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KODAI MATH. J.
25 (2002), 309–320

UNIQUENESS OF ENTIRE FUNCTIONS AND FIXED POINTS

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Abstract

Let f be a nonconstant entire function. If f, f' and f'' have the same fixed points, then $f \equiv f'$.

1. Introduction

Let f be a nonconstant meromorphic function in the whole complex plane. We use the following standard notation of value distribution theory,

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \dots$$

(see Hayman [1], Yang [6]). We denote by $S(r,f)$ any function satisfying

$$S(r,f) = o\{T(r,f)\},$$

as $r \to +\infty$, possibly outside a set of r of finite linear measure.

Let g be a meromorphic function, and let a, b be two complex numbers. If $g(z) = b$ whenever $f(z) = a$, then we denote it by $f(z) = a \Rightarrow g(z) = b$. Thus $f(z) = a \Leftrightarrow g(z) = a$ means $f(z) = a$ if and only if $g(z) = a$.

In 1986, Jank-Muse-Volkmann [3] proved the following result.

Theorem A. *Let f be a nonconstant entire function, and a be a nonzero value. If $f(z) = a \Leftrightarrow f'(z) = a$, and $f'(z) = a \Rightarrow f''(z) = a$, then $f \equiv f'$.*

In this paper, we extend Theorem A as follows.

Theorem 1. *Let f be a nonconstant entire function, and let a, c be two nonzero constants. If $f(z) = a \Rightarrow f'(z) = a$, $f'(z) = a \Rightarrow f''(z) = c$, then either $f(z) = Ae^{cz/a} + (ac - a^2)/c$ or $f(z) = Ae^{cz/a} + a$, where A is a nonzero constant.*

2000 *Mathematics Subject Classification*: 30D35.

Keywords and phrases: Meromorphic function, entire function, uniqueness.

† Supported by the NNSF of China (Grant No. 10071038) and "Qinglan Project" of the Educational Department of Jiangsu Province.

Received February 18, 2002.

Corollary 2. *Let f be a nonconstant entire function, and let a be a nonzero constant. If $f(z) = a \Rightarrow f'(z) = a$, $f'(z) = a \Rightarrow f''(z) = a$, then either $f \equiv f'$ or $f \equiv f' + a$.*

Remark. In Corollary 2, the case $f \equiv f' + a$ occurs.

Let $f(z) = a + Ae^z$. Then $f'(z) = Ae^z$, $f''(z) = Ae^z$. Obviously, f satisfies $f(z) = a \Rightarrow f'(z) = a$, $f'(z) = a \Rightarrow f''(z) = a$, and $f \equiv f' + a$.

The main result of this paper is the following

Theorem 3. *Let f be a nonconstant entire function. If $f(z) = z \Leftrightarrow f'(z) = z$, and $f'(z) = z \Rightarrow f''(z) = z$, then $f \equiv f'$.*

If $f(z_0) = z_0$, then z_0 is called a fixed point of f.

Corollary 4. *Let f be a nonconstant entire function. If f, f' and f'' have the same fixed points, then $f \equiv f'$.*

2. Proof of Theorem 1

Set

$$\psi(z) = \frac{cf'(z) + af''(z)}{f(z) - a} - \frac{2cf''(z)}{f'(z) - a}. \tag{2.1}$$

Let $f(z_0) = a$. Then by the assumptions we may suppose that, near z_0

$$f(z) = a + a(z - z_0) + \frac{c}{2}(z - z_0)^2 + b(z - z_0)^3 + O((z - z_0)^4), \tag{2.2}$$

where $b = f^{(3)}(z_0)/6$ is a constant. Thus we have

$$\begin{aligned} f'(z) &= a + c(z - z_0) + 3b(z - z_0)^2 + O((z - z_0)^3), \\ f''(z) &= c + 6b(z - z_0) + O((z - z_0)^2). \end{aligned} \tag{2.3}$$

Hence

$$\frac{cf'(z) + af''(z)}{f(z) - a} = \frac{2c}{z - z_0} + 6b + O(z - z_0),$$

$$\frac{2cf''(z)}{f'(z) - a} = \frac{2c}{z - z_0} + 6b + O(z - z_0).$$

Thus we obtain

$$\psi(z_0) = 0. \tag{2.4}$$

Next we consider two cases.

