

EIGENMODES OF LENS AND PRISM SPACES

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Abstract

Cosmologists are taking a renewed interest in multiconnected spherical 3-manifolds (spherical spaceforms) as possible models for the physical universe. To understand the formation of large scale structures in such a universe, cosmologists express physical quantities, such as density fluctuations in the primordial plasma, as linear combinations of the eigenmodes of the Laplacian, which can then be integrated forward in time. This need for explicit eigenmodes contrasts sharply with previous mathematical investigations, which have focused on questions of isospectrality rather than eigenmodes. The present article provides explicit orthonormal bases for the eigenmodes of lens and prism spaces.

1. Introduction

In recent years cosmologists have taken a renewed interest in multiply connected 3-manifolds as possible models for the universe [1, 2, 3], motivated by upcoming opportunities to determine the topology of the real universe using satellite measurements of the microwave background [4] and galaxy catalogs [5]. Cosmologists initially focused on closed hyperbolic 3-manifolds, favored by the low observed matter density in the universe, as well as the more easily understood flat 3-manifolds. But since 1998 it has become clear that the modest amount of matter in the universe (30%) is complemented by a large amount of exotic energy (70%). This extra energy implies that the observable universe is approximately flat, or perhaps slightly spherical [6]. Cosmologists have therefore shifted their interest from hyperbolic 3-manifolds to flat and spherical ones. Beyond the data's very slight preference for a spherical universe, the cosmologists' new interest in multiply connected spherical 3-manifolds (spherical spaceforms) is due to the fact that the volume of a spherical 3-manifold decreases as the topology gets more complicated, unlike hyperbolic 3-manifolds whose volumes increase as the topology gets more complicated. Thus even though both hyperbolic and spherical 3-manifolds are consistent with an approximately flat observable universe, the spherical topologies would be more easily detectable observationally [7].

To understand and simulate microwave background measurements in a multiply connected spherical universe, cosmologists must first understand the density fluctuations in the primordial plasma (see [8] for a review). Such density

fluctuations are expressed as linear combinations of the eigenmodes (eigenfunctions) of the Laplacian, just as the vibration of a drumhead may be expressed as a linear combination of the drumhead's eigenmodes. But unlike mathematicians' studies of isospectrality [9], where the spectrum was the primary object of interest ("Can you hear the shape of a lens space?") and the eigenmodes were secondary, the cosmologists' research puts the eigenmodes at center stage. More specifically, for each wave number k (corresponding to eigenvalue $k(k+2)$), the cosmologists want an explicit orthonormal basis for the corresponding space of eigenmodes. The present paper provides such an eigenbasis for all lens spaces (Theorem 2, Section 9) and prism spaces (Theorem 3, Section 10).

2. Toroidal coordinates

The determination of the eigenmodes of a lens space (resp. prism space) is elementary and constructive. Visualize such a manifold as the 3-sphere S^3 under the action of a cyclic (resp. binary dihedral) group Γ of covering transformations. The key to simplicity is to choose a coordinate system that respects the covering transformations Γ . A *toroidal coordinate system* meets our needs perfectly (Figure 1).

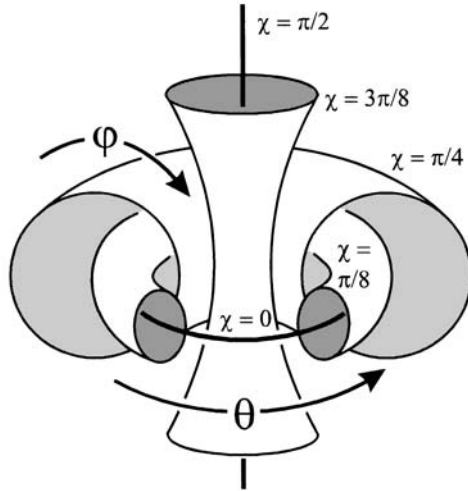


FIGURE 1. Toroidal coordinates. Nested tori fill the 3-sphere like layers of an onion. Just as the layers of an onion collapse to a line at the onion's core, the nested tori collapse to two circles, one at $\chi = 0$ and the other at $\chi = \pi/2$.

Let x , y , z and w be the usual coordinates in \mathbf{R}^4 , so the 3-sphere S^3 is defined by $x^2 + y^2 + z^2 + w^2 = 1$. The coordinates χ , θ and φ parameterize the 3-sphere as

$$\begin{aligned}
 (1) \quad x &= \cos \chi \cos \theta \\
 y &= \cos \chi \sin \theta \\
 z &= \sin \chi \cos \varphi \\
 w &= \sin \chi \sin \varphi
 \end{aligned}$$

for

$$\begin{aligned}
 (2) \quad 0 &\leq \chi \leq \pi/2 \\
 0 &\leq \theta \leq 2\pi \\
 0 &\leq \varphi \leq 2\pi.
 \end{aligned}$$

For each fixed value of $\chi \in (0, \pi/2)$, the θ and φ coordinates sweep out a torus. Taken together, these tori almost fill S^3 . The exceptions occur at the endpoints $\chi = 0$ and $\chi = \pi/2$, where the stack of tori collapses to the circles $x^2 + y^2 = 1$ and $z^2 + w^2 = 1$, respectively.

3. The Laplacian in toroidal coordinates

The coordinates χ , θ and φ are everywhere orthogonal to each other. Thus the metric on the 3-sphere may be written as

$$(3) \quad ds^2 = h_\chi^2 d\chi^2 + h_\theta^2 d\theta^2 + h_\varphi^2 d\varphi^2$$

where

$$\begin{aligned}
 (4) \quad h_\chi &= 1 \\
 h_\theta &= \cos \chi \\
 h_\varphi &= \sin \chi.
 \end{aligned}$$

The Laplacian is just the divergence of the gradient of a function Ψ , and the divergence is, in turn, just the “net outflow per unit volume”. So by visualizing the small volume element $d\chi d\theta d\varphi$ we may write down the Laplacian for any orthogonal coordinate system as

$$(5) \quad \nabla^2 = \frac{1}{h_\chi h_\theta h_\varphi} \left\{ \frac{\partial}{\partial \chi} \frac{h_\theta h_\varphi}{h_\chi} \frac{\partial}{\partial \chi} + \frac{\partial}{\partial \theta} \frac{h_\chi h_\varphi}{h_\theta} \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \varphi} \frac{h_\chi h_\theta}{h_\varphi} \frac{\partial}{\partial \varphi} \right\}$$

with no calculation required. In the present case, substituting (4) into (5) gives the Laplacian in toroidal coordinates

$$(6) \quad \nabla^2 = \frac{1}{\cos \chi \sin \chi} \left\{ \frac{\partial}{\partial \chi} \cos \chi \sin \chi \frac{\partial}{\partial \chi} + \frac{\partial}{\partial \theta} \frac{\sin \chi}{\cos \chi} \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \varphi} \frac{\cos \chi}{\sin \chi} \frac{\partial}{\partial \varphi} \right\}.$$

4. The Helmholtz equation in toroidal coordinates

The wave number k parameterizes the eigenmodes of the Laplacian on the 3-sphere S^3 . Each integer wave number $k > 0$ corresponds to an eigen-

value $-k(k+2)$ with multiplicity $(k+1)^2$ [10, 11]. The Helmholtz equation thus takes the form

$$(7) \quad \nabla^2 \Psi = -k(k+2)\Psi.$$

We will look for solutions that factor as

$$(8) \quad \Psi(\chi, \theta, \varphi) = X(\chi)\Theta(\theta)\Phi(\varphi).$$

We have no *a priori* guarantee that all solutions must take this form, but in Section 6 we'll see that the number of independent solutions of this form does indeed equal the dimension $(k+1)^2$ of the full eigenspace.

