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A LOCAL LIMIT THEOREM FOR RANDOM WALK DEFINED ON A FINITE MARKOV CHAIN WITH ABSORBING BARRIERS

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Abstract

Let $\{\xi_n\}_{n \ge 0}$ denote an ergodic Markov chain with a finite state space $\Xi = \{1, 2, \dots, s\}$. For each $j, k \in \Xi$, let $\{Y_n^{jk}\}_{n \ge 1}$ be a sequence of i.i.d. $\{-1, 1\}$-valued random variables which are independent of $\{\xi_n\}$. We define the process $\{S_n\}_{n \ge 0}$ by $S_0 = 0$ and $S_n = S_{n-1} + Y_n^{\xi_{n-1}\xi_n}$ for $n \ge 1$. Let a be a positive integer. We denote by T_x the first exit time of the process from the interval $[-x, a-x]$ for each $x = 0, 1, \dots, a$. We give an asymptotic behaviour of the transition functions $P_{jk}^{(n)}(x, y) = \boldsymbol{P}\{x + S_n = y; T_x > n; \xi_n = k \mid \xi_0 = j\}$ as $n \to \infty$ for each $x, y \in [0, a]$ and all $j, k \in \Xi$.

1. Preliminaries and the main result

Let $(\Omega, \mathscr{F}, \boldsymbol{P})$ be a probability space. Let $\{\xi_n\}_{n \ge 0}$ denote a Markov chain with a finite state space $\Xi = \{1, 2, \dots, s\}$ with a transition matrix $Q = (q_{jk})_{j,k \in \Xi}$. Let $\{Y_n^{jk}\}_{n \ge 1}^{j,k \in \Xi}$ be a family of independent random variables which are independent of $\{\xi_n\}$. We assume that $\{Y_n^{jk}\}_{n \ge 1}$ is the sequence of identically distributed random variables for each $j, k \in \Xi$. We set

$$S_0 = 0, \quad S_n = S_{n-1} + Y_n^{\xi_{n-1}\xi_n} \quad \text{for } n \ge 1.$$

The process $\{S_n\}_{n \ge 0}$ is known as *a random walk defined on a finite Markov chain*. Limit theorems for such processes were treated, e.g., by Keilson and Wishart [1], and by Miller [2]. Takenami [5] proved local limit theorems for a class of periodic Markov chains, realizing it as such a process.

Let a be a positive integer, and denote by $[0, a]$ the interval consisting of the integers $0, 1, 2, \dots, a$. We set $T_x = \inf\{n > 0 \mid x + S_n \notin [0, a]\}$ for each $x \in [0, a]$, the first exit time from the interval $[0, a]$. Set

$$P_{jk}^{(n)}(x, y) = \boldsymbol{P}\{x + S_n = y; T_x > n; \xi_n = k \mid \xi_0 = j\} \tag{1.1}$$

for each $x, y \in [0, a]$ and all $j, k \in \Xi$. We are interested in an asymptotic behaviour of (1.1) as $n \to \infty$. A corresponding result for simple random walk may be found, e.g., in P21.2 of Spitzer [4].

We will consider our problem under the following assumptions.

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Assumption 1. Q is ergodic, that is, irreducible and aperiodic.

Assumption 2. The random variables Y_1^{jk} take only two values $+1$ and -1 with positive probabilities α_{jk} and $\beta_{jk} = 1 - \alpha_{jk}$, respectively, for all $j, k \in \Xi$.

Assumption 3. There exists a positive constant c, which does not depend on j, such that $c = \boldsymbol{P}\{Y_1^{\xi_0\xi_1} = 1 \mid \xi_0 = j\}$ for all $j \in \Xi$.

Set $A = (q_{jk}\alpha_{jk})_{j,k\in\Xi}$ and $B = (q_{jk}\beta_{jk})_{j,k\in\Xi}$. By Assumptions 2 and 3 we see that

$$c = \sum_{k\in\Xi} q_{jk}\alpha_{jk} \quad \text{and} \quad 1 - c = \sum_{k\in\Xi} q_{jk}\beta_{jk} \tag{1.2}$$

for all $j \in \Xi$. We define the $s \times s$ orthogonal matrix

$$U = (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_s),$$

of which the column vectors are given by $\boldsymbol{u}_1 = \sqrt{1/s}(\overbrace{1, 1, \ldots, 1}^{s})^*$ and

$$\boldsymbol{u}_{2+l} = \sqrt{\frac{1}{(s-l)(s-l-1)}}(\overbrace{0, 0, \ldots, 0}^{l}, -s+l+1, \overbrace{1, 1, \ldots, 1}^{s-l-1})^*$$

