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# ON PARALLEL CONFORMAL CONNECTIONS

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**Introduction.** Riemannian manifolds endowed with a parallel conformal connection $\nabla_{p.c}$ have been defined by the present author in [1]. In this paper one studies in the first section a type of such manifolds for which the principal field $X$ associated with $\nabla_{p.c}$ is *parallel*. In this case $X$ is an *infinitesimal homothety* of the volume element of $M_c$ and is an invariant section of the cannonical form in the set of 2-frames $\mathcal{O}^2(M_c)$. If $M_c$ is of even dimension $2m$, then the connection $\nabla_{p.c}$ defines on $M_c$ a *conformal symplectic form* $\varphi$ and the dual field of the principal 1-form $\alpha$ ($\alpha$ is the dual form of $X$ with respect to the metric of $M_c$) with respect to $\varphi$ is a *Killing field*. Finally it is shown that $M_c$ is of constant scalar curvature and is *Ricci flat* in the direction of $X$. In the second section, making use of some notions introduced by K. Yano and S. Ishihara in [5] and by J. Klein in [7] one studies different properties of the tangent bundle manifold $TM_c$. Thus the *complete lift* $\varphi^c$ of $\varphi$, on $TM_c$ is a homogenous form of degree 1 and is *also conformal symplectic*. If $V$ is the *canonical field* on $TM_c$, then the Lie bracket $[V, X]$ is an *infinitesimal automorphism of* $\varphi^c$. Further some properties involving the *canonical symplectic form* $\Omega$ on $TM_c$ ($\Omega$ is a *Finslerian form*) and a second conformal symplectic form $\Theta$, which is homogenous of degree 2, are discussed. In the last section one considers a regular *mechanical system* (in the sense of J. Klein [8]), $\mathcal{M}=\{M_c, T, \pi\}$ such that the *kinetic energy* $T$ is homogenous of degree 2 and the *dynamical system* $Z$ associated with $\mathcal{M}$ is a *spray* on $M_c$.

**1.** $M_c$ **manifold.** Let $M$ be an $n$-dimensional $C^\infty$-Riemannian manifold and let $\mathcal{O}(M)$ be the bundle of orthonormal frames of $M$. If $\mathcal{O}\in\mathcal{O}(M)$ is such a frame, let $\{e_i\}$, $\{\omega^i\}$ and $\omega^i_k=\mathcal{I}^i_{kj}\omega^j$, $i, k, j=1, \cdots, n$, be the vectorial and dual basis and the connection forms associated with $\mathcal{O}$ respectively. Then the line element $dp$ ($p\in M$), the connection equations and the structure equations (E. Cartan) are respectively

$$dp=\omega^i\otimes e_i\,, \tag{1.1}$$

$$\nabla e_i=\omega^k_i\otimes e_k\,, \tag{1.2}$$

$$d\wedge\omega^i=\Omega^i+\omega^k\wedge\omega^i_k\,, \tag{1.3}$$

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(1.4) $$d\wedge\omega_k^i=\Omega_k^i+\omega_k^j\wedge\omega_j^i\,,$$

where $\Omega^i$ and $\Omega_k^i$ are the *torsion* and the *curvature 2-forms* respectively.

A connection $\nabla$ such that

(1.5) $$\omega_k^i=t_k\omega^i-t_i\omega^k\,;\ t_i\in C^\infty(M)$$

has been called in [1], a *parallel conformal connection*, and denoted by $\nabla_{p.c}$. If $T_p(M)$ is the tangent space at $p\in M$ we shall call

(1.6) $$X=\sum_i t_ie_i\in T_p(M)$$

the *principal field* (p. f.) associated with $\nabla_{c.p}$ and if $\mathcal{J}$ is the canonical isomorphisme (with respect the metric of $M$) the Pfaffian

(1.7) $$\mathcal{J}X=\alpha=\sum_i t_i\omega^i$$

is the *principal Pfaffian* (*p. P.*) associated with $\nabla_{p.c}$.

