J. Math. Soc. Japan
Vol. 73, No. 1 (2021) pp. 185–209
doi: 10.2969/jmsj/83278327

$\mathbb{Z}_k$-code vertex operator algebras

By Tomoyuki Arakawa, Hiromichi Yamada and Hiroshi Yamauchi

(Received Sep. 8, 2019)

Abstract. We introduce a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type associated with a $\mathbb{Z}_k$-code for $k \geq 2$. Our argument is based on the $\mathbb{Z}_k$-symmetry among the simple current modules for the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$. We show that it is naturally realized as the commutant of a certain subalgebra in a lattice vertex operator algebra. Furthermore, we construct all the irreducible modules inside a module for the lattice vertex operator algebra.

1. Introduction.

The parafermion vertex operator algebra $K(\mathfrak{g}, k)$ associated with a finite dimensional simple Lie algebra $\mathfrak{g}$ and a positive integer k is by definition the commutant of the Heisenberg vertex operator algebra generated by the Cartan subalgebra of $\mathfrak{g}$ in $L_{\widehat{\mathfrak{g}}}(k, 0)$, where $L_{\widehat{\mathfrak{g}}}(k, 0)$ is the simple affine vertex operator algebra associated with the affine Kac–Moody Lie algebra $\widehat{\mathfrak{g}}$ at level k. In the case where $\mathfrak{g} = \mathfrak{sl}_2$ and $k \geq 2$, $K(\mathfrak{sl}_2, k)$ is isomorphic to a minimal series principal W-algebra of type A which is a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type [**2**], and has exactly k simple currents M^j, $j \in \mathbb{Z}_k$, with $\mathbb{Z}_k$-symmetry. That is, those simple currents form a cyclic group of order k with respect to the fusion product, $M^i \boxtimes_{M^0} M^j = M^{i+j}$ for $i, j \in \mathbb{Z}_k$ with $M^0 = K(\mathfrak{sl}_2, k)$.

In this article we introduce a vertex operator algebra M_D associated with a $\mathbb{Z}_k$-code D of lenght ℓ. Here, a $\mathbb{Z}_k$-code D is an additive subgroup of $(\mathbb{Z}_k)^\ell$. For each codeword $\xi = (\xi_1, \ldots, \xi_\ell) \in D$, we associate the tensor product $M_\xi = M^{\xi_1} \otimes \cdots \otimes M^{\xi_\ell}$ of simple current $K(\mathfrak{sl}_2, k)$-modules M^{ξ_r}, $1 \leq r \leq \ell$. Then the direct sum

$$M_D = \bigoplus_{\xi \in D} M_\xi$$

has a structure of an abelian intertwining algebra [**14**, Theorem 4.1]. Furthermore, M_D becomes a vertex operator algebra if each M_ξ has integral conformal weight [**14**, Theorem 4.2]. Being a D-graded simple current exrension of $M_{\mathbf{0}} = K(\mathfrak{sl}_2, k)^{\otimes \ell}$, the vertex operator algebra M_D is simple, self-dual, rational, C_2-cofinite, and of CFT-type with central charge $2\ell(k-1)/(k+2)$ (Theorem 7.3). Such a construction of M_D was initiated in [**35**] for the case $k = 2$, and the properties of the vertex operator algebra M_D for $k = 2$ have been studied extensively, see [**6**], [**31**], [**36**], [**37**] and the references

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2010 *Mathematics Subject Classification.* Primary 17B69; Secondary 17B67.

Key Words and Phrases. vertex operator algebra, parafermion algebra, simple current, code.
The first author was partially supported by JSPS KAKENHI grant No.17H01086 and No.17K18724. The third author was partially supported by JSPS KAKENHI grant No.19K03409.

therein. The vertex operator algebra M_D for $k = 3$ was constructed by a slightly different method in [**23**], and its irreducible modules were studied in [**25**].

We realize the vertex operator algebra M_D inside a vertex operator algebra V_{Γ_D} associated with a certain positive definite even lattice Γ_D. Moreover, every irreducible M_D-module is explicitly described inside a module for the lattice vertex operator algebra V_{Γ_D}.

More precisely, consider the lattice vertex operator algebra $V_{\sqrt{2}A_{k-1}}$, which is an extension of the vertex operator algebra $K(\mathfrak{sl}_2, k) \otimes K(\mathfrak{sl}_k, 2)$. There are cosets $N^{(j)}$, $j \in \mathbb{Z}_k$, of $\sqrt{2}A_{k-1}$ in the dual lattice $(\sqrt{2}A_{k-1})^\circ$ such that $N^{(i)} + N^{(j)} = N^{(i+j)}$, and $V_{N^{(j)}}$ contains M^j. For $\xi = (\xi_1, \dots, \xi_\ell) \in D$, we consider a coset $N(\xi) = N^{(\xi_1)} \times \cdots \times N^{(\xi_\ell)}$ of $(\sqrt{2}A_{k-1})^\ell$ in $((\sqrt{2}A_{k-1})^\circ)^\ell$. The union Γ_D of those cosets is a positive definite even lattice if and only if $(\xi|\xi) = 0$ for all $\xi \in D$ (Lemma 7.1), where $(\,\cdot\,|\,\cdot\,)$ is the standard inner product on $(\mathbb{Z}_k)^\ell$. Then M_D is realized as the commutant of $K(\mathfrak{sl}_k, 2)^{\otimes \ell}$ in the lattice vertex operator algebra V_{Γ_D} (Equation (7.4)).

We also consider a necessary and sufficient condition on the code D for which Γ_D is a positive definite odd lattice, and M_D is a vertex operator superalgebra.

Using the representation theory of simple current extensions (Section 2.2), we construct all the irreducible M_D-modules inside $V_{(\Gamma_D)^\circ}$, where $(\Gamma_D)^\circ$ is the dual lattice of Γ_D (Theorems 8.7, 8.9, and 8.10). Any linear character χ of the finite abelian group D naturally induces an automorphism of the vertex operator algebra M_D. We discuss irreducible χ-twisted M_D-modules as well. In particular, we obtain the number of inequivalent irreducible χ-twisted M_D-modules (Theorem 8.12). We also study the irreducible M_D-modules in the case where M_D is a vertex operator superalgebra (Theorem 9.1).

The construction of M_D as a commutant of $K(\mathfrak{sl}_k, 2)^{\otimes \ell}$ in the lattice vertex operator algebra V_{Γ_D} was previously discussed in [**3**]. However, the treatment of the simple current $K(\mathfrak{sl}_2, k)$-modules M^j in $V_{N^{(j)}}$, $j \in \mathbb{Z}_k$, was slightly different, and the method there is not suitable for all the irreducible $K(\mathfrak{sl}_2, k)$-modules in $V_{(\sqrt{2}A_{k-1})^\circ}$. In the present paper, we use decompositions of certain irreducible $V_{\sqrt{2}A_{k-1}}$-modules (Proposition 6.3), from which we know how the irreducible $K(\mathfrak{sl}_2, k)$-modules appear in $V_{(\sqrt{2}A_{k-1})^\circ}$ (Proposition 6.4), and it enables us to describe the irreducible M_D-modules inside $V_{(\Gamma_D)^\circ}$.

This paper is organized as follows. Section 2 is devoted to preliminaries, where we recall the representation theory of simple current extensions. In Section 3, we review the properties of the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ for later use. In Sections 4, 5, and 6, we describe the cosets of $N = \sqrt{2}A_{k-1}$ in $N^\circ = (\sqrt{2}A_{k-1})^\circ$, and study how irreducible $K(\mathfrak{sl}_2, k)$-modules appear in the irreducible V_N-modules. The vertex operator algebra M_D is defined in Section 7. In Section 8, we study the irreducible twisted and untwisted modules for M_D, including the classification of irreducible modules, and realizations of the irreducible modules in $V_{(N^\circ)^\ell}$. In Section 9, we discuss the irreducible M_D-modules in the case where M_D is a vertex operator superalgebra. Finally, in Section 10, we mention some known examples of M_D. We calculate the minimal norm of elements in each coset of N in N° in Appendix A.

As to the $P(z)$-tensor product $\boxtimes_{P(z)}$ of [**19**] for a vertex operator algebra V, we only use it with $z = 1$. We write $\boxtimes_V$ for $\boxtimes_{P(1)}$, and call it the fusion product. We also use $\otimes$ to denote the tensor product of vertex operator algebras and their modules as in [**15**].

ACKNOWLEDGEMENTS. The authors would like to thank Ching Hung Lam and Hiroki Shimakura for stimulating discussions and helpful advice.

2. Preliminaries.

In this section, we recall some basic properties of simple current extensions of vertex operator algebras and their irreducible modules. Our notations for vertex operator algebras and their modules are standard [**15**], [**16**], [**32**].

2.1. Simple current modules.

Let V be a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type. Then a fusion product $M \boxtimes_V N$ over V of any V-modules M and N exists [**20**], [**34**]. The fusion product is commutative and associative [**18**, Theorem 3.7].

