M. C. BELTRAMETTI, K. A. CHANDLER AND A. J. SOMMESE
KODAI MATH. J.
25 (2002), 139–150

REDUCIBLE HYPERPLANE SECTIONS, II

M. C. Beltrametti, K. A. Chandler and A. J. Sommese

Abstract

Let $\hat{X}$ be a smooth connected subvariety of complex projective space $\boldsymbol{P}^n$. The question was raised in [2] of how to characterize $\hat{X}$ if it admits a reducible hyperplane section $\hat{L}$. In the case in which $\hat{L}$ is the union of $r \geq 2$ smooth normal crossing divisors, each of sectional genus zero, classification theorems were given for $\dim \hat{X} \geq 5$ or $\dim \hat{X} = 4$ and $r = 2$.

This paper restricts attention to the case of two divisors on a threefold, whose sum is ample, and which meet transversely in a smooth curve of genus at least 2. A finiteness theorem and some general results are proven, when the two divisors are in a restricted class including $\boldsymbol{P}^1$-bundles over curves of genus less than two and surfaces with nef and big anticanonical bundle. Next, we give results on the case of a projective threefold $\hat{X}$ with hyperplane section $\hat{L}$ that is the union of two transverse divisors, each of which is either $\boldsymbol{P}^2$, a Hirzebruch surface $\boldsymbol{F}_r$, or $\widetilde{\boldsymbol{F}_2}$.

Introduction

This paper is a sequel of [2], which initiated the study of a connected submanifold $\hat{X}$ of complex projective space that has a reducible hyperplane section $\hat{L}$. As $\dim \hat{X}$ increases so does the simplicity of the characterization. In [2] a description is given of $(\hat{X}, \hat{L})$ for which $\hat{L}$ decomposes as $\hat{A}_1 + \cdots + \hat{A}_r$ into $r \geq 2$ smooth components with normal crossings under the hypothesis that $h^1(\mathcal{O}_{\hat{A}_i})$ is equal to the sectional genus of $\hat{A}_i$ for each i. A complete result for the cases $n = 4$ and $r = 2$; and for $n \geq 5$ was obtained. Further, in the case of $n = 3$ and $r = 2$ the situation in which the curve $A_1 \cap A_2$ has genus at most 1 was thoroughly analyzed. Here we investigate the more delicate issues presented by the following specialization of the question.

Problem. Let $\hat{L}$ be a very ample line bundle on a projective threefold $\hat{X}$. Suppose that $\hat{L}$ decomposes as a divisor into a sum $\hat{L} = \hat{A} + \hat{B}$, where $\hat{A}$ and

---

2000 *Mathematics Subject Classification*: Primary 14N30, 14M99; Secondary 14J99.
Keywords and phrases: Adjunction theory, smooth complex polarized threefold, very ample line bundle, reducible surface section, normal crossing divisors, Hirzebruch surface, scroll.
Received August 30, 2001.

$\hat{B}$ are smooth connected surfaces meeting transversely along a smooth curve $h = \hat{A} \cap \hat{B}$. Assume that each of $\hat{A}$, $\hat{B}$ is either $\boldsymbol{P}^2$ or $\boldsymbol{F}_r$. Then describe $(\hat{X}, \hat{L})$.

We call h the *hinge curve*. The curve h is connected [2, Corollary 2.3].

In this paper we shall focus on the situation when h has genus $g(h) \geq 2$: we refer to [2, Theorems 3.10, 3.11] for the cases when $g(h) \leq 1$. We also refer to [2, 5, 7] for related results.

The organization of the paper is as follows. In Section 2, we present a general finiteness theorem for a threefold $\hat{X}$ with an ample divisor $\hat{L}$ of the form $\hat{A} + \hat{B}$, where $\hat{A}$, $\hat{B}$ are in a restricted class $\mathscr{C}$ of surfaces and meet transversely in a smooth curve of genus ≥ 2. The class $\mathscr{C}$ includes surfaces with nef and big anticanonical bundle; and $\boldsymbol{P}^1$-bundles over either $\boldsymbol{P}^1$ or an elliptic curve. The finiteness theorem asserts that there is an $\varepsilon > 0$ such that the Kodaira dimension of $K_{\hat{X}} + (1/2 + \varepsilon)\hat{L}$ is $-\infty$. By a result of Fujita, this implies that $(\hat{X}, \hat{L})$ is a birational transform of members of an explicit list of very special pairs.

In Section 3, it is shown that if the divisors $\hat{A}$, $\hat{B}$ are $\boldsymbol{P}^2$ or scrolls over $\boldsymbol{P}^1$, then the restriction of the bundle $K_{\hat{X}} + \hat{L}$ to the divisors in big.

In Section 4, the Hodge Index type theorem for reducible divisors leads to the elimination of the cases in which both $\hat{A}$ and $\hat{B}$ are among $\boldsymbol{P}^2$ and the singular quadric $\widetilde{\boldsymbol{F}_2}$ with an isolated singularity.

Finally, in Section 5 we study the case when $\hat{A}$ is $\boldsymbol{P}^2$, the Hirzebruch surface $\boldsymbol{F}_r$, or the singular quadric with isolated singularity $\widetilde{\boldsymbol{F}_2}$; and $\hat{B} = \boldsymbol{F}_s$, under the extra assumption that $(\hat{A}, \hat{L}_{\hat{A}})$, $(\hat{B}, \hat{L}_{\hat{B}})$ are scrolls.

The first author would like to thank the University of Notre Dame for its support. The third author would like to thank the Duncan Chair of the University of Notre Dame for its support.

