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JØRGENSEN GROUPS OF PARABOLIC TYPE III (UNCOUNTABLY INFINITE CASE)

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To the memory of Professor Nobuyuki Suita

Abstract

Jørgensen groups of parabolic type parametrized by three real parameters are divided into three types: finite type, countably infinite type and uncoutably infinite type. In the previous papers we found all Jørgensen groups of finite type and of countably infinite type. In this paper we find all Jørgensen groups of uncoutably infinite type. Consequently, the problem finding all Jørgensen groups of these parabolic type has been completely solved.

0. Introduction

0.1. It is one of the most important problems in the theory of Kleinian groups to decide whether or not a subgroup G of the Möbius transformation group is discrete. For this problem there are two important and uesful theorems: One is Poincaré's polyhedron theorem, which gives a sufficient condition for G to be discete. The other is Jørgensen's inequality, which gives a necessary condition for a two-generator Möbius transformation group to be discrete.

0.2. Let Möb denote the set of all linear fractional transformations (Möbius tranformations)

$$A(z) = \frac{az+b}{cz+d}$$

of the extended complex plane $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, where a, b, c, d are complex numbers and the determinant $ad - bc = 1$. There is an isomorphism between Möb and $PSL(2, \mathbf{C})$. We always write elements of Möb as matrices with

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determinant 1 in this paper. We recall that Möb ($= PSL(2, \mathbf{C})$) acts on the upper half space H^3 of $\mathbf{R}^3$ as the group of conformal isometries of hyperbolic 3-space.

In this paper we use a Kleinian group in the same meaning as a discrete group. Namely, a Kleinian group is a discrete subgroup of Möb. A Kleinian group G is of *the first kind* if the limit set $\Lambda(G)$ of G is all of the extended complex plane $\hat{\mathbf{C}}$ and it is of *the second kind* otherwise. A subgroup G of Möb is said to be *elementary* if there exists a finite G-orbit in $\hat{\mathbf{R}}^3$.

0.3. The *trace* $\mathrm{tr}(A)$ of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (ad - bc = 1)$$

in $SL(2, \mathbf{C})$ is defined by $\mathrm{tr}(A) = a + d$. We remark that the trace of an element of Möb ($= PSL(2, \mathbf{C})$) is not well-defined, but Jørgensen number (defined later) is still well-defined after choosing matrix representatives.

0.4. In 1976 Jørgensen obtained the following important theorem, which gives a necessary condition for a non-elementary Möbius transformation group $G = \langle A, B \rangle$ to be discrete.

Theorem A (Jørgensen [**1**]). *Suppose that the Möbius transformations A and B generate a non-elementary discrete group. Then*

$$J(A, B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$

The lower bound 1 *is best possible.*

0.5. Here we will state some definitions.

Definition 1. Let A and B be Möbius transformations. The *Jørgensen number* $J(A, B)$ for the ordered pair (A, B) is defined by

$$J(A, B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|.$$

Definition 2. A subgroup G of Möb is called a *Jørgensen group* if G satiafies the following four conditions:

(1) G is a two-generator group.
(2) G is a discrete group.
(3) G is a non-elementary group.
(4) There exist generators A and B of G such that $J(A, B) = 1$.

0.6. Jørgensen and Kiikka showed the following.

Theorem B (Jørgensen-Kiikka [**2**]). *Let $\langle A, B \rangle$ be a non-elementary discrete group with $J(A, B) = 1$. Then A is elliptic of order at least seven or A is parabolic.*

If $\langle A, B\rangle$ is a Jørgensen group such that A is parabolic and $J(A, B) = 1$, then we call it a *Jørgensen group of parabolic type*. There are infinite number of Jørgensen groups (Jørgensen-Lascurain-Pignataro [**3**], Sato [**6**]).

Now it gives rise to the following problem.

Problem 1. Find all Jørgensen groups of parabolic type.

0.7. Let $\langle A, B\rangle$ be a marked two-generator group such that A is parabolic. Then we can normalize A and B as follows:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := B_{\sigma,\mu} = \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix}$$

where $\sigma \in \mathbf{C}\backslash\{0\}$ and $\mu \in \mathbf{C}$ (see [**4**] for the detail).

We denote by $G_{\sigma,\mu}$ the marked group generated by A and $B_{\sigma,\mu}$: $G_{\sigma,\mu} = \langle A, B_{\sigma,\mu}\rangle$. We say that $(\sigma,\mu) \in \mathbf{C}\backslash\{0\} \times \mathbf{C}$ *is the point representing a marked group* $G_{\sigma,\mu}$ *and that* $G_{\sigma,\mu}$ *is the marked group coresponding to a point* (σ,μ).

0.8. In [**6**], Sato considered the case of $\mu = ik$ ($k \in \mathbf{R}$). Namely, he considered marked two-generator groups $G_{\sigma,ik} = \langle A, B_{\sigma,ik}\rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\sigma,ik} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix}$$

where $\sigma \in \mathbf{C}\backslash\{0\}$ and $k \in \mathbf{R}$.

Now we have the following conjecture.

Conjecture. For any Jørgensen group G of parabolic type there exists a marked group $G_{\sigma,ik}$ ($\sigma \in \mathbf{C}\backslash\{0\}, k \in \mathbf{R}$) such that $G_{\sigma,ik}$ is conjugate to G in Möb.

If this conjecture is true, then it is suffucient to consider the case of $\mu = ik$ in order to find all Jørgensen groups of parabolic type. In this paper we only consider the case of $\mu = ik$.

0.9. Let C be the following cylinder:

$$C = \{(\sigma, ik) \mid |\sigma| = 1, k \in \mathbf{R}\}.$$

Theorem C (Sato [**6**]). *If a marked two-generator group* $G_{\sigma,ik}$ ($\sigma \in \mathbf{C}\backslash\{0\}, k \in \mathbf{R}$) *is a Jørgensen group, then the point* (σ, ik) *representing* $G_{\sigma,ik}$ *lies on the cylinder* C.

If (σ, ik) is a point on the cylinder C, then we set $\sigma = -ie^{i\theta}$ $(0 \leq \theta \leq 2\pi)$. For simplicity we write $B_{\theta,k}$ and $G_{\theta,k}$ for $B_{\sigma,ik}$ and $G_{\sigma,ik}$ with $\sigma = -ie^{i\theta}$, respectively. If $G_{\theta,k}$ is a Jørgensen group, then we call the group a *Jørgensen group of parabolic type* (θ, k).

Now it gives rise to the following problem.

Problem 2. Find all Jørgensen groups of parabolic type (θ, k).

