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PLANAR GEODESIC SUBMANIFOLDS IN COMPLEX SPACE FORMS

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Let M^n and $\bar{M}^{n+p}$ be connected complete Riemannian manifolds of dimension n and $n+p$ respectively. An isometric immersion of M^n into $\bar{M}^{n+p}$ is called a *planar geodesic immersion* when every geodesic in M^n is mapped locally into 2-dimensional totally geodesic submanifold of $\bar{M}^{n+p}$. When the ambient manifold $\bar{M}^{n+p}$ is a space form of constant curvature $\tilde{c}$, K. Sakamoto [7] has showed that such an immersion is an isotropic immersion in the sence of B. O'Neill [6] with parallel second fundamental tensor. Using this fact, he reduced planar geodesic immersions into space forms to full, minimal and planar geodesic immersions of compact rank one symmetric spaces into spheres and obtained

THEOREM A. *Let $f: M^n \longrightarrow S^{n+q}(\tilde{c})$ be a planar geodesic immersion. Then the simply connected Riemannian covering manifold of M^n is a sphere, a complex projective space, a quaternionic projective space or a Cayley projective plane. The immersion is rigid.*

A submanifold M^n of a complex space form $\bar{M}^{n+p}(\tilde{c})$ with constant holomorphic sectional curvature $\tilde{c}$ is called *complex* or *invariant* (resp. *totolly real*) if each tangent space of M^n is mapped into itself (resp. the normal space) by the complex structure of $\bar{M}^{n+p}(\tilde{c})$. A complex submanifold of a Kaehler manifold is also a Kaehler manifold. K. Ogiue [5] has showed that if $M^n(c)$ is a Kaehler submanifold immersed in $\bar{M}^{n+p}(\tilde{c})$ and if the second fundamental form of the immersion is parallel, then either $c=\tilde{c}$ (i. e., $M^n(c)$ is totally geodesic in $\bar{M}^{n+p}(\tilde{c})$) or $c=\tilde{c}/2$, the latter case arising only when $\tilde{c}>0$. Moreover the immersion is rigid. When $\tilde{c}\leqq 0$, E. Calabi [1] proved that if $M^n(c)$ is imbedded in $\bar{M}^{n+p}(\tilde{c})$ as a Kaehler submanifold, then $M^n(c)$ is totally geodesic in $\bar{M}^{n+p}(\tilde{c})$.

In this paper we study a planar geodesic immersion $f: M^n \longrightarrow \bar{M}^{n+p}(\tilde{c})$ of a connected complete Riemannian manifold of real dimension n into a complex space form of real dimension $n+p$ with constant holomorphic sectional curvature $\tilde{c}\neq 0$. When the immersion f is complex or totally real, it is an isotropic immersion with parallel second fundamental tensor. Moreover, if the immersion f is totally real and not totally geodesic, we can reduce the immersion to a full, minimal, planar geodesic immersion of M^n into a real projective space $RP^{n+p}(\tilde{c}/4)$ (resp. a real hyperbolic space $H^{n+p}(\tilde{c}/4)$) when $\bar{M}^{n+p}(\tilde{c})$ is a complex projective

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space $CP^m(\tilde{c})$ (resp. a complex hyperbolic space $D^m(\tilde{c})$), where $m=n+p/2$. Roughly speaking, our results are due to Calabi-Ogiue's results in the case where M^n is complex and due to Theorem A in the case where M^n is totally real.

Manifolds, tensor fields, geometric objects and mappings we consider are assumed to be differentiable and of class C^∞.

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§ 1. Preliminaries.