1. Background material

We work over the complex field $\boldsymbol{C}$. Throughout the paper we deal with projective varieties V, and follow the usual notation of algebraic geometry. The book [1] is a good reference for standard results and notation of adjunction theory.

For a line bundle L on an irreducible normal variety V of dimension n the *sectional genus*, $g(L) = g(V, L)$, of (V, L) is defined by $2g(L) - 2 = (K_V + (n-1)L) \cdot L^{n-1}$.

By $\boldsymbol{F}_r$ with $r \geq 0$ we denote the unique $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$ with a section E taking on the minimal self intersection $E^2 = -r$ on the surface. By $\widetilde{\boldsymbol{F}_2}$ we denote $\boldsymbol{F}_2$ with the section, which has self intersection -2, blown down. Note that $\widetilde{\boldsymbol{F}_2}$ is isomorphic to any quadric hypersurface $Q \subset \boldsymbol{P}^3$ that has a single isolated singularity.

Let V be a normal r-Gorenstein (i.e., rK_V is a Cartier divisor) projective variety of dimension n and let D be a $\boldsymbol{Q}$-Cartier divisor on V such that $\kappa(D) = n$. We define the *unnormalized spectral value* of the pair (V, D) as

$$u(V, D) := \sup\{t \in \boldsymbol{Q} \mid \kappa(K_V + tD) = -\infty\}.$$

We refer to [1, §7.1] for details.

The following result follows immediately from [2, §2].

Lemma 1.1. *Let $\hat{L}$ be an ample line bundle on a smooth projective 3-fold $\hat{X}$. Assume that there are two smooth connected divisors $\hat{A}$, $\hat{B}$ on $\hat{X}$. Assume that $\hat{A} + \hat{B} \in |\hat{L}|$, that $\hat{A}$ and $\hat{B}$ are rational, and that $\hat{A}$, $\hat{B}$ intersect transversely in a smooth curve h. Then h is connected, and $h^1(\mathcal{O}_{\hat{X}}) = h^2(\mathcal{O}_{\hat{X}}) = 0$.*

2. A finiteness theorem

In this section we prove a general finiteness theorem for pairs $(\hat{X}, \hat{L})$ consisting of an ample line bundle on a smooth projective threefold $\hat{X}$, with $|\hat{L}|$ containing a divisor $D = \hat{A} + \hat{B}$, having two irreducible components from a large class $\mathscr{C}$ of negative Kodaira dimension surfaces. The class $\mathscr{C}$ *consists of the normal connected Gorenstein projective surfaces S with the property that given any smooth connected Cartier divisor C on S, it follows that either $h^1(\mathcal{O}_C) \leq 1$ or $K_S \cdot C \leq -1$.*

Lemma 2.1. *The class $\mathscr{C}$ includes*:

1. *normal Gorenstein surfaces with $-K_S$ nef and big*; *or*
2. $\boldsymbol{F}_r$, $r \geq 0$, *the r-th Hirzebruch surface*; *or*
3. *a $\boldsymbol{P}^1$-bundle over an elliptic curve.*

In cases 1 and 2, smooth connected Cartier divisors C with $h^1(\mathcal{O}_C) \geq 2$ satisfy $K_S \cdot C \leq -3$.

Proof. Let C be a smooth connected Cartier divisor of S, i.e., let C be a curve on S with C contained in S_{reg}, the smooth points of S. We assume that we are in the situation that $h^1(\mathcal{O}_C) \geq 2$, since otherwise there is nothing to show.

First assume that $-K_S$ is nef and big, and that the result is false, i.e., that $-K_S \cdot C \leq 2$. We know that $-K_S \cdot C = 0, 1, 2$. If $-K_S \cdot C = 0$, then we conclude, using the Hodge Index Theorem, that $C^2 \leq 0$, which contradicts $h^1(\mathcal{O}_C) \geq 2$. If $-K_S \cdot C = 1$, then we conclude that $C^2 \geq 3$, which contradicts the Hodge Index Theorem, i.e., $C^2 \leq C^2 K_S^2 \leq 1$. If $-K_S \cdot C = 2$, then we conclude that $C^2 \geq 4$, which gives equality in the Hodge Index Theorem, i.e., $4 \leq C^2 \leq C^2 K_S^2 \leq 4$. This implies that numerically $C \sim -K_S$, which implies the contradiction $K_S + C \sim 0$.

For S a Hirzebruch surface the result is a straightforward check.

Assume finally that S is a $\boldsymbol{P}^1$-bundle over an elliptic curve Y. In this case the section σ of minimal self-intersection satisfies $e := -\sigma^2 \geq -1$, and K_S is numerically equal to $-2\sigma - ef$ for a fiber of the induced projection $\pi : S \to Y$. Since we are assuming that $h^1(\mathcal{O}_C) \geq 2$, we know that numerically $C = k\sigma + tf$ where $k \geq 2$. Moreover $K_S \cdot C \geq 0$ gives $ke - 2t = 2ke - ek - 2t \geq 0$. Since $C^2 = -ek^2 + 2kt$, we have the absurdity that $2 \leq 2g(C) - 2 = K_S \cdot C + C^2 = (1-k)(ke - 2t) \leq 0$. Q.E.D.

One main result of the paper is the Finiteness Theorem 2.2. This theorem shows that, if the hinge curve h has genus $g(h) \geq 2$, the pair $(\hat{X}, \hat{L})$ belongs to an explicit list of very special cases described by Fujita (see [3, 4] and also [1, 7.8.1]).

Note in the following that the hypothesis that h is connected is automatically satisfied if $\hat{A}$ and $\hat{B}$ are connected [2, Corollary 2.3].