Substituting the expression (6) for ∇^2 and the factorization (8) of Ψ into the Helmholtz equation (7) gives

$$(9) \quad \frac{\Theta\Phi}{\cos\chi\sin\chi} \frac{\partial}{\partial\chi} \cos\chi\sin\chi \frac{\partial X}{\partial\chi} + \frac{X\Phi}{\cos^2\chi} \frac{\partial^2\Theta}{\partial\theta^2} + \frac{X\Theta}{\sin^2\chi} \frac{\partial^2\Phi}{\partial\varphi^2} = -k(k+2)X\Theta\Phi.$$

Multiplying through by $\cos^2\chi\sin^2\chi/(X\Theta\Phi)$ isolates the Θ and Φ factors

$$(10) \quad \frac{\cos\chi\sin\chi}{X} \frac{d}{d\chi} \cos\chi\sin\chi \frac{dX}{d\chi} + \sin^2\chi \left(\frac{1}{\Theta} \frac{d^2\Theta}{d\theta^2} \right) + \cos^2\chi \left(\frac{1}{\Phi} \frac{d^2\Phi}{d\varphi^2} \right) \\ = -k(k+2)\cos^2\chi\sin^2\chi.$$

The expressions in Θ and Φ must each be constant, and to allow a periodic solution the constants must be negative,

$$(11) \quad \frac{1}{\Theta} \frac{d^2\Theta}{d\theta^2} = -\ell^2$$

$$(12) \quad \frac{1}{\Phi} \frac{d^2\Phi}{d\varphi^2} = -m^2.$$

The solutions are the usual circular harmonics

$$(13) \quad \Theta_\ell(\theta) = \cos|\ell|\theta \quad \text{or} \quad \sin|\ell|\theta$$

and

$$(14) \quad \Phi_m(\varphi) = \cos|m|\varphi \quad \text{or} \quad \sin|m|\varphi.$$

By convention, nonnegative ℓ indicates $\cos|\ell|\theta$ while negative ℓ indicates $\sin|\ell|\theta$, and similarly for m .

Substituting (11) and (12) into the Helmholtz equation (10) reduces it to a second order ordinary differential equation for X

$$(15) \quad \frac{\cos\chi\sin\chi}{X} \frac{d}{d\chi} \cos\chi\sin\chi \frac{dX}{d\chi} - \ell^2 \sin^2\chi - m^2 \cos^2\chi = -k(k+2)\cos^2\chi\sin^2\chi.$$

For integers k , ℓ and m satisfying $|\ell| + |m| \leq k$ and $\ell + m \equiv k \pmod{2}$, equation (15) is a close relative of the Jacobi equation and admits the solution

$$(16) \quad X_{k\ell m}(\chi) = \cos^{|\ell|} \chi \sin^{|m|} \chi P_d^{(|m|, |\ell|)}(\cos 2\chi)$$

where $P_d^{(|m|, |\ell|)}$ is the Jacobi polynomial

$$(17) \quad P_d^{(|m|, |\ell|)}(u) = \frac{1}{2^d} \sum_{i=0}^d \binom{|m|+d}{i} \binom{|\ell|+d}{d-i} (u+1)^i (u-1)^{d-i}$$

and

$$(18) \quad d = \frac{k - (|\ell| + |m|)}{2}.$$

5. The eigenmodes of S^3 in trigonometric and polynomial forms

Substituting the expressions for X , Θ and Φ from (16), (13) and (14) gives the eigenmode

$$(19) \quad \begin{aligned} \tilde{\Psi}_{k\ell m}(\chi, \theta, \varphi) &= \cos^{|\ell|} \chi \sin^{|m|} \chi P_d^{(|m|, |\ell|)}(\cos 2\chi) \\ &\quad \times (\cos|\ell|\theta \text{ or } \sin|\ell|\theta) \\ &\quad \times (\cos|m|\varphi \text{ or } \sin|m|\varphi) \end{aligned}$$

where as usual the choice of $\cos|\ell|\theta$ or $\sin|\ell|\theta$ (resp. $\cos|m|\varphi$ or $\sin|m|\varphi$) depends on the sign of ℓ (resp. m).

The Jacobi polynomial may be expanded as a homogeneous polynomial of degree $2d$ in x , y , z and w ,

$$(20) \quad \begin{aligned} P_d^{(|m|, |\ell|)}(\cos 2\chi) &= \frac{1}{2^d} \sum_{i=0}^d \binom{|m|+d}{i} \binom{|\ell|+d}{d-i} (\cos 2\chi + 1)^i (\cos 2\chi - 1)^{d-i} \\ &= \sum_{i=0}^d \binom{|m|+d}{i} \binom{|\ell|+d}{d-i} (\cos^2 \chi)^i (-\sin^2 \chi)^{d-i} \\ &= \sum_{i=0}^d \binom{|m|+d}{i} \binom{|\ell|+d}{d-i} (x^2 + y^2)^i (-(z^2 + w^2))^{d-i}. \end{aligned}$$

The $\cos^{|\ell|} \chi$ factor combines felicitously with the $\cos|\ell|\theta$ or $\sin|\ell|\theta$ factor to create a homogeneous polynomial of degree $|\ell|$ in x and y , for example,

$$(21) \quad \begin{aligned} \cos^{|\ell|} \chi \cos|\ell|\theta &= \cos^{|\ell|} \chi \sum_{0 \leq i \leq |\ell|/2} (-1)^i \binom{|\ell|}{2i} \cos^{|\ell|-2i} \theta \sin^{2i} \theta \\ &= \sum_{0 \leq i \leq |\ell|/2} (-1)^i \binom{|\ell|}{2i} (\cos \chi \cos \theta)^{|\ell|-2i} (\cos \chi \sin \theta)^{2i} \\ &= \sum_{0 \leq i \leq |\ell|/2} (-1)^i \binom{|\ell|}{2i} x^{|\ell|-2i} y^{2i}. \end{aligned}$$

Similarly, $\sin^{|m|} \chi$ combines with $\cos|m|\varphi$ or $\sin|m|\varphi$ to create a degree $|m|$ polynomial in z and w . Multiplied together, these factors express $\tilde{\Psi}_{k\ell m}$ as a homogeneous degree k harmonic polynomial in (x, y, z, w) coordinates. For example, when $k = 7$, $\ell = 3$ and $m = -2$ we have

$$(22) \quad \begin{aligned} \tilde{\Psi}_{7,3,-2}(\chi, \theta, \varphi) &= [P_1^{(2,3)}(\cos 2\chi)][\cos^3 \chi \cos 3\theta][\sin^2 \chi \sin 2\varphi] \\ &= [3(x^2 + y^2) - 4(z^2 + w^2)][x^3 - 3xy^2][2zw]. \end{aligned}$$

The fact that each $\tilde{\Psi}_{k\ell m}$ may be expressed as a polynomial proves that the $\tilde{\Psi}_{k\ell m}$ are smooth even along the circles $\chi = 0$ and $\chi = \pi/2$, where the toroidal coordinate system collapses.

