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# ENTIRE FUNCTIONS OF SMALL GROWTH THAT SHARE ONE VALUE WITH ITS LINEAR DIFFERENTIAL POLYNOMIALS*†‡

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**Abstract**

In this paper, we investigate the relationship between an entire function of small growth $f$ and its linear differential polynomial $L(f)$ when they share one value by applying value distribution theory and complex oscillation theory. As consequences of the main result we can get the precise form of $f$.

## 1. Introduction and main results

Let $f(z)$ and $g(z)$ denote some non-constant meromorphic functions and $a$ be a finite value. We say $f(z) = a \to g(z) = a$ if $z_n$ $(n = 1, 2, \ldots)$ are the zeros of $f(z) - a$ with multiplicities $v(n)$, and $z_n$ $(n = 1, 2, \ldots)$ are also zeros of $g(z) - a$ with multiplicities at least $v(n)$. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that $f$ and $g$ share $a$ CM. By $S(r, f)$ we denote any quantity satisfying

$$S(r, f) = o(T(r, f)),$$

as $r \to \infty$, possibly outside a set of $r$ with finite linear measure. Then a meromorphic function $\alpha(z)$ is said a small function of $f$ if $T(r, \alpha) = S(r, f)$. In addition, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory (e.g. see [5] or [7]). Especially, we use $\sigma(f)$ to denote the order of growth of $f(z)$.

On the problem of uniqueness of an entire function and its derivative that share one value, the following results have been obtained.

Theorem A ([12]). *Let $f$ be a non-constant entire function, $k$ be a positive integer. If $f$ and $f^{(k)}$ share the value 1 CM, and if*

$$\overline{N}\left(r, \frac{1}{f'}\right) < (\lambda + o(1)) T(r, f)$$

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*for some real constant* $\lambda \in (0, 1/4)$, *then*

$$\frac{f^{(k)} - 1}{f - 1} = c,$$

*for some non-zero constant* $c$.

THEOREM B ([11]). *Let* $f$ *be a non-constant entire function of finite order, and let* $a \neq 0$ *be a finite constant,* $k$ *be a positive integer. If* $f$ *and* $f^{(k)}$ *share* $a$ *CM, then*

$$\frac{f^{(k)} - a}{f - a} = c,$$

*for some non-zero constant* $c$.

However, there are no corresponding results about the uniqueness of an entire function and its linear differential polynomial that share one value. In this paper, we note the precise result about growth of an entire function of small growth (whose order is less than 1/2), so we have a try by applying complex oscillation theory. In the sequel, we set

$$L(f) = a_k(z)f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_0(z)f, \quad (k \geq 1) \tag{1.1}$$

where $a_j(z)$ $(j = 0, 1, \dots, k)$ are polynomials and $a_k(z) \not\equiv 0$. Indeed, we shall prove the following theorems:

THEOREM 1. *Let* $f$ *be a non-constant entire function of order* $\sigma(f) < 1/2$ *and* $b(z)$ *be a non-zero small function of* $f$. *If* $f - b(z) = 0 \to L(f) - b(z) = 0$, *then*

$$\frac{L(f) - b(z)}{f - b(z)} = Q(z), \tag{1.2}$$

*where* $Q(z)$ *is a non-zero polynomial.*

THEOREM 2. *Let* $f$ *and* $b(z)$ *be as in Theorem* 1. *If* $f - b(z) = 0 \to L(f) - b(z) = 0$, *then the following conclusions hold*:

(a) *If* $(\deg a_j - \deg a_i)/(j - i) \leq 1/2$ *for any* $i \neq j$ $(i, j \in \{1, \dots, k\})$, *then*

$$\frac{L(f) - b(z)}{f - b(z)} = Q(z), \tag{1.3}$$

*where the non-zero polynomial* $Q(z)$ *satisfies* $\deg Q \leq \max\{\deg a_j \mid j = 0, \dots, k\}$.

(b) *If* $b(z) \equiv b \neq 0$ *and* $a_j$ $(j = 0, 1, \dots, k)$ *are constants, then* $f = b_m z^m + \cdots + b_1 z + b_0$ $(0 < m \leq k)$ *where* $b_m \neq 0$, $b_i$ $(i = 1, \dots, m)$ *are constants with* $a_j = 0$ $(j = 1, \dots, m - 1)$ *and* $m! a_m b_m = b(1 - a_0)$.

THEOREM 3. *Let* $f$ *be a non-constant entire function of finite order satisfying* $\sigma(f) \neq 1 + n/k$ *for any positive integer* $n$, *and let* $a \neq 0$ *be a finite constant. If* $f = a \to f^{(k)} = a$ *and*

$$\overline{\lim_{r\to\infty}} \frac{\log(N(r, f^{(k)} = a) - N(r, f = a))}{\log r} < \frac{1}{2},$$

*then*

(1.4)
$$\frac{f^{(k)} - a}{f - a} = c,$$

*for some non-zero constant c.*

In fact, $f$ satisfying the hypothesis of Theorem 1 must be a solution of the following equation