Theorem 2.2 (Finiteness Theorem). *Let $\hat{L}$ be an ample line bundle on a smooth projective 3-fold $\hat{X}$. Assume that there are two divisors $\hat{A}$, $\hat{B}$ on $\hat{X}$ from the class $\mathscr{C}$. Assume that $\hat{A} + \hat{B} \in |\hat{L}|$ and that $\hat{A}$, $\hat{B}$ intersect transversely in a smooth connected curve h of genus $g(h) \geq 2$. Then $u(\hat{X}, \hat{L}) > 1/2$. In particular, $\hat{X}$ is of Kodaira dimension $-\infty$, and thus satisfies $h^3(\mathcal{O}_{\hat{X}}) = 0$.*

Proof. For simplicity of notation, we omit ˆ's in this proof. The genus formula yields

$$(K_X + L) \cdot h = (K_X + A + B) \cdot A \cdot B = 2g(h) - 2, \tag{1}$$

or $(K_A + B_A) \cdot B_A = 2g(h) - 2$, and therefore, by definition of class $\mathscr{C}$, one has $B_A \cdot B_A \geq 2g(h) - 1$, and similarly $A_B \cdot A_B \geq 2g(h) - 1$. Then (1) gives

$$K_X \cdot h \leq -2g(h). \tag{2}$$

Now compute, for any real number ε, $0 < \varepsilon < 1/(4g(h) - 2)$,

$$\begin{aligned}
\left(K_X + \left(\frac{1}{2} + \varepsilon\right)L\right) \cdot h &= \left(K_X + L - \left(\frac{1}{2} - \varepsilon\right)L\right) \cdot h \\
&= 2g(h) - 2 - \left(\frac{1}{2} - \varepsilon\right)L \cdot h \\
&\leq 2g(h) - 2 - \left(\frac{1}{2} - \varepsilon\right)(4g(h) - 2) \\
&= -1 + \varepsilon(4g(h) - 2) < 0.
\end{aligned}$$

Finally, for $h = A \cap B$ on X, we have the normal bundle decomposition $N_{h/X} = N_{h/A} \oplus N_{h/B}$ and $\deg(N_{h/A}) = B^2 \cdot A = B_A \cdot B_A \geq 2g(h) - 1$ by the above. It follows that $h^1(N_{h/A}) = 0$ and $N_{h/A}$ has not identically zero sections. Similarly for $N_{h/B}$. Then $N_{h/X}$ is generically spanned by its global sections and $h^1(N_{h/X}) = 0$. Thus general deformation theory implies that the union of the deformations of h on X contains an open set. Therefore the inequality $(K_X + (1/2 + \varepsilon)L) \cdot h < 0$ proved above shows that $u(X, L) > 1/2$, cf. [1, 7.6.4]. Q.E.D.

A little more can be said on the case of $\boldsymbol{P}^1$-bundles over $\boldsymbol{P}^1$ or surfaces with nef and big anticanonical bundle.

Proposition 2.3. *Let $\hat{L}$ be an ample line bundle on a smooth projective 3-fold $\hat{X}$. Assume that there are two smooth divisors $\hat{A}$, $\hat{B}$ on $\hat{X}$ each of which is either a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$ or a surface with nef and big anticanonical bundle.*

Assume that $\hat{A} + \hat{B} \in |\hat{L}|$ *and that* $\hat{A}$, $\hat{B}$ *intersect transversely in a smooth connected curve* h. *Then* $H^0(K_{\hat{X}} + \hat{L}) \to H^0(K_h) \to 0$.

Proof. Tensor the Koszul complex

$$0 \to \mathcal{O}_{\hat{X}} \to \hat{A} \oplus \hat{B} \to \hat{L} \to \hat{L}_h \to 0$$

with $K_{\hat{X}}$. Using the hypercohomology spectral sequence, we see that the desired result will follow if we show that $H^2(K_{\hat{X}}) = H^1(K_{\hat{X}} + \hat{A}) = H^1(K_{\hat{X}} + \hat{B}) = 0$.

The assertion $H^2(K_{\hat{X}}) = 0$ follows from Lemma 1.1. To see that $H^1(K_{\hat{X}} + \hat{A}) = 0$ consider the exact sequence

$$0 \to K_{\hat{X}} \to K_{\hat{X}} + \hat{A} \to K_{\hat{A}} \to 0.$$

Now use Lemma 1.1 and the fact that $\hat{A}$ is rational. The argument for $H^1(K_{\hat{X}} + \hat{B}) = 0$ is identical. Q.E.D.

One consequence of Proposition 2.3 is that, under the same hypotheses with the added assumption that $g(h) \geq 2$, it follows that the Kodaira dimension of $K_{\hat{X}} + \hat{L}$ is at least one. This implies that the Kodaira dimension of $K_{\hat{X}} + 2\hat{L}$ is three, and also that the restriction of $K_{\hat{X}} + 2\hat{L}$ to $\hat{A}$ (or $\hat{B}$) is nontrivial. Therefore [2, Theorems 3.6, 3.8] specialize to the following result.