6. The eigenmodes form a basis

For each k , the set of $\tilde{\Psi}_{k\ell m}$ forms a basis for the space of eigenfunctions on S^3 with wave number k . More precisely, define the basis

$$(23) \quad B_k = \{\tilde{\Psi}_{k\ell m} \mid |\ell| + |m| \leq k \text{ and } \ell + m \equiv k \pmod{2}\}.$$

To prove that B_k is a basis, we must show that the $\tilde{\Psi}_{k\ell m}$ it contains are linearly independent and span the full eigenspace.

Linear independence. The inner product of two elements $\tilde{\Psi}_{k\ell m}$ and $\tilde{\Psi}_{k\ell' m'}$ of B_k is

$$(24) \quad \begin{aligned} \langle \tilde{\Psi}_{k\ell m}, \tilde{\Psi}_{k\ell' m'} \rangle &= \int_{S^3} \tilde{\Psi}_{k\ell m} \tilde{\Psi}_{k\ell' m'} dV \\ &= \int_{\chi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{2\pi} (X_{k\ell m} \Theta_{\ell} \Phi_m)(X_{k\ell' m'} \Theta_{\ell'} \Phi_{m'}) \cos \chi \sin \chi d\varphi d\theta d\chi \\ &= \left(\int_{\chi=0}^{\pi/2} X_{k\ell m} X_{k\ell' m'} \cos \chi \sin \chi d\chi \right) \left(\int_{\theta=0}^{2\pi} \Theta_{\ell} \Theta_{\ell'} d\theta \right) \\ &\quad \times \left(\int_{\varphi=0}^{2\pi} \Phi_m \Phi_{m'} d\varphi \right). \end{aligned}$$

If $\ell \neq \ell'$ (resp. $m \neq m'$), then the orthogonality of the circular harmonics $\langle \Theta_{\ell}, \Theta_{\ell'} \rangle = 0$ (resp. $\langle \Phi_m, \Phi_{m'} \rangle = 0$) immediately implies $\langle \tilde{\Psi}_{k\ell m}, \tilde{\Psi}_{k\ell' m'} \rangle = 0$, proving that $\tilde{\Psi}_{k\ell m}$ and $\tilde{\Psi}_{k\ell' m'}$ are orthogonal. Because the $\tilde{\Psi}_{k\ell m}$ in B_k are non-zero and pairwise orthogonal, they must be linearly independent.

Span. We have shown that the $\tilde{\Psi}_{k\ell m}$ in B_k are linearly independent. To prove that they span the full eigenspace, it suffices to check that the number of elements of B_k equals the dimension of the full eigenspace, which is known to be $(k+1)^2$. The set $B_0 = \{\tilde{\Psi}_{0,0,0}\}$ has $(0+1)^2 = 1$ element, and the set $B_1 = \{\tilde{\Psi}_{1,+1,0}, \tilde{\Psi}_{1,-1,0}, \tilde{\Psi}_{1,0,+1}, \tilde{\Psi}_{1,0,-1}\}$ has $(1+1)^2 = 4$ elements, as required. For

the remaining B_k , with $k \geq 2$, we proceed by induction, assuming that the set B_{k-2} is already known to contain $((k-2)+1)^2 = (k-1)^2$ elements. Each element $\tilde{\Psi}_{k-2,\ell,m} \in B_{k-2}$ corresponds to an element $\tilde{\Psi}_{k,\ell,m} \in B_k$. The set B_k also contains the additional elements $\tilde{\Psi}_{k,0,\pm k}, \tilde{\Psi}_{k,\pm 1,\pm(k-1)}, \dots, \tilde{\Psi}_{k,\pm(k-1),\pm 1}, \tilde{\Psi}_{k,\pm k,0}$. Taking into account the plus-or-minus signs, this gives $2 + 4 + \dots + 4 + 2 = 2 + 4(k-1) + 2 = 4k$ additional elements. Adding these to the $(k-1)^2$ elements corresponding to B_{k-2} , we get a total of $(k-1)^2 + 4k = (k+1)^2$ elements, as required.

This completes the proof that B_k is a basis for the space of eigenfunctions on S^3 with wave number k .

7. Normalization

The $\tilde{\Psi}_{k\ell m}$ are already mutually orthogonal (Section 6), so if we normalize them to unit length they will form an orthonormal basis for the eigenspace. An orthonormal basis is convenient in cosmological applications, because it makes it easy to construct an unbiased random density fluctuation with wave number k .

To compute the norm of a given $\tilde{\Psi}_{k\ell m}$, set $\ell' = \ell$ and $m' = m$ in equation (24), giving

$$(25) \quad \langle \tilde{\Psi}_{k\ell m}, \tilde{\Psi}_{k\ell m} \rangle = \left(\int_{\chi=0}^{\pi/2} X_{k\ell m}^2 \cos \chi \sin \chi \, d\chi \right) \left(\int_{\theta=0}^{2\pi} \Theta_{\ell}^2 \, d\theta \right) \left(\int_{\varphi=0}^{2\pi} \Phi_m^2 \, d\varphi \right).$$

The Θ and Φ integrals are easy to evaluate. Substituting in the solutions (13) and (14) immediately gives

$$(26) \quad \int_{\theta=0}^{2\pi} \Theta_{\ell}^2 \, d\theta = \pi \quad \text{and} \quad \int_{\varphi=0}^{2\pi} \Phi_m^2 \, d\varphi = \pi$$

when ℓ and m are nonzero, along with the special cases

$$(27) \quad \int_{\theta=0}^{2\pi} \Theta_0^2 \, d\theta = 2\pi \quad \text{and} \quad \int_{\varphi=0}^{2\pi} \Phi_0^2 \, d\varphi = 2\pi.$$

The X integral seems daunting, but luckily the $\cos^{|\ell|} \chi \sin^{|m|} \chi$ factor in the expression (16) for X provides exactly the standard weighting function relative to which the Jacobi polynomials are normalized!