So by 1.3 and 1.5 we readly get

(1.8) $$d\wedge\omega^i=\Omega^i+\alpha\wedge\omega^i\,.$$

Assume now that $X$ is *parallel*, that is,

(1.9) $$\nabla X=0\,.$$

By using 1.2 and 1.5 we obtain from 1.9

(1.10) $$dt_i=t_i\alpha-t^2\omega^i\,;\ t^2=\|X\|^2\,.$$

Taking account of 1.7 one finds instantly

(1.10′) $$t^2=\text{const.}$$

Next exterior differentiation of (1.10) gives

(1.11) $$\Omega^i=0$$

and so by an easy argument follows

(1.12) $$d\wedge\alpha=0\,.$$

Hence if the *p.f.* $X$ is parallel then the connection $\nabla_{p.c}$ is necessarily *torsionless* and the *p.P.* $\alpha$ is *closed*. In the following the manifolds under consideration will be of even dimension ($n=2m$) and structured by $\nabla_{p.c}$ connection with parallel principal field. Such manifolds will be denoted by $M_c$.

We have shown in [1] that if $M$ is of even dimension ($n=2m$) then the connection $\nabla_{p.c}$ defines on $M$ a *conformal symplectic structure* $CSp(m\,;R)$. Thus if we consider the almost symplectic form

(1.13) $$\varphi=\omega^1\wedge\omega^2+\cdots+\omega^{2m-1}\wedge\omega^{2m},$$

then by

(1.14) $$d\wedge\omega^i=\alpha\wedge\omega^i,$$

one gets at once

(1.15) $$d\wedge\varphi=2\alpha\wedge\varphi.$$

Thus we see that $2\alpha$ is the *co-vector of Lee* of the structure $CSp(m; R)$. Let now $\mu_\varphi: Z\to -i_Z\varphi$ be the isomorphism defined by $\varphi$. An easy calculation gives

(1.16) $$\mu_\varphi^{-1}(\alpha)=X_a=-t_2e_1+t_1e_2-\cdots+t_{2m-1}e_{2m}$$

and $X_a$ will be called the *associated field* of $X$. Taking the *star operator* $*$ of $\alpha$ one has

(1.17) $$*\alpha=\sum_i(-1)^{i-1}\omega^1\wedge\cdots\wedge\hat{\omega}^i\wedge\cdots\wedge\omega^{2m}$$

(the "roof" indicates the missing terms).

Making use of 1.10 and 1.13 one finds from (1.17) $\delta\alpha=\operatorname{div}X=-t^2=$const. and so $X$ is an *infinitesimal homothety* of the volume element of $M$.

Put now $\mu_\varphi(X)=\alpha_a$ and call $\alpha_a$ the *associated* 1-*form* of $\alpha$. Denoting by $\tilde{*}$ the *symplectic adjoint operator* [2] one has

(1.18) $$*\alpha_a=\tilde{*}\alpha=\frac{\alpha}{(m-1)!}\lambda(\lambda^{m-1}\varphi).$$

Making use of (1.12) and (1.15) we readly see that $\delta\alpha_a=0$, that is $\operatorname{div}X_a=0$.

On the other hand $Z(Z^i)\in T_p(M_c)$ being any vector field, we derive from (1.2), (1.5) and (1.10)

(1.19) $$\nabla_Z e_i=t_iZ-Z^iX;\ \nabla_Z:\text{ covariant derivative.}$$

Now with the aid of (1.17) and since $\langle X, X_a\rangle=0$, one finds by a straightforward calculation

(1.20) $$\langle\nabla_Z X_a, Z'\rangle+\langle\nabla_{Z'}X_a, Z\rangle=0,$$

where $Z$ and $Z'$ are arbitrary vector fields. The above relation proves that $X_a$ is a *Killing vector field*, So the equation

(1.21) $$\mathcal{L}_{X_a}\alpha=0;\ \mathcal{L}_Z=i_Zd\wedge+d\wedge i_Z:\text{ Lie derivative}$$

together with (1.18) are in accordance (if $M$ is campact) with Bochner's theorem. Further by using (1.19) and taking account of (1.11) one finds

(1.22) $$\mathcal{L}_{X_a}X=0.$$

Denote like usual by $R^i_{jkl}$ the Riemann curvature tensor, that is, $\Omega^i_j=(1/2)R^i_{jkl}\omega^k\wedge\omega^l$. By (1.4), (1.5) and (1.10) one finds

$$R^k_{iik}=t^2-t_i^2-t_k^2$$

(1.23) $$R^k_{iil}=-t_k t_l$$

$$R^k_{ijl}=0\,;\ k\neq i\neq j\neq l\,.$$

From the above expressions we derive the components of the Ricci tensor as follows

(1.24) $$R_{ii}=(n-2)(t^2-t_i^2)\,,\qquad R_{ik}=-(n-2)t_i t_k\,.$$

From 1.24 and taking account of (1.10) one quickly finds that the scalar curvature of $M$ is constant, that is

(1.25) $$R=(n-1)(n-2)t^2\,.$$

Next denote by Ric$(X)$ the *Ricci curvature* in the direction $X$.