Theorem 2.4. *Let* $\hat{L}$ *be an ample line bundle on a smooth projective* 3*-fold* $\hat{X}$. *Assume that there are two smooth divisors* $\hat{A}$, $\hat{B}$ *on* $\hat{X}$ *each of which is either a* $\boldsymbol{P}^1$*-bundle over* $\boldsymbol{P}^1$ *or a surface with nef and big anticanonical bundle. Assume that* $\hat{A} + \hat{B} \in |\hat{L}|$ *and that* $\hat{A}$, $\hat{B}$ *intersect transversely in a smooth connected curve* h *of genus* ≥ 2. *Then there is a surjective morphism* $\phi : \hat{X} \to X$, *where* X *is a smooth projective* 3*-fold, such that*:

1. ϕ *expresses* $\hat{X}$ *as the blowup of* X *at a finite set* $\mathscr{F}$, *and there is an ample line bundle* L *on* X *such that* $\hat{L} \cong \phi^* L - \phi^{-1}(\mathscr{F})$;
2. $K_{\hat{X}} + 2\hat{L} \cong \phi^*(K_X + 2L)$ *where* $K_X + 2L$ *is ample*;
3. $K_X + L$ *is either nef and big, or* (X, L) *is a conic fibration over a surface* Y *in the sense of adjunction theory* [1], *i.e., there exists a morphism* $\nu : X \to Y$ *with* $K_X + L \cong \nu^* H$ *for an ample line bundle* H *on a normal surface* Y;
4. ϕ *is an embedding in a neighborhood of* h; *and*
5. $L = A + B$ *where* $A := \phi(\hat{A})$ *and* $B := \phi(\hat{B})$ *are Cartier divisors meeting transversely in* $\phi(h)$ *and each having at most one point contained in the set* $\mathscr{F}$.

From now on we usually abuse notation, and let h to denote $\phi(h)$. We also write h_A (respectively h_B) to emphasize that we view h as a curve on A (respectively on B).

Lemma 2.5. *Let* $(\hat{X}, \hat{L})$, (X, L), $\hat{A}$, $\hat{B}$, A, B *be as in Theorem* 2.4. *Then*

1. $h^{i,0}(X) = 0$, $i = 1, 2, 3$;
2. $h^i(K_X + A) = h^i(K_X + B) = 0$ *for all* $i \geq 0$; *and*
3. *the restriction map gives the following isomorphisms*

$$H^0(K_X + L) \cong H^0(K_A + h_A) \cong H^0(K_B + h_B) \cong H^0(K_h).$$

Proof. Noting that the first reduction morphism, ϕ, of Theorem 2.4 is birational, the first assertion follows immediately from Lemma 1.1 and Theorem 2.2.

To prove 2, consider the exact sequence

$$0 \to K_X + B \to K_X + L \to K_A \to 0.$$

By the assumption on A, $h^0(K_A) = h^1(K_A) = 0$, $h^2(K_A) = 1$, $h^3(K_A) = 0$. Thus from the cohomology sequence associated to the sequence above we infer that $h^i(K_X + B) = 0$ (and by symmetry $h^i(K_X + A) = 0$) for all $i \geq 0$.

Item 3 follows immediately from the first two assertions. Q.E.D.

Theorem 2.6. *Let $\hat{L}$ be an ample line bundle on a smooth projective 3-fold $\hat{X}$. Assume that there are two smooth divisors $\hat{A}$, $\hat{B}$ on $\hat{X}$ each of which is either a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$ or a surface with nef and big anticanonical bundle. Assume that $\hat{A} + \hat{B} \in |\hat{L}|$ and that $\hat{A}$, $\hat{B}$ intersect transversely in a smooth connected curve h of genus $g(h) \geq 2$. Let X, A, B, L be as in Theorem 2.4. Then $H^0(K_X + L)$ spans $K_X + L$ in a neighborhood of $A + B$.*

Proof. By Lemma 2.5, the desired spannedness of $K_X + L$ will follow from the spannedness of $K_A + h_A$ and $K_B + h_B$. From Theorem 2.4 we know that $K_X + L$ is nef (and hence $K_A + h_A$ and $K_B + h_B$ are also).

First assume that $\hat{A}$ is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$. Either the map ϕ of Theorem 2.4 is an isomorphism on $\hat{A}$, in which case A is also a $\boldsymbol{P}^1$-bundle, or, by [2, Theorem 3.6, 2.], $\phi_{\hat{A}}$ expresses $\hat{A}$ as the blowup of A at one point. In this latter case, $\hat{A}$ is the Hirzebruch surface $\boldsymbol{F}_1$, and $A := \phi(\hat{A}) = \boldsymbol{P}^2$ (note that $\boldsymbol{F}_1$ is the only Hirzebruch surface with a -1-curve). Since $K_X + L$ is nef, $K_A + h_A$ is nef, and for either $\boldsymbol{P}^2$ or $\boldsymbol{P}^1$-bundles over $\boldsymbol{P}^1$, nef line bundles are spanned.

Now assume that $-K_{\hat{A}}$ is nef and big. Note that $-K_A$ is also nef and big. Indeed, going to the first reduction map we have a birational morphism $\phi_{\hat{A}} : \hat{A} \to A$ where some disjoint -1 curves are collapsed. Writing $-K_{\hat{A}} = K_{\hat{A}} + 2(-K_{\hat{A}})$, we see from the basepoint free theorem that $-NK_{\hat{A}}$ is spanned for $N \gg 0$. Thus $-NK_A$ is spanned off the finite set equal to the image of the exceptional curves. This implies $-K_A$ is nef. Since $K_A^2 > K_{\hat{A}}^2$, bigness is clear.

Consider the line bundle h_A. We would like to show by Reider's Theorem [6] that $K_A + h_A$ is spanned. Note that $h_A^2 = 2g(h_A) - 2 - K_A \cdot h_A \geq 2 + 3 = 5$ by the hypothesis $g(h_A) \geq 2$ and Lemma 2.1. Since h_A is a smooth curve of positive genus, and $K_A \cdot h_A < 0$, we conclude that h_A is nef and big. Therefore by Reider's Theorem, either $K_A + h_A$ is spanned, or there exists an effective Cartier divisor $\ell \subset A$ such that either $h_A \cdot \ell = 0$ with $\ell^2 = -1$, or $h_A \cdot \ell = 1$ with $\ell^2 = 0$.