0.10. We devide Jørgensen groups of this type into three parts as follows:
Part 1. $|k| \le \sqrt{3}/2,\ 0 \le \theta \le 2\pi$ (finite case).
Part 2. $\sqrt{3}/2 < |k| \le 1,\ 0 \le \theta \le 2\pi$ (countably infinte case).
Part 3. $1 < |k|,\ 0 \le \theta \le 2\pi$ (uncountably infinte case).
By some lemmas in [**7**], it suffices to consider the case of $0 \le \theta \le \pi/2$ and $k \ge 0$ for solvoing Problem 2.

In the previous papers [**4**, **5**] we found all Jørgensen groups of finite case and of countably infinite case. Namely we obtained the following theorem.

Theorem D (Li–Oichi–Sato [**4**, **5**]). (i) *There are sixteen Jørgensen groups in the region* $D_1 = \{(\theta, k) \in \mathbf{R} \mid 0 \le \theta \le \pi/2, 0 \le k \le \sqrt{3}/2\}$. *Nine of them are Kleinian groups of the first kind and seven groups are of the second kind.*

(ii) *There are countably infinite Jørgensen groups in the region* $D_2 = \{(\theta, k) \in \mathbf{R} \mid 0 \le \theta \le \pi/2, \sqrt{3}/2 < k \le 1\}$. *One of them is a Kleinian group of the first kind and others are of the second kind.*

In this paper we find all Jørgensen groups of uncountably infinite case. Consequently, Problem 2 has been completely solved.

Thanks are due to the referees for their careful reading and very valuable suggestions.

1. Preliminary

In this section we will state Poincaré's polyhedron theorem following Maskit [**6**, pp. 73–78] and some properties of an isometric hemi-sphere.

1.1. Poincaré's polyhedron theorem gives a sufficient condition for a subgroup of the Möbius transformation group to be discrete. See Maskit [**6**] for notation and terminologies, for example, a side pairing transformation, a cycle transformation and a cycle of edges.

Theorem E (Poincaré's Polyhedron Theorem (Maskit [**6**, p. 73])). *Let P be a polyhedron with side pairing transformations satisfying the following conditions* (1) *through* (6). *Then, G, the group generated by the side pairing transformations, is discrete and P is a fundamental polyhedron for G, and the reflection relations and cycle relations form a complete set of relations for G*:

(1) *For each side s of P, there is a side s' and there is an element $g_s \in G$ satisfing $g_s(s) = s'$ and $g_{s'} = g_s^{-1}$.*

(2) $g_s(P) \cap P = \emptyset$.

(3) *For every point $z \in P^*$, $p^{-1}(z)$ is a finite set. Here P^* is the space of*

equivalence classes so that the projection $p : \bar{P}$ *(the closure of* P*)* $\to P^*$ *is continuous and open.*

(4) *Let* e *be an edge and let* h *be the cycle transformation at* e*. Then for each edge* e*, there is a positive integer* t *such that* $h^t = 1$*.*

(5) *Let* $\{e_1, e_2, \dots, e_m\}$ *be any cycle of edges of* P *and let* $\alpha(e_k)$ $(k = 1, 2, \dots, m)$ *be the angle measursed from inside* P *at the edge* e_k*. Let* q *be the smallest positive integer such that* $h^q = 1$*, where* h *is the cycle transformation at* e_k*. Then the equality*

$$\sum_{k=1}^{m} \alpha(e_k) = 2\pi/q$$

holds.

(6) P^* *is complete.*

For simplicity, in this paper we only say a relation for both a cycle relation and a refrection relation.

1.2. Isometric circles and isometric hemi-spheres. Let X be a Möbius transformation represented by the following matrix:

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1 \quad (a, b, c, d \in \mathbf{C}, c \neq 0).$$

The *isometric circle* of the transformation X is the set:

$$\{z \in \mathbf{C} : |cz + d| = 1\}.$$

The *isometric hemi-sphere* of the transformation X is the hemi-sphere in the upper half space H^3 that has the same center and the same radius as those of the isometric circle of X. We call this isometric hemi-sphere the isometric hemishere *associated with* the isometric circle. The transformation X maps the isometric hemi-sphere of X onto the isometric hemi-sphere of X^{-1}. This property is helpful to us when we find a fundamental polyhedron for a discrete group.

1.3. Notation. Let F_X and $F_{X^{-1}}$ be two faces of a polyhedron P such that F_X is mapped onto $F_{X^{-1}}$ by the side pairing transformation X.

We denote by $e_{(m,n),\theta}$ the n-th edge of the m-th cycle transfomation such that the angle measured from the polyhedron P at the edge is θ.

For simplicity we use the following diagram to represent the m-th cycle transformation:

$$e_{(m,1),\theta_1} \xrightarrow{X_1} e_{(m,2),\theta_2} \xrightarrow{X_2} \cdots \xrightarrow{X_{n-1}} e_{(m,n),\theta_n} \xrightarrow{X_n} \circlearrowleft_{\theta}^{p}$$

This diagram means the following: The initial edge $e_{(m,1),\theta_1}$ is mapped to the second edge $e_{(m,2),\theta_2}$ by the side pairing transformation X_1 and then the edge $e_{(m,2),\theta_2}$ is mapped to the edge $e_{(m,3),\theta_3}$ by the side pairing transformation X_2 and so on. The symbol $e_{(m,n),\theta_n} \overset{X_n}{\to} \circlearrowleft_{\theta}^{p}$ means that the final edge $e_{(m,n),\theta_n}$ is mapped to the initial edge $e_{(m,1),\theta_1}$ by the side pairing transformation X_n and the sum of all angles at the edges in this sequence is equal to θ, that is, $\theta = \theta_1 + \theta_2 + \cdots + \theta_n$. The cycle transformation $X_n X_{n-1} \cdots X_1$ is either the identity transformation or an elliptic transformation. The number p is the order of the cycle transformation, that is, if $X_n X_{n-1} \cdots X_1$ is the identity transformation, then $p = 1$ and if $X_n X_{n-1} \cdots X_1$ is an elliptic transformation of order q, then $p = q$.

2. Main Theorem

In this section we will state our main theorem. Let $V(G_{\theta,k})$ denote the volume of 3-orbifold for a Kleinian group $G_{\theta,k}$ of the first kind and let $L(\theta)$ denote the *Lobachevskiĭ function*:

$$L(\theta) = -\int_0^\theta \log|2 \sin u| \, du.$$

A Riemann surface with signature $(g; m_1, \ldots, m_n, \infty)$ means a Riemann surface of genus g with n branch points of orders $m_1, \ldots, m_n$ and one puncture.

Main Theorem (uncountably infinite case).
The group $G_{\theta,k}$ with $0 \le \theta \le \pi/2$ and $k > 1$ is a Jørgensen group if and only if one of the following conditions holds.

(a) $\theta = 0$ *and* $k > 1$. *In this case,* $G_{0,k}$ *is a Kleinian group of the second kind, and* $\Omega(G_{0,k})/G_{0,k}$ *is a union of two Riemann surfaces with signatures* $(0; 2, 3, \infty)$ *and* $(0; 2, 2, 2, 3)$.