for $0 \le l \le s-2$, where the superscript $*$ denotes the transpose of a matrix or a vector. Set

$$C = U^*AU \quad \text{and} \quad D = U^*BU. \tag{1.3}$$

Then by (1.2) the matrices have the forms

$$C = \begin{array}{c@{}l} \begin{array}{cc} \overbrace{\phantom{c_{11}}}^{1} & \overbrace{\phantom{C_{22}}}^{s-1} \end{array} & \\ \left(\begin{array}{c|c} c_{11} & C_{12} \\ \hline 0 & \\ \vdots & C_{22} \\ 0 & \end{array}\right) & \begin{array}{l} \}1 \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} s-1 \end{array} \end{array} \quad \text{and} \quad D = \begin{array}{c@{}l} \begin{array}{cc} \overbrace{\phantom{d_{11}}}^{1} & \overbrace{\phantom{D_{22}}}^{s-1} \end{array} & \\ \left(\begin{array}{c|c} d_{11} & D_{12} \\ \hline 0 & \\ \vdots & D_{22} \\ 0 & \end{array}\right) & \begin{array}{l} \}1 \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} s-1 \end{array} \end{array}.$$

Here C_{12} and D_{12} are $(s-1)$-dimensional row vectors, C_{22} and D_{22} are $(s-1) \times (s-1)$ matrices. Note that $c_{11} = c$ and $d_{11} = 1 - c$. We introduce the following matrices:

$$G_{11} = \begin{array}{c@{}c} \begin{array}{cccccc} 0 & 1 & 2 & & a-1 & a \end{array} & \\ \begin{pmatrix} 0 & c_{11} & 0 & \cdots & 0 & 0 \\ d_{11} & 0 & c_{11} & \cdots & 0 & 0 \\ 0 & d_{11} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & c_{11} \\ 0 & 0 & 0 & \cdots & d_{11} & 0 \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ \\ a-1 \\ a \end{array} \end{array},$$

$$G_{12} = \begin{array}{c} \begin{array}{ccccc} 0 & 1 & & a-1 & a \end{array} \\ \left(\begin{array}{ccccc} \mathbf{0} & C_{12} & \cdots & \mathbf{0} & \mathbf{0} \\ D_{12} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_{12} & \cdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & C_{12} \\ \mathbf{0} & \mathbf{0} & \cdots & D_{12} & \mathbf{0} \end{array}\right) \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \\ a-1 \\ a \end{array}$$

and

$$G_{22} = \begin{array}{c} \begin{array}{ccccc} 0 & 1 & & a-1 & a \end{array} \\ \left(\begin{array}{ccccc} O & C_{22} & \cdots & O & O \\ D_{22} & O & \cdots & O & O \\ O & D_{22} & \cdots & O & O \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ O & O & \cdots & O & C_{22} \\ O & O & \cdots & D_{22} & O \end{array}\right) \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \\ a-1 \\ a \end{array},$$

where $\mathbf{0}$ denotes the $(s-1)$-dimensional row vector with all entries equal to zero and O denotes the $(s-1)\times(s-1)$ null matrix. Note that G_{12} is an $(a+1)\times(s-1)(a+1)$ matrix and G_{22} is an $(s-1)(a+1)\times(s-1)(a+1)$ matrix. Denote by $g_{11}(x,y)$ the (x,y) entry of G_{11}, and by $G_{12}(x,y)$ and $G_{22}(x,y)$ the (x,y) submatrix in G_{12} and G_{22}, respectively. Let μ denote the Perron-Frobenius eigenvalue of G_{11}. Then we see that

$$\mu = 2\lambda_0 c_{11}^{1/2} d_{11}^{1/2}, \tag{1.4}$$

where λ_0 is given in (2.1). See, e.g., P21.1 of Spitzer [4]. Set

$$H_{22} = \left(I - \frac{1}{\mu}G_{22}\right)^{-1}. \tag{1.5}$$

By (3.5), we see that the right hand side of (1.5) exists if Assumptions 1 through 3 are satisfied. We shall represent it in the following form:

$$H_{22} = \begin{pmatrix} H_{22}(0,0) & H_{22}(0,1) & \cdots & H_{22}(0,a) \\ H_{22}(1,0) & H_{22}(1,1) & \cdots & H_{22}(1,a) \\ \cdots & \cdots & \cdots & \cdots \\ H_{22}(a,0) & H_{22}(a,1) & \cdots & H_{22}(a,a) \end{pmatrix},$$