We denote by $\mathrm{Irr}(V)$ the set of equivalence classes of irreducible V-modules. Then

$$M^1 \boxtimes_V M^2 = \sum_{M^3 \in \mathrm{Irr}(V)} \dim I_V \binom{M^3}{M^1 \ M^2} M^3$$

for $M^1, M^2 \in \mathrm{Irr}(V)$, where $I_V\binom{M^3}{M^1 \ M^2}$ is the set of all intertwining operators of type $\binom{M^3}{M^1 \ M^2}$. An irreducible V-module A is called a simple current if $A \boxtimes_V X$ is an irreducible V-module for any $X \in \mathrm{Irr}(V)$. A set $\{A^\alpha \mid \alpha \in D\}$ of simple current V-modules indexed by a finite abelian group D is said to be D-graded if A^α, $\alpha \in D$, are inequivalent to each other with $A^0 = V$ and $A^\alpha \boxtimes_V A^\beta = A^{\alpha+\beta}$, $\alpha, \beta \in D$. The set $\mathrm{Irr}(V)_{\mathrm{sc}}$ of equivalence classes of simple current V-modules is graded by a finite abelian group [**31**, Corollary 1]. The inverse of $A \in \mathrm{Irr}(V)_{\mathrm{sc}}$ with respect to the fusion product is its contragredient module A'. The fusion product by $A \in \mathrm{Irr}(V)_{\mathrm{sc}}$ induces a permutation

$$X \mapsto A \boxtimes_V X \tag{2.1}$$

on $\mathrm{Irr}(V)$. For a V-module X, we denote its conformal weight by $h(X)$, which is a rational number [**10**, Theorem 11.3]. We define a map $b_V : \mathrm{Irr}(V)_{\mathrm{sc}} \times \mathrm{Irr}(V) \to \mathbb{Q}/\mathbb{Z}$ by

$$b_V(A, X) = h(A \boxtimes_V X) - h(A) - h(X) + \mathbb{Z} \tag{2.2}$$

for $A \in \mathrm{Irr}(V)_{\mathrm{sc}}$ and $X \in \mathrm{Irr}(V)$. The map b_V was introduced in [**14**, Section 3] in the case where $\mathrm{Irr}(V)_{\mathrm{sc}} = \mathrm{Irr}(V)$, see also [**38**, Section 2]. A proof of the following lemma can be found in [**42**, Section 2].

LEMMA 2.1. *Let A, $B \in \mathrm{Irr}(V)_{\mathrm{sc}}$, and $X \in \mathrm{Irr}(V)$.*
(1) $b_V(A \boxtimes_V B, X) = b_V(A, X) + b_V(B, X)$.
(2) $b_V(A, B \boxtimes_V X) = b_V(A, B) + b_V(A, X)$.

2.2. Representations of simple current extensions.

Let V be a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type. Let $\{V^\alpha \mid \alpha \in D\}$ be a D-graded set of simple current V-modules for a finite abelian group D with $V^0 = V$ and $h(V^\alpha) \in (1/2)\mathbb{Z}$ for all $\alpha \in D$. Then the direct sum $V_D = \bigoplus_{\alpha \in D} V^\alpha$ has a structure of either a simple vertex operator algebra or a simple vertex operator superalgebra which extends the V-module structure on V_D [**5**, Theorem 3.12], see also the references therein. Such a simple vertex operator (super)algebra structure on V_D is unique [**12**, Proposition 5.3]. The vertex operator (super)algebra V_D is called a D-graded simple current extension of V. In this section, we only consider the case in which $h(V^\alpha) \in \mathbb{Z}$ for all $\alpha \in D$, and V_D is a vertex operator algebra. It is known that V_D is simple, self-dual, rational, C_2-cofinite, and of CFT-type [**43**, Theorem 2.14].

We recall the representation theory of V_D from [**24**], [**43**]. As to the notion of a g-twisted module for a vertex operator algebra with respect to its automorphism g, we adopt the definition in [**10**]. Thus a g-twisted module in [**43**] means a g^{-1}-twisted module in this paper.

Let $D^* = \mathrm{Hom}(D, \mathbb{C}^\times)$ be the character group of D. For $\chi \in D^*$, a scalar multiplication by $\chi(\alpha)$ on V^α, $\alpha \in D$, is an automorphism of the vertex operator algebra V_D. That is, D^* naturally acts on V_D, and we can regard D^* as a subgroup of $\mathrm{Aut}\, V_D$. Let M be a χ-twisted V_D-module for $\chi \in D^*$. We say M is D-graded if there is a decomposition $M = \bigoplus_{\alpha\in D} M^\alpha$ as a V-module such that $0 \neq V^\alpha \cdot M^\beta \subset M^{\alpha+\beta}$ for α, $\beta \in D$, where we set $V^\alpha \cdot S = \mathrm{span}\{a_{(n)}v \mid a \in V^\alpha, v \in S, n \in \mathbb{Q}\}$ for a subset S of M.

We consider the action of D on $\mathrm{Irr}(V)$ in (2.1). Let $\mathrm{Irr}(V) = \bigcup_{i\in I} \mathscr{O}_i$ be the D-orbit decomposition. Using the map b_V in (2.2), we define a map $\chi_X : D \to \mathbb{C}^\times$ by

$$\chi_X(\alpha) = \exp(2\pi\sqrt{-1}\, b_V(V^\alpha, X))$$

for $X \in \mathrm{Irr}(V)$. The map χ_X is a linear character of D by (1) of Lemma 2.1. For a D-orbit $\mathscr{O}_i$, (2) of Lemma 2.1 implies that χ_X is independent of the choice of $X \in \mathscr{O}_i$, as $h(V^\alpha) \in \mathbb{Z}$ for all $\alpha \in D$. Thus χ_X is uniquely determined by $\mathscr{O}_i$, so we can write χ_i for χ_X.

We summarize [**24**, Theorem 4.4] and [**43**, Lemma 2.11, Theorems 2.14, 2.19, 3.2, 3.3] as follows.

Theorem 2.2. *Let V_D be a D-graded simple current extension of V, and let $X \in \mathrm{Irr}(V)$.*

(1) *There exists a unique structure of a D-graded χ_X-twisted V_D-module on the space $V_D \boxtimes_V X = \bigoplus_{\alpha\in D} V^\alpha \boxtimes_V X$ which contains $V^0 \boxtimes_V X \cong X$ as a V-submodule.*

(2) *If $M = \bigoplus_{\alpha\in D} M^\alpha$ is a D-graded χ_X-twisted V_D-module such that $X \subset M^\alpha$ as a V-submodule for some $\alpha \in D$, then $V_D \cdot X$ is isomorphic to the D-graded χ_X-twisted V_D-module $V_D \boxtimes_V X$ in the assertion* (1), *where $V_D \cdot X = \mathrm{span}\{a_{(n)}v \mid a \in V_D, v \in X, n \in \mathbb{Q}\} \subset M$.*

(3) *Let $\sigma \in \mathrm{Aut}\, V_D$ such that σ is the identity on V. Assume that there is a σ-twisted V_D-module containing X as a V-submodule. Then $\sigma = \chi_X$, and there exists a surjective V_D-homomorphism from $V_D \boxtimes_V X$ onto $V_D \cdot X$.*

For a D-orbit $\mathscr{O}_i$ in $\mathrm{Irr}(V)$, the structure of a D-graded χ_X-twisted V_D-module on the space $V_D \boxtimes_V X$ in (1) of the above theorem is independent of the choice of $X \in \mathscr{O}_i$, and it is uniquely determined by $\mathscr{O}_i$. The χ_X-twisted V_D-module $V_D \boxtimes_V X$ is not necessarily irreducible. The assertion (3) of the above theorem implies that $V_D \cdot X$ is isomorphic to a direct summand of $V_D \boxtimes_V X$.

Since any irreducible χ-twisted V_D-module for $\chi \in D^*$ is isomorphic to a direct summand of the χ_X-twisted V_D-module $V_D \boxtimes_V X$ with $\chi = \chi_X$ for some $X \in \mathrm{Irr}(V)$ by Theorem 2.2, the study of χ-twisted V_D-modules is reduced to the study of the χ_X-twisted V_D-module $V_D \boxtimes_V X$.

Let $D_X = \{\alpha \in D \mid V^\alpha \boxtimes_V X \cong X\}$ be the stabilizer of $X \in \mathrm{Irr}(V)$ for the action of D on $\mathrm{Irr}(V)$ in (2.1). For a D-orbit $\mathscr{O}_i$, the stabilizer D_X is independent of the choice of $X \in \mathscr{O}_i$, and it is uniquely determined by $\mathscr{O}_i$. Hence we can write D_i for D_X.

In the case where $D_X = 0$, the following assertion holds [**39**, Proposition 3.8].

Proposition 2.3. *If $D_X = 0$, then $V_D \boxtimes_V X$ is an irreducible χ_X-twisted V_D-module.*

If D_X is non-trivial, then the χ_X-twisted V_D-module $V_D \boxtimes_V X$ is reducible, and we need to take some 2-cocycles of D_X into account to obtain its irreducible decomposition as discussed in [**24**], [**43**]. Let $X \in \mathrm{Irr}(V)$, and assume that $D_X \neq 0$. We consider the D_X-graded simple current extension $V_{D_X} = \bigoplus_{\alpha \in D_X} V^\alpha$ of V. Set $V_{\beta+D_X} = \bigoplus_{\alpha\in\beta+D_X} V^\alpha$ for a coset $\beta + D_X \in D/D_X$. Then $V_D = \bigoplus_{\beta+D_X\in D/D_X} V_{\beta+D_X}$ is a D/D_X-graded simple current extension of V_{D_X}. Note that $V_{D_X} \boxtimes_V X \cong X^{\oplus |D_X|}$ as V-modules. Set $Q = \mathrm{Hom}_V(X, V_{D_X} \boxtimes_V X)$. Then $\dim Q = |D_X|$, and we have a canonical isomorphism

$$V_{D_X} \boxtimes_V X \cong X \otimes Q. \tag{2.3}$$

It is shown in [**24**, Theorem 3.10] and [**43**, Theorems 2.14, 2.19] that there exists a 2-cocycle $\epsilon \in Z^2(D_X, \mathbb{C}^\times)$ such that the space Q carries a structure of a module for a twisted group algebra $\mathbb{C}^\epsilon[D_X]$ associated with ϵ [**22**, Chapter 2]. Indeed, Q is isomorphic to the regular representation of $\mathbb{C}^\epsilon[D_X]$. If R is a $\mathbb{C}^\epsilon[D_X]$-submodule of Q, then the subspace $X \otimes R$ of $X \otimes Q$ in (2.3) is a V_{D_X}-submodule of $V_{D_X} \boxtimes_V X$. Thus the irreducible decomposition of $V_{D_X} \boxtimes_V X$ as a V_{D_X}-module is obtained by the irreducible decomposition of Q as a $\mathbb{C}^\epsilon[D_X]$-module.