Let M^n and $\bar{M}^{n+p}$ be connected complete Riemannian manifolds with real dimension n ($\geqq 2$) and $n+p$ respectively and let $f: M^n \longrightarrow \bar{M}^{n+p}$ an isometric immersion. We denote by $\bar{\nabla}$ the covariant differentiation with respect to the Riemannian metric of $\bar{M}^{n+p}$. Then we may write

$$(1.1) \qquad \bar{\nabla}_X Y = \nabla_X Y + H(X, Y)$$

for arbitrary tangent vector fields X and Y on M, where $\nabla_X Y$ and $H(X, Y)$ denote the components of $\bar{\nabla}_X Y$ tangent and normal to M^n respectively. Then ∇ becomes the covariant differentiation of the Riemannian manifold M^n. The symmetric bilinear form H valued in the normal space is called the *second fundamental form* of the immersion f. For a normal vector field C on a neighborhood of $P \in M^n$, we write

$$(1.2) \qquad \bar{\nabla}_X C = -A_C X + \nabla_X^{\perp} C,$$

$-A_C X$ and $\nabla_X^{\perp} C$ being the components of $\bar{\nabla}_X C$ tangent and normal to M^n respectively, where $\nabla^{\perp}$ is the covariant differentiation with respect to the induced connection in the normal bundle NM which will be called the *normal connection*. Denoting the inner product of vectors with respect to the Riemannian metric of $\bar{M}^{n+p}$ by $\langle\ ,\ \rangle$, we find that the tangential component $-A_C X$ of $\bar{\nabla}_X C$ and the second fundamental form H are related by

$$(1.3) \qquad \langle A_C X, Y\rangle = \langle H(X, Y), C\rangle$$

for any vector Y contained in the tangent space $T_P M^n$. Thus A_C is a symmetric linear transformation of $T_P M^n$. Given an orthonormal normal frame $\{C_{n+1}, \cdots, C_{n+p}\}$, we write $A_\alpha = A_{C_\alpha}$ ($\alpha = n+1, \cdots, n+p$). In the sequel, indices α, β and γ run over the range $\{n+1, \cdots, n+p\}$.

Let $'\nabla$ be the covariant differentiation with respect to the induced connection in the direct sum (tangent bundle TM) $\oplus$ (normal bundle NM). For the second fundamental form H, we find

$$(1.4) \qquad ('\nabla_X H)(Y, Z) = \nabla_X^{\perp}(H(Y, Z)) - H(\nabla_X Y, Z) - H(Y, \nabla_X Z)$$

for all tangent vector fields X, Y and Z on M.

Let the ambient manifold $\bar{M}^{n+p}$ be a complete, simply connected complex space form of real dimension $n+p$ with constant holomorphic sectional curvature $\tilde{c}\neq 0$. The complex dimension will be denoted by $m=(n+p)/2$. Thus $\bar{M}^{n+p}$ will be complex projective space $CP^m(\tilde{c})$ or complex hyperbolic space $D^m(\tilde{c})$ according as $\tilde{c}>0$ or $\tilde{c}<0$. If $\bar{J}$ denotes the complex structure, the Riemannian curvature tensor of the complex space form $\bar{M}^m(\tilde{c})$ is given by

$$\bar{R}(\bar{X}, \bar{Y})\bar{Z}=(\tilde{c}/4)(\langle\bar{Y}, \bar{Z}\rangle\bar{X}-\langle\bar{X}, \bar{Z}\rangle Y+\langle\bar{J}\bar{Y}, \bar{Z}\rangle\bar{J}\bar{X} -\langle\bar{J}\bar{X}, \bar{Z}\rangle\bar{J}\bar{Y}-2\langle\bar{J}\bar{X}, \bar{Y}\rangle\bar{J}\bar{Z}) \tag{1.5}$$

for all $\bar{X}$, $\bar{Y}$, $\bar{Z}$ tangent to $\bar{M}^m(\tilde{c})$.