In the former case, $K_A \cdot \ell < 0$, since $K_A \cdot \ell \le 0$ and $K_A \cdot \ell + \ell^2$ is even. This contradicts the nefness of $K_A + h_A$.

Finally consider the case $h_A \cdot \ell = 1$ with $\ell^2 = 0$. Note that since ℓ is effective, we cannot have $-K_A \cdot \ell = 0$ by the usual Hodge index relation. Thus we have $K_A \cdot \ell < 0$. Since $K_A \cdot \ell + \ell^2$ is even, we have that $K_A \cdot \ell \le -2$. This implies that $(K_A + h_A) \cdot \ell \le -1$, which contradicts nefness of $K_A + h_A$. Q.E.D.

3. Some birationality results

3.1 (Working assumptions). Let $\hat{L}$ be a very ample line bundle on a 3-fold $\hat{X}$. Assume that there are two smooth transverse divisors $\hat{A}$, $\hat{B}$ on $\hat{X}$ with $\hat{A} + \hat{B} \in |\hat{L}|$ and $\hat{A}, \hat{B} \in \{\boldsymbol{P}^2, \boldsymbol{F}_r\}$. Assume that the hinge curve $h = \hat{A} \cap \hat{B}$ has genus $g(h) \ge 2$.

From Theorem 2.4, we know that there exists the first reduction (X, L), $\phi : \hat{X} \to X$, with $K_X + 2L$ ample and $K_X + L$ nef. If $A = \phi(\hat{A})$, $B = \phi(\hat{B})$, then $A + B \in |L|$ and $A, B \in \{\boldsymbol{P}^2, \boldsymbol{F}_r\}$. Furthermore we know by 5 of Theorem 2.4, that neither $\hat{A}$ nor $\hat{B}$ is a fiber of ϕ and that A, B meet transversely along the curve $\phi(h)$ isomorphic to h.

Lemma 3.2. *Assumptions and notation as in* 3.1. *The complete linear systems $|K_A + h_A|$ and $|K_B + h_B|$ map h generically one-to-one. In particular, $K_A + h_A$, $K_B + h_B$, and $K_X + L$ are nef and big.*

Proof. By Lemma 2.5, we see that $|K_X + L|$ maps h generically one-to-one provided that each of the complete linear systems $|K_A + h_A|$ and $|K_B + h_B|$ map h generically one-to-one.

Let us see that each of the linear systems map h generically one-to-one. From 3 of Theorem 2.4, the restriction of $K_X + L$ of one of the divisors A, B is nef and big. (By ampleness of $A + B$ either A or B surjects on the base.) Assume for simplicity, that $K_B + h_B \approx (K_X + L)_B$ is nef and big. If $B = \boldsymbol{P}^2$ or $\boldsymbol{F}_0$, the line bundle $K_B + h_B$ is ample, and indeed very ample.

Thus we may restrict attention to the hypothesis that $B = \boldsymbol{F}_r$, $r \ge 1$. Let $\mathscr{E} := E + rf$. Then either $K_B + h_B = a\mathscr{E} + bf$ is very ample or $b = 0$ and $K_B + h_B = a\mathscr{E}$. Thus $|K_B + h_B|$ maps h generically one-to-one.

Next, we verify that $K_X + L$ is nef and big, using part 3 of Theorem 2.4, together with its notation. If not, v maps h two-to-one onto a curve $v(h)$ with all restrictions of elements of $H^0(K_X + L)$ to h the pullbacks of sections of $H_{v(h)}$. This is a contradiction to the assertion that $|K_X + L|$ maps h generically one-to-one onto its image.

Finally, to see that $(K_X + L)_A \approx K_A + h_A$ is nef and big, observe that the map given by $|K_A + h_A|$ is generically one-to-one on h_A, and the genus of the curve h_A is not zero. Q.E.D.

The following is a corollary of the preceding lemma.

Lemma 3.3. *Assumptions and notation as in* 3.1. *Assume* $A = \boldsymbol{F}_r$ *and let* $h = aE + bf$ *on* A. *Then* $a \geq 3$.

Proof. Note that $a = h \cdot f \geq 0$, and $a \neq 1$ since $g(h) > 0$. Assume $a = 2$. Then $(K_A + h_A) \cdot f = -2 + 2 = 0$ and hence $|(K_X + L)_A| = |K_A + h_A|$ collapses A along the ruling f. This contradicts Lemma 3.2. Q.E.D.

4. The cone cases

The main result in this section is that the situation of a reducible ample divisor $L = A + B$ with both of A and B in $\{\boldsymbol{P}^2, \widetilde{\boldsymbol{F}_2}\}$ is very restricted. The proof of this is based on the usual Hodge Index type theorem for ample divisors, which yields in our case

$$[(A + B) \cdot A \cdot A][(A + B) \cdot B \cdot B] \leq [(A + B) \cdot A \cdot B]^2 \tag{3}$$

with equality if and only if A is a rational multiple of B as homology class.

Lemma 4.1. *Let* L *be an ample line bundle on a smooth connected projective threefold* X. *Assume that* A, B *are two reduced divisors on* X *that meet transversely in a smooth curve* h *of genus* $g(h)$. *Assume that* $A + B \in |L|$, *and that* $A, B \in \{\boldsymbol{P}^2, \widetilde{\boldsymbol{F}_2}\}$. *Then* $g(h) \leq 1$.

Proof. Assume without loss of generality that $g := g(h) \geq 2$. In this case the degree of h on A (respectively, on B) is uniquely determined by g.