(1.5)
$$\sum_{j=1}^{k} a_j(z) f^{(j)} + (a_0(z) - Q(z)) f = b(z)(1 - Q(z)),$$

where $a_j(z)$ $(j = 0, 1, \dots, k)$, $Q(z)$ and $b(z)$ are as in Theorem 1. Hence, it is natural to ask if there always exists any transcendental entire function of small growth satisfying (1.5). Now, we give an affirmative answer. For example, let $f = 1 + \sum_{n=1}^{\infty} z^n/(3n)!$ and $g(z) = (1/2)(\cos z^{1/4} + \cos iz^{1/4}) = 1 + \sum_{n=1}^{\infty} z^n/(4n)!$, then $\sigma(f) = 1/3$ and $\sigma(g) = 1/4$. Moreover, they also respectively satisfy

$$27z^3 f''' + 54z^2 f'' + 6zf' - zf = 0,$$

$$64z^4 g^{(4)} + 288z^3 g''' + 204z^2 g'' + 6zg' - \frac{1}{4} zg = 0.$$

Set $L_1(f) = 27z^3 f''' + 54z^2 f'' + 6zf' - (z-1)f$, $L_2(f) = 27z^3 f''' + 54z^2 f'' + 6zf' + c(z)f$, $L_3(g) = 64z^4 g^{(4)} + 288z^3 g''' + 204z^2 g'' + 6zg' - ((1/4)z - 1)g$ and $L_4(g) = 64z^4 g^{(4)} + 288z^3 g''' + 204z^2 g'' + 6zg' + c(z)g$ where $c(z)$ is any polynomial. From this, we have

$$\frac{L_1(f) - a(z)}{f - a(z)} = 1, \quad \frac{L_2(f)}{f} = c(z) + z, \quad \frac{L_3(g) - d(z)}{g - d(z)} = 1, \quad \frac{L_4(g)}{g} = c(z) + \frac{1}{4}z,$$

where $a(z)$ is any small function of $f$, $d(z)$ is any small function of $g$. This example also shows that the assumption of $\deg a_j$ in case (a) of Theorem 2 is sharp. In general, we also can obtain the following

THEOREM 4. *Let $p$ and $q$ be positive integers. Suppose that $f = 1 + \sum_{n=1}^{\infty} z^{qn}/(pn)!$, then $f$ satisfies the equation as*

(1.6)
$$A_p z^p f^{(p)} + A_{p-1} z^{p-1} f^{(p-1)} + \cdots + A_1 z f' = z^q f,$$

*where $A_j$ $(j = 1, 2, \dots, p)$ are constants that depend only on $p$ and $q$.*

Clearly, $f$ in Theorem 4 is an entire function of $\sigma(f) = q/p$. Take the right $p$ and $q$ such that $\sigma(f) < 1/2$, from (1.6) we know that there really exist transcendental entire functions of small growth satisfying (1.5).

## 2. Preliminary lemmas

In this section, we present some lemmas which are necessary in this paper.

Lemma 1 ([9]). *Let $f(z)$ be a transcendental meromorphic function, and $\alpha > 1$ be a given constant. Then there exist a set $E_1 \subset (1, +\infty)$ of finite logarithmic measure and a constant $B > 0$ that depends only on $\alpha$ and $(m, n)$, ($m, n$ are integers with $0 \le m < n$) such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \le B\left(\frac{T(\alpha r, f)}{r}(\log^{\alpha} r) \log T(\alpha r, f)\right)^{n-m}.$$

Lemma 2 ([1]). *Let $f(z)$ be an entire function of order $\sigma(f) = \sigma < 1/2$ and denote $A(r) = \inf_{|z|=r} \log|f(z)|$, $B(r) = \sup_{|z|=r} \log|f(z)|$. If $\sigma < \alpha < 1$, then*

$$\underline{\log dens}\{r : A(r) > (\cos \pi\alpha)B(r)\} \ge 1 - \frac{\sigma}{\alpha},$$

*where*

$$\underline{\log dens}\, E = \varliminf_{r\to\infty}\left(\int_1^r (\chi_E(t)/t)\, dt\right)\bigg/\log r$$

$$\overline{\log dens}\, E = \varlimsup_{r\to\infty}\left(\int_1^r (\chi_E(t)/t)\, dt\right)\bigg/\log r$$

*and $\chi_E(t)$ is the characteristic function of a set $E$.*

Lemma 3 ([2]). *Suppose that $w(z)$ is a meromorphic function with $\sigma(w) = \beta < \infty$. Then for any given $\varepsilon > 0$, there is a set $E_2 \subset (1, +\infty)$ that has finite logarithmic measure, such that*

$$|w(z)| \le \exp\{r^{\beta+\varepsilon}\}$$

*holds for $|z| = r \notin [0, 1] \cup E_2$, $r \to \infty$.*

Lemma 4 ([4]). *Suppose that $T(r)$ is a continuous non-decreasing positive function on $[r_0, \infty)$ $(r_0 \ge 1)$ which satisfies $T(r) \to \infty$ $(r \to \infty)$. If there exists an increasing sequence $\{r_n\}$, $r_n \uparrow \infty$ $(n \to \infty)$, such that $\lim_{n\to\infty} \log T(r_n)/\log r_n \le \mu < +\infty$, then for any given $\tau_1 (> 1)$ and $\tau_2 (> 1)$, we have*