We denote by Proj_{TM} and Proj_{NM} the projections of $T_P\bar{M}^m(\tilde{c})$ to the tangent space T_PM^n and the normal space N_PM^n respectively and put $J=\mathrm{Proj}_{TM}\circ\bar{J}|TM$, $J_N=\mathrm{Proj}_{NM}\circ\bar{J}|TM$, $J_T=\mathrm{Proj}_{TM}\circ\bar{J}|NM$ and $J^{\perp}=\mathrm{Proj}_{NM}\circ\bar{J}|NM$. Then we can write

$$\bar{J}X=J\bar{X}+J_NX, \quad \bar{J}C=J_TC+J^{\perp}C \tag{1.6}$$

for all X tangent to M^n. Taking account of $\bar{J}^2=-I$, we see that these tensors satisfy

$$J^2+J_TJ_N=-I, \quad J_NJ+J^{\perp}J_N=0, \qquad J^{\perp 2}+J_NJ_T=-I, \quad JJ_T+J_TJ^{\perp}=0, \tag{1.7}$$

I denoting the identity transformation, and also we find

$$\langle J_NX, C\rangle=-\langle X, J_TC\rangle \tag{1.8}$$

with the help of $\langle\bar{J}X, Y\rangle=-\langle X, \bar{J}Y\rangle$.

Differentiating covariantly the left hand side of (1.6), we have

$$\bar{\nabla}_Y\bar{J}X=(\nabla_YJ)X+J(\nabla_YX)-A_{J_NX}Y+('\nabla_YJ_N)X+J_N(\nabla_YX)+H(JX, Y)$$

because of (1.1) and (1.2). On the other hand, using $\bar{\nabla}\bar{J}=0$ and (1.6) itself, we also have

$$\bar{\nabla}_Y\bar{J}X=J(\nabla_YX)+J_TH(X, Y)+J_N(\nabla_YX)+J^{\perp}H(X, Y),$$

from which

$$(\nabla_YJ)X=A_{J_NX}Y+J_TH(X, Y), \qquad ('\nabla_YJ_N)X=J^{\perp}H(X, Y)-H(JX, Y). \tag{1.9}$$

Similarly, from the right hand side of (1.6), we also obtain

$$('\nabla_XJ_T)C=A_{J^{\perp}C}X-JA_CX, \qquad ('\nabla_XJ^{\perp})C=-J_NA_CX-H(X, J_TC). \tag{1.10}$$

Let's denote curvature tensors for the connections ∇ and $\nabla^{\perp}$ by R and $R^{\perp}$ respectively. Then, using (1.5), we can easily see that the structure equations of Gauss are given by

$$
\begin{aligned}
(1.11)\qquad R(X, Y)Z=&(\tilde{c}/4)(\langle Y, Z\rangle X-\langle X, Z\rangle Y+\langle JY, Z\rangle JX\\
&-\langle JX, Z\rangle JY-2\langle JX, Y\rangle JZ)\\
&+\sum_{\alpha}(\langle A_\alpha Y, Z\rangle A_\alpha X-\langle A_\alpha X, Z\rangle A_\alpha Y),
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
(1.11)'\qquad \langle R(X, Y)Z, W\rangle=&(\tilde{c}/4)(\langle Y, Z\rangle\langle X, W\rangle-\langle X, Z\rangle\langle Y, W\rangle\\
&+\langle JY, Z\rangle\langle JX, W\rangle-\langle JX, Z\rangle\langle JY, W\rangle\\
&-2\langle JX, Y\rangle\langle JZ, W\rangle)\\
&+\langle H(Y, Z), H(X, W)\rangle-\langle H(X, Z), H(Y, W)\rangle,
\end{aligned}
$$

and those of Coddazi and Ricci respectively by

$$
\begin{aligned}
(1.12)\qquad &('\nabla_X H)(Y, Z)-('\nabla_Y H)(X, Z)\\
&=(\tilde{c}/4)(\langle JY, Z\rangle J_N X-\langle JX, Z\rangle J_N Y\\
&\qquad\qquad -2\langle JX, Y\rangle J_N Z),
\end{aligned}
$$

$$
\begin{aligned}
(1.13)\qquad R^{\perp}(X, Y)C=&(\tilde{c}/4)(\langle J_N Y, C\rangle J_N X-\langle J_N X, C\rangle J_N Y\\
&-2\langle JX, Y\rangle J^{\perp}C)+\sum_{\alpha}\langle [A_C, A_\alpha]X, Y\rangle C_\alpha.
\end{aligned}
$$

Therefore, if the submanifold M^n is complex or totally real, that is, $J_N=0$ or $J=0$, then

$$
(1.14)\qquad ('\nabla_X H)(Y, Z)-('\nabla_Y H)(X, Z)=0
$$

with the help of (1.12). Conversely if the above equation is verified at every point of M^n, then M^n is complex or totally real. Hence 2-dimensional complete totally geodesic submanifolds in $CP^m(\tilde{c})$ are $CP^1(\tilde{c})$ or $RP^2(\tilde{c}/4)$.