In consequence of (1.24) and (1.7) a short calculation gives

(1.26) $$\mathrm{Ric}\,(X)=0\,.$$

Hence the manifold $M$ is *Ricci flat* in the direction $X$.

On the other hand referring to (1.5) and (1.10) one finds that both $\omega^i$ and $\omega^i_k$ are *invariant* by $X$; that is $\mathcal{L}_X\omega^i=0$, $\mathcal{L}_X\omega^i_k=0$.

Therefore one may say that $X$ is an *invariant section* for the canonical form $\omega^i\otimes e_i+\sum_{i,k}\omega^i_k\otimes e^k_i$ of the set of 2-frames $\mathcal{O}^2(M_c)$ (frames of second order).

Finally coming back to the structure $CSp(m\,;\,R)$ defined by (1.14) we have

(1.26) $$i_X\varphi=-\mu_\circ(X)=-\alpha_a=-t_2\omega^1+t_1\omega^2-\cdots+t_{2m-1}\omega^{2m}\,.$$

By (1.20) and (1.14) a short computation gives

(1.27) $$d\wedge\alpha_a=\alpha\wedge\alpha_a+2t^2\varphi\,.$$

Since $t^2$ is constant, this equation proves as is known [3] that $X$ is a *conformal symplectic infinitesimal transformation* of $\varphi$. From the preceding discussion we may state the

THEOREM. *Let $M_c$ be a Riemannian manifold of even dimension* $2m$ *structured by a $\nabla_{p.c}$ connection with principal field $X$ and let $X_a$ and $\eta$ be the associated field of $X$ and the volume element of $M_c$ respectively. Then*:

(i) *the connection $\nabla_{p.c}$ defines on $M_c$ a conformal symplectic structure $CSp(m\,;\,R)=(\varphi,2\alpha)$ having (up to a constant factor) the dual form of $X$ as covector of Lee,*

(ii) *the field* $X$ *has the following properties: it is an infinitesimal homothety of* $\eta$, *it is an invariant section of the canonical form of the set of* 2-*frames* $\mathcal{O}^2(M_c)$, *it is a conformal symplectic infinitesimal transformation of* $CSp(m; R)$;

(iii) *the field* $X_a$ *has the following properties it is an infinitesimal automorphism of* $\eta$, *it is a Killing field;*

(iv) $M_c$ *is of constant scalar curvature and is Ricci flat in the direction of* $X$.

**2. Tangent bundle manifold** $TM_c$. $M_c$, being of constant scalar curvature, is as is known ($n \geqq 3$) endowed with a conformal flat structure. Therefore referring to (1.14) we may get

$$\alpha = -df/f\,;\; f \in C^\infty(M_c) \tag{2.1}$$

and call $f$ the *integrating factor* associated with $\nabla_{p.c}$. Denote by $TM_c$ the tangent bundle manifold having $M_c$ as basis and by $V(v^i)$ the *canonical field* (the field of Liouville) on $TM_c$. Thus we may consider the set $B^* = \{\omega^i, dv^i\}$ as a co-vectorial basis of $TM_c$.

Denote (like usual) by $d_v$ and $i_v$ the *vertical differentiation* and the *vertical derivation operators* respectively taken with respect to $B^*$ ($d_v$ is an antiderivation of degree 1 of $\wedge(TM)$ and $i_v$ is a derivation of degree 0 of $\wedge(TM)$ [4]).

Put

$$l = fv \in C^\infty(TM_c)\,, \tag{2.2}$$

where

$$v = \frac{1}{2}\sum_i (v^i)^2\,. \tag{2.3}$$

One has

$$d_v l = f\sum_i v^i \omega^i = \lambda \in \wedge^1(TM_c) \tag{2.4}$$

and by (2.1) and (1.14) we get

$$d \wedge d_v l = f\sum_i dv^i \wedge \omega^i = \Omega\,. \tag{2.5}$$

Clearly $\Omega$ is an exact symplectic form which will be called the *canonical symplectic form* on $TM_c$.