(b) (1) $\theta = \pi/6$ *and* $k = \sqrt{3}n/2$ $(n = 2, 4, 6, \ldots)$. *In this case,* $G_{\pi/6,k}$ *is a Kleinian group of the first kind, and* $V(G_{\pi/6,\sqrt{3}n/2}) = 3L(\pi/3)$.

(2) $\theta = \pi/6$ *and* $k = \sqrt{3}n/2$ $(n = 3, 5, 7, \ldots)$. *In this case,* $G_{\pi/6,k}$ *is a Kleinian group of the first kind, and* $V(G_{\pi/6,\sqrt{3}n/2}) = 6L(\pi/3)$.

(c) (1) $\theta = \pi/4$ *and* $k = 3/2$. *In this case,* $G_{\pi/4,k}$ *is a Kleinian group of the first kind, and* $V(G_{\pi/4,3/2}) = 3V(G_{\pi/2,1/2})$.

(2) $\theta = \pi/4$ *and* $k = 1 + \sqrt{2}/2$. *In this case,* $G_{\pi/4,k}$ *is a Kleinian group of the first kind, and* $V(G_{\pi/4,1+\sqrt{2}/2}) = V(G_{\sqrt{2}\pi/2,1/2})/2 + 2V(G_{\pi/2,1/2})$.

(3) $\theta = \pi/4$ *and* $k = (5 + \sqrt{5})/4$. *In this case,* $G_{\pi/4,k}$ *is a Kleinian group of the first kind, and* $V(G_{\pi/4,1+(1+\sqrt{5})/4}) = V(G_{\pi/4,(1+\sqrt{5})/4})/2 + 2V(G_{\pi/2,1/2})$.

(4) $\theta = \pi/4$ *and* $k = 1 + \sqrt{3}/2$. *In this case,* $G_{\pi/4,k}$ *is a Kleinian group of the first kind, and* $V(G_{\pi/4,1+\sqrt{3}/2}) = V(G_{\pi/4,\sqrt{3})/2})/2 + 2V(G_{\pi/2,1/2})$.

(5) $\theta = \pi/4$ *and* $k = 1 + \cos(\pi/n)$ $(n = 7, 8, \ldots)$. *In this case,* $G_{\pi/4,k}$ *are Kleinian groups of the second kind, and* $\Omega(G_{\pi/4,k})/G_{\pi/4,k}$ *is a Riemann surface with signature* $(0; 2, 3, n)$.

(6) $\theta = \pi/4$ *and* $k = 2$. *In this case,* $G_{\pi/4,k}$ *is a Kleinian group of the second kind, and* $\Omega(G_{\pi/4,k})/G_{\pi/4,k}$ *is a Riemann surface with signature* $(0; 2, 3, \infty)$.

(7) $\theta = \pi/4$ *and* $k > 2$. *In this case,* $G_{\pi/4,k}$ *is a Kleinian group of the second kind, and* $\Omega(G_{\pi/4,k})/G_{\pi/4,k}$ *is a Riemann surface with signature* $(0; 2, 2, 2, 3)$.

(d) $\theta = \pi/3$ *and* $k = \sqrt{3}n/2$ $(n = 2, 3, \ldots)$. *In this case,* $G_{\pi/3,k}$ *is a Kleinian group of the first kind, and* $V(G_{\pi/3,\sqrt{3}n/2}) = 3L(\pi/3)$.

(e) (Sato-Yamada [**8**]) $\theta = \pi/2$ *and* $k > 1$. *In this case,* $G_{\pi/2,k}$ *is a Kleinian group of the second kind, and* $\Omega(G_{\pi/2,k})/G_{\pi/2,k}$ *is a Riemann surface with signature* $(0; 2, 2, 3, 3)$.

Corollary. *There are uncountably infinite Jørgensen groups in the region* $\{(\theta, k) \mid 0 \le \theta \le \pi/2, k > 1\}$.

3. Proofs

In this section, we will give the proof of our main theorem. In order to prove that the group $G_{\theta,k}$ is discrete, we will construct a fundamental polyhedron for $G_{\theta,k}$ for applying Poincaré's polyhedron theorem. On the other hand, in order to prove that $G_{\theta,k}$ is not discrete, we will find a pair of elements of $G_{\theta,k}$ whose Jørgensen number is less than one.

3.1. The case of $\theta \ne 0, \pi/6, \pi/4, \pi/3$ and $\pi/2$

In these cases, we can prove by the same methods as in our previous paper [**4**] that $G_{\theta,k}$ are not discrete and so not Jørgensen groups for all $k \in \mathbf{R}$. We omit the proofs here.

3.2. The case of $\theta = 0$

For simplicity we write B_k and G_k for $B_{0,k}$ and $G_{0,k}$, respectively. We set S_k and T_k as follows:

$$S_k := B_k A^{-1} B_k A B_k^{-1} A^{-1} B_k = \begin{pmatrix} ik & -1 + k^2 \\ 1 & -ik \end{pmatrix},$$

$$T_k := A^{-1} B_k A B_k^{-1} A^{-1} B_k = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

We note that the transformations S_k and T_k are elliptic of order two and the fixed points of S_k (resp. T_k) are $i(1 + k)$ and $i(-1 + k)$ (resp. 0 and ∞). In the left-hand side of Figure 1 we can see all isometric circles of radius one near the origin. To obtain a fundamental polyhedron for G_k we cut the isometric hemispheres associated with the isometric circles along the dotted lines in the left-hand side of Figure 1. Then we have a fundamental polyhedron for G_k as in the right-hand side of Figure 1.

In the left-hand side of Figure 2 we can see the side pairing transformations: $A(F_A) = F_{A^{-1}}$, $S(F_{S_k}) = F_{S_k^{-1}}$, $T(F_{T_K}) = F_{T_k^{-1}}$, where in the figure we write S and T for S_k and T_k, respectively. We set $G_k^* = \langle A, S_k, T_k \rangle$. In the right-hand side of Figure 2 we can see the edges of the polyhedron P. In this case we have the