§ 2. Planar geodesic submanifolds.

We consider an isometric immersion $f: M^n \longrightarrow \bar{M}^m(\tilde{c})$ such that every geodesic $\sigma:(a, b)\longrightarrow M^n$ on M^n is locally contained in a 2-dimensional totally geodesic submanifold of $\bar{M}^m(\tilde{c})$, that is, for each $t\in(a, b)$, there exist an open interval I, $t\in I\subset(a, b)$ and 2-dimensional totally geodesic submanifold $M^2{}_t$ such that $f(\sigma(I))\subset M^2{}_t$. In this case the immersion f is called a *planar geodesic immersion* and the manifold M^n called a *planar geodesic submanifold*. Then we have

LEMMA 2.1. (*S. L. Hong* [3] *and K. Sakamoto* [7]). *Let X and Y are orthonormal vectors at $P\in M^n$ in TM. Then*

$$\langle H(X, X), H(X, Y)\rangle=0. \tag{2.1}$$

The equation (2.1) is equivalent to the condition that f is isotropic, i. e.,

$$\|H(X, X)\|^2=\lambda^2 \tag{2.2}$$

for all unit vector X tangent to M^n, where λ is a differentiable function on M^n. In fact, in this case the function λ^2 is given by

$$\lambda^2=1/n(n+2)\sum_\alpha \{(\text{trace } A_\alpha)^2+\text{trace } A_\alpha{}^2\}$$

(See K. Sakamoto [7]). We first prove

LEMMA 2.2. *The function λ^2 is constant on M^n.*

Proof. Let P be arbitrary fixed point of M^n. Take normal coordinate neighborhood U around P in M^n. Let X be any unit vector tangent to M^n at P and Y an unit vector orthogonal to X at P. Let σ be a geodesic with unit speed such that $\sigma(0)=P$ and $\dot\sigma(0)=Y$. Then, using (1.1), we find

$$\bar{\nabla}_{f\dot\sigma}f\dot\sigma=f\nabla_{\dot\sigma}\dot\sigma+H(\dot\sigma, \dot\sigma)=H(\dot\sigma, \dot\sigma).$$

We assume now $H(Y, Y)=H(\dot\sigma(0), \dot\sigma(0))\neq 0$. Then, since M^2 is totally geodesic submanifold in $\bar{M}^m(\tilde c)$, we have $\bar{\nabla}_{f\dot\sigma}H(\dot\sigma, \dot\sigma)\in T_{f\cdot\sigma}M^2$ and hence

$$\bar{\nabla}_{f\dot\sigma}H(\dot\sigma, \dot\sigma)=a(t)f\dot\sigma+b(t)H(\dot\sigma, \dot\sigma)$$

for some differentiable functions $a(t)$ and $b(t)$. Translate X and Y parallely to each point of U along the unique geodesic from P to that point. Then we have vector fields defined on U denoted by $\bar X$ and $\bar Y$ which extend X and Y respectively and satisfy

$$\nabla_X\bar X=\nabla_Y\bar Y=\nabla_X\bar Y=\nabla_Y\bar X=0 \quad \text{at} \quad P.$$