$$\underline{\log dens}\, E_3 \ge 1 - \mu\frac{\log \tau_1}{\log \tau_2},$$

*where $E_3 = \{r : T(\tau_1 r) \le \tau_2 T(r)\}$.*

Lemma 5. *Let $f$ be a non-constant entire function of order $\sigma(f) = \mu < +\infty$, and $a(z), b(z)$ be small functions of $f$. Set $F(z) = f(z) + a(z)$. Then for any given $t$ $(0 < t < 1)$, there is a set $E_4 \subset (1, +\infty)$ satisfying $\underline{\log dens}\, E_4 > t$, such that*

$$\frac{M(r,b)}{M(r,F)} \to 0$$

*holds for* $|z| = r \in E_4, \ r \to \infty$.

*Proof.* According to the hypothesis of $f$ and by Lemma 4, there exists a set $H_1 = \{r \,|\, T(4r, f) \le 2^k T(r, f)\}$ $(k \ge \max\{4, [2\mu + 1]\})$ with $\underline{\log dens}\, H_1 \ge 1 - 2\mu/k$ where $k$ is an integer. It is obvious that $H_1$ is a closed set. Set $r_1 = \min\{H_1 \cap [1, +\infty)\}, r_2 = \min\{H_1 \cap [2r_1, +\infty)\}, \dots r_v = \min\{H_1 \cap [2r_{v-1}, +\infty)\}, \dots$. We can get a sequence $\{r_v\}$ $(r_v \to +\infty)$ and $H_1 \subset \bigcup_{v \ge 1}^{\infty} [r_v, 2r_v]$, so

$$\underline{\log dens}\, H_1 \le \underline{\log dens} \left\{ \bigcup_{v=1}^{\infty} [r_v, 2r_v] \right\}. \tag{2.1}$$

By Lemma 3 and the definition of small function, there exists a set $H_2 \subset (1, +\infty)$ with finite linear measure, such that

$$\frac{T(r,a)}{T(r,f)} \to 0, \quad \frac{T(r,b)}{T(r,f)} \to 0, \tag{2.2}$$

and $a(z), b(z)$ have finite moduli for $|z| = r \notin [0,1] \cup H_2, \ r \to \infty$. Set $H_3 = H_1 \backslash H_2$, then $\underline{\log dens}\, H_3 = \underline{\log dens}\, H_1$.

Let $a_s$ $(s = 1, 2, \dots, n(3r_v, a))$ and $b_m$ $(m = 1, 2, \dots, n(3r_v, b))$ denote the poles of $a(z)$ and the poles of $b(z)$ in $|z| \le 3r_v$ respectively. By the Boutroux-Cartan Theorem, we have

$$\prod_{s=1}^{n(3r_v, a)} |z - a_s| \ge \left(\frac{r_v}{2^k e}\right)^{n(3r_v, a)}, \quad \prod_{m=1}^{n(3r_v, b)} |z - b_m| \ge \left(\frac{r_v}{2^k e}\right)^{n(3r_v, b)} \tag{2.3}$$

except some $z$ in two groups of disks $(\gamma_1) + (\gamma_2)$, and the sum of their radii is no larger than $r_v/2^{k-2}$. Therefore, there exist $|z| = \rho$ that have no intersection with $(\gamma_1) + (\gamma_2)$ in $r_v \le |z| \le 2r_v$, then we have (2.3) on $|z| = \rho$. Let $E_v^*$ denote the set of those values of $\rho$, then mes $E_v^* \ge (1 - 1/2^{k-3}) r_v$. Applying the Poisson-Jensen formula (see [5]), we have

$$\log^+ |a(z)| \le \frac{3r_v + \rho}{3r_v - \rho} m(3r_v, a) + \sum_{|a_s| \le 3r_v} \log \left| \frac{(3r_v)^2 - \overline{a_s} z}{3r_v (z - a_s)} \right| \tag{2.4}$$

where $|z| = \rho \in E_v^*$. Substituting (2.3) into (2.4), we obtain

$$\begin{aligned} \log^+ |a(z)| &\le \frac{3r_v + \rho}{3r_v - \rho} m(3r_v, a) + n(3r_v, a) \log(3 \cdot 2^{k+1} e) \\ &\le CT(4r_v, a) \le CT(4r_v, f) \le C2^k T(r_v, f) \le C2^k \log^+ M(\rho, f) \end{aligned}$$

where $C$ is some positive constant and $|z| = \rho \in E_v^* \backslash H_2, \ v \to \infty$. From the above inequality and (2.2), it is easy to see

$$\frac{\log^+ M(\rho,a)}{\log^+ M(\rho,f)} \le 3\cdot 2^k \frac{T(4r_v,a)}{T(4r_v,f)} \to 0, \tag{2.5}$$

where $|z| = \rho \in E_v^*\backslash H_2$, $v\to\infty$. In fact, since $r_v \le \rho \le 2r_v$ and $\rho \in E_v^*\backslash H_2$, we have

$$\log^+ M(\rho,a) \le \frac{4r_v+\rho}{4r_v-\rho} T(4r_v,a) \le \frac{6r_v}{2r_v} T(4r_v,a)$$