Let T be an irreducible V_{D_X}-submodule of $V_{D_X} \boxtimes_V X$. Then T is also a direct sum of some copies of X as a V-module, and $V_{\beta+D_X} \boxtimes_V T$, $\beta + D_X \in D/D_X$, are inequivalent irreducible V_{D_X}-modules. Hence the χ_X-twisted V_D-module $V_D \boxtimes_{V_{D_X}} T$ is irreducible by Proposition 2.3. The χ_X-twisted V_D-module structure of $V_D \boxtimes_{V_{D_X}} T$ is uniquely determined by T. Therefore, the irreducible decomposition of $V_D \boxtimes_V X$ as a χ_X-twisted V_D-module is in one-to-one correspondence with the irreducible decomposition of Q in (2.3) as a $\mathbb{C}^\epsilon[D_X]$-module.

The determination of the 2-cocycle ϵ requires more information on the associativity constraints of the fusion products of V-modules [**24**], [**43**]. However, we will only deal with the case where D_X can be regarded as a binary code in this paper. So we make the following assumption on D_X.

Hypothesis 2.4. (1) M^0 is a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type.

(2) M^1 is a self-dual simple current M^0-module such that the $\mathbb{Z}_2$-graded simple current extension $M^0 \oplus M^1$ of M^0 is either a simple vertex operator algebra with $h(M^1) \in \mathbb{Z}$ or a simple vertex operator superalgebra with $h(M^1) \in \mathbb{Z} + 1/2$.

(3) For any irreducible M^0-module P, the direct sum $P^0 \oplus P^1$ with $P^0 = P$ and $P^1 = M^1 \boxtimes_{M^0} P$ has a unique structure of a $\mathbb{Z}_2$-graded either untwisted or $\mathbb{Z}_2$-twisted $M^0 \oplus M^1$-module.

(4) $V = (M^0)^{\otimes n}$ for some $n > 0$.

(5) $X \in \mathrm{Irr}(V)$ with $D_X \neq 0$. Moreover, D_X has a structure of a binary code of length n, and $V^\alpha \cong M^{\alpha_1} \otimes \cdots \otimes M^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in D_X$. In particular,

$$V_{D_X} = \bigoplus_{\alpha=(\alpha_1,\ldots,\alpha_n)\in D_X} M^{\alpha_1} \otimes \cdots \otimes M^{\alpha_n} \subset (M^0 \oplus M^1)^{\otimes n}$$

as an extension of $V = (M^0)^{\otimes n}$.

Suppose V_{D_X} satisfies Hypothesis 2.4. Under this assumption, we can describe the 2-cocycle $\epsilon \in Z^2(D_X, \mathbb{C}^\times)$ explicitly. We divide our argument into two cases.

Case 1. Suppose $M^0 \oplus M^1$ is a simple vertex operator algebra with $h(M^1) \in \mathbb{Z}$. By (3) of Hypothesis 2.4, the 2-cocycle $\epsilon \in Z^2(D_X, \mathbb{C}^\times)$ is cohomologous to a 2-coboundary by [**22**, Chapter 2, Corollary 2.5]. Hence Q is the regular representation of an ordinary group algebra $\mathbb{C}[D_X]$, so that Q is a direct sum of $|D_X|$ inequivalent irreducible $\mathbb{C}[D_X]$-modules. Therefore, $V_{D_X} \boxtimes_V X$ decomposes into a direct sum of $|D_X|$ inequivalent irreducible V_{D_X}-submodules. By considering V_D as a D/D_X-graded simple current extension of V_{D_X}, we see that the irreducible decomposition of $V_D \boxtimes_V X$ as a χ_X-twisted V_D-module is as follows.

Proposition 2.5. *Suppose $D_X \neq 0$ and V_{D_X} satisfies Hypothesis* 2.4. *Suppose further that $M^0 \oplus M^1$ in* (2) *of Hypothesis* 2.4 *is a simple vertex operator algebra with $h(M^1) \in \mathbb{Z}$. Then the irreducible decomposition of the χ_X-twisted V_D-module $V_D \boxtimes_V X$ is given as*

$$V_D \boxtimes_V X = \bigoplus_{j=1}^{|D_X|} U^j,$$

where U^j, $1 \leq j \leq |D_X|$, are inequivalent irreducible χ_X-twisted V_D-modules. Furthermore, $U^j \cong \bigoplus_{W \in \mathscr{O}_i} W$ as V-modules, where $\mathscr{O}_i$ is the D-orbit in $\mathrm{Irr}(V)$ *containing X.*

Case 2. Suppose $M^0 \oplus M^1$ is a simple vertex operator superalgebra with $h(M^1) \in \mathbb{Z} + 1/2$. In this case, D_X is an even binary code, as the conformal weight of $V^\alpha \cong M^{\alpha_1} \otimes \cdots \otimes M^{\alpha_n}$ is an integer for $\alpha = (\alpha_1, \ldots, \alpha_n) \in D_X$. By (3) of Hypothesis 2.4, we can find the 2-cocycle ϵ inside $Z^2(D_X, \{\pm 1\})$ which satisfies

$$\epsilon(\alpha, \alpha) = (-1)^{\mathrm{wt}(\alpha)/2}, \quad \epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha|\beta)} \tag{2.4}$$

for $\alpha, \beta \in D_X$, where $\mathrm{wt}(\alpha)$ is the Hamming weight of α, and $(\cdot|\cdot)$ is the standard inner product on $(\mathbb{Z}_2)^n$ [**31**, Section 4.1], see also [**35**], [**36**]. The conditions above uniquely determine the class of ϵ in $H^2(D_X, \{\pm 1\})$ [**16**, Proposition 5.3.3].

It is shown in [**16**, Theorem 5.5.1] that each irreducible representation of $\mathbb{C}^\epsilon[D_X]$ is induced from an irreducible representation of its maximal commutative subalgebra, and the equivalence classes of irreducible $\mathbb{C}^\epsilon[D_X]$-modules are distinguished by their central characters. Let $D_X^\perp = \{\alpha \in (\mathbb{Z}_2)^n \mid (\alpha|D_X) = 0\}$ be the dual code of the binary code D_X, and let E be a maximal self-orthogonal subcode of D_X. It follows from (2.4) that the center of $\mathbb{C}^\epsilon[D_X]$ is $\mathbb{C}^\epsilon[D_X \cap D_X^\perp]$, and $\mathbb{C}^\epsilon[E]$ is a maximal commutative subalgebra of $\mathbb{C}^\epsilon[D_X]$. Since $\mathbb{C}^\epsilon[D_X \cap D_X^\perp] \cong \mathbb{C}[D_X \cap D_X^\perp]$ is an ordinary group algebra, the number of inequivalent irreducible representations of $\mathbb{C}^\epsilon[D_X]$ is equal to that of $\mathbb{C}[D_X \cap D_X^\perp]$, which coincides with the order $|D_X \cap D_X^\perp|$ of $D_X \cap D_X^\perp$. Each irreducible $\mathbb{C}^\epsilon[D_X]$-module has dimension $[D_X : E] = [E : D_X \cap D_X^\perp]$, namely, $[D_X : D_X \cap D_X^\perp]^{1/2}$ [**16**, Theorem 5.5.1]. Since the space Q in (2.3) is isomorphic to the regular representation of $\mathbb{C}^\epsilon[D_X]$, the irreducible decomposition of $V_D \boxtimes_V X$ as a χ_X-twisted V_D-module is as follows.

Proposition 2.6. *Suppose $D_X \neq 0$ and V_{D_X} satisfies Hypothesis 2.4. Suppose further that $M^0 \oplus M^1$ in (2) of Hypothesis 2.4 is a simple vertex operator superalgebra with $h(M^1) \in \mathbb{Z} + 1/2$. Then the irreducible decomposition of the χ_X-twisted V_D-module $V_D \boxtimes_V X$ is given as*

$$V_D \boxtimes_V X = \bigoplus_{j=1}^{|D_X \cap D_X^\perp|} (U^j)^{\oplus m},$$

where $m = [D_X : D_X \cap D_X^\perp]^{1/2}$, and U^j, $1 \leq j \leq |D_X \cap D_X^\perp|$, are inequivalent irreducible χ_X-twisted V_D-modules. Furthermore, $U^j \cong \bigoplus_{W \in \mathscr{O}_i} W^{\oplus m}$ as V-modules, where $\mathscr{O}_i$ is the D-orbit in $\mathrm{Irr}(V)$ containing X.

3. Parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$.

In this section, we recall the properties of the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ for $2 \leq k \in \mathbb{Z}$. If $k = 2$, then $K(\mathfrak{sl}_2, 2)$ is isomorphic to the Virasoro vertex operator algebra $L(1/2, 0)$ of central charge $1/2$. So we assume that $k \geq 3$ for the rest of this section.

Let $\{h, e, f\}$ be a standard Chevalley basis of the Lie algebra $\mathfrak{sl}_2$. Let $L_{\widehat{\mathfrak{sl}}_2}(k, 0)$ be the simple affine vertex operator algebra associated with $\widehat{\mathfrak{sl}}_2$ and level k. Then $K(\mathfrak{sl}_2, k)$ is defined to be the commutant of the Heisenberg vertex operator algebra generated by $h(-1)\mathbf{1}$ in $L_{\widehat{\mathfrak{sl}}_2}(k, 0)$ [**7**], [**8**], [**9**].

We follow the notaions in [**8**, Section 4]. Let $L = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_k$ with $\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j}$ and $\gamma = \alpha_1 + \cdots + \alpha_k$. Let H, E, and $F \in V_L$ be as in [**8**, Section 4]. Then the component operators $H_{(n)}$, $E_{(n)}$, $F_{(n)}$, $n \in \mathbb{Z}$, give a level k representation of $\widehat{\mathfrak{sl}}_2$ under the correspondence $h(n) \leftrightarrow H_{(n)}$, $e(n) \leftrightarrow E_{(n)}$, $f(n) \leftrightarrow F_{(n)}$, and the subalgebra V^{aff} of the vertex operator algebra $V_L \cong L_{\widehat{\mathfrak{sl}}_2}(1, 0)^{\otimes k}$ generated by H, E, and F is isomorphic to $L_{\widehat{\mathfrak{sl}}_2}(k, 0)$. We identify V^{aff} with $L_{\widehat{\mathfrak{sl}}_2}(k, 0)$. We also identify $H_{(n)}$, $E_{(n)}$, and $F_{(n)}$ with $h(n)$, $e(n)$, and $f(n)$, respectively. Let

$$M^j = \{v \in L_{\widehat{\mathfrak{sl}}_2}(k, 0) \mid H_{(n)}v = -2j\delta_{n,0}v \text{ for } n \geq 0\}.$$

Then $M^0 = K(\mathfrak{sl}_2, k)$, and $L_{\widehat{\mathfrak{sl}}_2}(k, 0) = \bigoplus_{j=0}^{k-1} M^j \otimes V_{\mathbb{Z}\gamma - j\gamma/k}$ as $M^0 \otimes V_{\mathbb{Z}\gamma}$-modules [**8**, Lemma 4.2]. The index j of M^j can be considered to be modulo k.