We can compute $(X\cdot\lambda^2)(P)$ as followings:

$$\begin{aligned}(X\cdot\lambda^2)(P)&=X\cdot\langle H(\bar Y, \bar Y), H(\bar Y, \bar Y)\rangle|P=2\langle\bar{\nabla}_{\bar X}(H(\bar Y, \bar Y)), H(\bar Y, \bar Y)\rangle|P\\ &=2\langle\nabla^\perp_{\bar X}(H(\bar Y, \bar Y)), H(\bar Y, \bar Y)\rangle|P\\ &=2\langle('\nabla_{\bar X}H)(\bar Y, \bar Y)+2H(\nabla_{\bar X}\bar Y, \bar Y), H(\bar Y, \bar Y)\rangle|P\\ &=2\langle('\nabla_{\bar X}H)(\bar Y, \bar Y), H(\bar Y, \bar Y)\rangle|P\\ &=2\langle('\nabla_{\bar Y}H)(\bar X, \bar Y)+(\tilde c/4)(\langle J\bar Y, \bar Y\rangle J_N\bar X-\langle J\bar X, \bar Y\rangle J_N\bar Y\\ &\qquad -2\langle J\bar X, \bar Y\rangle J_N\bar Y), H(Y, Y)\rangle|P\end{aligned}$$

$$=2\langle\nabla^{\perp}_{\bar{Y}}(H(\bar{X}, \bar{Y})), H(\bar{Y}, \bar{Y})\rangle|P$$
$$-(3/2)\tilde{c}\langle JX, Y\rangle\langle J_N Y, H(Y, Y)\rangle$$
$$=2\langle\bar{\nabla}_{\bar{Y}}(H(\bar{X}, \bar{Y})), H(\bar{Y}, \bar{Y})\rangle|P$$
$$-(3/2)\tilde{c}\langle JX, Y\rangle\langle J_N Y, H(Y, Y)\rangle$$
$$=-2\langle H(\bar{X}, \bar{Y}), \bar{\nabla}_{\bar{Y}}(H(\bar{Y}, \bar{Y}))\rangle|P$$
$$-(3/2)\tilde{c}\langle JX, Y\rangle\langle J_N Y, H(Y, Y)\rangle$$

with the help of (1.12), (2.1) and (2.2). Since $\bar{\nabla}_{\bar{Y}}(H(\bar{Y}, \bar{Y}))|P=\bar{\nabla}_{f\dot{\sigma}}(H(\dot{\sigma}, \dot{\sigma}))|_{t=0}$ $=a(0)fY+b(0)H(Y, Y)$, the first term of the last equation vanishes. Hence we obtain

$$(X\cdot\lambda^2)(P)=(-3/2)\tilde{c}\langle JX, Y\rangle\langle J_N Y, H(Y, Y)\rangle.$$

Since P is taken arbitrary in M^n, for orthonormal vectors X and Y

$$(X\cdot\lambda^2)=(-3/2)\tilde{c}\langle \bar{J}X, Y\rangle\langle \bar{J}Y, H(Y, Y)\rangle.$$

If M^2 is invariant, then $\bar{J}Y\in T_P M^2$ and hence $H(Y, Y)=\pm\lambda\bar{J}Y$. Thus we have $X\cdot\lambda^2=0$. If M^2 is totally real, then $\bar{J}Y$ is orthogonal to $T_P M^2$ and consequently $\bar{J}Y$ is orthogonal to $H(Y, Y)$. Thus we have $X\cdot\lambda^2=0$. Therefore we have proved $(X\cdot\lambda^2)(P)=0$ for arbitrary X if $\lambda^2(P)\neq 0$. When $\lambda^2(P)=0$, λ^2 takes minimum at P. Thus $(X\cdot\lambda^2)(P)=0$ for any X. These complete the proof.

Q. E. D.

From the above proof we have

$$H(X, X)=\pm\lambda\bar{J}X \quad \text{or} \quad H(X, X)\perp\bar{J}X$$

for each unit vector X tangent to M^n. But if we regard $\langle H(X, X), \bar{J}X\rangle$ as a function on the unit sphere bundle, then this function is clearly differentiable (continuous). Hence the above relations are established for all unit vectors X. Thus we have

LEMMA 2.3. *Every geodesic is contained in a totally geodesic submanifold* $CP^1(\tilde{c})$ *or* $RP^2(\tilde{c}/4)$.