First let us do the case of $A = B = \boldsymbol{P}^2$. Then $h_A \in |\mathcal{O}_{\boldsymbol{P}^2}(d)|$ and $h_B \in |\mathcal{O}_{\boldsymbol{P}^2}(d)|$ where $2g - 2 = d(d - 3)$. Note that $d^2 = h_A^2 = B \cdot B \cdot A = h_B \cdot N_{B/X}$. Thus $N_{B/X} = \mathcal{O}_{\boldsymbol{P}^2}(d)$, and similarly $N_{A/X} = \mathcal{O}_{\boldsymbol{P}^2}(d)$. Plugging into equation (3), we get equality. Thus $A = \lambda B$ as homology classes for some $\lambda \in \boldsymbol{Q}$. Since $A^2 \cdot B = d^2 = B^2 \cdot A$, we see that $\lambda = 1$. Thus since L is ample and since $L = 2A = 2B$ in homology, it follows that A, B are ample. The Lefschetz theorem yields $\mathrm{Pic}(X) = \mathrm{Pic}(A) = \boldsymbol{Z}[\mathcal{O}_{\boldsymbol{P}^2}(1)]$. Therefore $K_X \approx \mathcal{O}_X(c)$, $\mathcal{O}_X(A) \approx \mathcal{O}_X(a)$, where $a \geq 1$ by ampleness. Then $(K_X + A)_A \approx K_A \approx \mathcal{O}_{\boldsymbol{P}^2}(-3)$ gives $K_X + A \approx \mathcal{O}_X(c + a) \approx \mathcal{O}_X(-3)$. Therefore $1 + c \leq a + c = -3$, or $c \leq -4$. So $X = \boldsymbol{P}^3$ by the Kobayashi-Ochiai Theorem [1, 3.1.6] and $g = 0$.

The case of $A = B = \widetilde{\boldsymbol{F}_2}$ proceeds in the same way, except that one of the possibilities allowed by the Kobayashi-Ochiai Theorem [1, 3.1.6] is (X, L) is $(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(4))$. In this case $g = 1$.

Finally, consider the case when one of A, B is $\boldsymbol{P}^2$ and the other is $\widetilde{\boldsymbol{F}_2}$. By renaming if necessary we may assume that $A = \boldsymbol{P}^2$ and $B = \widetilde{\boldsymbol{F}_2}$. Letting $h_B = A_B = \mathcal{O}_B(\delta)$ and $h_A = B_A = \mathcal{O}_A(d)$, we have that $A^2 \cdot B = 2\delta^2$, $B^2 \cdot A = d^2$. Also from $d^2 = h_A^2 = B \cdot B \cdot A = h_B \cdot N_{B/X}$ we conclude that $N_{B/X} = \mathcal{O}_B(d^2/2\delta)$. Similarly we conclude that $N_{A/X} = \mathcal{O}_A(2\delta^2/d)$. Thus $A^3 = 4\delta^4/d^2$ and $B^3 = d^4/4\delta^2$. Again by equation (3), we conclude that A, B are positive multiples of L in homology and hence ample. Using the argument from the case when both are $\boldsymbol{P}^2$, we see that $X = \boldsymbol{P}^3$. In this case $g = 0$. Q.E.D.

5. The cone and scroll cases

We keep again our working assumption as in 3.1. In this section we consider the remaining case when both A and B are Hirzebruch surfaces, under the *extra assumption that* $(\hat{A}, \hat{L}_{\hat{A}})$, $(\hat{B}, \hat{L}_{\hat{B}})$ *are scrolls*, i.e., $\hat{A}$, $\hat{B}$ are both scrolls with respect to $\hat{L}$.

We start with the following general lemma.

Lemma 5.1. *Let $\hat{L}$ be a very ample line bundle on a 3-fold $\hat{X}$. Let $\hat{A} + \hat{B} \in |\hat{L}|$, where $\hat{A}$, $\hat{B}$ are two smooth divisors on $\hat{X}$ meeting transversely in a smooth curve h of genus $g(h) > 0$. Assume that each of $(\hat{A}, \hat{L}_{\hat{A}})$ and $(\hat{B}, \hat{L}_{\hat{B}})$ is a scroll or a cone (from a vertex not contained in h) over a smooth curve $C_{\hat{A}}$, $C_{\hat{B}}$, respectively; with scroll (or cone) projections $p_{\hat{A}} : \hat{A} \to C_{\hat{A}}$, $p_{\hat{B}} : \hat{B} \to C_{\hat{B}}$ respectively. Then $\pi = (p_{\hat{A}}, p_{\hat{B}}) : h \to C_{\hat{A}} \times C_{\hat{B}}$ maps h isomorphically onto a smooth curve.*

Proof. Let $\xi \subset h$ be a subscheme of degree 2 (i.e., a pair of distinct points, or a tangent subscheme supported at a single point). We show that π separates ξ. If not, ξ is contained in a fiber of π. Hence ξ belongs to a fiber of $p_{\hat{A}}$ and one of $p_{\hat{B}}$, in which case the same holds true for the line ℓ spanned by ξ. But since the intersection $\hat{A} \cap \hat{B}$ is transverse and connected, it follows that $\hat{A} \cap \hat{B} = \ell$. This contradicts the hypothesis that $g(h) > 0$. Q.E.D.

Theorem 5.2. *Let $\hat{L}$ be a very ample line bundle on a smooth projective threefold $\hat{X}$. Assume that there exists two irreducible divisors $\hat{A}$, $\hat{B}$ on $\hat{X}$ meeting transversely in a smooth curve h, and such that $\hat{A} + \hat{B} \in |\hat{L}|$. Assume further that*

1. *$(\hat{A}, \hat{L}_{\hat{A}})$ is $(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(1))$, or $(Q, \mathcal{O}_{\boldsymbol{P}^3}(1)_Q)$ with $Q \subset \boldsymbol{P}^3$ the singular quadric $\widetilde{\boldsymbol{F}_2}$;*

and

2. *$(\hat{B}, \hat{L}_{\hat{B}})$ is a scroll over $\boldsymbol{P}^1$.*

Then $g(h) = 0$.