where $H_{22}(x,y)$ is an $(s-1)\times(s-1)$ matrix for each $x,y\in[0,a]$. We set, for each $x,y\in[0,a]$,

$$h_1(x,y) = 2v_0(x)v_0(y)c_{11}^{(y-x)/2}d_{11}^{(-y+x)/2}$$

and

$$(h_2(x,y),\dots,h_s(x,y)) = \sum_{x',x''\in[0,a]} h_1(x,x')\frac{1}{\mu}G_{12}(x',x'')H_{22}(x'',y),$$

where $v_0(x)$ is given in (2.2). Define the s-dimensional row vector

$$\boldsymbol{h}(x, y) = (h_1(x, y), h_2(x, y), \ldots, h_s(x, y))$$

for each $x, y \in [0, a]$.

Now we state our main theorem.

Theorem 1. *Suppose that Assumptions* 1 *through* 3 *are satisfied. Then we have, for each* $x, y \in [0, a]$ *and all* $j, k \in \Xi$,

$$P_{jk}^{(n)}(x, y) = \begin{cases} \mu^n(\sqrt{1/s}\,\boldsymbol{h}(x, y)\boldsymbol{u}_k + o(1)) & \text{if } n + y - x \text{ is even}, \\ 0 & \text{otherwise}. \end{cases} \tag{1.6}$$

Here the inner products $\boldsymbol{h}(x, y)\boldsymbol{u}_k$ *are strictly positive for each* $x, y \in [0, a]$ *and all* $j, k \in \Xi$, *and* $o(1)$ *tends to zero as* $n \to \infty$.

In Section 2, in order to prove Theorem 1, we introduce the sequence of lemmas. In Section 3, we prove Theorem 1.

2. Some lemmas for Theorem 1

In this section, we introduce some lemmas on which our proof of Theorem 1 is based.

We set, for each $x, y \in [0, a]$ and $n \geq 1$,

$$\Gamma_n(x, y) = \left\{(t_1, \ldots, t_n) \in \{-1, 1\}^n \mid x + \sum_{m'=1}^{m} t_{m'} \in [0, a], 1 \leq m \leq n - 1; \right.$$
$$\left. x + \sum_{m'=1}^{n} t_{m'} = y\right\}$$

and $\gamma_n(x, y) = \#\Gamma_n(x, y)$. Put

$$\lambda_z = \cos\frac{z+1}{a+2}\pi \tag{2.1}$$

and

$$v_z(z') = \sqrt{\frac{2}{a+2}}\sin\frac{(z+1)(z'+1)}{a+2}\pi \tag{2.2}$$

for each $z, z' \in [0, a]$. Using P21.1 of Spitzer [4], we have the following lemma.

Lemma 1.

$$\gamma_n(x, y) = \sum_{z \in [0, a]} (2\lambda_z)^n v_z(x) v_z(y)$$

for each $x, y \in [0, a]$ *and* $n \geq 1$.

Lemma 2. *Let* κ *be any eigenvalue of* G_{22}. *Then* $\mu > |\kappa|$, *where* μ *is the Perron-Frobenius eigenvalue of* G_{11} *given in* (1.4).

Proof. Define the $s(a+1) \times s(a+1)$ matrix

$$P = \begin{array}{cc} \begin{array}{cccccc} 0 & 1 & 2 & & a-1 & a \end{array} & \\ \begin{pmatrix} O & A & O & \cdots & O & O \\ B & O & A & \cdots & O & O \\ O & B & O & \cdots & O & O \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ O & O & O & \cdots & O & A \\ O & O & O & \cdots & B & O \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ \\ a-1 \\ a \end{array} \end{array},$$

where the O is $s \times s$ null matrix. By Assumption 1 we may see that P is $s(a+1) \times s(a+1)$ non-negative irreducible matrix with period 2. For each $z \in [0, a]$, we define the s-dimensional column vector $\boldsymbol{w}_z = (d_{11}/c_{11})^{z/2} v_0(z) \cdot (1, 1, \ldots, 1)^*$ and $s(a+1)$-dimensional column vector

$$\boldsymbol{w} = \begin{pmatrix} \boldsymbol{w}_0 \\ \boldsymbol{w}_1 \\ \vdots \\ \boldsymbol{w}_a \end{pmatrix}$$