In the sequel, we denote by (C) the case where $H(X, X)=\lambda\bar{J}X$ holds on the unit sphere bundle and by (R) the case where $H(X, X)\perp\bar{J}X$ always holds on the unit sphere bundle.

Finally we prepare the following lemma.

LEMMA 2.4. *For all vectors X, Y and Z tangent to* M^n

$$('\nabla_X H)(Y, Z)=(-\tilde{c}/4)(\langle JX, Y\rangle J_N Z+\langle JX, Z\rangle J_N Y). \tag{2.3}$$

Proof. Let X be any unit vector at any point $P\in M^n$. Let σ be a unit speed geodesic such that $\sigma(0)=P$ and $\dot\sigma(0)=X$. We may assume that $\lambda\neq 0$. Since $\|H(\dot\sigma, \dot\sigma)\|^2=\lambda^2=$constant, and $\lambda\neq 0$, we can see that $\bar\nabla_{f\dot\sigma}(H(\dot\sigma, \dot\sigma))=-\lambda^2 f\dot\sigma$. On the other hand, we also have

$$\bar\nabla_{f\dot\sigma}(H(\dot\sigma, \dot\sigma))=-fA_{H(\dot\sigma,\dot\sigma)}\dot\sigma+\nabla^{\perp}_{\dot\sigma}(H(\dot\sigma, \dot\sigma)).$$

Thus $\nabla^{\perp}_{\dot\sigma}(H(\dot\sigma, \dot\sigma))=0$ and $A_{H(\dot\sigma,\dot\sigma)}\dot\sigma=\lambda^2\dot\sigma$. These imply

$$('\nabla H)(X, X, X)=0 \quad \text{and} \quad A_{H(X,X)}X=\lambda^2 X\langle X, X\rangle$$

for any vector X. Therefore we obtain

$$\mathfrak{S}('\nabla H)(X, Y, Z)=0$$

and

$$\mathfrak{S}\sum_{\alpha}\langle A_\alpha X, Y\rangle A_\alpha Z=\lambda^2\mathfrak{S}\langle X, Y\rangle Z,$$

where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z. The latter is the isotropic condition. The former reduces to

$$('\nabla_X H)(Y, Z)+('\nabla_Y H)(Z, X)+('\nabla_Z H)(X, Y)=0,$$

from which, taking account of (1.12), we have

$$\begin{aligned}0=&('\nabla_X H)(Y, Z)+('\nabla_X H)(Y, Z)+(\tilde c/4)(\langle JX, Z\rangle J_N Y\\ &-\langle JY, Z\rangle J_N X-2\langle JY, X\rangle J_N Z)+('\nabla_X H)(Y, Z)\\ &+(\tilde c/4)(\langle JX, Y\rangle J_N Z-\langle JZ, Y\rangle J_N X-2\langle JZ, X\rangle J_N Y)\\ =&3('\nabla_X H)(Y, Z)+(3\tilde c/4)(\langle JX, Y\rangle J_N Z+\langle JX, Z\rangle J_N Y),\end{aligned}$$

and hence (2.3). Q. E. D.

Therefore we see that if the submanifold M^n is complex or totally real, then the second fundamental form is parallel.

§3. The case (C).

In §§3 and 4 we shall study only planar geodesic submanifolds of complex projective space $CP^m(\tilde c)$ and those of complex hyperbolic space $D^m(\tilde c)$.

First we consider the case (C), that is, the case where $H(X, X)=\lambda\bar J X$ holds on the unit sphere bundle. In this case $\lambda=0$ and consequently we have

PROPOSITION 1. *The case* (C) *occurs only if M^n is totally geodesic, $M^n=RP^n(\tilde c/4)$ or $CP^{n/2}(\tilde c)$ and the immersion is rigid.*

§ 4. The case (R).

