wedge\phi+w_kd''\phi\\&=d''w_k\wedge\phi .\end{aligned}$$

Hence, by Lemma 3.1, we have

$$\|\bar\partial(w_k\phi)\|^2<\frac{nA^2}{k^2}\|\phi\|^2 .$$

Putting $\phi_k=w_k\phi$, we we have

$$\bar\partial\phi_k \longrightarrow 0 \quad (k\longrightarrow\infty) \qquad \text{(strong)} .$$

On the other hand, $\phi_k\to\phi$ $(k\to\infty)$ (strong). Since $\bar\partial$ is a closed operator, we see that ϕ is in $D_{\bar\partial}^{0,s}$ and $\bar\partial\phi=0$. This proves $\phi\in Z_{\bar\partial}^{0,s}(M)$. In the same way, we have $N_{\delta''}^{0,s}\cap L_2^{0,s}(M)\subset Z_{\bar\theta}^{0,s}(M)$. This proves Proposition 3.1.

DEFINITION 3.1. The Laplacian acting on $\wedge^{0,*}(M)$ is defined by

$$\square=-(d''\delta''+\delta''d'').$$

For any $\phi\in L_2^{0,s}(M)\cap\wedge^{0,s}(M)$, we have

$$(d''\phi,\ d''\alpha)_{B(k)}+(\delta''\phi,\ \delta''\alpha)_{B(k)}=(-\square\phi,\ \alpha)_{B(k)} \tag{3.1}$$

for all $\alpha\in\wedge_{B(k)}^{0,s}(M)$, where $\wedge_{B(k)}^{0,s}(M)$ is the space of all forms of type $(0, s)$ with compact support contained in $B(k)$ and $B(k)$ is an open tube of radius k of the leaf through the fixed point o in M. For $\alpha=w_k^2\phi$, we have

$$d''\alpha=w_k^2d''\phi+2w_kd''w_k\wedge\phi$$

$$\delta''\alpha=w_k^2\delta''\phi+(-1)^{qs+q+1}*''(2w_kd''w_k\wedge*''\phi).$$

Substituting in (3.1), we have

$$\begin{aligned}&\|w_kd''\phi\|_{B(k)}^2+\|w_k\delta''\phi\|_{B(k)}^2\\&\leqq|(\square\phi,\ w_k^2\phi)_{B(k)}|+|(d''\phi,\ 2w_kd''w_k\wedge\phi)_{B(k)}|\\&\quad+|(\delta''\phi,\ *''(2w_kd''w_k\wedge*''\phi))_{B(k)}|.\end{aligned} \tag{3.2}$$

On the other hand, the Schwartz inequality gives the following

$$|(d''\phi,\ 2w_kd''w_k\wedge\phi)_{B(k)}|\leqq\frac{1}{2}(\|w_kd''\phi\|_{B(k)}^2+4\|d''w_k\wedge\phi\|_{B(k)}^2)$$

$$|(\delta''\phi,\ *''(2w_kd''w_k\wedge*''\phi))_{B(k)}|\leqq\frac{1}{2}(\|w_k\delta''\phi\|_{B(k)}^2+4\|d''w_k\wedge*''\phi\|_{B(k)}^2)$$

and

$$|(\square\phi,\ w_k^2\phi)_{B(k)}|\leqq\frac{1}{2}\left(\frac{1}{\sigma}\|w_k\phi\|_{B(k)}^2+\sigma\|\square\phi\|_{B(k)}^2\right)$$

for every $\sigma>0$.

Substituting in (3.2),

$$\begin{aligned}&\|w_kd''\phi\|_{B(k)}^2+\|w_k\delta''\phi\|_{B(k)}^2\\&\qquad<\sigma\|\square\phi\|_{B(k)}^2+\left(\frac{1}{\sigma}+\frac{8nA^2}{k^2}\right)\|\phi\|_{B(k)}^2.\end{aligned}$$

Letting $k\to\infty$, we have

$$\|d''\phi\|^2+\|\delta''\phi\|^2<\sigma\|\square\phi\|^2+\frac{1}{\sigma}\|\phi\|^2$$

for every $\sigma>0$. In particular, setting $\square\phi=0$ and letting $\sigma\to\infty$, we have

LEMMA 3.2. *Let the bundle-like metric on M be complete and all leaves be compact. If $\phi\in L_2^{0,s}(M)\cap\wedge^{0,s}(M)$ such that $\square\phi=0$, then $d''\phi=0$ and $\delta''\phi=0$,* i.e. $\phi\in N_{d''}^{0,s}(M)\cap N_{\delta''}^{0,s}(M)$.

From Proposition 3.1 and Lemma 3.2, we have the following theorem.

THEOREM 3.1. *Let the bundle-like metric on M be complete and all leaves be compact. If* $\phi \in L_2^{0,s}(M) \cap \wedge^{0,s}(M)$ *such that* $\square\phi=0$, *then* $\phi \in H_2^{0,s}(M)$.

Remark. The bundle-like metric can be deformed to a continuous complete metric if all leaves are compact, but the deformed metric can not be C^∞-metric (cf. [4, 5]).

Remark. It is well known by I. Vaisman [8] that for a compact oriented foliated riemannian manifold M, the space $\mathcal{H}^{r,s}(M)$ of foliated harmonic forms is a subspace of the de Rham cohomology space $H^{r,s}(M)$.

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