Then μ is an eigenvalue of P, and $\boldsymbol{w}$ is a right eigenvector associated with μ. Since μ is positive and $\boldsymbol{w}$ is non-negative, μ is the Perron-Frobenius eigenvalue of P (See, e.g., Seneta [3] p. 23). Define the $s(a+1) \times s(a+1)$ matrix J by

$$\begin{array}{cc} \begin{array}{cccccc} 0 & 1 & 2 & & a-1 & a \end{array} & \\ \begin{pmatrix} U & O & O & \cdots & O & O \\ O & U & O & \cdots & O & O \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ O & O & O & \cdots & U & O \\ O & O & O & \cdots & O & U \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ \\ a-1 \\ a \end{array} \end{array}.$$

Then we have

$$J^* P J = \begin{array}{cc} \begin{array}{cccccc} 0 & 1 & 2 & & a-1 & a \end{array} & \\ \begin{pmatrix} O & C & O & \cdots & O & O \\ D & O & C & \cdots & O & O \\ O & D & O & \cdots & O & O \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ O & O & O & \cdots & O & C \\ O & O & O & \cdots & D & O \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ \\ a-1 \\ a \end{array} \end{array}. \tag{2.3}$$

By an appropriate permutation of the rows and the columns, the right hand side of (2.3) may be reduced to the form

$$\begin{pmatrix} G_{11} & G_{12} \\ O & G_{22} \end{pmatrix},$$

where the O is $(s-1)(a+1) \times (a+1)$ null matrix. Since μ and $-\mu$ are eigenvalues of G_{11} and μ is the Perron-Frobenius eigenvalue of P, we may see that $\mu > |\kappa|$. □

LEMMA 3. *Suppose that Assumptions* 1 *and* 2 *are satisfied. Then there exist positive constants* n_1 *and* c_1, $0 < c_1 < 1$, *such that, for each* $x, y \in [0, a]$, *all* j, k *and* $k' \in \Xi$,

$$c_1 < \frac{P_{jk'}^{(n)}(x, y)}{P_{jk}^{(n)}(x, y)} < \frac{1}{c_1}$$

when $n > n_1$ *and* $n + y - x$ *is even.*

Proof. By Assumption 1, there exists a positive constant n_1' such that, for all $j, k \in \Xi$, $q_{jk}^{(n)} > 0$ when $n > n_1'$. Set $n_1 = \max\{n_1', a\}$ and

$$c_1 = \min\{P_{jk}^{(n_1)}(x, y) \mid x, y \in [0, a]; n_1 + y - x \text{ is even}; j, k \in \Xi\}. \tag{2.4}$$

Then c_1 is strictly positive. Thus by (2.4)

$$\begin{aligned} P_{jk}^{(n)}(x, y) &= \sum_{j' \in \Xi} \sum_{\substack{y' \in [0, a] \\ n_1 + y - y' \text{ is even}}} P_{jj'}^{(n-n_1)}(x, y') P_{j'k}^{(n_1)}(y', y) \\ &\geq c_1 \boldsymbol{P}\{T_x > n - n_1 \mid \xi_0 = j\} \end{aligned}$$

when $n > n_1$ and $n + y - x$ is even. Clearly, $P_{jk}^{(n)}(x, y) \leq \boldsymbol{P}\{T_x > n - n_1 \mid \xi_0 = j\}$. Therefore the proof is complete. □

3. Proof of Theorem 1

We define the $s \times s$ matrix-valued function, for $t \in \{-1, 1\}$,

$$F(t) = \begin{cases} C & \text{if } t = 1, \\ D & \text{if } t = -1. \end{cases}$$

We may write, for $t \in \{-1, 1\}$,

$$F(t) = \begin{array}{c} \begin{array}{cc} \overbrace{\phantom{f_{11}(t)}}^{1} & \overbrace{\phantom{F_{12}(t)}}^{s-1} \end{array} \\ \left(\begin{array}{c|c} f_{11}(t) & F_{12}(t) \\ \hline 0 & \\ \vdots & F_{22}(t) \\ 0 & \end{array}\right) \end{array} \begin{array}{l} \}1 \\ \left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\} s-1 \end{array},$$

where $F_{12}(t)$ is an $(s-1)$-dimensional row vector-valued function and $F_{22}(t)$ is an $(s-1)\times(s-1)$ matrix-valued function.