$$(28) \quad \begin{aligned} & \int_{\chi=0}^{\pi/2} X_{k\ell m}^2 \cos \chi \sin \chi \, d\chi \\ &= \int_{\chi=0}^{\pi/2} [\cos^{|\ell|} \chi \sin^{|m|} \chi P_d^{(|m|, |\ell|)}(\cos 2\chi)]^2 \cos \chi \sin \chi \, d\chi \\ &= \int_{\chi=0}^{\pi/2} (\cos^2 \chi)^{|\ell|} (\sin^2 \chi)^{|m|} [P_d^{(|m|, |\ell|)}(\cos 2\chi)]^2 \cos \chi \sin \chi \, d\chi \end{aligned}$$

then, changing the variable to $u = \cos 2\chi$,

$$(29) \quad = \frac{1}{2^{|\ell|+|m|+2}} \int_{u=-1}^{+1} (1+u)^{|\ell|} (1-u)^{|m|} [P_d^{(|m|,|\ell|)}(u)]^2 du$$

and using the standard normalization for Jacobi polynomials [12, 13], we get

$$(30) \quad = \frac{(|\ell|+d)! (|m|+d)!}{2(k+1)d!(|\ell|+|m|+d)!}.$$

Define the *normalized eigenmodes* $\Psi_{k\ell m}$ to be

$$(31) \quad \Psi_{k\ell m} = \frac{\tilde{\Psi}_{k\ell m}}{\sqrt{\langle \tilde{\Psi}_{k\ell m}, \tilde{\Psi}_{k\ell m} \rangle}} = \frac{\tilde{\Psi}_{k\ell m}}{\sqrt{2^{\hat{\ell}+\hat{m}} \pi} \sqrt{\frac{(|\ell|+d)! (|m|+d)!}{2(k+1)d!(|\ell|+|m|+d)!}}}$$

where $\hat{\ell}$ is 1 when ℓ is 0, and $\hat{\ell}$ is 0 when ℓ is nonzero, and similarly for \hat{m} , to accommodate the special cases in equations (26) and (27).

The results of this section and the previous one together prove

THEOREM 1. *The $\Psi_{k\ell m}$, taken over all integers k , ℓ and m satisfying $|\ell| + |m| \leq k$ and $\ell + m \equiv k \pmod{2}$, comprise an orthonormal basis for the eigenspace of the Laplacian on the 3-sphere.*

8. The action of an isometry of S^3 on the space of eigenmodes

An arbitrary orientation-preserving isometry of the 3-sphere has matrix

$$(32) \quad \begin{pmatrix} \cos \Delta\theta & -\sin \Delta\theta & 0 & 0 \\ \sin \Delta\theta & \cos \Delta\theta & 0 & 0 \\ 0 & 0 & \cos \Delta\varphi & -\sin \Delta\varphi \\ 0 & 0 & \sin \Delta\varphi & \cos \Delta\varphi \end{pmatrix}$$

relative to an appropriate orthonormal (x, y, z, w) coordinate system on \mathbf{R}^4 . In toroidal coordinates (χ, θ, ϕ) , the same isometry may be described as

$$(33) \quad \begin{aligned} \chi &\rightarrow \chi \\ \theta &\rightarrow \theta + \Delta\theta \\ \varphi &\rightarrow \varphi + \Delta\varphi. \end{aligned}$$

The action of this isometry on the space of eigenfunctions is much simpler in toroidal coordinates than in traditional polar coordinates. For example, if $\ell > 0$ and $-m < 0$, then the isometry (33) maps

$$\begin{aligned} \Psi_{k,+\ell,-m}(\chi, \theta, \varphi) &\rightarrow \Psi_{k,+\ell,-m}(\chi, \theta + \Delta\theta, \varphi + \Delta\varphi) \\ &= X_{k\ell m}(\chi) \Theta_{+\ell}(\theta + \Delta\theta) \Phi_{-m}(\varphi + \Delta\varphi) \\ &= X_{k\ell m}(\chi) \cos \ell(\theta + \Delta\theta) \sin m(\varphi + \Delta\varphi) \end{aligned}$$

$$\begin{aligned}
&= X_{k\ell m}(\chi)(\cos \ell\Delta\theta \cos \ell\theta - \sin \ell\Delta\theta \sin \ell\theta) \\
&\quad \times (\sin m\Delta\varphi \cos m\varphi + \cos m\Delta\varphi \sin m\varphi) \\
&= \cos \ell\Delta\theta \sin m\Delta\varphi X_{k\ell m}(\chi)\Theta_{+\ell}(\theta)\Phi_{+m}(\varphi) \\
&\quad + \cos \ell\Delta\theta \cos m\Delta\varphi X_{k\ell m}(\chi)\Theta_{+\ell}(\theta)\Phi_{-m}(\varphi) \\
&\quad - \sin \ell\Delta\theta \sin m\Delta\varphi X_{k\ell m}(\chi)\Theta_{-\ell}(\theta)\Phi_{+m}(\varphi) \\
&\quad - \sin \ell\Delta\theta \cos m\Delta\varphi X_{k\ell m}(\chi)\Theta_{-\ell}(\theta)\Phi_{-m}(\varphi) \\
&= \cos \ell\Delta\theta \sin m\Delta\varphi \Psi_{k,+\ell,+m}(\chi, \theta, \varphi) \\
&\quad + \cos \ell\Delta\theta \cos m\Delta\varphi \Psi_{k,+\ell,-m}(\chi, \theta, \varphi) \\
&\quad - \sin \ell\Delta\theta \sin m\Delta\varphi \Psi_{k,-\ell,+m}(\chi, \theta, \varphi) \\
&\quad - \sin \ell\Delta\theta \cos m\Delta\varphi \Psi_{k,-\ell,-m}(\chi, \theta, \varphi)
\end{aligned}$$

(34) ... and similarly for the images of $\Psi_{k,+\ell,+m}$, $\Psi_{k,-\ell,+m}$ and $\Psi_{k,-\ell,-m}$.

Thus the subspace spanned by the $\Psi_{k,\pm\ell,\pm m}$ is invariant (setwise but not necessarily pointwise) under the action of the isometry (33). Typically this subspace is 4-dimensional, but when ℓ or m is zero it is 2-dimensional, or only 1-dimensional when both ℓ and m are zero.

Thus the complete eigenspace factors into orthogonal 1-, 2- and 4-dimensional invariant subspaces, each spanned by a set $\Psi_{k,\pm\ell,\pm m}$. To understand the full action of the isometry, it suffices to understand its action on each invariant subspace.

CASE 1. $\ell = m = 0$.

The 1-dimensional invariant subspace spanned by $\Psi_{k,0,0}$ is pointwise fixed by every isometry of the form (33).

CASE 2. $\ell = 0$ or $m = 0$ (but not both).