Let $L^\circ = (1/2)L$ be the dual lattice of L, and let v^i, $0 \le i \le k$, and $v^{i,j}$, $0 \le j \le i$, be as in [**8**, Section 4]. Then the V^{aff}-submodule $V^{\mathrm{aff}} \cdot v^i$ of V_{L° generated by v^i is isomorphic to an irreducible $L_{\widehat{\mathfrak{sl}}_2}(k,0)$-module $L_{\widehat{\mathfrak{sl}}_2}(k,i)$ with top level $\mathrm{span}\{v^{i,j} \mid 0 \le j \le i\}$ of conformal weight $i(i+2)/4(k+2)$ [**17**], [**32**, Section 6.2]. Let

$$M^{i,j} = \{v \in V^{\mathrm{aff}} \cdot v^i \mid H_{(n)}v = (i-2j)\delta_{n,0}v \text{ for } n \ge 0\}$$

for $0 \le i \le k$, $0 \le j \le k-1$. Then

$$L_{\widehat{\mathfrak{sl}}_2}(k,i) = \bigoplus_{j=0}^{k-1} M^{i,j} \otimes V_{\mathbb{Z}\gamma + (i-2j)\gamma/2k} \tag{3.1}$$

as $M^0 \otimes V_{\mathbb{Z}\gamma}$-modules [**8**, Lemma 4.3]. The index j of $M^{i,j}$ can be considered to be modulo k. Note that $M^{0,j} = M^j$.

The -1 isometry of the lattice L lifts to an automorphism θ of the vertex operator algebra V_L of order 2. Actually, $\theta(H) = -H$, $\theta(E) = F$, and $\theta(F) = E$.

We summarize the properties of $M^0 = K(\mathfrak{sl}_2, k)$ [**1**], [**2**], [**7**], [**8**], [**13**].

(1) M^0 is a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type with central charge $2(k-1)/(k+2)$.

(2) $\mathrm{ch}\, M^0 = 1 + q^2 + 2q^3 + \cdots$.

(3) M^0 is generated by its conformal vector ω and a primary vector W^3 of weight 3.

(4) The automorphism group $\mathrm{Aut}\, M^0$ of M^0 is generated by θ, and $\theta(W^3) = -W^3$.

(5) The irreducible M^0-modules $M^{i,j}$'s are not always inequivalent. In fact,

$$M^{i,j} \cong M^{k-i,j-i}, \quad 0 \le i \le k,\ 0 \le j \le k-1. \tag{3.2}$$

(6) $M^{i,j}$, $0 \le j < i \le k$, form a complete set of representatives of the equivalence classes of irreducible M^0-modules.

(7) The top level of $M^{i,j}$ is a one dimensional space $\mathbb{C}v^{i,j}$, and its weight is

$$h(M^{i,j}) = \frac{1}{2k(k+2)}\big(k(i-2j) - (i-2j)^2 + 2k(i-j+1)j\big) \tag{3.3}$$

for $0 \le j \le i \le k$. Note that (3.3) is valid even when $j = i$. Any irreducible M^0-module except for M^0 itself has positive conformal weight.

(8) The automorphism θ of M^0 induces a permutation $M^{i,j} \mapsto M^{i,j} \circ \theta \cong M^{i,i-j}$ on the irreducible M^0-modules for $0 \le i \le k$, $0 \le j \le k-1$.

(9) M^j, $0 \le j \le k-1$, are the simple currents with $h(M^j) = j(k-j)/k$, and

$$M^{j'} \boxtimes_{M^0} M^{i,j} = M^{i,j+j'}, \quad 0 \le i \le k,\ 0 \le j,\ j' \le k-1. \tag{3.4}$$

The following lemma is a consequence of (3.2) and (3.4).

Lemma 3.1. *$M^{j'} \boxtimes_{M^0} M^{i,j} \cong M^{i,j}$ if and only if $j' = 0$, or k is even and $j' = i = k/2$.*

Let

$$N = \{\alpha \in L \mid \langle \alpha, \gamma \rangle = 0\}.$$

Then $M^0 = \mathrm{Com}_{V^{\mathrm{aff}}}(V_{\mathbb{Z}\gamma}) \subset \mathrm{Com}_{V_L}(V_{\mathbb{Z}\gamma}) = V_N$. The commutant of V^{aff} in V_L is isomorphic to the parafermion vertex operator algebra $K(\mathfrak{sl}_k, 2)$ [**26**]. We denote it by T. Thus $T = \mathrm{Com}_{V_L}(V^{\mathrm{aff}}) = \mathrm{Com}_{V_N}(M^0) \cong K(\mathfrak{sl}_k, 2)$.

4. Cosets $N(j, a)$ of N in N°.

We keep the notations in Section 3. In this section, we describe the cosets of N in its dual lattice N°. For $\boldsymbol{a} = (a_1, \dots, a_k) \in \{0,1\}^k$, set $\delta_{\boldsymbol{a}} = (1/2)\sum_{p=1}^k a_p\alpha_p$. Then $L^\circ = \bigcup_{\boldsymbol{a}\in\{0,1\}^k}(L + \delta_{\boldsymbol{a}})$ is the coset decomposition of L° by L. Let $\beta_p = \alpha_p - \alpha_{p+1}$, $1 \le p \le k-1$, so $\{\beta_1, \dots, \beta_{k-1}\}$ is a $\mathbb{Z}$-basis of N. Set $R = N \oplus \mathbb{Z}\gamma$. Then $R \subset L \subset L^\circ \subset R^\circ$ with $R^\circ = N^\circ \oplus (\mathbb{Z}\gamma)^\circ$ and $(\mathbb{Z}\gamma)^\circ = \mathbb{Z}\gamma/2k$. Let

$$\lambda_k = \frac{1}{2k}(\beta_1 + 2\beta_2 + \cdots + (k-1)\beta_{k-1}) = \frac{1}{2k}\gamma - \frac{1}{2}\alpha_k.$$

Then $\langle \beta_p, \lambda_k \rangle = \delta_{p,k-1}$, $1 \le p \le k-1$, and $\langle \lambda_k, \lambda_k \rangle = 1/2 - 1/2k$. The following lemma holds.

Lemma 4.1. (1) *$\{\beta_2/2, \dots, \beta_{k-1}/2, \lambda_k\}$ is a $\mathbb{Z}$-basis of N°.*
(2) *The coset decomposition of N° by N is given as*

$$N^\circ = \bigcup_{\substack{0\le i\le 2k-1\\ d_2,\dots,d_{k-1}\in\{0,1\}}} (N + d_2\beta_2/2 + \cdots + d_{k-1}\beta_{k-1}/2 + i\lambda_k).$$

(3) $N^\circ/N \cong \mathbb{Z}_2^{k-2} \times \mathbb{Z}_{2k}$.

We consider another $\mathbb{Z}$-basis of N°. Let

$$\lambda_p = \lambda_k - \frac{1}{2}\beta_p - \cdots - \frac{1}{2}\beta_{k-1} = \frac{1}{2k}\gamma - \frac{1}{2}\alpha_p, \quad 1 \le p \le k-1.$$

Then $\lambda_p \in N^\circ$ and $2\lambda_p \equiv 2\lambda_k \pmod{N}$. Note that

$$\lambda_1 + \cdots + \lambda_k = 0. \tag{4.1}$$

Lemma 4.1 implies the next lemma.

Lemma 4.2. (1) *$\{\lambda_2, \dots, \lambda_{k-1}, \lambda_k\}$ is a $\mathbb{Z}$-basis of N°.*
(2) *The coset decomposition of N° by N is given as*

$$N^\circ = \bigcup_{\substack{0\le i\le 2k-1\\ d_2,\dots,d_{k-1}\in\{0,1\}}} \big(N + d_2\lambda_2 + \cdots + d_{k-1}\lambda_{k-1} + i\lambda_k\big).$$

The coset decomposition of L by R is given as

$$L = \bigcup_{j=0}^{k-1} (R - j\alpha_k) = \bigcup_{j=0}^{k-1} \Big(R + 2j\lambda_k - \frac{j}{k}\gamma \Big), \tag{4.2}$$

and $L/R \cong \mathbb{Z}_k$. Moreover, the coset decomposition of R° by L° is given as

$$R^\circ = \bigcup_{j=0}^{k-1} \Big(L^\circ - \frac{j}{2k}\gamma \Big),$$

and $R^\circ/L^\circ \cong \mathbb{Z}_k$.