and

$$\log^+ M(\rho,f) \ge T(r_v,f) \ge \frac{1}{2^k} T(4r_v,f).$$

Similarly, we obtain

$$\frac{\log^+ M(\rho,b)}{\log^+ M(\rho,f)} \to 0, \tag{2.6}$$

where $|z| = \rho \in E_v^*\backslash H_2$, $v\to\infty$. Since $f$ is a non-constant entire function, we have $M(r,f)\to\infty$ $(r\to\infty)$. Considering (2.5), (2.6) and this, we have

$$\frac{M(\rho,a)}{M(\rho,f)} \to 0, \quad \frac{M(\rho,b)}{M(\rho,f)} \to 0, \tag{2.7}$$

where $|z| = \rho \in E_v^*\backslash H_2$, $v\to\infty$. We know $F(z) = f(z)+a(z)$, so $M(r,F) \ge M(r,f) - M(r,a)$ for $r \notin H_2$. From above argument, for $\rho \in E_v^*\backslash H_2$, $v\to\infty$, we obtain

$$\frac{M(\rho,b)}{M(\rho,F)} \le \frac{M(\rho,b)}{(1+o(1))M(\rho,f)} \to 0. \tag{2.8}$$

Set $E_4 = \bigcup_{v=1}^{\infty} E_v^*\backslash H_2$, then $\underline{\log dens}\, E_4 = \underline{\log dens} \bigcup_{v=1}^{\infty} E_v^*$. Moreover, there exists a sequence $\{r_n'\}$, $r_n'\uparrow$ $(n\to\infty)$ such that

$$\underline{\log dens}\, E_4 = \varliminf_{n\to\infty} \left(\int_1^{r_n'} (\chi_{E_4}(t)/t)\,dt\right) \Big/ \log r_n'.$$

For every $E_v^*$, we have

$$\int_{r_v}^{2r_v} (\chi_{E_v^*}(t)/t)\,dt \ge \log 2 - \int_{r_v}^{r_v+(1/2^{k-3})r_v} \frac{1}{t}\,dt = \log\frac{2}{1+1/2^{k-3}}. \tag{2.9}$$

Now we discuss the following two cases.

Case 1. Suppose that $r_n' \in [r_{v_n}, 2r_{v_n}]$ for some $v_n$. Clearly we have

$$\frac{\int_1^{r_n'} (\chi_{E_4}(t)/t)\,dt}{\log r_n'} \ge \frac{\int_{r_1}^{r_{v_n}} (\chi_{E_4}(t)/t)\,dt}{\log 2r_{v_n}}. \tag{2.10}$$

Case 2. Suppose that $r'_n \notin \bigcup_{v=1}^{\infty}[r_v, 2r_v]$. Let $r_{v_n}$ be the closest to $r'_n$ of $\{r_v\}$ and $r_{v_n} \geq r'_n$, then

$$\frac{\int_1^{r'_n}(\chi_{E_4}(t)/t)\,dt}{\log r'_n} \geq \frac{\int_{r_1}^{r_{v_n}}(\chi_{E_4}(t)/t)\,dt}{\log r_{v_n}}. \tag{2.11}$$

Set $mes\, H_2 = \delta$, from (2.9) we have

$$\int_{r_1}^{r_{v_n}}(\chi_{E_4}(t)/t)\,dt \geq \sum_{v=1}^{v_n-1}\int_{r_v}^{2r_v}(\chi_{E_v^*}(t)/t)\,dt - \frac{\delta}{r_1}$$

$$\geq \frac{1}{\log 2}\log\frac{2}{1+1/2^{k-3}}\sum_{v=1}^{v_n-1}\int_{r_v}^{2r_v}(\chi_{H_3}(t)/t)\,dt - \frac{\delta}{r_1}$$

Combining this with (2.10) and (2.11), we obtain

$$\underline{\log dens}\, E_4 \geq \left(1-\frac{2\mu}{k}\right)\frac{1}{\log 2}\log\frac{2}{1+1/2^{k-3}}$$

We know that $\phi(x) = (1/\log 2)\log(2/(1+1/2^{x-3}))(1-2\mu/x)$ is continuous on $[3,\infty)$ and tends to 1 $(x\to\infty)$. Hence, for a given $t$ $(0<t<1)$, there must exist $N^*$ such that $\phi(x) > t$ for $x > N^*$. When $k \geq [N^*+1]$, then $\underline{\log dens}\, E_4 > t$.

Lemma 6 ([9]). *Let $f(z)$ be a transcendental meromorphic function with finite order $\rho$, and let $\Gamma = \{(k_1, j_1), (k_2, j_2), \ldots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers satisfying $k_i \geq j_i \geq 0$ for $i = 1, \ldots, q$. For any given constant $\varepsilon > 0$, then there exists a set $E_5 \subset [0, 2\pi)$ that has linear measure zero such that if $\psi_0 \in [0, 2\pi) - E_5$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for any $z$ satisfying* $\arg z = \psi_0$ *and $|z| \geq R_0$, and for any $(k, j) \in \Gamma$ we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}.$$

For the next lemma, denote $\delta(p,\theta) = \alpha\cos n\theta - \beta\sin n\theta$, where $p(z) = (\alpha + i\beta)z^n + \cdots$ is a polynomial with $\alpha$ and $\beta$ real.