Next, we consider the case (R), that is, the case that $H(X, X) \perp \bar{J}X$ holds on the unit sphere bundle. Then we have $\langle H(X, X), J_N X\rangle = 0$ for any vector X. Therefore, symmetrizing this equation, we have

LEMMA 4.1. *In the case* (R) *the following equations hold*:

$$\mathfrak{S}\langle H(X, Y), J_N Z\rangle = 0.$$

We first assume that the planar geodesic submanifold M^n is a Kaehler submanifold. Then the second fundamental form is parallel because of Lemma 2.4. On the other hand, since the immersion f is isotropic, M^n is of constant holomorphic sectional curvature $\tilde{c}-2\lambda^2$. In fact, using (1.11)′, we find

$$\langle R(X, JX)JX, X\rangle = \tilde{c} - 2\|H(X, X)\|^2 = \tilde{c} - 2\lambda^2$$

for arbitrary unit vector X. Since the second fundamental form is parallel, we can also see that $\tilde{c}-2\lambda^2=\tilde{c}/2$ (i.e., $\lambda^2=\tilde{c}/4$). This conclusion is due to Ogiue [5]. He obtained his results by establishing an equation of Simon's type. Thus we have

PROPOSITION 2. *If a planar geodesic submanifold M^n of $CP^m(\tilde{c})$ is a Kaehler submanifold, then it is $CP^{n/2}(\tilde{c}/2)$ or $CP^{n/2}(\tilde{c})$, i.e., a complex Veronese manifold or totally geodesic. The immersion is rigid.*

We next consider the case where the planar geodesic submanifold M^n is totally real. We then have $A_{J_N X}Y + J_T H(X, Y) = 0$ which is a direct consequence of $J=0$ and (1.9). Therefore $\langle A_{J_N X}Y, Z\rangle = -\langle J_T H(X, Y), Z\rangle$ and consequently $\langle J_N X, H(Y, Z)\rangle = -\langle X, J_T H(Y, Z)\rangle = \langle A_{J_N Z}Y, X\rangle = \langle J_N Z, H(X, Y)\rangle$. Thus we find $\langle J_N X, H(Y, Z)\rangle = \langle J_N Z, H(X, Y)\rangle = \langle J_N Y, H(Z, X)\rangle$, from which and Lemma 4.1,

$$\langle J_N X, H(Y, Z)\rangle = 0. \tag{4.1}$$

Moreover the second fundamental form is parallel too. Using these facts, we shall reduce the codimension to the dimension of the first normal space. The first normal space $N_1(P)$ at P is defined as the subspace in the normal space $N_P M^n$ which is spanned by the set $\{H(X, Y) : X, Y \in T_P M^n\}$. Denoting by $Sp\{\ \}$ the vector space spanned by the set $\{\ \}$ of vectors, we can see (cf. Sakamoto [7])

$$N_1(P) = Sp\{C : A_C = 0\}_P^{\perp}$$

and

$$N_1(P) \approx Sp\{A_C : C \in N_P M^n\}$$

in the vector space consisting of symmetric linear transformations on $T_P M^n$, where $^\perp$ means the orthogonal complement in $N_P M^n$.

LEMMA 4.1. (*Sakamoto* [7]). *If a planar geodesic submanifold M^n of $CP^m(\tilde{c})$ is totally real and of type* (R), *then the first normal space N_1 is parallel with respect to the normal connection and is orthogonal to $J_N(TM)$ in the normal bundle, i.e., $N_1 \perp J_N(TM)$.*

Proof. Let P and Q be arbitrary two points of M^n and σ a curve from P to Q in M^n. Let $\{X_1, \cdots, X_n\}$ be an orthonormal base of $T_P M^n$. Then $N_1(P)=Sp\{H(X_i, X_j) : i, j=1, 2, \cdots, n\}$. Translate parallely this orthonormal base from P to Q along σ with respect to the Riemannian connection of M^n. Then we have orthonormal frame field parallel along σ, which will be also denoted by $\{X_1, \cdots, X_n\}$. Thus $H(X_i, X_j)$ is parallel along σ with respect to the induced connection in the normal bundle for each i and j, because

$$\nabla^\perp_{\dot\sigma} H(X_i, X_j)=('\nabla_{\dot\sigma} H)(X_i, X_j)+H(\nabla_{\dot\sigma} X_i, X_j)+H(X_i, \nabla_{\dot\sigma} X_j)=0,$$

where we have used $'\nabla H=0$. It follows that the parallel displacement along σ from P to Q with respect to the induced connection in the normal bundle gives an isomorphism of $N_1(P)$ to $N_1(Q)$. Hence the dimension of N_1 is constant and N_1 is invariant by the parallel displacement with respect to $\nabla^\perp$ and hence $N_1 \perp J_N(TM)$ with the help of (4.1). Q.E.D.