For $\boldsymbol{a} = (a_1, \dots, a_k) \in \{0,1\}^k$, the support $\mathrm{supp}(\boldsymbol{a})$ is the set of p, $1 \le p \le k$, for which $a_p \neq 0$, and the Hamming weight $\mathrm{wt}(\boldsymbol{a})$ is the number of nonzero entries a_p. Then

$$\delta_{\boldsymbol{a}} = -\sum_{p=1}^{k} a_p \lambda_p + \frac{\mathrm{wt}(\boldsymbol{a})}{2k}\gamma.$$

For $\boldsymbol{a} = (a_1, \dots, a_k) \in \{0,1\}^k$, let

$$N(j, \boldsymbol{a}) = N - \sum_{p=1}^{k} a_p \lambda_p + 2j\lambda_k, \quad 0 \le j \le k-1. \tag{4.3}$$

Since $2k\lambda_k \in N$, we can consider j to be modulo k. We have

$$N(j, \boldsymbol{a}) + N(j', \boldsymbol{a}') = N(j + j' - (\mathrm{wt}(\boldsymbol{a}) + \mathrm{wt}(\boldsymbol{a}') - \mathrm{wt}(\boldsymbol{a} + \boldsymbol{a}'))/2, \boldsymbol{a} + \boldsymbol{a}'),$$

where $\boldsymbol{a}+\boldsymbol{a}'$ is the sum of $\boldsymbol{a}$ and $\boldsymbol{a}'$ as elements of $(\mathbb{Z}_2)^k$, that is, the symmetric difference as subsets of $\{0,1\}^k$. By the definition of λ_p, we also have

$$N(j, \boldsymbol{a}) = N + \frac{1}{2}\sum_{p=1}^{k} a_p \alpha_p - j\alpha_k + \frac{2j - \mathrm{wt}(\boldsymbol{a})}{2k}\gamma. \tag{4.4}$$

Since $2\lambda_k - \gamma/k = -\alpha_k$, this equation implies that

$$R + \delta_{\boldsymbol{a}} + 2j\lambda_k - \frac{j}{k}\gamma = N(j, \boldsymbol{a}) + \Big(\mathbb{Z}\gamma + \frac{\mathrm{wt}(\boldsymbol{a}) - 2j}{2k}\gamma \Big)$$

as subsets of $R^\circ = N^\circ \oplus (\mathbb{Z}\gamma)^\circ$. Hence it follows from (4.2) that

$$L + \delta_{\boldsymbol{a}} = \bigcup_{j=0}^{k-1} \Big(N(j, \boldsymbol{a}) + \Big(\mathbb{Z}\gamma + \frac{\mathrm{wt}(\boldsymbol{a}) - 2j}{2k}\gamma \Big) \Big). \tag{4.5}$$

Lemma 4.3. (1) *For $0 \le j, j' \le k-1$ and $\boldsymbol{a}, \boldsymbol{a}' \in \{0,1\}^k$, we have $N(j, \boldsymbol{a}) = N(j', \boldsymbol{a}')$ if and only if one of the following conditions holds.*

(i) $j \equiv j' \pmod k$ *and* $\boldsymbol{a} = \boldsymbol{a}'$.

(ii) $j' \equiv j - \mathrm{wt}(\boldsymbol{a}) \pmod k$ *and* $\boldsymbol{a} + \boldsymbol{a}' = (1, \dots, 1)$.

(2) *$N(j, \boldsymbol{a})$, $0 \le j \le k-1$, $\boldsymbol{a} \in \{0,1\}^k$ with $j < \mathrm{wt}(\boldsymbol{a})$, are the distinct cosets of N in N°.*

PROOF. Clearly, $N(j,\boldsymbol{a}) = N(j',\boldsymbol{a}')$ if the condition (i) holds. Suppose the condition (ii) holds. Then $N(j,\boldsymbol{a}) = N(j',\boldsymbol{a}')$ by (4.1) and (4.3). Set $i = \mathrm{wt}(\boldsymbol{a})$ and $i' = \mathrm{wt}(\boldsymbol{a}')$, and assume that $j < i$. Then $0 \le j < i \le k$ and $0 \le i' \le j' < k$. The number of pairs $(j,\boldsymbol{a})$ with $0 \le j \le k-1$ and $\boldsymbol{a} \in \{0,1\}^k$ is $2^k k$. Since $|N^\circ/N| = 2^{k-1}k$, we see that $N(j,\boldsymbol{a}) = N(j',\boldsymbol{a}')$ only if j, j', $\boldsymbol{a}$, and $\boldsymbol{a}'$ satisfy the conditions (i) or (ii). Hence the assertions (1) and (2) hold. $\square$

REMARK 4.4. In Case (ii) of Lemma 4.3 (1), we have $(\mathrm{wt}(\boldsymbol{a}') - 2j') - (\mathrm{wt}(\boldsymbol{a}) - 2j) \equiv k \pmod{2k}$. This agrees with the fact that $N(j,\boldsymbol{a}) + (\mathbb{Z}\gamma + (\mathrm{wt}(\boldsymbol{a}) - 2j)\gamma/2k)$, $0 \le j \le k-1$, $\boldsymbol{a} \in \{0,1\}^k$, in (4.5) are the distinct cosets of R in L°.

The next lemma also holds.

LEMMA 4.5. *The -1 isometry $N^\circ \to N^\circ$; $\alpha \mapsto -\alpha$ transforms $N(j,\boldsymbol{a})$ into $N(\mathrm{wt}(\boldsymbol{a}) - j, \boldsymbol{a})$.*

5. Decomposition of $V_{N(j,\boldsymbol{a})}$.

We keep the notations in Sections 3 and 4. In this section, we study a decomposition of the irreducible V_N-module $V_{N(j,\boldsymbol{a})}$ as a direct sum of irreducible modules for a tensor product of $k-1$ Virasoro vertex operator algebras and M^0. Let

$$c_m = 1 - \frac{6}{(m+2)(m+3)}$$

for $m = 1, 2, \ldots$, and let

$$h^m_{r,s} = \frac{(r(m+3) - s(m+2))^2 - 1}{4(m+2)(m+3)}$$

for $1 \le r \le m+1$, $1 \le s \le m+2$. Then $h^m_{r,s} = h^m_{m+2-r,m+3-s}$, and $L(c_m, h^m_{r,s})$, $1 \le s \le r \le m+1$, form a complete set of representatives of the equivalence classes of irreducible modules for the Virasoro vertex operator algebra $L(c_m, 0)$ [**41**]. We denote the conformal vector of $L(c_m, 0)$ by ω^m.

Recall that ω is the conformal vector of M^0. Let ω_T be the conformal vector of $T = \mathrm{Com}_{V_N}(M^0)$. Then the conformal vector $\omega_N = \omega_T + \omega$ of V_N is a sum of mutually orthogonal Virasoro vectors $\omega^1, \ldots, \omega^{k-1}$, and ω [**11**], [**29**] with $\omega_T = \omega^1 + \cdots + \omega^{k-1}$. The vector ω^m generates $L(c_m, 0)$, so $T \supset L(c_1, 0) \otimes \cdots \otimes L(c_{k-1}, 0)$. The following decomposition is known [**21**], [**27**], [**40**].

LEMMA 5.1. *For $\boldsymbol{a} = (a_1, \ldots, a_k) \in \{0,1\}^k$,*

$$V_{L+\delta_{\boldsymbol{a}}} = \bigoplus_{\substack{0 \le i_s \le s \\ i_s \equiv b_s \pmod 2 \\ 1 \le s \le k}} L(c_1, h^1_{i_1+1, i_2+1}) \otimes \cdots \otimes L(c_{k-1}, h^{k-1}_{i_{k-1}+1, i_k+1}) \otimes L_{\widehat{\mathfrak{sl}}_2}(k, i_k)$$

as $L(c_1, 0) \otimes \cdots \otimes L(c_{k-1}, 0) \otimes L_{\widehat{\mathfrak{sl}}_2}(k, 0)$-modules, where $b_s = \sum_{p=1}^{s} a_p$.

Combining the decomposition (3.1) with Lemma 5.1, we have

$$V_{L+\delta_{\boldsymbol{a}}} = \bigoplus_{j=0}^{k-1}\Big(\bigoplus_{\substack{0\le i_s\le s\\ i_s\equiv b_s \pmod 2\\ 1\le s\le k}} L(c_1,h^1_{i_1+1,i_2+1})\otimes\cdots\otimes L(c_{k-1},h^{k-1}_{i_{k-1}+1,i_k+1}) \otimes M^{i_k,j}\otimes V_{\mathbb{Z}\gamma+(i_k-2j)\gamma/2k}\Big) \tag{5.1}$$

as $L(c_1,0)\otimes\cdots\otimes L(c_{k-1},0)\otimes M^0\otimes V_{\mathbb{Z}\gamma}$-modules.

Since $b_k=\mathrm{wt}(\boldsymbol{a})$, (4.5) implies that

$$V_{L+\delta_{\boldsymbol{a}}} = \bigoplus_{j=0}^{k-1} V_{N(j,\boldsymbol{a})}\otimes V_{\mathbb{Z}\gamma+(b_k-2j)\gamma/2k} \tag{5.2}$$

as $V_N\otimes V_{\mathbb{Z}\gamma}$-modules.

As $V_{\mathbb{Z}\gamma}$-modules, $V_{\mathbb{Z}\gamma+(b_k-2j)\gamma/2k}\cong V_{\mathbb{Z}\gamma+(i_k-2q)\gamma/2k}$ if and only if $q\equiv j+(i_k-b_k)/2 \pmod k$. Here, note that i_k on the right hand side of (5.1) satisfies $i_k\equiv b_k \pmod 2$. Comparing (5.1) and (5.2), we have the following theorem, see [**28**, Proposition 3.4].

Theorem 5.2. *For $0\le j\le k-1$ and $\boldsymbol{a}=(a_1,\dots,a_k)\in\{0,1\}^k$, the irreducible V_N-module $V_{N(j,\boldsymbol{a})}$ decomposes as a direct sum*

$$V_{N(j,\boldsymbol{a})} = \bigoplus_{\substack{0\le i_s\le s\\ i_s\equiv b_s \pmod 2\\ 1\le s\le k}} L(c_1,h^1_{i_1+1,i_2+1})\otimes\cdots\otimes L(c_{k-1},h^{k-1}_{i_{k-1}+1,i_k+1})\otimes M^{i_k,j+(i_k-b_k)/2} \tag{5.3}$$

of irreducible $L(c_1,0)\otimes\cdots\otimes L(c_{k-1},0)\otimes M^0$-modules, where $b_s=\sum_{p=1}^s a_p$.

The next remark is a restatement of [**28**, Proposition 3.5].