In addition we shall call $l$ and $\lambda$ the *Liouville function* and the *Liouville form* respectively on $TM_c$.

If $\iota: \wedge^1(M) \to C^\infty(TM)$ is the operator of K. Yano and S. Ishihara [3] one has (with respect to $B^*$)

$$\iota\alpha = \sum_i t_i v^i \tag{2.6}$$

and so

(2.7) $$d_v(\iota\alpha)=\alpha\,,$$

If $\partial_i$ denotes the Pfaffian derivativ with respect $\omega^i$, then according to [5], *complete lift* $\alpha^c$ of $\alpha$ is defined by

(2.8) $$\alpha^c=(\partial t_i,\ t_i)\,;\ \partial=\sum v^i\partial_i\,.$$

With the help of (1.10) one finds

(2.9) $$\alpha^c=(\iota\alpha)\alpha-(t^2/f)\lambda+\beta\,,$$

where

(2.10) $$\beta=\sum_i t_i dv^i\,.$$

One obtains

(2.11) $$d_v\alpha^c=0\,;\ i_v\alpha^c=\alpha=i_v\beta$$

and so by (2.7) and remarking that $\alpha^c=d(\iota\alpha)$, one checks $(d\wedge d_v+d_v d\wedge)(\iota\alpha)=0$. On the other hand since

(2.12) $$\iota_v(\iota\alpha)=0$$

one checks $[i_v,\ d]=d_v$.

The *complete lift* $X^c$ of $X$ is as is known

(2.13) $$X^c=\begin{pmatrix} t_i \\ \partial_{t_i}\end{pmatrix}=X+(\iota\alpha)X^V-t^2V$$

where $X^V=\begin{pmatrix} 0 \\ t_i\end{pmatrix}$ and $V$ are the *vertical lift* of $X$ and the *canonical field* respectively. Referring to (2.4) and (2.5) we find at once

(2.14) $$i_V\Omega=\lambda\,,\qquad \iota_{X^V}\Omega=f\alpha\,,\qquad \iota_X\Omega=-f\beta\,.$$

On the other hand taking account of (2.5), exterior differentiation of (2.10) gives

(2.15) $$d\wedge\beta=\alpha\wedge\beta+t^2/f\Omega\,.$$

Now making use of (2.15) we derive from (2.14) the following equations

(2.16) $$\mathcal{L}_V\Omega=\Omega\,,\qquad \mathcal{L}_{X^V}\Omega=0\,,\qquad \mathcal{L}_X\Omega=-t^2\Omega\,.$$

These equations assert that $\Omega$ is *homogenous of rank* 1 [7] and that $X^V$ and $X$ are an *infinitesimal automorphism* and an *infinitesimal homothety* of $\Omega$ respectively.

Further by (2.13) and (2.14) we get

(2.17) $$i_{X^c}\Omega=(\iota\alpha)f\alpha-f\beta-t^2\lambda$$

and therefore

$$\mathcal{L}_{X^c}\Omega=\alpha^c\wedge f\alpha-2t^2\Omega\,. \tag{2.18}$$

But $\alpha^c$ being exact (as $\alpha$) we quickly obtain

$$d\wedge(\mathcal{L}_{X^c}\Omega)=0 \tag{2.19}$$

and this proves that $\Omega$ is a *relatively invariant* 2-form of $X^c$[6].

Next making use of the vertical derivation operator $i_v$ one finds $i_v\Omega=0$, and so by virtue of the definition given in [7] one may say that $\Omega$ is a *Finslerian form.*

According to [5] the complete lift $\varphi^c$ of $\varphi$ (with respect to $B^*$) is expressed by

$$\varphi^c=dv^1\wedge\omega^2+\cdots+dv^{2m-1}\wedge\omega^{2m}+\omega^1\wedge dv^2+\cdots+\omega^{2m-1}\wedge dv^{2m}\,. \tag{2.20}$$

By virtue of (1.14) a short calculation gives

$$d\wedge\varphi^c=\alpha\wedge\varphi^c \tag{2.21}$$

and so $\varphi^c$ defines on $TM_c$ a conformal symplectic structure $CSp(2m\,;\,R)$.

From 2.20 we obtain

$$i_V\varphi^c=-v^2\omega^1+v^1\omega^2-\cdots-v^{2m}\omega^{2m-1}+v^{2m-1}\omega^{2m}\,. \tag{2.22}$$

Thus

$$\mathcal{L}_V\varphi^c=\varphi^c \tag{2.23}$$

that is, $\varphi^c$ is *homogenous of degree* 1.