Lemma 7 ([3]). *Let $p(z)$ be a polynomial of degree $n \geq 1$, $w(z)$ $(\not\equiv 0)$ be a meromorphic function of order less than $n$. Set $g = we^p$, then there exists a set $H_1 \subset [0, 2\pi)$ of linear measure zero, such that if $\theta \in [0, 2\pi) - (H_1 \cup H_2)$, we have*

*(1) if $\delta(p,\theta) > 0$, then there exists an $r(\theta) > 0$, such that for any $r \geq r(\theta)$,*

$$|g(re^{i\theta})| > \exp\left(\frac{1}{2}\delta(p,\theta)r^n\right),$$

(2) *if* $\delta(p,\theta) < 0$, *then there exists an* $r(\theta) > 0$, *such that for any* $r \geq r(\theta)$,

$$|g(re^{i\theta})| < \exp\left(\frac{1}{2}\delta(p,\theta)r^n\right),$$

*where* $H_2 = \{\theta : \delta(p,\theta) = 0, 0 \leq \theta \leq 2\pi\}$ *is a set of linear measure zero.*

*Proof.* Writing $p(z) = (\alpha + i\beta)z^n + p_{n-1}(z)$, we see that $g(z) = h(z)e^{(\alpha+i\beta)z^n}$, where $h(z)$ is a meoromorphic function with $\sigma(h) = s < n$. By lemma 6, there exists a set $H_1$ of linear measure zero such that for any given $\varepsilon > 0$ and $\theta \in [0, 2\pi) - H_1$, when $r \geq r_1(\theta)$ we have

$$\left|\frac{h'(re^{i\theta})}{h(re^{i\theta})}\right| \leq r^{(s-1+\varepsilon)}.$$

Since

$$\log h(re^{i\theta}) = \int_{r_0}^{r} \frac{h'(te^{i\theta})}{h(te^{i\theta})}\, dt + \log h(r_0e^{i\theta}),$$

so $|\log h(re^{i\theta})| \leq r^{s+\varepsilon} + c$ where $c$ is a constant. When $r > r_2(\theta) \geq r_1(\theta)$, we have

$$|\log|h(re^{i\theta})|| \leq |\log h(re^{i\theta})| \leq r^{s+2\varepsilon}.$$

Take $s + 2\varepsilon < n$. Note that for $z = re^{i\theta}$ we have $|e^{(\alpha+i\beta)z^n}| = e^{\delta(p,\theta)r^n}$, so when $r > r_2(\theta)$

$$\exp(-r^{s+2\varepsilon} + \delta(p,\theta)r^n) \leq |g(re^{i\theta})| \leq \exp(r^{s+2\varepsilon} + \delta(p,\theta)r^n).$$

It is easy to see that the conclusions hold from the above inequality.

Lemma 8 ([8]). *If* $g$ *is an entire function of order* $\sigma$, *then*

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\log^+ v_g(r)}{\log r},$$

*where* $v_g(r)$ *is the central index of* $g$.

## 3. Proof of Theorem 1

Suppose that $f$ is a polynomial, then $L(f)$ is also a polynomial. We know that the small function of a polynomial is constant, so the result holds clearly. Therefore we may assume that $f$ is transcendental in the following argument.

Under the hypothesis of Theorem 1 and by the Hadamard factorization theorem, it is easy to get

$$\frac{L(f) - b(z)}{f - b(z)} = Q(z), \tag{3.1}$$

where $Q(z)$ is an entire function of order $\sigma(Q) = \mu < 1/2$. Hence, by Lemma 2, for any $\alpha$ satisfying $\mu < \alpha < 1/2$, there exists a set $E_1$ with $\underline{\log dens}\, E_1 \geq 1 - \mu/\alpha$ such that

$$|Q(re^{i\theta})| \geq M(r, Q)^{\xi}, \tag{3.2}$$

for $|z| = r \in E_1$, where $\xi = \cos \pi\alpha > 0$. Set $F(z) = f(z) - b(z)$, from (3.1) we have

$$a_k F^{(k)} + a_{k-1} F^{(k-1)} + \cdots + a_1 F' + (a_0 - Q)F = b_0(z), \tag{3.3}$$

where $b_0(z) = -\sum_{j=0}^{k} b^{(j)} a_j + b$ is a small function of $f$. Rewrite (3.3) as

$$a_k \frac{F^{(k)}}{F} + a_{k-1} \frac{F^{(k-1)}}{F} + \cdots + a_1 \frac{F'}{F} + (a_0 - Q) = \frac{b_0(z)}{F}. \tag{3.4}$$

By Lemma 1, there are a set $E_2 \subset (1, \infty)$ of a finite logarithmic measure and a constant $B > 0$ such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E_2$, we have

$$\left|\frac{F^{(j)}(z)}{F(z)}\right| \leq Br[T(2r, F)]^{j+1}, \quad (j = 1, 2, \ldots, k). \tag{3.5}$$

Take $\varepsilon$ to satisfy $0 < 2\varepsilon < 1 - \mu/\alpha$. By Lemma 5, there is a set $E_3$ with $\underline{\log dens}\, E_3 \geq \mu/\alpha + \varepsilon$ such that

$$\frac{M(r, b_0)}{M(r, F)} \to 0 \tag{3.6}$$

holds for $|z| = r \in E_3$, $r \to \infty$.