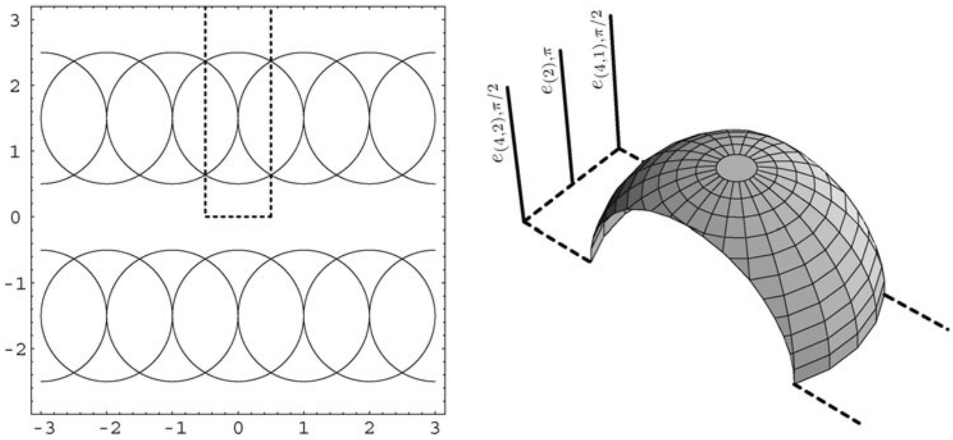

FIGURE 1. Isometric circles and a fundamental polyhedron ($\theta = 0$)

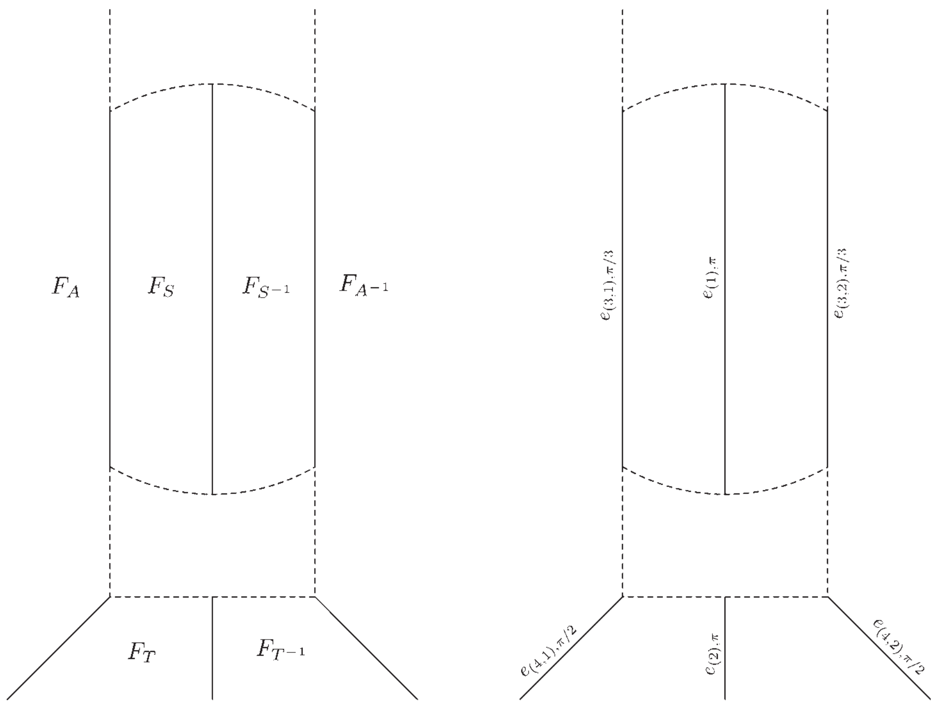

FIGURE 2. Side pairings and cycle relations ($\theta = 0$)

following four cycle transformations: (1) $e_{(1),\pi} \xrightarrow{S} \circlearrowleft^2_{2\pi/2}$, (2) $e_{(2),\pi} \xrightarrow{T} \circlearrowleft^2_{2\pi/2}$, (3) $e_{(3,1),\pi/3} \xrightarrow{A} e_{(3,2),\pi/3} \xrightarrow{S} \circlearrowleft^3_{2\pi/3}$, (4) $e_{(4,1),\pi/2} \xrightarrow{A} e_{(4,2),\pi/2} \xrightarrow{T} \circlearrowleft^2_{2\pi/2}$, where we write S and T for S_k and T_k, respectively. These form a complete set of relations for G_k^*. Namely, the relations are as follows: $S_k^2 = I$, $T_k^2 = I$, $(S_kA)^3 = I$ and $(T_kA)^2 = I$, where I is the identity transformation.

By Poincaré's polyhedron theorem we can see that G_k^* is a discrete group. Since $B_k = S_kT_k^{-1}$, we have $G_k = G_k^*$. Thus G_k is a discrete group (a Kleinian group of the 2-nd kind) and so a Jørgensen group.

We can see by the same method as in [**8**] that $\Omega(G_k)/G_k$ is a union of two Riemann surfaces with signatures $(0; 2, 3, \infty)$ and $(0; 2, 2, 2, 3)$. We omit the proof.

3.3. The case of $\theta = \pi/2$

The proof of this case is written in Sato-Yamada [**8**], but for the completeness we give the proof.

In the case of $\theta = \pi/2$ we have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B_k := B_{\pi/2,k} = \begin{pmatrix} ik & -1-k^2 \\ 1 & ik \end{pmatrix} \quad (k \in \mathbf{R}).$$

We set matrices S_k, T_k and U_k as follows:

$$S_k := A^{-1}B_kA^{-1}B_k^{-1}A^{-1} = \begin{pmatrix} ik & -1+k^2 \\ 1 & -ik \end{pmatrix},$$

$$T_k := A^{-1}B_k^{-1}A^{-1}B_kA^{-1} = \begin{pmatrix} -ik & -1+k^2 \\ 1 & ik \end{pmatrix},$$

$$U_k := B_k^{-1}A^{-1}B_kA^{-1}B_k^{-1}A^{-1} = \begin{pmatrix} 1 & -2ik \\ 0 & 1 \end{pmatrix}.$$

By a similar method to the case of $\theta = 0$ we have side pairing transformations A, S_k, T_k, and U_k (see the left-hand side of Figure 4, where we write S, T, and U for S_k, T_k, and U_k, respectively). In Figure 3 we can see all isometric circles of radius one near the origin and a fundamental polyhedron P for $G_k^* = \langle A, S_K, T_k, U_k \rangle$. In this case we have the following six cycle transformations (see the right-hand side of Figure 4): (1) $e_{(1),\pi} \xrightarrow{S} \circlearrowleft^2_{2\pi/2}$, (2) $e_{(2),\pi} \xrightarrow{T} \circlearrowleft^2_{2\pi/2}$, (3) $e_{(3,1),\pi/3} \xrightarrow{A} e_{(3,2),\pi/3} \xrightarrow{S^{-1}} \circlearrowleft^3_{2\pi/3}$, (4) $e_{(4,1),\pi/3} \xrightarrow{A} e_{(4,2),\pi/3} \xrightarrow{T^{-1}} \circlearrowleft^3_{2\pi/3}$, (5) $e_{(5,1),\pi/2} \xrightarrow{A} e_{(5,2),\pi/2} \xrightarrow{U} e_{(5,3),\pi/2} \xrightarrow{A^{-1}} e_{(5,4),\pi/2} \xrightarrow{U^{-1}} \circlearrowleft^1_{2\pi/1}$, (6) $e_{(6,1),\pi/2} \xrightarrow{S} e_{(6,2),\pi/2} \xrightarrow{U} e_{(6,3),\pi/2} \xrightarrow{T^{-1}} e_{(6,4),\pi/2} \xrightarrow{U^{-1}} \circlearrowleft^1_{2\pi/1}$, where we write S, T and U for S_k, T_k and U_k, respectively. These form a complete set of relations for G_k^*: $S_k^2 = I$, $T_k^2 = I$, $(S_k^{-1}A)^3 = I$, $(T_k^{-1}A)^3 = I$, $U_k^{-1}A^{-1}U_kA = I$, $U_k^{-1}T_k^{-1}U_kS_k = I$. By Poincaré's polyhedron theorem the group G_k^* is discrete. Since $B_k = S_kU_k^{-1}$, we can easily see $G_k^* = G_k$. Hence G_k is discrete (a Kleinian group of the 2-nd kind) and so a Jørgensen group.