If we put

$$i_X\varphi^c=-t_2dv^1+t_1dv^2-\cdots+t_{2m-1}dv^{2m}=-\beta_a=-\mu_{\varphi^c}(X)\,. \tag{2.24}$$

we obtain

$$i_v\beta_a=\alpha_a \tag{2.25}$$

and one checks $(i_Vd_v+d_vi_V)\beta_a=i_v\beta_a$.

Exterior differentiation of (2.24) gives

$$d\wedge\beta_a=\alpha\wedge\beta_a+t^2\varphi^c \tag{2.26}$$

and from (2.24) and (2.25) we find

$$\mathcal{L}_V\beta_a=\beta_a\,, \tag{2.27}$$

that is, $\beta_a$ is homogenous of degree 1.

From (2.20) we also have

(2.28) $$i_X\varphi^c = i_X\varphi = -\alpha_a\,.$$

Now by (2.23), (2.27) and (2.28) we infer

(2.29) $$i_{[V,X]}\varphi^c = \mathcal{L}_V i_X\varphi^c - i_X\mathcal{L}_V\varphi^c = 0\,.$$

Clearly $i_{[V,X]}\alpha = 0$, and so referring to 2.21 we finally may write

(2.30) $$\mathcal{L}_{[V,X]}\varphi^c = 0$$

that is, the Lie bracket $[V, X]$ is an *infinitesimal automorphism* of $\varphi^c$.

Consider now the almost symplectic form

(2.31) $$\Theta = (\iota\alpha)(\alpha\wedge\lambda + \Omega) \in \wedge^2(TM_c)\,.$$

By 2.4 and 2.5 exterior differentiation of $\Theta$ gives

(2.32) $$d\wedge\Theta = \left(\frac{\alpha^c}{\iota\alpha} - \alpha\right)\wedge\Theta$$

and so $\Theta$ defines a second conformal structure on $TM_c$ having $\dfrac{\alpha^c}{\iota\alpha} - \alpha$ as co-vector of Lee.

One has

(2.33) $$d_v\Theta = \alpha\wedge\Omega\,, \qquad i_v\Theta = 0$$

and with the help of (2.11) and (2.32), one checks $d_v\Theta = [i_v, d_v]\Theta$.

Now making use of 2.16 we derive from 2.31 and 2.32

(2.34) $$\mathcal{L}_V\Theta = 2\Theta\,, \qquad \mathcal{L}_X\Theta = -t^2\Theta\,, \qquad \mathcal{L}_{X^V}\Theta = \frac{t^2}{\iota\alpha}\Theta\,.$$

Hence $\Theta$ is *homogenous of degree* 2, $X$ is an infinitesimal homothety of $\Theta$ and $X^V$ is an infinitesimal conformal transformation of $\Theta$.

We may formulate the preceding results as follows:

THEOREM. *Let $TM_c$ be the tangent bundle manifold having as basis the manifold $M_c$ of section* 1. *Let $V$, $\lambda$, $\Omega$ and $\iota$ be the canonical field on $TM_c$, the Liouville form, the symplectic canonical form and the operator which assigns to* 1-*forms on $M_c$ functions on $TM_c$ respectively. Then.*

(i) *$\Omega$ is a Finslerian form, $X$ is an infinitesimal homothety of $\Omega$, the vertical lift $X^V$ of $X$ is an infinitesimal automorphisme of $\Omega$, and $\Omega$ is a relatively invariant form of the complete lift $X^c$ of $X$;*

(ii) *the complete lift $\varphi^c$ of the conformal symplectic form $\varphi$ on $M_c$ is a conformal symplectic form on $TM_c$ and the Lie bracket $[V, X]$ is an infinitesimal automorphism of $\varphi^c$;*

(iii) *the form $\Theta = (\iota\alpha)(\alpha\wedge\lambda + \Omega)$ is homogenous of degree* 2 *and defines a second conformal symplectic structure on $TM_c$ having $\dfrac{\alpha^c}{\iota\alpha} - \alpha$ as co-vector of Lee and $X$ is an infinitesimal homothety of $\Theta$.*

Note. Let $S_{X_a}(M_c)$ be the *cross section* determined on $TM_c$ by the associated vector field $X_a$ of $X$. In consequence of (1.22) (that is the Lie derivate of $X$ with respect to $X_a$ vanishes) and of the theorem stated in [5] one may say that $X^c$ is tangent to the cross-section $S_{X_a}(M_c)$.