Remark 5.3. $N(j,\boldsymbol{a})=N(j',\boldsymbol{a}')$ for j', $\boldsymbol{a}'$ in Case (ii) of Lemma 4.3 (1) corresponds to the following properties of the highest weights $h^m_{p,q}$ for $L(c_m,0)$ and the irreducible modules $M^{i,j}$ for $K(sl_2,k)$.

(1) $h^m_{p,q}=h^m_{m+2-p,m+3-q}$ for $1\le p\le m+1$, $1\le q\le m+2$.

(2) $M^{i,j}\cong M^{k-i,j-i}$ as $K(sl_2,k)$-modules for $0\le i\le k$, $j\in\mathbb{Z}_k$.

We note that for a given $\boldsymbol{a}\in\{0,1\}^k$, the $L(c_1,0)\otimes\cdots\otimes L(c_{k-1},0)$-modules

$$L(c_1,h^1_{i_1+1,i_2+1})\otimes\cdots\otimes L(c_{k-1},h^{k-1}_{i_{k-1}+1,i_k+1}),$$

$0\le i_s\le s$, $i_s\equiv b_s \pmod 2$, $1\le s\le k$, in (5.3) are inequivalent to each other.

6. Irreducible $K(\mathfrak{sl}_2,k)$-modules in $V_{N(j,\boldsymbol{a})}$.

In this section, we discuss how irreducible $K(\mathfrak{sl}_2,k)$-modules $M^{i,j}$ appear on the right hand side of (5.3). Since $h^s_{p,q}=0$ if and only if $(p,q)=(1,1)$ or $(s+1,s+2)$, the following lemma holds.

LEMMA 6.1. *Let $1 \le m < k$. Then for $a_1, \dots, a_{m+1} \in \{0,1\}$ and $0 \le i_s \le s$, $1 \le s \le m+1$, the two conditions $i_s \equiv b_s \pmod 2$, $1 \le s \le m+1$, and $h^s_{i_s+1, i_{s+1}+1} = 0$, $1 \le s \le m$, hold only if* (i) *$a_s = 0$ and $i_s = 0$, $1 \le s \le m+1$, or* (ii) *$a_s = 1$ and $i_s = s$, $1 \le s \le m+1$.*

For an arbitrarily given $a_1 \in \{0,1\}$, each coset of N in N° is uniquely expressed as $N(j, \boldsymbol{a})$, $j \in \mathbb{Z}_k$, $\boldsymbol{a} = (a_1, a_2, \dots, a_k)$, $a_2, \dots, a_k \in \{0,1\}$ by Lemma 4.3. For the rest of this section, we take $a_1 = 0$. For simplicity of notation, we omit $\mathbf{1} \otimes \cdots \otimes \mathbf{1}$ in an equation as

$$\{v \in V_N \mid \omega^s_{(1)} v = 0, 1 \le s \le k-1\} = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes M^0.$$

The following two propositions are clear from Theorem 5.2 and Lemma 6.1.

PROPOSITION 6.2. *For $j \in \mathbb{Z}_k$, we have*

$$\{v \in V_{N(j,(0,\dots,0))} \mid \omega^s_{(1)} v = 0, 1 \le s \le k-1\} = M^j.$$

PROPOSITION 6.3. *For $j \in \mathbb{Z}_k$ and $d \in \{0,1\}$, we have*

$$\begin{aligned} &\{v \in V_{N(j,(0,\dots,0,d))} \mid \omega^s_{(1)} v = 0, 1 \le s \le k-2\} \\ &\quad = \bigoplus_{\substack{0 \le i \le k \\ i \equiv d \pmod 2}} L(c_{k-1}, h^{k-1}_{1,i+1}) \otimes M^{i, j+(i-d)/2}. \end{aligned} \tag{6.1}$$

The next proposition is a consequence of (3.2).

PROPOSITION 6.4. *Let $d \in \{0,1\}$.*

(1) *If k is odd, then $M^{i,j+(i-d)/2}$, $j \in \mathbb{Z}_k$, $0 \le i \le k$, $i \equiv d \pmod 2$, are inequivalent to each other, and they are the $k(k+1)/2$ inequivalent irreducible modules $M^{i,j}$, $0 \le j < i \le k$.*

(2) *If k is even, then $M^{i,j+(i-d)/2}$, $j \in \mathbb{Z}_k$, $0 \le i \le k$, $i \equiv d \pmod 2$, cover twice the set of inequivalent irreducible modules $M^{i,j}$, $0 \le j < i \le k$ with $i \equiv d \pmod 2$. There are $k(k+2)/4$ (resp. $k^2/4$) inequivalent irreducible modules $M^{i,j}$, $0 \le j < i \le k$ with $i \equiv 0 \pmod 2$ (resp. $i \equiv 1 \pmod 2$). Moreover, for a fixed $j \in \mathbb{Z}_k$, the irreducible modules $M^{i,j+(i-d)/2}$, $0 \le i \le k$, $i \equiv d \pmod 2$, are inequivalent to each other.*

7. $\boldsymbol{\Gamma_D}$ and $\boldsymbol{M_D}$ for a $\mathbb{Z}_k$-code $\boldsymbol{D}$.

In this section, we define a vertex operator algebra or a vertex operator superalgebra M_D for a $\mathbb{Z}_k$-code D. The arguments are essentially the same as in Section 3 of [**3**].

Let ℓ be a fixed positive integer. A $\mathbb{Z}_k$-code of length ℓ means an additive subgroup of $(\mathbb{Z}_k)^\ell$. We denote by $(\,\cdot\,|\,\cdot\,)$ the standard inner product $(\xi|\eta) = \xi_1\eta_1 + \cdots + \xi_\ell\eta_\ell \in \mathbb{Z}_k$ for $\xi = (\xi_1, \dots, \xi_\ell)$, $\eta = (\eta_1, \dots, \eta_\ell) \in (\mathbb{Z}_k)^\ell$.

For simplicity of notation, set $N^{(j)} = N(j, (0, \dots, 0)) = N + 2j\lambda_k$, $j \in \mathbb{Z}_k$. We consider a coset $N(\xi)$ of N^ℓ in $(N^\circ)^\ell$ defined by

$$N(\xi) = \{(x_1, \dots, x_\ell) \mid x_r \in N^{(\xi_r)}, 1 \le r \le \ell\} \subset (N^\circ)^\ell \tag{7.1}$$

for $\xi = (\xi_1, \dots, \xi_\ell) \in (\mathbb{Z}_k)^\ell$. Since $\langle \alpha, \beta \rangle \in -2ij/k + 2\mathbb{Z}$ for $\alpha \in N^{(i)}$, $\beta \in N^{(j)}$, we have

$$\langle \alpha, \beta \rangle \in -\frac{2}{k}(\xi|\eta) + 2\mathbb{Z} \quad \text{for } \alpha \in N(\xi), \beta \in N(\eta). \tag{7.2}$$

Let D be a $\mathbb{Z}_k$-code of length ℓ. We consider two cases.

Case A: $(\xi|\xi) = 0$ for all $\xi \in D$.

Case B: k is even, $(\xi|\eta) \in \{0, k/2\}$ for all $\xi, \eta \in D$, and $(\xi|\xi) = k/2$ for some $\xi \in D$.

Let

$$\Gamma_D = \bigcup_{\xi \in D} N(\xi) \subset (N^\circ)^\ell, \tag{7.3}$$

which is a sublattice of $(N^\circ)^\ell$, as $N(\xi) + N(\eta) = N(\xi + \eta)$ and D is an additive subgroup of $(\mathbb{Z}_k)^\ell$. The following lemma holds by (7.2).

Lemma 7.1. (1) *Γ_D is a positive definite even lattice if and only if D is in Case* A.
(2) *Γ_D is a positive definite odd lattice if and only if k is even and D is in Case* B.

If D is in Case A, then V_{Γ_D} is a vertex operator algebra. If k is even and D is in Case B, we set

$$D^0 = \{\xi \in D \mid (\xi|\xi) = 0\}, \quad D^1 = \{\xi \in D \mid (\xi|\xi) = k/2\}.$$

We also set $\Gamma_{D^p} = \bigcup_{\xi \in D^p} N(\xi)$, $p = 0, 1$. Then D^0 is a subgroup of the additive group D of index two, and $D = D^0 \cup D^1$ is the coset decomposition of D by D^0. Moreover, $\Gamma_{D^p} = \{\alpha \in \Gamma_D \mid \langle \alpha, \alpha \rangle \in p + 2\mathbb{Z}\}$, $p = 0, 1$, and $\Gamma_D = \Gamma_{D^0} \cup \Gamma_{D^1}$ with Γ_{D^0} an even sublattice. We have that $V_{\Gamma_D} = V_{\Gamma_{D^0}} \oplus V_{\Gamma_{D^1}}$ is a vertex operator superalgebra.

It follows from (7.1) that $V_{N(\xi)} = V_{N^{(\xi_1)}} \otimes \cdots \otimes V_{N^{(\xi_\ell)}} \subset (V_{N^\circ})^\ell$. We also have $V_{\Gamma_D} = \bigoplus_{\xi \in D} V_{N(\xi)}$ by (7.3). Let

$$M_\xi = \{v \in V_{N(\xi)} \mid (\omega_{T^{\otimes \ell}})_{(1)} v = 0\},$$

where $\omega_{T^{\otimes \ell}}$ is the conformal vector of the vertex operator subalgebra $T^{\otimes \ell}$ of $(V_N)^{\otimes \ell}$. Then $M_\xi = M^{\xi_1} \otimes \cdots \otimes M^{\xi_\ell}$ for $\xi = (\xi_1, \dots, \xi_\ell) \in (\mathbb{Z}_k)^\ell$ by Proposition 6.2, which is a simple current for $M_{\mathbf{0}} = (M^0)^{\otimes \ell}$ with $\mathbf{0} = (0, \dots, 0)$ the zero codeword. Since $u_{(n)} v \in V_{N(\xi+\eta)}$ for $u \in V_{N(\xi)}$, $v \in V_{N(\eta)}$, $n \in \mathbb{Z}$, we have $u_{(n)} v \in M_{\xi+\eta}$ for $u \in M_\xi$, $v \in M_\eta$, $n \in \mathbb{Z}$. Thus $M_\xi \boxtimes_{M_{\mathbf{0}}} M_\eta = M_{\xi+\eta}$ for $\xi, \eta \in (\mathbb{Z}_k)^\ell$, and $\mathrm{Irr}(M_{\mathbf{0}})_{\mathrm{sc}} = \{M_\xi \mid \xi \in (\mathbb{Z}_k)^\ell\}$ is $(\mathbb{Z}_k)^\ell$-graded. The top level of M_ξ is one dimensional with $h(M_\xi) = \left(\sum_{r=1}^{\ell} \xi_r\right) - (\xi|\xi)/k$, as $h(M^j) = j - j^2/k$, where ξ_r and $(\xi|\xi)$ are considered to be nonnegative integers.