Case 1. $f''/f' \equiv c/a$, that is $cf' \equiv af''$. Then we get

$$f'(z) = \frac{cA}{a} e^{(c/a)z}, \quad f(z) = Ae^{(c/a)z} + B, \tag{2.5}$$

where $A \neq 0, B$ are two constants.

If there exists z_0 satisfying $f(z_0) = a$, then by the assumptions and (2.5) we obtain

$$f(z) = Ae^{(c/a)z} + \frac{ac - a^2}{c}.$$

If there doesn't exist z_0 satisfying $f(z_0) = a$, then by (2.5) we get

$$f(z) = a + Ae^{(c/a)z},$$

where A is a nonzero constant.

Case 2. $f''/f' \not\equiv c/a$. Then by the assumptions we have

$$\begin{aligned} N\left(r, \frac{1}{f' - a}\right) &\leq N\left(r, \frac{1}{f''/f' - c/a}\right) \leq T\left(r, \frac{f''}{f'}\right) + O(1) \\ &= N\left(r, \frac{f''}{f'}\right) + S(r, f) = \overline{N}\left(r, \frac{1}{f'}\right) + S(r, f). \end{aligned} \tag{2.6}$$

In the following, we consider two subcases.

Case 2.1. $\psi \not\equiv 0$. Then by (2.1) and (2.4) we get

$$\begin{aligned} N\left(r, \frac{1}{f - a}\right) &\leq N\left(r, \frac{1}{\psi}\right) \leq T(r, \psi) + O(1) \\ &\leq N_0\left(r, \frac{1}{f' - a}\right) + S(r, f), \end{aligned} \tag{2.7}$$

where $N_0(r, 1/(f' - a))$ is the counting function for those zero points of $f'(z) - a$ which are not zero points of $f(z) - a$.

Thus by the assumption and (2.7) we obtain

$$2N\left(r, \frac{1}{f - a}\right) \leq N\left(r, \frac{1}{f' - a}\right) + S(r, f). \tag{2.8}$$

On the other hand, by Nevanlinna first fundamental theorem we have

$$\begin{aligned} m\left(r, \frac{1}{f - a}\right) &\leq m\left(r, \frac{1}{f'}\right) + S(r, f) \\ &\leq T(r, f') - N\left(r, \frac{1}{f'}\right) + S(r, f) \\ &\leq T(r, f) - N\left(r, \frac{1}{f'}\right) + S(r, f) \end{aligned}$$

$$= T\left(r, \frac{1}{f-a}\right) - N\left(r, \frac{1}{f'}\right) + S(r, f)$$

$$= m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) - N\left(r, \frac{1}{f'}\right) + S(r, f).$$

Thus

$$N\left(r, \frac{1}{f'}\right) \le N\left(r, \frac{1}{f-a}\right) + S(r, f). \tag{2.9}$$

Hence by (2.6), (2.8) and (2.9) we get

$$N\left(r, \frac{1}{f-a}\right) = N\left(r, \frac{1}{f'-a}\right) + S(r, f) = S(r, f). \tag{2.10}$$

By Milloux's inequality (see [1, 6])

$$T(r, f) \le N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f'-a}\right) + S(r, f). \tag{2.11}$$

Thus by (2.10) and (2.11) we get $T(r, f) = S(r, f)$, a contradiction.

Case 2.2. $\psi \equiv 0$. That is

$$\frac{cf'(z) + af''(z)}{f(z) - a} \equiv \frac{2cf''(z)}{f'(z) - a}. \tag{2.12}$$

Thus by $f(z) = a \Rightarrow f'(z) = a$ and (2.12) we deduce that $f(z) = a \Leftrightarrow f'(z) = a$. Hence, $f''(z) = 0 \Rightarrow f'(z) = 0$.

Set

$$\phi(z) = \frac{af''(z) - cf'(z)}{f(z) - a}. \tag{2.13}$$

Since $f''/f' \not\equiv c/a$, we get $\phi \not\equiv 0$.

Let $f'(z_0) = 0$ and $f''(z_0) \ne 0$. Then by (2.12) we get

$$f(z_0) = \frac{2ac - a^2}{2c}. \tag{2.14}$$

Differentiating the two sides of (2.12) we get

$$\frac{[cf''(z) + af'''(z)][f(z) - a] - f'(z)[cf'(z) + af''(z)]}{[f(z) - a]^2} \equiv \frac{2cf'''(z)[f'(z) - a] - 2c[f''(z)]^2}{[f'(z) - a]^2}. \tag{2.15}$$

Thus by (2.14), (2.15), $f'(z_0) = 0$ and $f''(z_0) \ne 0$, we obtain

$$f''(z_0) = c. \tag{2.16}$$

Hence we have

$$\phi(z_0) = -\frac{2c^2}{a}. \tag{2.17}$$

Next we divide two subcases.