**3. Regular mechanical system $\mathcal{M}=\{M_c, T, \pi\}$ on $mTM_c$.** Consider now on $TM_c$ the mechanical system $\mathcal{M}=\{M_c, T, \pi\}$, [8] such that the *kinetic energy* $T$ and *semi-basic* 1-form $\pi$ be defined respectively by

$$T=l \tag{3.1}$$

and

$$\pi=l\alpha\,. \tag{3.2}$$

Referring to (2.6), one has

$$d\wedge d_V T=\Omega \tag{3.3}$$

and so according to J. Klein's definition, equation 3.3 proves that the system $\mathcal{M}$ is *regular*. (it has as fundamental form the symplectic canonical form of $TM_c$).
On the other hand a short calculation gives

$$V(T)=2T\,, \tag{3.4}$$

Hence $T$ is *homogenous of degree* 2.
If $Z$ is the *dynamical system* associated with $\mathcal{M}$ it is as is known [4] well defined by

$$i_Z\Omega=d(T-V(T))+\pi\,. \tag{3.5}$$

Since $T$ is homogenous of degree 2, the following theorem of A. Lichnerowicz [9] holds: the form

$$\Omega-(dT-\pi)\wedge dt\in\lambda^2((TM_c)\times R) \tag{3.6}$$

is an *integral relation of invariance* for $Z+\dfrac{\partial}{\partial t}$.

Further one has

$$d_v\Pi=\lambda\wedge\alpha\,,\qquad i_v\Pi=0 \tag{3.6}$$

and

$$d\wedge\Pi=\frac{dv}{v}\wedge\Pi, \tag{3.7}$$

and so the equation $d_v\Pi=[i_v, d\wedge]\Pi$ is verified.
By 3.6 and 3.7 a short computation gives

(3.8) $$\mathcal{L}_V \Pi = 2\Pi .$$

Hence $\Pi$ is homogenous of degree 2 as the kinetic energy $T$. This fact proves according to a known Proposition [3] that the dynamical system $Z$ is a *spray* on $M_c$.

Thus we have the

THEOREM. *Let $TM_c$ be the tangent bundle manifold discussed in section* 2. *Consider on $TM_c$ the mechanical system $\mathcal{M} = \{M_c, T, \pi\}$ whose kinetic energy is the Liouville function $l$ on $TM_c$, and whose semi-basic* 1-*form is the product by $l$ of the principal* 1-*form on $M_c$. Then:*

(i) *$\mathcal{M}$ is regular and has as fundamental form the canonical symplectic form on $TM_c$,*

(ii) *the kinetic energy $T$ is homogenous of degree* 2 *and the dynamical system associated with $\mathcal{M}$ is a spray on $M_c$.*

## REFERENCES

[1] ROSCA, R., Connexions conformes plates parallèles, C. R. Acad. Sc. Paris, t. **281**, Série A, 1975, pp. 699-701.

[2] LIBERMANN, P., Sur le problème d'équivalence de certaines structures infinitéstimales. Ann. Mat. Para. Appl., **36** (1954) 27-120.

[3] LEFEBVRE, J., Propriétés des algébres d'automorphismes de certaines structures presque symplectiques. C. R. Acad. Sc. Paris, t. **266**, 1965, Serie A, 354, pp. 354-356.

[4] GODBILLON, C., Géométrie différentielle et mécanique analytique Hermann, Paris, 1969.

[5] YANO, K. AND S. ISHIHARA, Tangent and cotangent bundles, differential geometry, M. Dekker, Inc., N. Y. 1973.

[6] ABRAHAM, R., Foundations of Mechanics. W. A. Benjamin Inc, 1967.

[7] KLEIN J. AND A. Voutier, Formes extérieures génératrices de sprays. Ann. Inst. Fourier, Grenoble, 18, 1 (1968) pp. 241-268.

[8] KLEIN, J., Espaces variationnels et mécanique. Ann. Inst. Fourier **12**, 1962, pp. 1-124.

[9] LICHNEROWICZ, A., Les relations intégrales d'invariance et leurs applications à la dynamique. Bull. Sc. Math. **70**, 1946, pp. 82-95.

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