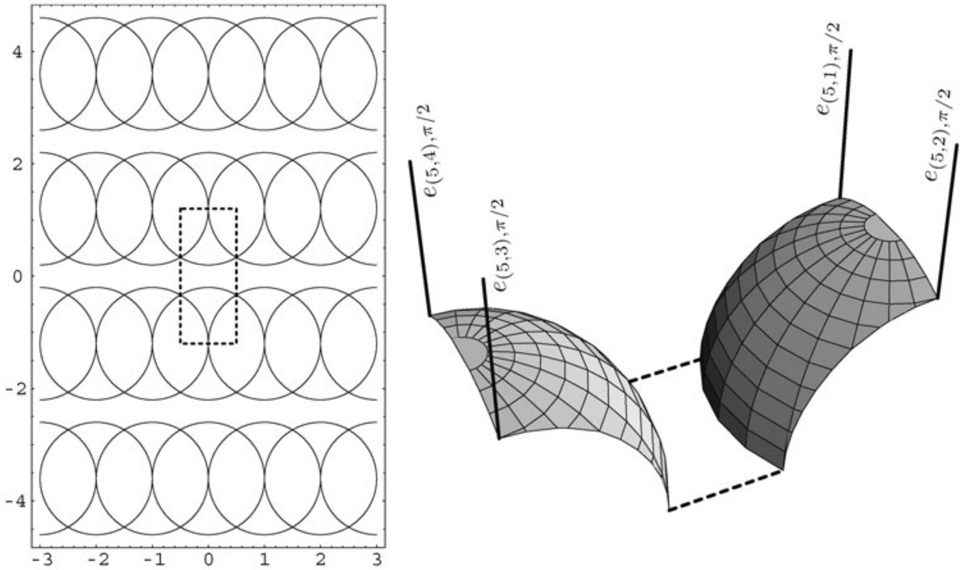

FIGURE 3. Isometric circles and a fundamental polyhedron ($\theta = \pi/2$)

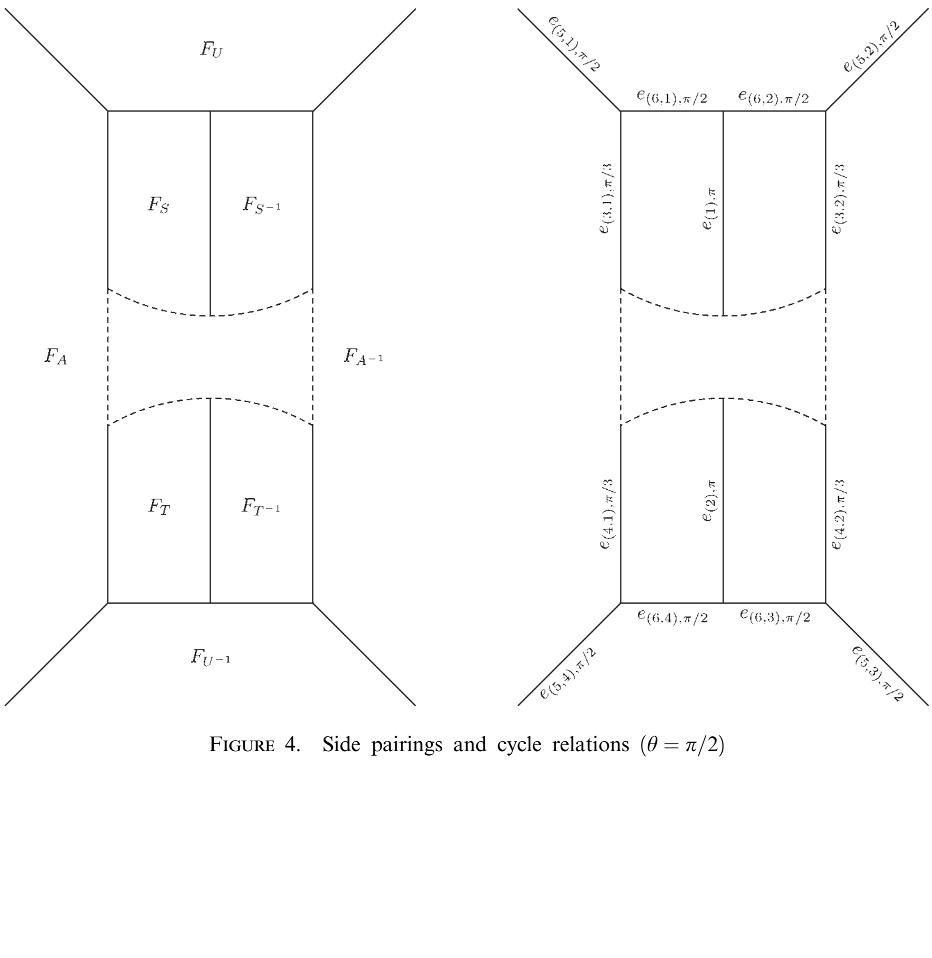

FIGURE 4. Side pairings and cycle relations ($\theta = \pi/2$)

We can see by the same method as in [**8**] that $\Omega(G_k)/G_k$ is a single Riemann surface with signature $(0; 2, 2, 3, 3)$. We omit the proof.

3.4. The case of $\theta = \pi/6$

In this section we set $\theta = \pi/6$. Then the group $G_k := G_{\pi/6,k}$ is generated by the following transformations A and $B_k := B_{\pi/6,k}$:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B_k = \begin{pmatrix} ke^{\pi i/6} & ik^2 e^{\pi i/6} - ie^{-\pi i/6} \\ -ie^{\pi i/6} & ke^{\pi i/6} \end{pmatrix},$$

that is, $G_k = \langle A, B_k \rangle$.