We assert that $E_1$ intersects $E_3$ with $\overline{\log dens}(E_1 \cap E_3) > 0$. In fact, if not, we obtain

$$1 + \varepsilon \leq \underline{\log dens}\, E_1 + \underline{\log dens}\, E_3 \leq \underline{\log dens}(E_1 \cup E_3) \leq 1$$

a contradiction. Moreover,

$$\begin{aligned} &\underline{\log dens}\, E_1 + \underline{\log dens}\, E_3 - \overline{\log dens}(E_1 \cap E_3) \\ &\quad \leq \underline{\log dens}(E_1 - (E_1 \cap E_3)) + \underline{\log dens}\, E_3 \leq 1. \end{aligned}$$

Clearly, from this we have $\overline{\log dens}(E_1 \cap E_3) \geq \varepsilon > 0$. From (3.2) to (3.6), we know that for $r \in (E_1 \cap E_3) - (E_2 \cup [0, 1])$, we have

$$M(r, Q)^{\xi} \leq kBr^{A}[T(2r, F)]^{k+2}, \tag{3.7}$$

where $A = 1 + \max\{\deg a_j, j = 0, \ldots, k\}$. In fact, (3.7) and $\sigma(F) < \infty$ imply that there exists a sequence $r_n \to +\infty$ such that

$$\log M(r_n, Q) = O(\log r_n), \quad n \to \infty,$$

which shows that $Q$ cannot be transcendental.

## 4. Proof of Theorem 2

As in the proof Theorem 1, we can get

$$\frac{L(z)-b(z)}{f-b(z)} = Q(z) \tag{4.1}$$

where $Q(z)$ is a non-zero polynomial. When $f$ is a non-constant polynomial, it is easy to see the conclusion (a) holds. Therefore, we may assume that $f$ is transcendental in the following.

It follows from (1.1) and (4.1) that

$$a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_1 f' + (a_0 - Q) f = b(z)(1 - Q(z)). \tag{4.2}$$

Rewrite (4.2) as

$$a_k \frac{f^{(k)}}{f} + a_{k-1}\frac{f^{(k-1)}}{f} + \cdots + a_1 \frac{f'}{f} + (a_0 - Q) = \frac{b(z)(1-Q(z))}{f}. \tag{4.3}$$

It is easy to see that $b(z)(1-Q(z))$ is a small function of $f$. Therefore, by Lemma 5 there exists a set $E_1$ with $\underline{\log dens}\, E_1 > 0$, such that

$$\frac{M(r, b(z)(1-Q))}{M(r,f)} \to 0, \tag{4.4}$$

for $|z| = r \in E_1$, $r \to \infty$. From the Wiman-Valiron Theory (see [6], [8] or [10]), we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{v_f(r)}{z}\right)^j (1+o(1)), \quad (j = 1, 2, \ldots, k), \tag{4.5}$$

where $|z| = r$, $|f(z)| = M(r,f)$, $r \notin E_2$ which has a finite logarithmic measure. Substituting (4.5) and (4.4) into (4.3), we obtain

$$\begin{aligned} d_k z^{n_k}\left(\frac{v_f(r)}{z}\right)^k (1+o(1)) + d_{k-1} z^{n_{k-1}} \left(\frac{v_f(r)}{z}\right)^{k-1} (1+o(1)) \\ + \cdots + d_0 z^{n_0}(1+o(1)) = o(1) \end{aligned} \tag{4.6}$$

where $a_0 - Q = d_0 z^{n_0}(1+o(1))$ and $a_j = d_j z^{nj}(1+o(1))$, $d_j$ $(j = 0, 1, \cdots, k)$ are constants and $d_k \neq 0$, $v_f(r)$ is the central index of $f$. Since any solution of an algebraic equation is a continuous function of the coefficients, therefore $v_f(r)$ is asymptotically equal to a solution of the equation

$$d_k (v_f(r))^k z^{n_k - k} + d_{k-1}(v_f(r))^{k-1} z^{n_{k-1}-(k-1)} + \cdots + d_0 z^{n_0} = 0. \tag{4.7}$$