For sake of discussion, assume $\ell > 0$ and $m = 0$. Relative to the basis

$$\begin{aligned}
(35) \quad \mathbf{d}_1 &= \Psi_{k,+\ell,0} \\
\mathbf{d}_2 &= \Psi_{k,-\ell,0}
\end{aligned}$$

the isometry (33) acts as a rotation through an angle $\ell\Delta\theta$, with matrix

$$(36) \quad \begin{pmatrix} \cos \ell\Delta\theta & \sin \ell\Delta\theta \\ -\sin \ell\Delta\theta & \cos \ell\Delta\theta \end{pmatrix}.$$

A quick computation similar to (34) verifies that matrix (36) is correct. Thus if $\ell\Delta\theta \equiv 0 \pmod{2\pi}$ the whole subspace is fixed pointwise; otherwise only the origin is fixed. Similar conclusions hold when $\ell = 0$ and $m > 0$.

CASE 3. $\ell > 0$ and $m > 0$.

Computation (34) expresses the isometry's action on each of the $\Psi_{k,\pm\ell,\pm m}$ as a linear combination of the $\Psi_{k,\pm\ell,\pm m}$ themselves. For best results use the rotated basis

$$(37) \quad \begin{aligned} \mathbf{e}_1 &= \sqrt{\frac{1}{2}}(\Psi_{k,+\ell,+m} + \Psi_{k,-\ell,-m}) = \sqrt{\frac{1}{2}}X_{k\ell m} \cos(\ell\theta - m\varphi) \\ \mathbf{e}_2 &= \sqrt{\frac{1}{2}}(\Psi_{k,-\ell,+m} - \Psi_{k,+\ell,-m}) = \sqrt{\frac{1}{2}}X_{k\ell m} \sin(\ell\theta - m\varphi) \\ \mathbf{e}_3 &= \sqrt{\frac{1}{2}}(\Psi_{k,+\ell,+m} - \Psi_{k,-\ell,-m}) = \sqrt{\frac{1}{2}}X_{k\ell m} \cos(\ell\theta + m\varphi) \\ \mathbf{e}_4 &= \sqrt{\frac{1}{2}}(\Psi_{k,-\ell,+m} + \Psi_{k,+\ell,-m}) = \sqrt{\frac{1}{2}}X_{k\ell m} \sin(\ell\theta + m\varphi) \end{aligned}$$

relative to which the action has matrix

$$(38) \quad \begin{pmatrix} \cos(\ell\Delta\theta - m\Delta\varphi) & \sin(\ell\Delta\theta - m\Delta\varphi) & 0 & 0 \\ -\sin(\ell\Delta\theta - m\Delta\varphi) & \cos(\ell\Delta\theta - m\Delta\varphi) & 0 & 0 \\ 0 & 0 & \cos(\ell\Delta\theta + m\Delta\varphi) & \sin(\ell\Delta\theta + m\Delta\varphi) \\ 0 & 0 & -\sin(\ell\Delta\theta + m\Delta\varphi) & \cos(\ell\Delta\theta + m\Delta\varphi) \end{pmatrix}.$$

Clearly the subspace spanned by $\{\mathbf{e}_1, \mathbf{e}_2\}$ (resp. $\{\mathbf{e}_3, \mathbf{e}_4\}$) is pointwise fixed if and only if $\ell\Delta\theta \equiv m\Delta\varphi \pmod{2\pi}$ (resp. $\ell\Delta\theta \equiv -m\Delta\varphi \pmod{2\pi}$). If $\ell\Delta\theta \equiv m\Delta\varphi \equiv 0$ or π , then the whole 4-dimensional subspace is pointwise fixed.

In summary, an isometry (33) fixes the subspace spanned by

$$(39) \quad \begin{array}{ll} \{\Psi_{k,0,0}\} & \text{always} \\ \{\Psi_{k,+\ell,0}, \Psi_{k,-\ell,0}\} & \text{iff } \ell\Delta\theta \equiv 0 \pmod{2\pi} \\ \{\Psi_{k,0,+m}, \Psi_{k,0,-m}\} & \text{iff } m\Delta\varphi \equiv 0 \pmod{2\pi} \\ \left\{ \begin{array}{l} \sqrt{\frac{1}{2}}(\Psi_{k,+\ell,+m} + \Psi_{k,-\ell,-m}), \\ \sqrt{\frac{1}{2}}(\Psi_{k,-\ell,+m} - \Psi_{k,+\ell,-m}) \end{array} \right\} & \text{iff } \ell\Delta\theta \equiv m\Delta\varphi \pmod{2\pi} \\ \left\{ \begin{array}{l} \sqrt{\frac{1}{2}}(\Psi_{k,+\ell,+m} - \Psi_{k,-\ell,-m}), \\ \sqrt{\frac{1}{2}}(\Psi_{k,-\ell,+m} + \Psi_{k,+\ell,-m}) \end{array} \right\} & \text{iff } \ell\Delta\theta \equiv -m\Delta\varphi \pmod{2\pi} \end{array}$$

and this is the complete description of the fixed point set. When working with fixed point sets, please keep in mind that the eigenmode $\Psi_{k\ell m}$ exists if and only if $|\ell| + |m| \leq k$ and $\ell + m \equiv k \pmod{2}$.

9. Eigenmodes of lens spaces

The lens space $L(p, q)$ is the quotient of the 3-sphere S^3 by the cyclic group whose generator g is the isometry (33) with $\Delta\theta = 2\pi/p$ and $\Delta\varphi = 2\pi q/p$. Each eigenmode of $L(p, q)$ lifts to a g -invariant eigenmode of S^3 , and conversely each g -invariant eigenmode of S^3 projects down to an eigenmode of $L(p, q)$. Thus the eigenmodes of $L(p, q)$ correspond to the g -invariant eigenmodes of S^3 .

Substituting $\Delta\theta = 2\pi/p$ and $\Delta\varphi = 2\pi q/p$ into (39) yields a set of simple integer conditions showing which eigenmodes of S^3 are g -invariant:

THEOREM 2. *The eigenspace of the Laplacian on the lens space $L(p, q)$ has an orthonormal basis that, when lifted to Z_p -invariant eigenmodes of the 3-sphere, comprises those eigenmodes in the left column for which the corresponding condition in the right column is satisfied, subject to the restriction that an eigenmode $\Psi_{k\ell m}$ exists if and only if the integers k , ℓ and m satisfy $|\ell| + |m| \leq k$ and $\ell + m \equiv k \pmod{2}$.*

		basis vectors	condition
(40)		$\Psi_{k,0,0}$	always
		$\Psi_{k,+\ell,0}, \Psi_{k,-\ell,0}$	$\ell \equiv 0 \pmod{p}$
		$\Psi_{k,0,+m}, \Psi_{k,0,-m}$	$qm \equiv 0 \pmod{p}$
		$\sqrt{\frac{1}{2}}(\Psi_{k,+\ell,+m} + \Psi_{k,-\ell,-m}), \sqrt{\frac{1}{2}}(\Psi_{k,-\ell,+m} - \Psi_{k,+\ell,-m})$	$\ell \equiv qm \pmod{p}$
		$\sqrt{\frac{1}{2}}(\Psi_{k,+\ell,+m} - \Psi_{k,-\ell,-m}), \sqrt{\frac{1}{2}}(\Psi_{k,-\ell,+m} + \Psi_{k,+\ell,-m})$	$\ell \equiv -qm \pmod{p}$

For a given lens space $L(p, q)$ and wave number k , the number of basis vectors gives the multiplicity of the eigenvalue $k(k+2)$ (see Table 1).