We consider the following transformation T_k:

$$\begin{aligned} G_k \ni T_k &= B_k^{-1} A^{-1} B_k A B_k^{-1} A B_k A^{-1} B_k^{-1} \\ &= \begin{pmatrix} 1 & (\sqrt{3} - 2k)i \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Then

$$G_{k+\sqrt{3}} \ni T_{k+\sqrt{3}} = \begin{pmatrix} 1 & -(\sqrt{3} + 2k)i \\ 0 & 1 \end{pmatrix}.$$

Thus we have

$$G_{k+\sqrt{3}} \ni T_{k+\sqrt{3}} B_{k+\sqrt{3}} T_{k+\sqrt{3}} = \begin{pmatrix} -ke^{\pi i/6} & ik^2 e^{\pi i/6} - ie^{-\pi i/6} \\ -ie^{\pi i/6} & -ke^{\pi i/6} \end{pmatrix} = B_{-k}.$$

Hence we have

$$G_{k+\sqrt{3}} = \langle A, B_{k+\sqrt{3}} \rangle \supset \langle A, B_{-k} \rangle = G_{-k}$$

Conversely, since

$$G_{-k} \ni T_{-k} = \begin{pmatrix} 1 & (\sqrt{3} + 2k)i \\ 0 & 1 \end{pmatrix},$$

we have

$$G_k \ni T_{-k} B_{-k} T_{-k} = \begin{pmatrix} (k + \sqrt{3})e^{\pi i/6} & i(k + \sqrt{3})^2 e^{\pi i/6} - ie^{-\pi i/6} \\ -ie^{\pi i/6} & (k + \sqrt{3})e^{\pi i/6} \end{pmatrix} = B_{k+\sqrt{3}}$$

and then

$$G_{k+\sqrt{3}} = \langle A, B_{k+\sqrt{3}} \rangle \subset \langle A, B_{-k} \rangle = G_{-k}.$$

Thus we have

$$G_{k+\sqrt{3}} = G_{-k}.$$

Now we recall the following theorem:

THEOREM F (Sato [**7**]). *The group G_k is discrete if and only if the group G_{-k} is discrete.*

By Theorem F we have the following lemma:

LEMMA. *The group $G_{k+\sqrt{3}}$ $(= G_{-k})$ is discrete if and only if the group G_k is discrete.*

In the previous paper Li–Oichi–Sato [**4**], we can see that G_k is discrete for $k = 0, \sqrt{3}/2$, and that G_k is not discrete for the other k in $0 \le k < \sqrt{3}$. Hence by the above lemma we have that $G_{\sqrt{3}n/2}$ is a discrete group only for $n = 0, 1, 2, 3, 4, \dots$.

Since $V(G_{\pi/6, k+(\sqrt{3}n/2)}) = V(G_{\pi/6,0})$ (resp. $V(G_{\pi/6, k+(\sqrt{3}n/2)}) = V(G_{\pi/6, \sqrt{3}/2})$ for even integers n (resp. for odd integers n), we have $V(G_{\pi/6, k+(\sqrt{3}n/2)}) = 3L(\pi/3)$ (resp. $V(G_{\pi/6, k+(\sqrt{3}n/2)}) = 6L(\pi/3)$) for even integers n (resp. for odd integers n) by [**4**]. We proved the assertions b(1) and b(2).

3.5. The case of $\theta = \pi/3$

Next, we will consider the case of $\theta = \pi/3$. In this case we will use two elliptic elements of order two with ∞ as one of the fixed points instead of the parabolic transformation T_k in §3.4. The group $G_k := G_{\pi/3, k}$ is generated by the following two matrices A and $B_k := B_{\pi/3, k}$, that is, $G_k = \langle A, B_k \rangle$:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B_k = \begin{pmatrix} ke^{\pi i/3} & ik^2e^{\pi i/3} - ie^{-\pi i/3} \\ -ie^{\pi i/3} & ke^{\pi i/3} \end{pmatrix}.$$

Then $B_{k+\sqrt{3}}$ and B_{-k} are as follows:

$$B_{k+\sqrt{3}} = \begin{pmatrix} (k+\sqrt{3})e^{\pi i/3} & i(k+\sqrt{3})^2 e^{\pi i/3} - ie^{-\pi i/3} \\ -ie^{\pi i/3} & (k+\sqrt{3})e^{\pi i/3} \end{pmatrix}$$

and

$$B_{-k} = \begin{pmatrix} -ke^{\pi i/3} & ik^2e^{\pi i/3} - ie^{-\pi i/3} \\ -ie^{\pi i/3} & -ke^{\pi i/3} \end{pmatrix}.$$

We consider the following transformations E_k^0 and E_k^1:

$$E_k^0 = B_k^{-1}AB_kAB_k^{-1}A^{-1}B_kA^{-1}B_k^{-1} = \begin{pmatrix} -i & \sqrt{3} \\ 0 & i \end{pmatrix}$$

and

$$E_k^1 = B_k^{-1}A^{-1}B_kA^{-1}B_k^{-1}AB_kAB_k^{-1} = \begin{pmatrix} i & \sqrt{3} \\ 0 & -i \end{pmatrix}.$$

Note that these transformations E_k^0 and E_k^1 are independent of k and hence $E_k^j \in G_k$ $(j = 0, 1)$ for any $k \in \mathbf{R}$.

We can see the following:

$$G_{k+\sqrt{3}} \ni E_k^1 B_{k+\sqrt{3}} E_k^0 = \begin{pmatrix} ke^{\pi i/3} & ik^2 e^{\pi i/3} - ie^{-\pi i/3} \\ -ie^{\pi i/3} & ke^{\pi i/3} \end{pmatrix} = B_k^{-1}.$$

Hence we have

$$G_{k+\sqrt{3}} = \langle A, B_{k+\sqrt{3}} \rangle \supset \langle A, B_k \rangle = G_k.$$

Conversely, since

$$G_k \ni E_k^0 B_k E_k^1 = (E_k^0)^2 B_{k+\sqrt{3}}^{-1} (E_k^1)^2 = B_{k+\sqrt{3}}^{-1},$$

we have

$$G_{k+\sqrt{3}} = \langle A, B_{k+\sqrt{3}} \rangle \subset \langle A, B_k \rangle = G_k.$$

Hence

$$G_{k+\sqrt{3}} = G_k.$$

Thus the assertion (d) follows from the main theorem in [4].

3.6. The case of $\theta = \pi/4$

Here we consider the case of $\theta = \pi/4$. We divide this case into the following four cases: (i) $k = 1 + \cos(\pi/n)$ $(n = 3, 4, 5, 6)$; (ii) $k = 1 + \cos(\pi/n)$ $(n = 7, 8, 9, \dots)$; (iii) $k \geq 2$; (iv) others.

For $\theta = \pi/4$ we have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_k := B_{\pi/4,k} = \begin{pmatrix} ke^{\pi i/4} & (ik^2 - 1)e^{\pi i/4} \\ -ie^{\pi i/4} & ke^{\pi i/4} \end{pmatrix} \quad (k \in \mathbf{R}).$$

We set $G_k = \langle A, B_k \rangle$.