From the argument used in [10, pp. 106–108], for sufficiently large $r$, we have

$$v_f \sim c_0 \cdot r^{\sigma}, \quad r \in E_1 - E_2 \tag{4.8}$$

where $c_0(>0)$ is constant and $\sigma$ is a rational number. It follows from (4.7) and (4.8) that the degrees (in $z$) of all terms of (4.7) are respectively

$$k(\sigma-1)+n_k, (k-1)(\sigma-1)+n_{k-1}, \dots, n_0. \tag{4.9}$$

If $(\deg a_j - \deg a_i)/(j-i) \le 1/2$ for $i \ne j$ $(i, j = 1, \dots, k)$, we know that any two of (4.9) except $n_0$ are distinct. In fact, if there exist $i_0$ and $j_0$ such that

$$i_0(\sigma-1)+n_{i_0} = j_0(\sigma-1)+n_{j_0}, \tag{4.10}$$

we have $\sigma = 1-(n_{j_0}-n_{i_0})/(j_0-i_0) \ge 1/2$, a contradiction. Hence, we can conclude that $n_0$ is equal to one of $\{k(\sigma-1)+n_k, \dots, (\sigma-1)+n_1\}$. We assume $n_0 = j_*(\sigma-1)+n_{j_*}$ $(1 \le j_* \le k)$. Set $Q(z) = \beta_0 z^h(1+o(1))$, $\beta_0$ is a non-zero constant. Now we discuss the following two subcases.

Subcase 1. Suppose $h \le \deg a_0$, then $h \le \max\{\deg a_j \mid j = 0, \dots, k\}$ holds clearly.

Subcase 2. Suppose $h > \deg a_0$, thus we have $h = n_0 = j_*(\sigma-1)+n_{j_*}$. Hence,

$$h < n_{j_*} = \deg a_{j_*} \le \max\{\deg a_j \mid j = 0, \dots, k\}.$$

Next, we consider the case (b). According to the case (a), clearly $Q(z)$ is a non-zero constant $c$. We assume that $f$ is transcendental. Rewrite (4.3) as

$$a_k\frac{f^{(k)}}{f}+a_{k-1}\frac{f^{(k-1)}}{f}+\cdots+a_1\frac{f'}{f}+(a_0-c)=\frac{b(1-c)}{f}. \tag{4.11}$$

From [8, pp. 33–35], we know that $v_f(r)$ is increasing, right-continuous and also tends to $+\infty$ as $r \to \infty$. In addition, it follows from Lemma 8 that $v_f(r) \le r^{1/2}$ for sufficiently large $r$. Therefore, we have

$$\left(\frac{v_f(r)}{z}\right)^m = o\left(\left(\frac{v_f(r)}{z}\right)^n\right), \quad (m > n). \tag{4.12}$$

Now we discuss the following two subcases.

Subcase 1. Suppose $\sigma(f) > 0$, then there exists a sequence $\{r_v\}$ $(r_v \to +\infty, r \notin E_2)$ satisfying

$$M(r_v, f) \ge (1+o(1))\exp(r_v^{\sigma(f)-\varepsilon}) \tag{4.13}$$

for sufficiently large $r_v$ and $\varepsilon > 0$. In fact, it is well known that $\sigma(f) = \overline{\lim}_{r\to\infty} \log\log M(r,f)/\log r$ since $f$ is entire. According to the definition of upper limit, we know that there exists a sequence $\{r_v'\}$ $(r_v' \to \infty)$ such that $\sigma(f) = \lim_{v\to\infty} \log\log M(r_v', f)/\log r_v'$. Set $\operatorname{lm} E_2 = \eta > 0$, where $\operatorname{lm} E_2$ denotes the logarithmic measure of $E_2$. We can take $r_v \in [r_v', e^\eta r_v'] \backslash E_2$, then

$$\frac{\log\log M(r_v, f)}{\log r_v} \ge \frac{\log\log M(r_v', f)}{\log e^\eta r_v'}.$$

Let $d_j$ $(0 \le j \le k)$ be the first non-zero complex number of $d_0 = a_0 - c, d_1 = a_1, d_2 = a_2, \dots, d_k = a_k$, and let $d_{j'}$ be the second non-zero complex number with $j' > j$. Substituting (4.5), (4.12) and (4.13) into (4.11), we have

$$|d_j|(1+o(1))\left(\frac{v_f(r_v)}{z}\right)^j \le \frac{|b(1-c)|}{(1+o(1))\exp(r_v^{\sigma(f)-\varepsilon})} \tag{4.14}$$

where $|z| = r_v$, $|f(z)| = M(r_v, f)$ and $r \notin E_2$. If $f$ is non-constant, the $v_f(r_v)$ must be unbound. When $c \ne 1$, from (4.14) we have $v_f(r_v) \to 0$. It is a contradiction. In the following, we treat the case $c = 1$. From the Wiman-Valiron Theory, we have

$$|d_{j'}|(1+o(1))r_v^{(j'-j)(\sigma(f)+\varepsilon-1)} \ge |d_j| \tag{4.15}$$

where $|z| = r_v \notin E_2$, $|f^{(j)}(z)| = M(r_v, f^{(j)})$. It is easy to see that (4.15) is absurd.

Subcase 2. Suppose $\sigma(f) = 0$, then we know that there exists a sequence $\{r_n\}$ tending to $\infty$ such that $M(r_n, f) \ge r^n$. Using the similar argument as above, we also get a result about $r_n$ like (4.15), which leads to a contradiction.

Hence, $f$ is a non-constant polynomial. From (4.11), if $a_0 \ne c$, clearly it is impossible. Therefore, $a_0 = c$ and $\deg f \le k$. Suppose $f = b_m z^m + \cdots + b_1 z + b_0$ $(0 < m \le k)$ where $b_i$ $(i = 0, 1, \dots, m)$ are constants and $b_m \ne 0$. It follows from (4.11) that $a_j = 0$ $(j = 1, \dots, m-1)$ and $m! a_m b_m = b(1 - a_0)$.