CASE 2.2.1. $\phi(z) \not\equiv -2c^2/a$. Then by (2.17),

$$\overline{N}\left(r, \frac{1}{f'}\right) - \overline{N}\left(r, \frac{1}{f''}\right) \le \overline{N}\left(r, \frac{1}{\phi + 2c^2/a}\right) \le T(r, \phi) + S(r, f). \tag{2.18}$$

Obviously, by (2.13), Logarithmic Derivative Lemma (see [1, 6]) and the assumptions we get

$$T(r, \phi) = S(r, f). \tag{2.19}$$

Thus we get

$$\overline{N}\left(r, \frac{1}{f'}\right) - \overline{N}\left(r, \frac{1}{f''}\right) = S(r, f). \tag{2.20}$$

By $f''(z) = 0 \Rightarrow f'(z) = 0$, (2.13) and (2.19) we have

$$\overline{N}\left(r, \frac{1}{f''}\right) \le \overline{N}\left(r, \frac{1}{\phi}\right) = T(r, \phi) + O(1) = S(r, f). \tag{2.21}$$

Hence

$$\overline{N}\left(r, \frac{1}{f'}\right) = S(r, f). \tag{2.22}$$

Thus by Milloux's inequality, (2.22) and (2.6), we get $T(r, f) = S(r, f)$, a contradiction.

CASE 2.2.2. $\phi \equiv -2c^2/a$. That is

$$af''(z) - cf'(z) + \frac{2c^2}{a}[f(z) - a] = 0, \tag{2.23}$$

for $z \in \mathbf{C}$.

If there exists z_0 such that $f''(z_0) = 0$, then by $f''(z) = 0 \Rightarrow f'(z) = 0$ and (2.23) we get $f'(z_0) = 0$ and $f(z_0) = a$, which contradicts $f(z) = a \Rightarrow f'(z) = a$. Hence, $f''(z) \neq 0$.

Solving the equation (2.23) we obtain

$$f(z) = c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + a, \tag{2.24}$$

where λ_1 and λ_2 are solutions of the equation $az^2 - cz + 2c^2/a = 0$, and c_1, c_2 are two constants.

Thus

$$f''(z) = c_1 \lambda_1^2 e^{\lambda_1 z} + c_2 \lambda_2^2 e^{\lambda_2 z}. \tag{2.25}$$

Since $f''(z) \neq 0$, we deduce from (2.25) that either $c_1 = 0$ or $c_2 = 0$. Without loss of generality, we assume that $c_2 = 0$, then

$$f(z) = c_1 e^{\lambda_1 z} + a. \tag{2.26}$$

Thus by $f'(z) = a \Rightarrow f''(z) = c$ and (2.26) we get

$$f(z) = Ae^{(c/a)z} + a.$$

The proof of Theorem 1 is complete.

3. Proof of Theorem 3

Firstly, we consider the case that f is a transcendental entire function. Obviously, we have

$$\begin{aligned} &m\left(r, \frac{1}{f-z}\right) + m\left(r, \frac{1}{f'-z}\right) \\ &\quad \leq m\left(r, \frac{1}{f''}\right) + m\left(r, \frac{1}{f''-1}\right) + S(r,f) \\ &\quad \leq m\left(r, \frac{1}{f''} + \frac{1}{f''-1}\right) + S(r,f) \\ &\quad \leq m\left(r, \frac{1}{f'''}\right) \leq T(r, f''') + S(r,f) \\ &\quad \leq T(r, f') + S(r,f). \end{aligned}$$

Hence by Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} &T(r, f-z) + T(r, f'-z) \\ &\quad \leq N\left(r, \frac{1}{f-z}\right) + N\left(r, \frac{1}{f'-z}\right) + T(r, f') + S(r,f). \end{aligned} \tag{3.1}$$

By $f(z) = z \Leftrightarrow f'(z) = z$, and $f'(z) = z \Rightarrow f''(z) = z$, it is easy to see that

$$N\left(r, \frac{1}{f'-z}\right) = N\left(r, \frac{1}{f-z}\right) + S(r,f). \tag{3.2}$$