10. Eigenmodes of prism spaces

The n^{th} prism space is the quotient S^3/D_n^* , where the binary dihedral group D_n^* is the extension of the binary cyclic group $Z_n^* = Z_{2n}$ by an order four Clifford translation along a perpendicular axis (Figure 2). The generator of the Z_{2n} action may be written in toroidal coordinates as

$$(41) \quad (\chi, \theta, \varphi) \rightarrow \left(\chi, \theta + \frac{2\pi}{2n}, \varphi + \frac{2\pi}{2n} \right)$$

TABLE 1. The number of eigenbasis vectors specified in Theorem 2 tells the multiplicity of each eigenvalue $k(k+2)$ in the spectrum of a lens space $L(p, q)$. Here are the results for $p \leq 9$ and $k \leq 14$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
S^3	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225
$L(2, 1)$	1	0	9	0	25	0	49	0	81	0	121	0	169	0	225
$L(3, 1)$	1	0	3	8	5	12	21	16	27	40	33	48	65	56	75
$L(4, 1)$	1	0	3	0	15	0	21	0	45	0	55	0	91	0	105
$L(5, 1)$	1	0	3	0	5	12	7	16	9	20	33	24	39	28	45
$L(5, 2)$	1	0	1	4	5	8	9	12	17	20	25	28	33	40	45
$L(6, 1)$	1	0	3	0	5	0	21	0	27	0	33	0	65	0	75
$L(7, 1)$	1	0	3	0	5	0	7	16	9	20	11	24	13	28	45
$L(7, 2)$	1	0	1	2	3	6	7	10	11	14	17	20	25	28	33
$L(8, 1)$	1	0	3	0	5	0	7	0	27	0	33	0	39	0	45
$L(8, 3)$	1	0	1	0	7	0	11	0	23	0	27	0	45	0	53
$L(9, 1)$	1	0	3	0	5	0	7	0	9	20	11	24	13	28	15
$L(9, 2)$	1	0	1	2	1	4	7	6	9	14	11	16	21	18	25

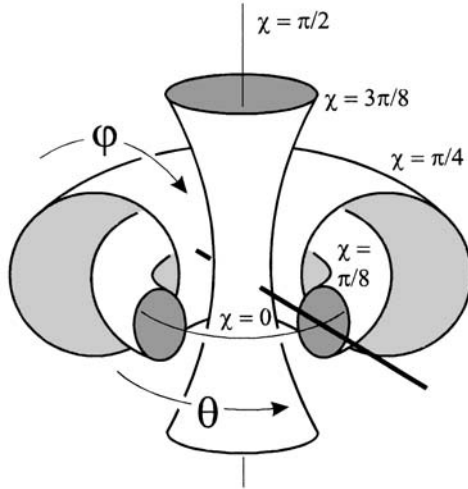


FIGURE 2. The binary dihedral group D_n^* has two generators. The first generator acts as a left-handed $2\pi/2n$ corkscrew motion preserving the toroidal layers just as in the lens space $L(2n, 1)$. The second generator acts as a left-handed $2\pi/4$ corkscrew motion along an orthogonal axis (drawn heavy in the figure), taking, for example, the torus at level $\chi = \pi/8$ to the one at $\chi = 3\pi/8$ and interchanging the roles of θ and φ .

while the generator of the order four Clifford translation may be written as

$$(42) \quad (\chi, \theta, \varphi) \rightarrow \left(\frac{\pi}{2} - \chi, -\varphi, \pi - \theta \right).$$

The eigenmodes of the prism space naturally correspond to the D_n^* -invariant eigenmodes of S^3 . That is, they correspond to the eigenmodes of S^3 that are invariant under both the Z_{2n} generator (41) and the Z_4 generator (42). Sections 8 and 9 have already determined the action of the Z_{2n} generator on the space of eigenmodes, with the fixed subspace specified by Theorem 2 with $p = 2n$ and $q = 1$. A similar computation shows how the Z_4 generator interchanges the roles of ℓ and m :

$$\begin{aligned}
 (43) \quad \Psi_{k\ell m}(\chi, \theta, \varphi) &\rightarrow \Psi_{k\ell m}(\pi/2 - \chi, -\varphi, \pi - \theta) \\
 &= X_{k\ell m}(\pi/2 - \chi) \Theta_\ell(-\varphi) \Phi_m(\pi - \theta) \\
 &= X_{k\ell m}(\pi/2 - \chi) \Phi_\ell(-\varphi) \Theta_m(\pi - \theta) \\
 &= [\pm X_{km\ell}(\chi)] [\pm \Theta_m(\theta)] [\pm \Phi_\ell(\varphi)] \\
 &= \pm \Psi_{km\ell}(\chi, \theta, \varphi).
 \end{aligned}$$

The first plus-or-minus sign in the penultimate line of (43) will be plus (resp. minus) when $d = \frac{k - (|\ell| + |m|)}{2}$ is even (resp. odd). [Proof: Substituting $\chi \rightarrow \frac{\pi}{2} - \chi$ into (16) and interchanging the roles of sine and cosine introduces a factor of $(-1)^d$.] The third plus-or-minus sign in (43) will be plus when ℓ is nonnegative, minus otherwise. The second plus-or-minus sign will be plus when m is either nonnegative and even or negative and odd, minus otherwise. Thus we may rewrite (43) as

$$\begin{aligned}
 (44) \quad \Psi_{k\ell m}(\chi, \theta, \varphi) &\rightarrow [\pm X_{km\ell}(\chi)] [\pm \Theta_m(\theta)] [\pm \Phi_\ell(\varphi)] \\
 &= \sigma_{k\ell m} \Psi_{km\ell}(\chi, \theta, \varphi)
 \end{aligned}$$

with

$$\sigma_{k\ell m} = (\pm)(\pm)(\pm)(\pm)$$

where

- the first \pm is + if and only if d is even
- the second \pm is + if and only if $\ell \geq 0$
- the third \pm is + if and only if $m \geq 0$
- the fourth \pm is + if and only if m is even.