## 5. Proof of Theorem 3

Under the assumption of Theorem 3 and by using the Hadamard Factorization Theorem, we easily get

$$\frac{f^{(k)} - a}{f - a} = Qe^p, \tag{5.1}$$

where $p(z)$ is a polynomial, and $Q(z)$ is an entire function of order $\sigma(Q) < 1/2$. Set $F(z) = f/a - 1$. From (5.1) we have

$$F^{(k)} - Qe^p F = 1. \tag{5.2}$$

If $p(z)$ is a non-constant polynomial, from (5.2) we can know that $F$ has infinite order by using the similar argument in [11] and Lemma 6. It leads to a contradiction. Hence $p(z)$ is a constant. By using the similar argument in the proof of Theorem 1, we can know that $Q(z)$ is a non-zero polynomial. Rewrite (5.2) as

$$\frac{F^{(k)}}{F} - c_0 Q = \frac{1}{F}, \tag{5.3}$$

where $c_0$ is a non-zero constant. From the Wiman-Valiron Theory and by using similar argument in the proof of Theorem 2, we obtain for $r \notin E_1$ which has a finite logarithmic measure.

(5.4) $$(v_f(r))^k z^{-k}(1+o(1)) + \beta z^n(1+o(1)) = o(1)$$

where $-c_0 Q(z) = \beta z^n(1+o(1))$, $\beta \neq 0$ is a constant. From (5.4) we deduce $\log v_f(r) = (n/k + 1 + o(1)) \log r$ for $r \notin E_1$. It thus follows $\sigma(f) = 1 + n/k$. On the other hand, we assume that $n$ is different from any positive integer. From this $n$ must be zero, so that $Q(z)$ is a constant, which completes the proof of Theorem 3.

## 6. Proof of Theorem 4

Since $f = 1 + \sum_{n=1}^{\infty} z^{qn}/(pn)!$, we have

$$f^{(j)} = \sum_{n=1}^{\infty} \frac{(qn)! z^{qn-j}}{(qn-j)!(pn)!} \quad (j = 1, 2, \dots, p).$$

Substituting this into (1.6), we get

(6.1) $$\sum_{n=1}^{\infty} \sum_{j=1}^{p} \frac{A_j (qn)! z^{qn}}{(qn-j)!(pn)!} \equiv \sum_{n=0}^{\infty} \frac{z^{q(n+1)}}{(pn)!} \equiv \sum_{n=1}^{\infty} \frac{z^{qn}}{(p(n-1))!}.$$

From this, there must be

(6.2) $$\sum_{j=1}^{p} \frac{A_j (qn)!}{(qn-j)!(pn)!} = \frac{1}{(p(n-1))!}, \quad (n = 1, 2, \dots).$$

It means

(6.3) $$\begin{aligned} &A_1 qn + A_2 qn(qn-1) + A_3 qn(qn-1)(qn-2) \\ &\qquad + \cdots + A_p qn(qn-1)(qn-2)\cdots(qn-p+1) \\ &= pn(pn-1)(pn-2)\cdots(pn-p+1). \end{aligned}$$

We can consider it as the comparison between two polynomials of degree $p$ in $n$. So clearly $A_p = (p/q)^p$, then take it into (6.3) we can get another comparison between two polynomials of degree $p-1$ in $n$. Similarly as above we can solve $A_j$ $(j = 1, 2, \dots, p-1)$.

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## References

[1] P. D. Barry, On a theorem of Besicovitch, Quart. J. Math. Oxford Ser. (2), **14** (1963), 293–302.

[2] Z. X. Chen, The zero, pole and order meromorphic solutions of differential equations with meromorphic coefficients, Kodai Math. J., **19** (1996), 341–354.

[3] S. A. Gao, Z. X. Chen and T. W. Chen, Theory of Complex Oscillation of Linear Differential Equations, Hua Zhong (Central China) University of Science and Technology Press, 1998. (in Chinese)
[4] W. Hayman, Angular value distribution of power series with gaps, Proc. London Math. Soc. (3), **24** (1972), 590–624.
[5] W. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.
[6] W. Hayman, The local growth of power series: a survey of the Wiman-Valiron method, Canad. Math. Bull., **17** (1974), 317–358.
[7] Y. Z. He and X. Z. Xiao, Algebroid Functions and Ordinary Differential Equations, Science Press, 1988. (in Chinese)
[8] G. Jank and L. Volkmann, Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differential gleichungen, Birkhäuser, Basel-Boston, 1985.
[9] G. Gundersen, Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates, J. London Math. Soc. (2), **37** (1988), 88–104.
[10] G. Valiron, Lectures on the General Theory of Integral Functions, Chelsea, New York, 1949.
[11] L. Z. Yang, Solution of a differential equation and its application, Kodai Math. J., **22** (1999), 458–464.
[12] Q. C. Zhang, The uniqueness of meromorphic functions with their derivatives, Kodai Math. J., **21** (1998), 179–184.

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