We have the next proposition by the properties of $M^0 = K(\mathfrak{sl}_2, k)$ in Section 3.

Proposition 7.2. *$M_{\mathbf{0}} = (M^0)^{\otimes \ell}$ is a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type with central charge $2\ell(k-1)/(k+2)$. Any irreducible $M_{\mathbf{0}}$-module except for $M_{\mathbf{0}}$ itself has positive conformal weight.*

Let M_D be the commutant of $T^{\otimes \ell}$ in V_{Γ_D}. Then

$$M_D = \{v \in V_{\Gamma_D} \mid (\omega_{T^{\otimes \ell}})_{(1)} v = 0\} = \bigoplus_{\xi \in D} M_\xi, \tag{7.4}$$

which is a D-graded simple current extension of $M_{\mathbf{0}}$. The following theorem holds.

THEOREM 7.3. (1) *If D is in Case* A, *then M_D is a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type with central charge $2\ell(k-1)/(k+2)$.*

(2) *If k is even and D is in Case* B, *then $M_D = M_{D^0} \oplus M_{D^1}$ is a simple vertex operator superalgebra, whose even part M_{D^0} and odd part M_{D^1} are given by $M_{D^p} = \bigoplus_{\xi \in D^p} M_\xi$, $p = 0, 1$, and $h(M_{D^1}) \in \mathbb{Z} + 1/2$.*

8. Irreducible M_D-modules: Case A.

Let $k \geq 2$, and let D be a $\mathbb{Z}_k$-code of length ℓ satisfying the condition of Case A in Section 7, that is, $(\xi|\xi) = 0$ for all $\xi \in D$. In this section, we classify the irreducible χ-twisted M_D-modules for $\chi \in D^*$. We construct all irreducible untwisted M_D-modules inside $V_{(\Gamma_D)^\circ}$ as well.

8.1. Linear characters of D.

Let

$$P(i,j) = k(i-2j) - (i-2j)^2 + 2k(i-j+1)j.$$

Then $h(M^{i,j}) = P(i,j)/2k(k+2)$ for $0 \leq j \leq i \leq k$ by (3.3). In the case where $0 \leq i \leq j < k$, we have $h(M^{i,j}) = P(k-i, j-i)/2k(k+2)$ by (3.2). We calculate the values of the map $b_{M^0} : \mathrm{Irr}(M^0)_{\mathrm{sc}} \times \mathrm{Irr}(M^0) \to \mathbb{Q}/\mathbb{Z}$ defined by

$$b_{M^0}(M^p, M^{i,j}) = h(M^p \boxtimes_{M^0} M^{i,j}) - h(M^p) - h(M^{i,j}) + \mathbb{Z},$$

where $M^p \boxtimes_{M^0} M^{i,j} = M^{i,j+p}$ by (3.4). If $0 \leq j < i \leq k$, then $0 \leq j < j+1 \leq i \leq k$, and

$$P(i,j+1) - P(i,j) = 2(k+2)(i-2j-1),$$

whereas if $0 \leq i \leq j < k$, then $0 \leq j-i < j+1-i \leq k-i \leq k$, and

$$P(k-i, j+1-i) - P(k-i, j-i) = 2(k+2)(i-2j+k-1).$$

In both cases, we have $b_{M^0}(M^1, M^{i,j}) = (i-2j)/k + \mathbb{Z}$. Thus

$$b_{M^0}(M^p, M^{i,j}) = \frac{p(i-2j)}{k} + \mathbb{Z} \tag{8.1}$$

for $0 \leq i \leq k$, $0 \leq j < k$, and $0 \leq p < k$ by Lemma 2.1.

For $\mu = (\mu_1, \ldots, \mu_\ell)$ with $0 \leq \mu_r \leq k$, $1 \leq r \leq \ell$, and $\nu = (\nu_1, \ldots, \nu_\ell) \in (\mathbb{Z}_k)^\ell$, let

$$M_{\mu,\nu} = M^{\mu_1,\nu_1} \otimes \cdots \otimes M^{\mu_\ell,\nu_\ell}.$$

Then $M_{\mathbf{0},\xi} = M_\xi$ and

$$\mathrm{Irr}(M_{\mathbf{0}}) = \{M_{\mu,\nu} \mid \mu = (\mu_1, \ldots, \mu_\ell),\ \nu = (\nu_1, \ldots, \nu_\ell),\ 0 \le \nu_r < \mu_r \le k,\ 1 \le r \le \ell\}. \tag{8.2}$$

It follows from (3.4) that

$$M_\xi \boxtimes_{M_{\mathbf{0}}} M_{\mu,\nu} = M_{\mu,\nu+\xi}. \tag{8.3}$$

Let $b_{M_{\mathbf{0}}} : \mathrm{Irr}(M_{\mathbf{0}})_{\mathrm{sc}} \times \mathrm{Irr}(M_{\mathbf{0}}) \to \mathbb{Q}/\mathbb{Z}$ be a map defined by

$$b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu}) = h(M_\xi \boxtimes_{M_{\mathbf{0}}} M_{\mu,\nu}) - h(M_\xi) - h(M_{\mu,\nu}) + \mathbb{Z}$$

for $\mu = (\mu_1, \ldots, \mu_\ell)$ with $0 \le \mu_r \le k$, $1 \le r \le \ell$, $\nu = (\nu_1, \ldots, \nu_\ell) \in (\mathbb{Z}_k)^\ell$, and $\xi = (\xi_1, \ldots, \xi_\ell) \in (\mathbb{Z}_k)^\ell$. Then (8.1) implies that

$$b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu}) = \sum_{r=1}^{\ell} \frac{\xi_r(\mu_r - 2\nu_r)}{k} + \mathbb{Z}. \tag{8.4}$$

Although μ_r is an integer between 0 and k, we can treat μ_r modulo k on the right hand side of (8.4). Then (8.4) is written as

$$b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu}) = \frac{1}{k}(\xi|\mu - 2\nu) + \mathbb{Z}, \tag{8.5}$$

where $(\,\cdot\,|\,\cdot\,)$ is the standard inner product on $(\mathbb{Z}_k)^\ell$. In particular,

$$b_{M_{\mathbf{0}}}(M_\xi, M_\eta) = -\frac{2}{k}(\xi|\eta) + \mathbb{Z}. \tag{8.6}$$

Lemma 8.1. *Let $\xi, \eta, \nu \in (\mathbb{Z}_k)^\ell$, and let $\mu = (\mu_1, \ldots, \mu_\ell)$ with $0 \le \mu_r \le k$, $1 \le r \le \ell$.*
(1) $b_{M_{\mathbf{0}}}(M_\xi, M_\eta) = 0$ *if* $\xi, \eta \in D$.
(2) $b_{M_{\mathbf{0}}}(M_{\xi+\eta}, M_{\mu,\nu}) = b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu}) + b_{M_{\mathbf{0}}}(M_\eta, M_{\mu,\nu})$.
(3) $b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu+\eta}) = b_{M_{\mathbf{0}}}(M_\xi, M_\eta) + b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu})$.

Proof. Suppose $\xi, \eta \in D$. Then $\xi+\eta \in D$, so $(\xi+\eta|\xi+\eta) = 0$ by our assumption on D. Since $(\xi|\xi) = (\eta|\eta) = 0$, we have $2(\xi|\eta) = 0$. Thus the assertion (1) holds by (8.6). The assertions (2) and (3) are clear from (8.5), see also Lemma 2.1. □

For $\eta \in (\mathbb{Z}_k)^\ell$, let $\chi(\eta)$ be a linear character of the abelian group $(\mathbb{Z}_k)^\ell$ given by

$$\chi(\eta) : (\mathbb{Z}_k)^\ell \to \mathbb{C}^\times; \quad \xi \mapsto \exp(2\pi\sqrt{-1}(\xi|\eta)/k).$$

Then $(\mathbb{Z}_k)^\ell \to \mathrm{Hom}((\mathbb{Z}_k)^\ell, \mathbb{C}^\times)$; $\eta \mapsto \chi(\eta)$ is a group isomorphism. The linear character $\chi_{M_{\mu,\nu}} \in D^*$ is the restriction $\chi(\mu - 2\nu)|_D$ of $\chi(\mu - 2\nu)$ to D by (8.5). That is,

$$\chi_{M_{\mu,\nu}}(\xi) = \exp(2\pi\sqrt{-1}b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu})) = \exp(2\pi\sqrt{-1}(\xi|\mu - 2\nu)/k). \tag{8.7}$$

Let $D^\perp = \{\eta \in (\mathbb{Z}_k)^\ell \mid (D|\eta) = 0\}$. Then $|D||D^\perp| = |(\mathbb{Z}_k)^\ell|$, as $(\,\cdot\,|\,\cdot\,)$ is a non-degenerate bilinear form.

LEMMA 8.2. (1) *The map* $(\mathbb{Z}_k)^\ell \to D^*$; $\eta \mapsto \chi(\eta)|_D$ *is a surjective group homomorphism with kernel* $D^\perp$.

(2) *For any* $\chi \in D^*$, *there exists* $M_{\mu,\mathbf{0}} \in \mathrm{Irr}(M_{\mathbf{0}})$ *such that* $\chi = \chi_{M_{\mu,\mathbf{0}}}$.

(3) $\chi_{M_{\mu,\nu}} = 1$; *the principal character of* D *if and only if* $\mu - 2\nu \in D^\perp$.