When $\ell \neq m$ the action (44) preserves the 2-dimensional plane spanned by $\Psi_{k\ell m}$ and $\Psi_{km\ell}$. If the parities of ℓ and m agree (both even or both odd) then the action is either a *positive reflection* interchanging $\Psi_{k\ell m} \leftrightarrow \Psi_{km\ell}$ (Figure 3a) or a *negative reflection* interchanging $\Psi_{k\ell m} \leftrightarrow -\Psi_{km\ell}$ (Figure 3b), according to the sign of $\sigma_{k\ell m}$. If the parities of ℓ and m disagree (one even and the other odd) then the action is a *quarter turn* taking either $\Psi_{k\ell m} \rightarrow \Psi_{km\ell} \rightarrow -\Psi_{k\ell m} \rightarrow$

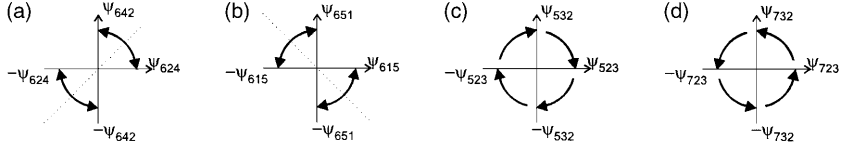


FIGURE 3. The Z_4 generator acts on each 2-dimensional subspace $\{\Psi_{k\ell m}, \Psi_{km\ell}\}$ as a reflection (if ℓ and m have the same parity) or a rotation (if ℓ and m have opposite parities).

$-\Psi_{km\ell} \rightarrow \Psi_{k\ell m}$ or the opposite (Figure 3cd). The positive and negative reflections each fix a 1-dimensional line, but the quarter turns fix only the origin.

When $\ell = m$ the situation is even simpler. The action (44) either fixes the 1-dimensional line spanned by $\Psi_{k\ell\ell}$ or inverts it, according to whether $\sigma_{k\ell\ell}$ is positive or negative.

The preceding two paragraphs have found the eigenmodes fixed by (44), i.e. the eigenmodes invariant under the action of the Z_4 generator (42). To find the eigenmodes of a prism manifold, we must check which of the modes invariant under the Z_4 generator (42) are invariant under the Z_{2n} generator (41) as well. This is straightforward, because Theorem 2, with $p = 2n$ and $q = 1$, already tells us which modes the Z_{2n} generator preserves. We know *a priori* that the eight basis vectors $\{\Psi_{k,\pm\ell,\pm m}, \Psi_{k,\pm m,\pm\ell}\}$ span a space that is setwise invariant under both the Z_{2n} and the Z_4 generators. Typically this space is 8-dimensional, but its dimension may be less. Consider the following special cases. Assume ℓ and m are distinct and positive unless otherwise indicated.

$\{\Psi_{k00}\}$ (1-dimensional)

The mode Ψ_{k00} exists if and only if k is even. When it exists, it is always fixed by the Z_{2n} generator, as indicated in the first line of the conditions (40) in Theorem 2. It is fixed by the Z_4 generator if and only if σ_{k00} is positive, which happens if and only if $d = k/2$ is even.

$\{\Psi_{k,\pm\ell,0}, \Psi_{k,0,\pm\ell}\}$ (4-dimensional)

According to the second and third lines of conditions (40), the Z_{2n} generator fixes this whole space pointwise when $\ell \equiv 0 \pmod{2n}$, and otherwise fixes nothing.

The Z_4 generator leaves the two 2-dimensional subspaces spanned by $\{\Psi_{k,\ell,0}, \Psi_{k,0,\ell}\}$ and by $\{\Psi_{k,-\ell,0}, \Psi_{k,0,-\ell}\}$ setwise invariant. If ℓ is odd, then the Z_4 generator acts on each 2-dimensional subspace as a quarter turn, fixing nothing but the origin. If ℓ is even (as it must be when $\ell \equiv 0 \pmod{2n}$), then the Z_4 generator acts on each subspace as a positive or negative reflection, fixing $\Psi_{k,\ell,0} + \Psi_{k,0,\ell}$ and $\Psi_{k,-\ell,0} - \Psi_{k,0,-\ell}$ if $\sigma_{k\ell 0}$ is positive (when d is even), or $\Psi_{k,\ell,0} - \Psi_{k,0,\ell}$ and $\Psi_{k,-\ell,0} + \Psi_{k,0,-\ell}$ if $\sigma_{k\ell 0}$ is negative (when d is odd).

$\{\Psi_{k,\pm\ell,\pm\ell}\}$ (4-dimensional)

This 4-dimensional subspace factors into two orthogonal 2-dimensional subspaces, spanned by $\{\Psi_{k,\ell,\ell} + \Psi_{k,-\ell,-\ell}, \Psi_{k,-\ell,\ell} - \Psi_{k,\ell,-\ell}\}$ and $\{\Psi_{k,\ell,\ell} -$

$\Psi_{k,-\ell,-\ell}, \Psi_{k,-\ell,\ell} + \Psi_{k,\ell,-\ell}$ respectively, each of which is setwise invariant under both the Z_{2n} and the Z_4 generators.

The subspace $\{\Psi_{k,\ell,\ell} + \Psi_{k,-\ell,-\ell}, \Psi_{k,-\ell,\ell} - \Psi_{k,\ell,-\ell}\}$ is always fixed by the Z_{2n} generator, according to the fourth line of conditions (40). It is fixed by the Z_4 generator as well if and only if $\sigma_{k\ell\ell}$ is positive, which happens if and only if $k \equiv 0 \pmod{4}$. (k must be even for the mode $\Psi_{k\ell\ell}$ to exist, so the only question is whether it is equivalent to 0 or 2 modulo 4.)

The subspace $\{\Psi_{k,\ell,\ell} - \Psi_{k,-\ell,-\ell}, \Psi_{k,-\ell,\ell} + \Psi_{k,\ell,-\ell}\}$ is fixed by the Z_{2n} generator if and only if $2\ell \equiv 0 \pmod{2n}$, according to the last line of (40). If $\sigma_{k\ell\ell}$ is positive the Z_4 generator fixes $\Psi_{k,\ell,\ell} - \Psi_{k,-\ell,-\ell}$ but not $\Psi_{k,-\ell,\ell} + \Psi_{k,\ell,-\ell}$, while if $\sigma_{k\ell\ell}$ is negative the opposite is true.