Proof. Let us focus on the case where $\hat{A} = \boldsymbol{P}^2$. The case of $\hat{A} = \widetilde{\boldsymbol{F}_2}$ is proved analogously. Let $N_{\hat{A}/\hat{X}} = \hat{A}_{\hat{A}} \cong \mathcal{O}_{\boldsymbol{P}^2}(d)$ denote the normal bundle of $\hat{A}$ in $\hat{X}$. Let $h_{\hat{A}} = \hat{B}_{\hat{A}} = \mathcal{O}_{\boldsymbol{P}^2}(\delta)$. Further, let denote by E a section of $\hat{B}$ with $E^2 = -r \le 0$, and by $\mathscr{E} = E + rf$ for a fiber f of the scroll projection. We have $\hat{B}_{\hat{B}} = M\mathscr{E} + Nf$ and $h_{\hat{B}} = \hat{A}_{\hat{B}} = a\mathscr{E} + bf$ for integers M, N, a, b. By Lemma 5.1 we have $g := g(h) = (a-1)(\delta-1)$. Further, the formulae for the genus on $\hat{A}$ and $\hat{B}$ yield the formulae $2g = (\delta-1)(\delta-2)$ and $2g = (a-1)(ar+2b-2)$. Assuming that $g \ge 1$, and hence that $\delta \ge 3$, $a \ge 2$, immediately gives $\delta = 2a$ and $4a = ar + 2b$.

Note that $d\delta = \hat{A}_{\hat{A}} \cdot \hat{B}_{\hat{A}} = \hat{B} \cdot \hat{A}^2 = \hat{A}_{\hat{B}}^2 = a(ar+2b)$. Combined with $\delta = 2a$ and $4a = ar + 2b$, we conclude that $d = \delta$. Since $\hat{L}_{\hat{A}} = \mathcal{O}_{\boldsymbol{P}^2}(d+\delta)$, we get a contradiction to $\hat{L}_{\hat{A}} \cong \mathcal{O}_{\boldsymbol{P}^2}(1)$. Q.E.D.

Let us now specialize Lemma 5.1 to the case when $\hat{A} = \boldsymbol{F}_r$, $\hat{B} = \boldsymbol{F}_s$. Denote by $\mathscr{E}_{\hat{A}} = E_{\hat{A}} + rf_{\hat{A}}$, $E_{\hat{A}}^2 = -r$, $f_{\hat{A}}$ a fiber of the ruling $\hat{A} = \boldsymbol{F}_r \to \boldsymbol{P}^1$; and similarly for $\hat{B}$. Write

$$h_{\hat{A}} = a(E_{\hat{A}} + rf_{\hat{A}}) + bf_{\hat{A}}; \quad h_{\hat{B}} = \alpha(E_{\hat{B}} + sf_{\hat{B}}) + \beta f_{\hat{B}}, \tag{4}$$

on $\hat{A}$, $\hat{B}$, respectively,

By Lemma 3.3 we may assume $a \geq 3$, $\alpha \geq 3$. Furthermore, since h is a positive genus curve on the Hirzebruch surfaces $\hat{A}$, $\hat{B}$, we may also assume $b \geq 0$, $\beta \geq 0$.

By Lemma 5.1, $\mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(\pi(h)) = \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(a, \alpha)$, and hence

$$g(h) = (a-1)(\alpha-1). \tag{5}$$

The genus formula also yields

$$2g(h) = (a-1)(ar+2b-2).$$

Therefore, from (5), we deduce that

$$2\alpha = ar + 2b. \tag{6}$$

Similarly we find

$$2a = \alpha s + 2\beta. \tag{7}$$

Combining (6) and (7), we have

$$2a\alpha = a(ar+2b) = \alpha(\alpha s + 2\beta). \tag{8}$$

Write

$$N_{\hat{A}/\hat{X}} = -\lambda(E_{\hat{A}} + rf_{\hat{A}}) + \rho f_{\hat{A}}; \quad N_{\hat{B}/\hat{X}} = -\mu(E_{\hat{B}} + sf_{\hat{B}}) + \sigma f_{\hat{B}},$$

for integers λ, μ, ρ, σ.

Note that on $\hat{A}$ one has $h_{\hat{B}}^2 = \hat{A}_{\hat{B}}^2 = \hat{A}^2 \cdot \hat{B} = \hat{A}_{\hat{A}} \cdot \hat{B}_{\hat{A}} = N_{\hat{A}/\hat{X}} \cdot h_{\hat{A}}$. Since

$$h_{\hat{B}}^2 = \alpha^2 s + 2\alpha\beta = \alpha(\alpha s + 2\beta) \quad \text{and} \quad N_{\hat{A}/\hat{X}} \cdot h_{\hat{A}} = -a\lambda r - \lambda b + \rho a,$$

we find that $\alpha(\alpha s + 2\beta) = -a\lambda r - \lambda b + \rho a$. Similarly, $a(ar+2b) = -\alpha\mu s - \mu\beta + \sigma\alpha$. Then by (8) we have

$$2a\alpha = -a\lambda r - \lambda b + \rho a = -\alpha\mu s - \mu\beta + \sigma\alpha. \tag{9}$$