We introduce the following matrices:

$$U_k := AB_kA^{-1}B_k^{-1}A^{-1}B_kAB_k^{-1}AB_kA^{-1}B_k^{-1} = \begin{pmatrix} -i & -2k \\ 0 & i \end{pmatrix},$$

$$V_k := AB_k^{-1}AB_kA^{-1}B_k^{-1}A^{-1}B_kAB_k^{-1}AB_k = \begin{pmatrix} i & -2k \\ 0 & -i \end{pmatrix},$$

$$S_k := AB_kAB_k^{-1}A^{-1}B_kA^{-1}B_k^{-1}A = \begin{pmatrix} -i(k-1) & -k(k-2) \\ -1 & i(k-1) \end{pmatrix},$$

$$T_k := AB_k^{-1}A^{-1}B_kA^{-1}B_k^{-1}AB_kA = \begin{pmatrix} i(k-1) & -k(k-2) \\ -1 & -i(k-1) \end{pmatrix}.$$

We note that U_k is an elliptic transformation of order 2 with fixed points ik and ∞; V_k is an elliptic transformation of order 2 with fixed points $-ik$ and ∞; S_k is an elliptic transformation of order 2 with fixed points ik and $i(k-2)$; T_k is an elliptic transformation of order 2 with fixed points $-ik$ and $-i(k-2)$.

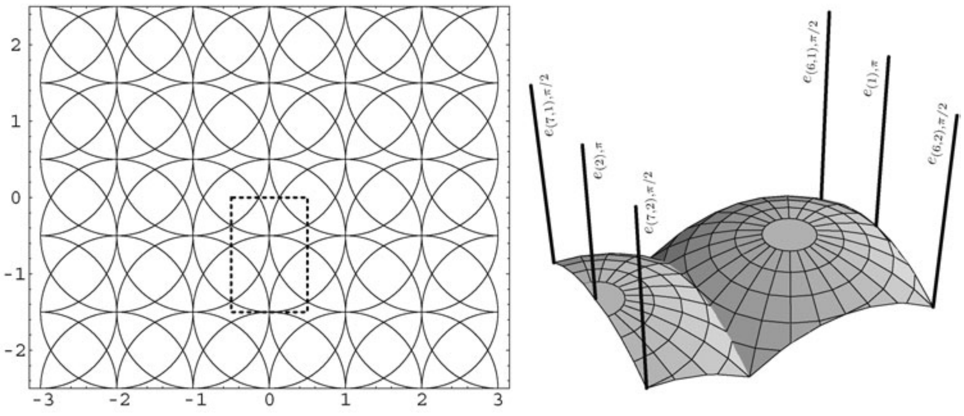

FIGURE 5. Isometric circles and a fundamental polyhedron ($(\theta, k) = (\pi/4, 3/2)$)

3.6.1. The case of $k = 1 + \cos(\pi/n)$ $(n = 3, 4, 5, 6)$

In this case we will show that all of these groups G_k are discrete (Kleinian groups of the first kind), and so are Jørgensen groups.

We write B, S, T, U and V for B_k, S_k, T_k, U_k and V_k, respectively. We set W, X and Y as follows:

$$W := STS = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X := WUB, \quad Y := WB.$$

Then we note that the transformation W is elliptic of order 2 whose fixed points are 0 and ∞.

In the left-hand side of Figure 5 we can see all isometric circles of radius one near the origin. In the right-hand side of Figure 5 we can see the polyhedron P over the rectangle cut along the dotted line in the left-hand side of Figure 5.

The sides of P are given in the left-hand side of Figure 6. The side pairing transformations are A, T, V, W, X and Y: $A(F_A) = F_{A^{-1}}$, $T(F_T) = F_{T^{-1}}$, $V(F_V) = F_{V^{-1}}$, $W(F_W) = F_{W^{-1}}$, $X(F_X) = F_{X^{-1}}$, $Y(F_Y) = F_{Y^{-1}}$. We set $G_k^* = \langle A, T, V, W, X, Y \rangle$.

In the right-hand side of Figure 6 we can see the edges of the polyhedron P. In this case we have the following eleven cycle transformations: (1) $e_{(1),\pi} \xrightarrow{W} \circlearrowleft^2_{2\pi/2}$, (2) $e_{(2),\pi} \xrightarrow{V} \circlearrowleft^2_{2\pi/2}$, (3) $e_{(3),\pi} \xrightarrow{T} \circlearrowleft^2_{2\pi/2}$, (4) $e_{(4),\pi} \xrightarrow{X} \circlearrowleft^2_{2\pi/2}$, (5) $e_{(5),\pi} \xrightarrow{Y} \circlearrowleft^2_{2\pi/2}$, (6) $e_{(6,1),\pi/2} \xrightarrow{A} e_{(6,2),\pi/2} \xrightarrow{W} \circlearrowleft^2_{2\pi/2}$, (7) $e_{(7,1),\pi/2} \xrightarrow{A} e_{(7,2),\pi/2} \xrightarrow{V} \circlearrowleft^2_{2\pi/2}$, (8) $e_{(8,1),\pi/3} \xrightarrow{A} e_{(8,2),\pi/3} \xrightarrow{T} \circlearrowleft^3_{2\pi/3}$, (9) $e_{(9,1),\pi/n} \xrightarrow{W} e_{(9,2),\pi/n} \xrightarrow{T} \circlearrowleft^n_{2\pi/n}$, (10) $e_{(10,1),\pi/2} \xrightarrow{V} e_{(10,2),\pi/2} \xrightarrow{Y} e_{(10,3),\pi} \xrightarrow{X} \circlearrowleft^1_{2\pi/1}$, (11) $e_{(11,1),\pi/3} \xrightarrow{A} e_{(11,2),\pi/3} \xrightarrow{Y} e_{(11,3),2\pi/3} \xrightarrow{T} e_{(11,4),2\pi/3} \xrightarrow{X} \circlearrowleft^1_{2\pi/1}$. The relations are as follows: $W^2 = I$, $V^2 = I$, $T^2 = I$, $X^2 = I$, $Y^2 = I$, $(WA)^2 = I$, $(VA)^2 = I$, $(TA)^3 = I$, $(TW)^n = I$, $XYV = I$, $XTYA = I$. These form a complete set of relations for G_k^*.