$\{\Psi_{k,\pm\ell,\pm m}, \Psi_{k,\pm m,\pm\ell}\}$ (8-dimensional)

This 8-dimensional subspace factors into four orthogonal 2-dimensional subspaces, spanned respectively by the bases

$$\begin{aligned} & \{\Psi_{k,+\ell, +m}, \Psi_{k, +m, +\ell}\} \\ & \{\Psi_{k,-\ell, -m}, \Psi_{k, -m, -\ell}\} \\ & \{\Psi_{k,+\ell, -m}, \Psi_{k, -m, +\ell}\} \\ & \{\Psi_{k,-\ell, +m}, \Psi_{k, +m, -\ell}\}. \end{aligned}$$

If the parities of ℓ and m do not match (one even and the other odd) then the Z_4 generator acts as a quarter turn on each of the four subspaces and fixes only the origin. If the parities of ℓ and m do match (both even or both odd), then the Z_4 generator acts as a reflection on each of those same four subspaces, fixing the following vectors, with the choice of signs (consistent as shown) depending on whether $\sigma_{k\ell m}$ is positive or negative:

$$\begin{aligned} & \Psi_{k,+\ell, +m} \pm \Psi_{k, +m, +\ell} \\ & \Psi_{k,-\ell, -m} \pm \Psi_{k, -m, -\ell} \\ & \Psi_{k,+\ell, -m} \mp \Psi_{k, -m, +\ell} \\ & \Psi_{k,-\ell, +m} \mp \Psi_{k, +m, -\ell}. \end{aligned}$$

These four vectors comprise a basis for the Z_4 generator's 4-dimensional fixed point space. Taking sums and differences of those vectors gives a new, more convenient basis for the same space:

$$(45) \quad \begin{aligned} & (\Psi_{k,+\ell, +m} + \Psi_{k,-\ell, -m}) \pm (\Psi_{k, +m, +\ell} + \Psi_{k, -m, -\ell}) \\ & (\Psi_{k,+\ell, +m} - \Psi_{k,-\ell, -m}) \pm (\Psi_{k, +m, +\ell} - \Psi_{k, -m, -\ell}) \\ & (\Psi_{k,+\ell, -m} + \Psi_{k,-\ell, +m}) \mp (\Psi_{k, -m, +\ell} + \Psi_{k, +m, -\ell}) \\ & (\Psi_{k,+\ell, -m} - \Psi_{k,-\ell, +m}) \mp (\Psi_{k, -m, +\ell} - \Psi_{k, +m, -\ell}). \end{aligned}$$

The fourth (resp. fifth) line of the conditions (40) shows that the Z_{2n} generator fixes the first and fourth (resp. second and third) vectors in (45) if and only if $\ell \equiv m \pmod{2n}$ (resp. $\ell \equiv -m \pmod{2n}$).

The above cases completely determine the D_n^* -invariant eigenmodes of S^3 , thus proving

THEOREM 3. *The eigenspace of the Laplacian on the prism space S^3/D_n^* , where D_n^* is the binary dihedral group of order $4n$, has an orthonormal basis that, when lifted to D_n^* -invariant eigenmodes of the 3-sphere, comprises those eigenmodes in the left column for which the corresponding condition in the right column is satisfied, subject to the restriction that an eigenmode $\Psi_{k\ell m}$ exists if and only if the integers k , ℓ and m satisfy $|\ell| + |m| \leq k$ and $\ell + m \equiv k \pmod{2}$.*

	basis vectors	condition
	$\Psi_{k,0,0}$	$k \equiv 0 \pmod{4}$
	$\sqrt{\frac{1}{2}}(\Psi_{k,\ell,0} + \Psi_{k,0,\ell})$ $\sqrt{\frac{1}{2}}(\Psi_{k,-\ell,0} - \Psi_{k,0,-\ell})$	$\ell \equiv 0 \pmod{2n}$ and d even
	$\sqrt{\frac{1}{2}}(\Psi_{k,\ell,0} - \Psi_{k,0,\ell})$ $\sqrt{\frac{1}{2}}(\Psi_{k,-\ell,0} + \Psi_{k,0,-\ell})$	$\ell \equiv 0 \pmod{2n}$ and d odd
	$\sqrt{\frac{1}{2}}(\Psi_{k,\ell,\ell} + \Psi_{k,-\ell,-\ell})$ $\sqrt{\frac{1}{2}}(\Psi_{k,-\ell,\ell} - \Psi_{k,\ell,-\ell})$	$k \equiv 0 \pmod{4}$
(46)	$\sqrt{\frac{1}{2}}(\Psi_{k,\ell,\ell} - \Psi_{k,-\ell,-\ell})$	$2\ell \equiv 0 \pmod{2n}$ and $k \equiv 0 \pmod{4}$
	$\sqrt{\frac{1}{2}}(\Psi_{k,-\ell,\ell} + \Psi_{k,\ell,-\ell})$	$2\ell \equiv 0 \pmod{2n}$ and $k \equiv 2 \pmod{4}$
	$\frac{1}{2}((\Psi_{k,+\ell,+m} + \Psi_{k,-\ell,-m}) + (\Psi_{k,+m,+\ell} + \Psi_{k,-m,-\ell}))$ $\frac{1}{2}((\Psi_{k,+\ell,-m} - \Psi_{k,-\ell,+m}) - (\Psi_{k,-m,+\ell} - \Psi_{k,+m,-\ell}))$	$\ell \equiv m \pmod{2n}$ and $\sigma_{k\ell m} > 0$
	$\frac{1}{2}((\Psi_{k,+\ell,+m} + \Psi_{k,-\ell,-m}) - (\Psi_{k,+m,+\ell} + \Psi_{k,-m,-\ell}))$ $\frac{1}{2}((\Psi_{k,+\ell,-m} - \Psi_{k,-\ell,+m}) + (\Psi_{k,-m,+\ell} - \Psi_{k,+m,-\ell}))$	$\ell \equiv m \pmod{2n}$ and $\sigma_{k\ell m} < 0$
	$\frac{1}{2}((\Psi_{k,+\ell,+m} - \Psi_{k,-\ell,-m}) + (\Psi_{k,+m,+\ell} - \Psi_{k,-m,-\ell}))$ $\frac{1}{2}((\Psi_{k,+\ell,-m} + \Psi_{k,-\ell,+m}) - (\Psi_{k,-m,+\ell} + \Psi_{k,+m,-\ell}))$	$\ell \equiv -m \pmod{2n}$ and $\sigma_{k\ell m} > 0$
	$\frac{1}{2}((\Psi_{k,+\ell,+m} - \Psi_{k,-\ell,-m}) - (\Psi_{k,+m,+\ell} - \Psi_{k,-m,-\ell}))$ $\frac{1}{2}((\Psi_{k,+\ell,-m} + \Psi_{k,-\ell,+m}) + (\Psi_{k,-m,+\ell} + \Psi_{k,+m,-\ell}))$	$\ell \equiv -m \pmod{2n}$ and $\sigma_{k\ell m} < 0$

Note that when k is odd, none of the above conditions are satisfied, and the eigenbasis is empty. This is not surprising, because when k is odd the eigenmodes correspond to odd-degree homogeneous polynomials (Section 5), which are anti-symmetric under the action of the antipodal map, and all groups D_n^* contain the antipodal map.

When k is even, the total number of eigenmodes for given D_n^* and k agrees with the multiplicities given by Ikeda's formulas $(2\bar{k} + 1)(\lceil \bar{k}/n \rceil + 1)$ (for \bar{k} even) and $(2\bar{k} + 1)\lceil \bar{k}/n \rceil$ (for \bar{k} odd) from Theorem 4.3 of [14], where $\bar{k} = k/2$ is half the wave number and $\lceil \bar{k}/n \rceil$ denotes the integer part of \bar{k}/n .

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