Thus by (3.1) and (3.2) we have

$$T(r,f) \leq 2N\left(r, \frac{1}{f-z}\right) + S(r,f). \tag{3.3}$$

Set

$$H(z) = \frac{f''(z)}{f'(z)-1} - \frac{z}{z-1}. \tag{3.4}$$

If $H \equiv 0$, then by (3.4) we get

$$\frac{f''(z)}{f'(z)-1} \equiv \frac{z}{z-1}.$$

Thus we have

$$f'(z) = 1 + C(z-1)e^z, \tag{3.5}$$

$$f(z) = z + C(z-2)e^z + A. \tag{3.6}$$

where A, C ($\neq 0$) are two constants.

Hence by (3.5), (3.6) and $f(z) = z \Leftrightarrow f'(z) = z$, we know that $f'(z) = z$ have the unique solution $z_0 = 2 - A$ with $z_0 \neq 0, 1$. But it is clear that $f'(z) = z$ have infinitely many solutions, a contradiction. Hence $H \not\equiv 0$, that is

$$\frac{f''(z)}{f'(z)-1} \not\equiv \frac{z}{z-1}.$$

Thus by the assumption of the theorem, we have

$$\begin{aligned} N\left(r, \frac{1}{f-z}\right) &\leq N\left(r, \frac{1}{H}\right) + O(\log r) \\ &\leq T(r, H) + S(r, f) \\ &\leq N\left(r, \frac{f''}{f'-1}\right) + S(r, f) \\ &\leq \overline{N}\left(r, \frac{1}{f'-1}\right) + S(r, f). \end{aligned} \tag{3.7}$$

Hence by (3.7) and (3.3) we get

$$T(r, f) \leq 2\overline{N}\left(r, \frac{1}{f'-1}\right) + S(r, f). \tag{3.8}$$

Set

$$\phi(z) = \frac{f'(z)-1}{f(z)-z} - \frac{f''(z)-1}{f'(z)-z}, \tag{3.9}$$

$$\psi(z) = \frac{(z-1)f''(z) - z[f'(z)-1]}{f(z)-z}. \tag{3.10}$$

Obviously, by Logarithmic Derivative Lemma (see [1, 6])

$$m(r, \phi) = S(r, f), \quad m(r, \psi) = S(r, f). \tag{3.11}$$

Let z_0 satisfy $f(z_0) = z_0$ and $z_0 \neq 0, 1$. Then by assumption we may assume that, near z_0

$$f(z) = z_0 + z_0(z - z_0) + \frac{z_0}{2}(z - z_0)^2 + \frac{f'''(z_0)}{6}(z - z_0)^3 + \cdots, \tag{3.12}$$

Thus we have

$$f'(z) = z_0 + z_0(z - z_0) + \frac{f'''(z_0)}{2}(z - z_0)^2 + \cdots, \tag{3.13}$$

$$f''(z) = z_0 + f'''(z_0)(z - z_0) + \cdots. \tag{3.14}$$

By (3.12)–(3.14) we get

$$N(r, \phi) = S(r, f), \quad N(r, \psi) = S(r, f). \tag{3.15}$$

Thus we have

$$T(r, \phi) = S(r, f), \quad T(r, \psi) = S(r, f). \tag{3.16}$$

By (3.12)–(3.14) and (3.9)–(3.10) we get

$$\phi(z_0) = \frac{f'''(z_0) - z_0}{2(1 - z_0)},$$

and

$$\psi(z_0) = f'''(z_0) - z_0 - 1.$$

Thus we obtain

$$2(z_0 - 1)\phi(z_0) + \psi(z_0) + 1 = 0.$$

If $2(z - 1)\phi(z) + \psi(z) + 1 \not\equiv 0$, then by (3.16)

$$\begin{aligned} N\left(r, \frac{1}{f - z}\right) &\le N\left(r, \frac{1}{2(z - 1)\phi + \psi + 1}\right) + O(\log r) \\ &\le T(r, \phi) + T(r, \psi) + S(r, f) \le S(r, f). \end{aligned} \tag{3.17}$$