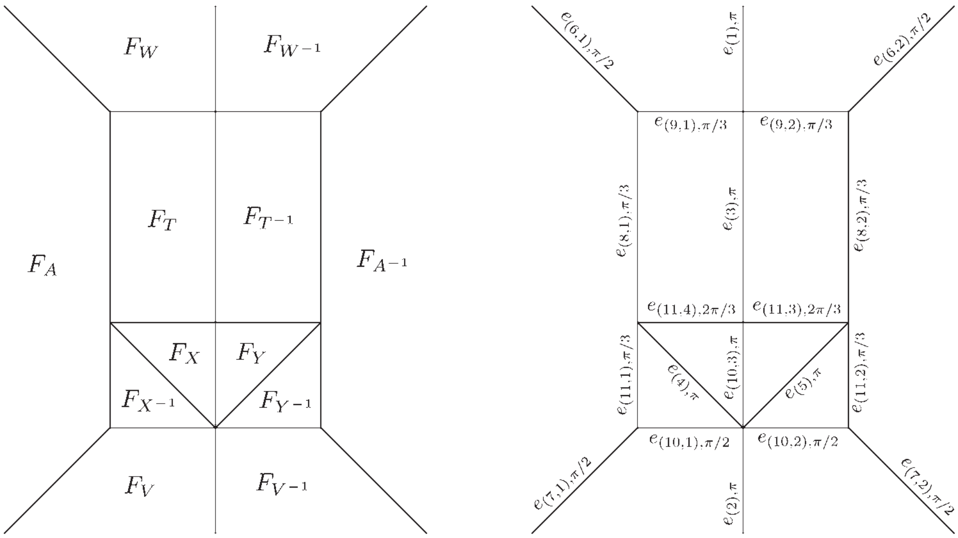

FIGURE 6. Side pairings and cycle relations ($(\theta, k) = (\pi/4, 3/2)$)

By Poincaré's polyhedron theorem the group G_k^* is discrete and the polyhedron P is a fundamental polyhedron for G_k^*. Since $G_k^* \ni W^{-1}Y = W^{-1}(WB) = B \in G_k$, we can see $G_k = G_k^*$. Hence G_k is discrete (a Kleinian group of the first kind) and so a Jørgensen group. We can easily calculate the volume of the polyhedron P, and so we omit it. We proved the assertions (c)(1) through (c)(4).

3.6.2. The case of $k = 1 + \cos(\pi/n)$ $(n = 7, 8, 9, \ldots)$

We can treat this case in the same way as in the above §3.6.1. That is, we have the same side pairing transformations, the same cycle transformations and the same relations. The only difference from the above is as follows: In this case G_k is a Kleinian group of the second kind. We can see by the same method as in Sato–Yamada [**8**] that $\Omega(G_k)/G_k$ is a Riemann surface with signature $(0; 2, 3, n)$. We proved the assertion (c)(5).

3.6.3. The case of $k \geq 2$

In this case we show that all of the groups $G_k := G_{\pi/4,k}$ are discrete (Kleinian groups of the second kind). A fundamental polyhedron for G_k is obtained by a similar method to the cases of $k = 1 + \cos(\pi/n)$ $(n = 3, 4, 5, \ldots)$. The six side pairing transformations A, T_k, V_k, W_k, X_k, Y_k are the same as in §3.6.1. We set $G_k^* = \langle A, T_k, V_k, W_k, X_k, Y_k \rangle$. The ten cycle relations (1)–(8), (10)–(11) and ten relations except (9) in §3.6.1 also hold in this case. These form a complete set of relations for G_k^*. By Poincaré's polyhedron theorem we have that $G_k = G_k^*$ is discrete and so a Jørgensen group. We can see by a similar way to in [**8**] that $\Omega(G_k)/G_k$ is a single Riemann surface with signature $(0; 2, 3, \infty)$ for

$k = 2$ and a single Riemann surface with signature $(0; 2, 2, 2, 3)$ for $k > 2$. We proved the assertions (c)(6) and (c)(7).

3.6.4. The case of other k

In this case we will show that all of the groups G_k are not discrete. We set $k = 1 + t$ $(0 < t < 1)$. We can easily calculate that

$$V_t := (S_k T_k)^{-1} = \begin{pmatrix} 2t^2 - 1 & 2it - 2it^3 \\ 2it & 2t^2 - 1 \end{pmatrix}.$$

We set

$$M_t = \left(\frac{1}{2\sqrt{1-t^2}}\right)^{1/2} \begin{pmatrix} 1 & -\sqrt{1-t^2} \\ 1 & \sqrt{1-t^2} \end{pmatrix} \quad (0 < t < 1).$$

Then we have

$$A_t^* := M_t A M_t^{-1} = \frac{1}{2\sqrt{1-t^2}} \begin{pmatrix} 2\sqrt{1-t^2} - 1 & 1 \\ -1 & 2\sqrt{1-t^2} + 1 \end{pmatrix}.$$

We set $V_t^* := M_t V_t M_t^{-1}$. Then we have

$$V_t^* = \begin{pmatrix} (2t^2 - 1) - 2it\sqrt{1-t^2} & 0 \\ 0 & (2t^2 - 1) + 2it\sqrt{1-t^2} \end{pmatrix}.$$

We set $\cos\theta = t$ $(0 < \theta < \pi/2)$. Then $e^{2i\theta} = (2t^2 - 1) + 2it\sqrt{1-t^2}$ and $e^{-2i\theta} = (2t^2 - 1) - 2it\sqrt{1-t^2}$. Thus we have

$$(*) \qquad J(A, V_t^n) = J(A_t^*, (V_t^*)^n) = \left|\frac{-1 + \cos(4n\theta)}{1 - \cos(2\theta)}\right|.$$

By straightforward calculations we have that if $0 < \cos\theta < \cos(\pi/3)$, then $0 \le J(A_t^*, V_t^*) < 1$. Furthermore we can easly see from the above equality $(*)$ that if $\cos(\pi/(2n-1)) < \cos\theta < \cos(\pi/(2n+1))$ $(n = 2, 3, 4, \ldots)$, then $0 \le J(A_t^*, (V_t^*)^n) < 1$.

We note that if $k = 1 + \cos(\pi/2n)$ $(n = 2, 3, 4, \ldots)$, then $G_k^n = \langle A, V_k^n \rangle$ are elementary groups. We can easily see that G_k^n are non-elementary groups for k with $1 < k < 3/2$ and $1 + \cos(\pi/(2n-1)) < k < 1 + \cos(\pi/(2n+1))$ $(k \ne 1 + \cos(\pi/2n))$ for $n = 2, 3, 4, 5 \ldots$. By Jørgensen's inequality theorem we can see that G_k^n are not discrete for k with $1 < k < 3/2$ and $1 + \cos(\pi/(2n-1)) < k < 1 + \cos(\pi/(2n+1))$ $(k \ne 1 + \cos(\pi/2n))$ for $n = 2, 3, 4, 5 \ldots$. Since G_k^n is a subgroup of G_k, G_k is not discrete and so not a Jørgensen group for k with $1 < k < 2$ $(k \ne 1 + \cos(\pi/2n))$ $(n = 2, 3, 4, 5, \ldots)$. Our proof is now complete.

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