For each $x, y \in [0, a]$ and $n \geq 1$, we define the $s \times s$ matrix-valued function

$$G^{(n)}(x, y) = \sum_{t \in \Gamma_n(x, y)} F(t_1) \cdots F(t_n).$$

Then it may be represented in the following form

$$G^{(n)}(x, y) = \begin{array}{cl} \begin{array}{cc} \overbrace{\phantom{g^{(n)}_{11}(x,y)}}^{1} & \overbrace{\phantom{G^{(n)}_{12}(x,y)}}^{s-1} \end{array} & \\ \left(\begin{array}{c|c} g^{(n)}_{11}(x, y) & G^{(n)}_{12}(x, y) \\ \hline 0 & \\ \vdots & G^{(n)}_{22}(x, y) \\ 0 & \end{array}\right) & \begin{array}{l} \}1 \\ \\ \left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\} s-1 \end{array} \end{array}. \tag{3.1}$$

Here

$$g^{(n)}_{11}(x, y) = \sum_{t \in \Gamma_n(x, y)} f_{11}(t_1) \cdots f_{11}(t_n),$$

$$G^{(n)}_{22}(x, y) = \sum_{t \in \Gamma_n(x, y)} F_{22}(t_1) \cdots F_{22}(t_n)$$

and

$$G^{(n)}_{12}(x, y) = \sum_{m=0}^{n-1} \sum_{x', x'' \in [0, a]} g^{(m)}_{11}(x, x') G_{12}(x', x'') G^{(n-1-m)}_{22}(x'', y), \tag{3.2}$$

where $g^{(0)}_{11}(x, y) = 1$ if $x = y$, 0 otherwise, and $G^{(0)}_{22}(x, y) = I$ if $x = y$, O otherwise. Thus by (1.1) and (1.3) we have

$$(P^{(n)}_{jk}(x, y))_{j, k \in \Xi} = UG^{(n)}(x, y)U^*. \tag{3.3}$$

Let $\|X\| = \max_{j,k}|x_{jk}|$ for a matrix $X = (x_{jk})$. By Lemma 1

$$\frac{1}{\mu^n} g^{(n)}_{11}(x, y) = h_1(x, y) + o(1) \tag{3.4}$$

as $n \to \infty$, when $n + y - x$ is even. By Lemma 2,

$$\left\| \frac{1}{\mu^n} G^{(n)}_{22}(x, y) \right\| = o(\rho^n) \tag{3.5}$$

as $n \to \infty$ for some ρ, $0 < \rho < 1$. Define the $(a+1) \times (s-1)(a+1)$ matrix $G^{(n)}_{12} = (G^{(n)}_{12}(x, y))_{x, y \in [0, a]}$ for $n \geq 1$. Then by (3.2) we have

$$G^{(n+1)}_{12} = \sum_{m=0}^{n} G^m_{11} G_{12} G^{n-m}_{22} = G^{(n)}_{12} G_{22} + G^n_{11} G_{12}. \tag{3.6}$$

By (3.2) and (3.5)

$$\lim_{n\to\infty} \frac{1}{\mu^n} G_{12}^{(n)} \text{ exists.} \tag{3.7}$$

Therefore by (3.5), (3.6) and (3.7) we have

$$\frac{1}{\mu^n} G_{12}^{(n)}(x, y) \to \sum_{x', x'' \in [0, a]} h_1(x, x') \frac{1}{\mu} G_{12}(x', x'') H_{22}(x'', y) \tag{3.8}$$

as $n \to \infty$, when $n + y - x$ is even. Therefore by (3.1), (3.3), (3.4), (3.5) and (3.8) the formula (1.6) holds.

We will show that $\boldsymbol{h}(x, y)\boldsymbol{u}_k$ are strictly positive for each $x, y \in [0, a]$ and all $k \in \Xi$. By formula (1.6), we may see that $\boldsymbol{h}(x, y)\boldsymbol{u}_k \geq 0$. Suppose that $\boldsymbol{h}(x, y)\boldsymbol{u}_k = 0$ for some $x, y \in [0, a]$, $k \in \Xi$. Since $\boldsymbol{h}(x, y) \neq \mathbf{0}$ and $\{\boldsymbol{u}_j\}_{j \in \Xi}$ is an orthogonal family in $\boldsymbol{R}^s$, $\boldsymbol{h}(x, y)\boldsymbol{u}_{k'} \neq 0$ for some $k' \in \Xi$. Thus

$$\frac{\boldsymbol{P}\{x + S_n = y; T_x > n; \xi_n = k' \mid \xi_0 = j\}}{\boldsymbol{P}\{x + S_n = y; T_x > n; \xi_n = k \mid \xi_0 = j\}} \to 0$$

as $n \to \infty$, when $n + y - x$ is even. This contradicts Lemma 3. Therefore $\boldsymbol{h}(x, y)\boldsymbol{u}_k > 0$ for each $x, y \in [0, a]$ and all $k \in \Xi$. The proof is complete. $\square$

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