Thus by (3.3) and (3.17) we get a contradiction: $T(r, f) = S(r, f)$. Hence

$$2(z - 1)\phi(z) + \psi(z) + 1 \equiv 0. \tag{3.18}$$

Now let z_1 satisfy $f'(z_1) = 1$ and $z_1 \ne 1$. Then by $f(z) = z \Leftrightarrow f'(z) = z$, we know that $f(z_1) \ne z_1$. Thus by (3.12) and (3.13) we have

$$\phi(z_1) = \frac{f''(z_1) - 1}{z_1 - 1}, \tag{3.19}$$

$$\psi(z_1) = \frac{(z_1 - 1)f''(z_1)}{f(z_1) - z_1}. \tag{3.20}$$

Hence by (3.18)–(3.20) we obtain

$$[2f(z_1) - z_1 - 1]f''(z_1) = f(z_1) - z_1.$$

If $2f(z_1) - z_1 - 1 = 0$, then $f(z_1) = z_1$, a contradiction. Hence $2f(z_1) - z_1 - 1 \neq 0$. Thus

$$f''(z_1) = \frac{f(z_1) - z_1}{2f(z_1) - z_1 - 1}. \tag{3.21}$$

Therefore by (3.19)–(3.21) we get

$$\phi(z_1) = \frac{1 - f(z_1)}{(z_1 - 1)[2f(z_1) - z_1 - 1]}, \tag{3.22}$$

$$\psi(z_1) = \frac{z_1 - 1}{2f(z_1) - z_1 - 1}. \tag{3.23}$$

By (3.9), (3.10) and (3.21) we get

$$\phi'(z_1) = \frac{f'''(z_1)}{z_1 - 1} + \frac{1}{2f(z_1) - z_1 - 1} + \frac{[f(z_1) - 1]^2}{(z_1 - 1)^2[2f(z_1) - z_1 - 1]^2}, \tag{3.24}$$

and

$$\psi'(z_1) = \frac{(z_1 - 1)f'''(z_1)}{f(z_1) - z_1} - \frac{z_1 - 1}{2f(z_1) - z_1 - 1}. \tag{3.25}$$

By (3.18) we get

$$2\phi(z) + 2(z - 1)\phi'(z) + \psi'(z) \equiv 0. \tag{3.26}$$

Thus we have

$$2\phi(z_1) + 2(z_1 - 1)\phi'(z_1) + \psi'(z_1) = 0. \tag{3.27}$$

By (3.22)–(3.24) and (3.27) we get

$$\frac{2[1 - f(z_1)]}{(z_1 - 1)^2}\psi(z_1) + 2f'''(z_1) + 2\psi(z_1) + \frac{2[f(z_1) - 1]^2}{(z_1 - 1)^3}\psi^2(z_1) + \psi'(z_1) = 0. \tag{3.28}$$

By (3.25) we get

$$f'''(z_1) = \frac{[\psi'(z_1) + \psi(z_1)][f(z_1) - z_1]}{z_1 - 1}. \tag{3.29}$$

Thus by (3.28) and (3.29) we have

$$\begin{aligned} &\frac{2[1 - f(z_1)]}{(z_1 - 1)^2}\psi(z_1) + \frac{2[\psi'(z_1) + \psi(z_1)][f(z_1) - z_1]}{z_1 - 1} \\ &\quad + 2\psi(z_1) + \frac{2[f(z_1) - 1]^2}{(z_1 - 1)^3}\psi^2(z_1) + \psi'(z_1) = 0. \end{aligned} \tag{3.30}$$

By (3.23) we get

$$f(z_1) = \frac{z_1+1}{2} + \frac{z_1-1}{2\psi(z_1)}. \tag{3.31}$$

Hence by (3.30) and (3.31), we have

$$2(z_1-1)\psi'(z_1) + \psi^3(z_1) + 2(z_1-1)\psi^2(z_1) + (2z_1-3)\psi(z_1) = 0. \tag{3.32}$$

Let

$$\Delta = 2(z-1)\psi'(z) + \psi^3(z) + 2(z-1)\psi^2(z) + (2z-3)\psi(z).$$

If $\Delta \not\equiv 0$, then

$$\overline{N}\left(r, \frac{1}{f'-1}\right) \le N\left(r, \frac{1}{\Delta}\right) + O(\log r) \le T(r,\Delta) + S(r,f) \le S(r,f). \tag{3.33}$$

Thus by (3.8) and (3.33) we get a contradiction: $T(r,f) = S(r,f)$.
Hence, $\Delta \equiv 0$, that is

$$2(z-1)\psi'(z) + \psi^3(z) + 2(z-1)\psi^2(z) + (2z-3)\psi(z) \equiv 0. \tag{3.34}$$