Since $(\hat{A}, \hat{L}_{\hat{A}})$ is a scroll, we also have $\hat{L}_{\hat{A}}$ $(=\hat{A}_{\hat{A}} + \hat{B}_{\hat{A}} = N_{\hat{A}/\hat{X}} + h_{\hat{A}}) = E_{\hat{A}} + jf_{\hat{A}}$. On the other hand, the coefficient of $E_{\hat{A}}$ in the expression for $N_{\hat{A}/\hat{X}} + h_{\hat{A}}$ is $-\lambda + a$. Therefore the last equality for $\hat{L}_{\hat{A}}$ implies $a - \lambda = 1$. Similarly the scroll condition for $(\hat{B}, \hat{L}_{\hat{B}})$ gives $\alpha - \mu = 1$. So from (9) we have

$$2a\alpha = -a(a-1)r - (a-1)b + \rho a = -\alpha(\alpha-1)s - (\alpha-1)\beta + \sigma\alpha.$$

Then in particular $\rho = 2\alpha + (a-1)r + b - \frac{b}{a}$. This implies $\rho \geq 6$ (with $\rho = 6$ giving $r = b = 0$), as well as a divides b, say, $b = ab'$.

In the same way, we find $\sigma = 2a + (\alpha - 1)s + \beta - \frac{\beta}{\alpha}$. So $\sigma \geq 6$ (with $\sigma = 6$ giving $s = \beta = 0$), as well as $\beta = \alpha\beta'$.

Thus formulas (6) and (7) become $2\alpha = a(r + 2b')$ and $2a = \alpha(s + 2\beta')$. From this we find

$$4 = (r + 2b')(s + 2\beta'). \tag{10}$$

Since $b', \beta' \geq 0$ it follows that $rs \leq 4$ and hence $r, s \in \{0, 1, 2, 3, 4\}$.

The following theorem summarizes the discussion above.

THEOREM 5.3. *Let $\hat{L}$ be a very ample line bundle on a 3-fold $\hat{X}$. Let $\hat{A} + \hat{B} \in |\hat{L}|$, where $\hat{A}$, $\hat{B}$ are two smooth divisors on $\hat{X}$ meeting transversely in a smooth curve h of genus $g(h) > 0$. Assume that $\hat{A} = \boldsymbol{F}_r$, $\hat{B} = \boldsymbol{F}_s$ are Hirzebruch surfaces. Further assume that $(\hat{A}, \hat{L}_{\hat{A}})$ and $(\hat{B}, \hat{L}_{\hat{B}})$ are scrolls over smooth curves. Then $r, s \in \{0, 1, 2, 4\}$ and the possible values of the coefficients $b = ab'$, $\beta = \alpha\beta'$ as in the expressions* (4) *of h as a curve of $\hat{A}$, $\hat{B}$ respectively are listed in the table below.*

$s \backslash r$	0	1	2	3	4
0	$b' = \beta' = 1$	$b' = 0,\ \beta' = 2$	$b' = 0,\ \beta' = 1$	$\surd$	$\surd$
1	$b' = 2,\ \beta' = 0$	$\surd$	$b' = 1,\ \beta' = 0$	$\surd$	$b' = \beta' = 0$
2	$b' = 1,\ \beta' = 0$	$b' = 0,\ \beta' = 1$	$b' = \beta' = 0$	$\surd$	$\surd$
3	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
4	$\surd$	$b' = \beta' = 0$	$\surd$	$\surd$	$\surd$

Proof. A purely numerical check, by using (10) and the symmetry between r and s, gives the possible values for the integers r, s, b', β' in the table (the symbol "$\surd$" means that the corresponding case does not occur). For example, if $r = 0$, equality (10) gives $2 = b'(s + 2\beta')$. This leads to the cases $(s, b', \beta') = (0, 1, 1), (1, 2, 0), (2, 1, 0)$ as in the first column. Thus we may assume $r, s \geq 0$. For example, if $r = 3$, equation (10) gives $4 = 3(s + 2\beta') + 2b'(s + 2\beta')$, so that $b' \neq 0$ and hence $b' > 0$, this giving again a numerical contradiction. Q.E.D.

REFERENCES

[1] M. C. BELTRAMETTI AND A. J. SOMMESE, The Adjunction Theory of Complex Projective Varieties, de Gruyter Expositions in Mathematics 16, Walter de Gruyter, Berlin, 1995.

[2] K. A. Chandler, A. Howard and A. J. Sommese, Reducible hyperplane sections I, J. Math. Soc. Japan, **51** (1999), 887–910.

[3] T. Fujita, On Kodaira energy and adjoint reduction of polarized manifolds, Manuscripta Math., **76** (1992), 59–84.

[4] T. Fujita, Notes on Kodaira energies of polarized varieties, preprint, 1992.

[5] A. Lanteri and A. L. Tironi, On reducible hyperplane sections of 4-folds, J. Math. Soc. Japan, **53** (2001), 559–563.

[6] I. Reider, Vector bundles of rank 2 and linear systems on algebraic surfaces, Ann. of Math. (2), **127** (1988), 309–316.

[7] A. L. Tironi, Reducible hyperplane sections of 4-folds: semi-nef decompositions with low sectional genera, Istit. Lombardo Accad. Sci. Lett. Rend. A, **134** (2001), 47–58.

Dipartimento di Matematica
Via Dodecaneso 35
I-16146 Genova, Italy
e-mail: beltrame@dima.unige.it

Department of Mathematics
Notre Dame, Indiana, 46556, U.S.A.
e-mail: kchandle@noether.math.nd.edu

Department of Mathematics
Notre Dame, Indiana, 46556, U.S.A.
e-mail: sommese@nd.edu
http://www.nd.edu/~sommese