Since $\langle J_N X, H(Y, Z)\rangle=0$, we see that $H(Y, Z)=0$, i.e., that M^n is totally geodesic, when $p=n$. Thus we have

PROPOSITION 3. *If a planar geodesic submanifold M^n of $CP^n(\tilde{c})$ is totally real and type* (R), *then M^n is totally geodesic.*

Now, we suppose $p>n$. Then we have

LEMMA 4.2. *The subspace $N_1 \oplus TM$ is totally real.*

Proof. From $\langle J_N X, H(Y, Z)\rangle=0$ we can easily see that $J_T N_1=0$. Differentiating covariantly the equation $\langle J_N X, H(Y, Z)\rangle=0$ with respect to $'\nabla$ and using $'\nabla H=0$ and (1.9), we have $J^\perp N_1 \perp N_1$ and so the assertion is followed from Lemma 4.1. Q.E.D.

LEMMA 4.3. *There exists a totally geodesic and totally real submanifold $RP^{n+q}(\tilde{c}/4)$ with constant sectional curvature $\tilde{c}/4$ such that $f(M^n) \subset RP^{n+q}(\tilde{c}/4)$ and the immersion $f: M^n \longrightarrow RP^{n+q}(\tilde{c}/4)$ is full, where $q=\dim N_1$.*

Proof. From Lemma 4.2 it follows that there exists a unique totally geodesic and totally real submanifold $RP^{n+q}(\tilde{c}/4)$ tangent to $N_1(P) \oplus T_P M^n$ at distinguished point $P \in M^n$. Let σ be a geodesic starting at P and ending at an arbitrary point $Q \in M^n$. Then $f\cdot\sigma$ is contained in RP^2 spanned by $f\dot\sigma(0)$ and

$H(\dot{\sigma}(0), \dot{\sigma}(0))$. Thus $f\cdot\sigma \subset RP^2 \subset RP^{n+q}(\tilde{c}/4)$. Clearly, the immersion $f: M^n \longrightarrow RP^{n+q}(\tilde{c}/4)$ is full. Q.E.D.

Therefore we can reduce our immersion to a full, planar geodesic immersion $f: M^n \longrightarrow RP^{n+q}(\tilde{c}/4)$. We may assume locally that the immersion $f: M^n \longrightarrow S^{n+q}(\tilde{c}/4)$ is planar and full. Thus, combining Theorem A, we have

PROPOSITION 4. *If a planar geodesic submanifold M^n of $CP^m(\tilde{c})$ is totally real and of type* (R), *then M^n is a compact irreducible symmetric space of rank one, i.e., M^n is one of a sphere, a real projective space, a complex projective space, a quaternionic projective space and a Cayley projective plane.*

Remark. Let the ambient manifold $\bar{M}^m(\tilde{c})$ $(m=(n+p)/2)$ is complex hyperbolic space $D^m(\tilde{c})$. When the case (C) is arised, we can also see that a planar geodesic submanifold M^n of $D^m(\tilde{c})$ is totally geodesic by means of Proposition 1. When the submanifold M^n is complex and of type (R), M^n is totally geodesic in $D^m(\tilde{c})$ because it is also of constant holomorphic sectional curvature. Finally when the immersion $f: M^n \longrightarrow D^m(\tilde{c})$ is totally real and of type (R), we can also reduce the immersion to a full, planar geodesic immersion $f: M^n \longrightarrow H^{n+q}(\tilde{c}/4)$. Hence, using Theorem A, we find that M^n is a sphere, a complex projective space, a quaternionic projective space or a Cayley projective plane and the immersion is rigid.

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