Obviously, by (3.34), ψ is an entire function. We claim that ψ is not transcendental. Indeed, if ψ is transcendental, then by (3.34) we have

$$\begin{aligned} 3T(r,\psi) &= 3m(r,\psi) = m(r,\psi^3) \\ &= m(r, 2(z-1)\psi^2 + (2z-3)\psi + 2(z-1)\psi')) \\ &\le m(r,\psi) + m\left(r, 2(z-1)\psi + (2z-3) + 2(z-1)\frac{\psi'}{\psi}\right) \\ &\le 2m(r,\psi) + S(r,\psi) = 2T(r,\psi) + S(r,\psi). \end{aligned}$$

Thus we get a contradiction: $T(r,\psi) = S(r,\psi)$. Hence ψ is a polynomial. Next, by simple computation, we deduce that either $\psi \equiv 0$ or $\psi \equiv -1$.

If $\psi \equiv 0$, Then by (3.10) we get

$$(z-1)f''(z) \equiv z[f'(z) - 1],$$

which means $H \equiv 0$, a contradiction.

If $\psi \equiv -1$, then by (3.18) we know $\phi \equiv 0$. Thus by (3.9) we have

$$\frac{f'(z)-1}{f(z)-z} \equiv \frac{f''(z)-1}{f'(z)-z}.$$

Next we can easily deduce that $f \equiv f'$.

Now we prove that f can not be a polynomial.

By simple computation, f can not be a polynomial with $\deg f \le 2$. Next we prove that f can not be a polynomial with $\deg f \ge 3$. Suppose that there exists such polynomial f with $f(z) = z \Leftrightarrow f'(z) = z$ and $f'(z) = z \Rightarrow f''(z) = z$, and $d = \deg f \ge 3$. Let $z_1, z_2, \ldots, z_n$ be the fixed points of f. Then we have

(3.35) $$f(z) = z + A(z - z_1)^{\alpha_1}(z - z_2)^{\alpha_2} \cdots (z - z_n)^{\alpha_n},$$

(3.36) $$f'(z) = z + B(z - z_1)^{\beta_1}(z - z_2)^{\beta_2} \cdots (z - z_n)^{\beta_n},$$

and

(3.37) $$f''(z) = z + C(z - z_1)^{\gamma_1}(z - z_2)^{\gamma_2} \cdots (z - z_n)^{\gamma_n} p(z),$$

where p ($\not\equiv 0$) is a polynomial, and A, B, C are three non-zero constants and $\{\alpha_j\}, \{\beta_j\}, \{\gamma_j\}$ $(j = 1, 2, \dots, n)$ are positive integers satisfying

(3.38) $$\sum_{j=1}^{n} \alpha_j = d, \quad \sum_{j=1}^{n} \beta_j = d - 1, \quad \sum_{j=1}^{n} \gamma_j + \deg p = d - 2.$$

From (3.35) and (3.36) we obtain

(3.39) $$1 + A\sum_{i=1}^{n} \alpha_i (z - z_i)^{\alpha_i - 1} \prod_{j \neq i} (z - z_j)^{\alpha_j} \equiv z + B\prod_{j=1}^{n} (z - z_j)^{\beta_j}.$$

If $\alpha_j \geq 2$, then by (3.39) we get $z_j = 1$. Similarly, we know that if $\beta_j \geq 2$, then $z_j = 1$. Without loss of generality, we assume that $j = 1$. Thus by (3.35)–(3.36) and (3.38) we have

(3.40) $$f(z) = z + A(z - 1)^{\alpha_1}(z - z_2) \cdots (z - z_n),$$

(3.41) $$f'(z) = z + B(z - 1)^{\alpha_1 - 1}(z - z_2) \cdots (z - z_n).$$

If $\alpha_1 \geq 3$, then by (3.40) we get $f(1) = 1$ and $f''(1) = 0$, which contradicts $f(z) = z \Rightarrow f''(z) = z$. Thus $\alpha_1 = 2$. Hence by (3.37)–(3.38), and (3.41) we have

(3.42) $$f'(z) = z + B(z - 1)(z - z_2) \cdots (z - z_n).$$

(3.43) $$f''(z) = z + C(z - 1)(z - z_2) \cdots (z - z_n).$$

Thus by (3.42) and (3.43) we get a contradiction: $\deg f' = \deg f''$. The proof of Theorem 3 is complete.

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