(4) $\chi_{M_{\mu,\nu}} = \chi_{M_{\mu',\nu'}}$ *if and only if* $\mu - 2\nu \equiv \mu' - 2\nu' \pmod{D^\perp}$.

PROOF. Non-degeneracy of the bilinear form $(\,\cdot\,|\,\cdot\,)$ implies the assertions (1) and (2). The assertions (3) and (4) are consequences of (8.7) and the definition of $D^\perp$. $\square$

8.2. Irreducible $M_{\mathbf{0}}$-modules in $V_{(N^\circ)^\ell}$.

Let

$$N(\eta, \delta) = \{(x_1, \dots, x_\ell) \mid x_r \in N(\eta_r, (0, \dots, 0, d_r)), 1 \le r \le \ell\} \subset (N^\circ)^\ell$$

for $\eta = (\eta_1, \dots, \eta_\ell) \in (\mathbb{Z}_k)^\ell$ and $\delta = (d_1, \dots, d_\ell) \in \{0, 1\}^\ell$.

PROPOSITION 8.3. (1) *Let* $\eta = (\eta_1, \dots, \eta_\ell) \in (\mathbb{Z}_k)^\ell$ *and* $\delta = (d_1, \dots, d_\ell) \in \{0, 1\}^\ell$. *Assume that* $\mu = (\mu_1, \dots, \mu_\ell)$ *with* $0 \le \mu_r \le k$, $1 \le r \le \ell$, *and* $\nu = (\nu_1, \dots, \nu_\ell) \in (\mathbb{Z}_k)^\ell$ *satisfy the conditions*

$$\mu_r \equiv d_r \pmod 2, \quad \nu_r = \eta_r + \frac{\mu_r - d_r}{2}, \quad 1 \le r \le \ell. \tag{8.8}$$

Then $V_{N(\eta,\delta)}$ *contains the irreducible* $M_{\mathbf{0}}$*-module* $M_{\mu,\nu}$.

(2) *Any irreducible* $M_{\mathbf{0}}$*-module is contained in* $V_{N(\eta,\delta)}$ *for some* η *and* δ. *If* k *is odd, then we can choose* δ *to be* $\delta = (0, \dots, 0)$.

PROOF. The assertions (1) and (2) hold by Propositions 6.3 and 6.4. $\square$

LEMMA 8.4. *Let* $\xi, \eta \in (\mathbb{Z}_k)^\ell$ *and* $\delta \in \{0, 1\}^\ell$. *Then* $\langle x, y\rangle \in (\xi|\delta - 2\eta)/k + \mathbb{Z}$ *for* $x \in N(\xi)$ *and* $y \in N(\eta, \delta)$.

PROOF. Since $\langle x, y\rangle \in p(d-2j)/k + \mathbb{Z}$ for $x \in N^{(p)}$ and $y \in N(j, (0, \dots, 0, d))$, the assertion holds. $\square$

PROPOSITION 8.5. *Let* $\mu = (\mu_1, \dots, \mu_\ell)$ *with* $0 \le \mu_r \le k$, $1 \le r \le \ell$, *and let* $\nu = (\nu_1, \dots, \nu_\ell) \in (\mathbb{Z}_k)^\ell$. *Take* $\eta \in (\mathbb{Z}_k)^\ell$ *and* $\delta \in \{0, 1\}^\ell$ *such that the conditions* (8.8) *hold. Then* $b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu}) = 0$ *for all* $\xi \in D$ *if and only if* $N(\eta, \delta) \subset (\Gamma_D)^\circ$.

PROOF. Since $\mu_r - 2\nu_r = d_r - 2\eta_r$ by (8.8), the assertion holds by (8.5) and Lemma 8.4. $\square$

8.3. Irreducible twisted M_D-modules in $V_{(N^\circ)^\ell}$.

Let $X \in \mathrm{Irr}(M_{\mathbf{0}})$. Then $X = M_{\mu,\nu}$ for some μ and ν by (8.2). Take η and δ such that the conditions (8.8) hold. Then $V_{N(\eta,\delta)}$ contains $M_{\mu,\nu}$ as an $M_{\mathbf{0}}$-submodule by Proposition 8.3. Since $M_\xi \subset V_{N(\xi)}$, and since $N(\xi) + N(\eta, \delta) = N(\xi + \eta, \delta)$, it follows that $V_{N(\xi+\eta,\delta)}$ contains $M_\xi \boxtimes_{M_{\mathbf{0}}} M_{\mu,\nu}$. For fixed η and δ, the cosets $N(\xi + \eta, \delta)$, $\xi \in D$, of N^ℓ in $(N^\circ)^\ell$ are all distinct. Hence the $\chi_{M_{\mu,\nu}}$-twisted M_D-module $M_D \cdot M_{\mu,\nu}$ generated by $M_{\mu,\nu}$ in $V_{(N^\circ)^\ell}$ is isomorphic to $M_D \boxtimes_{M_{\mathbf{0}}} M_{\mu,\nu}$ by (2) of Theorem 2.2.

Furthermore, if $\chi_{M_{\mu,\nu}}(\xi) = 1$ for all $\xi \in D$, then $N(\eta, \delta) \subset (\Gamma_D)^\circ$ by Proposition 8.5, so $M_D \cdot M_{\mu,\nu} \subset V_{(\Gamma_D)^\circ}$. Therefore, the following theorem holds.

Theorem 8.6. *Let $X \in \mathrm{Irr}(M_{\mathbf{0}})$.*

(1) *$V_{(N^\circ)^\ell}$ contains a χ_X-twisted M_D-module isomorphic to $M_D \boxtimes_{M_{\mathbf{0}}} X$.*

(2) *If $\chi_X = 1$, then $V_{(\Gamma_D)^\circ}$ contains an untwisted M_D-module isomorphic to $M_D \boxtimes_{M_{\mathbf{0}}} X$.*

Let W be an irreducible χ-twisted M_D-module for $\chi \in D^*$, and let X be an irreducible $M_{\mathbf{0}}$-submodule of W. Then W is isomorphic to a direct summand of $M_D \boxtimes_{M_{\mathbf{0}}} X$ with $\chi = \chi_X$ by (3) of Theorem 2.2. Thus Theorem 8.6 implies the following theorem.

Theorem 8.7. (1) *$V_{(N^\circ)^\ell}$ contains any irreducible χ-twisted M_D-module for $\chi \in D^*$.*

(2) *$V_{(\Gamma_D)^\circ}$ contains any irreducible untwisted M_D-module.*

Let $\mathrm{Irr}(M_{\mathbf{0}}) = \bigcup_{i \in I} \mathscr{O}_i$ be the D-orbit decomposition of $\mathrm{Irr}(M_{\mathbf{0}})$ for the action of D on $\mathrm{Irr}(M_{\mathbf{0}})$ in (8.3), and let $D_{M_{\mu,\nu}} = \{\xi \in D \mid M_\xi \boxtimes_{M_{\mathbf{0}}} M_{\mu,\nu} \cong M_{\mu,\nu}\}$ be the stabilizer of $M_{\mu,\nu}$. Lemma 3.1 implies the following lemma.

Lemma 8.8. *$M_\xi \boxtimes_{M_{\mathbf{0}}} M_{\mu,\nu} \cong M_{\mu,\nu}$ as $M_{\mathbf{0}}$-modules for some $\xi \neq \mathbf{0}$ if and only if k is even, $\xi = (\xi_1, \ldots, \xi_\ell) \in \{0, k/2\}^\ell$, and $\mu_r = k/2$ for $1 \le r \le \ell$ such that $\xi_r = k/2$.*

The next theorem is a restatement of Proposition 2.3.

Theorem 8.9. *Let $X \in \mathrm{Irr}(M_{\mathbf{0}})$. If $D_X = 0$, then $M_D \boxtimes_{M_{\mathbf{0}}} X$ is an irreducible χ_X-twisted M_D-module.*

Now, suppose $D_X \neq 0$. Then k is even and $D_X \subset \{0, k/2\}^\ell$ by Lemma 8.8. In order to apply the results in Section 2.2, we recall the previous arguments for a special case where the $\mathbb{Z}_k$-code is of length one consisting of two codewords (0) and $(k/2)$. Let $C = \{(0), (k/2)\}$ be such a $\mathbb{Z}_k$-code. Then $\Gamma_C = N \cup N^{(k/2)}$ with $N^{(k/2)} = N + k\lambda_k$, and $M_C = M^0 \oplus M^{k/2}$ is a $\mathbb{Z}_2$-graded simple current extension of M^0 by the self-dual simple current M^0-module $M^{k/2}$ with $h(M^{k/2}) = k/4$.

If $k \equiv 0 \pmod 4$, then $(k/2)^2 \equiv 0 \pmod k$. Hence the $\mathbb{Z}_k$-code C is in Case A in Section 7, and M_C is a simple vertex operator algebra with $h(M^{k/2}) \in \mathbb{Z}$. If $k \equiv 2 \pmod 4$, then $(k/2)^2 \equiv k/2 \pmod k$. Hence C is in Case B in Section 7, and M_C is a simple vertex operator superalgebra with $h(M^{k/2}) \in \mathbb{Z} + 1/2$. In both cases, there exists a unique structure of a $\mathbb{Z}_2$-graded either untwisted or $\mathbb{Z}_2$-twisted M_C-module on the space $P^0 \oplus P^1$ with $P^0 = P$ and $P^1 = M^{k/2} \boxtimes_{M^0} P$ for any irreducible M^0-module P.

Under the correspondence $0 \mapsto 0$ and $k/2 \mapsto 1$, we can regard any additive subgroup of $\{0, k/2\}^\ell \subset (\mathbb{Z}_k)^\ell$ as an additive subgroup of $(\mathbb{Z}_2)^\ell$. In the case where $k \equiv 2 \pmod 4$, this correspondence is the reduction modulo 2, and it in fact gives an isometry from $(\{0, k/2\}^\ell, (\cdot \mid \cdot))$ to $((\mathbb{Z}_2)^\ell, (\cdot \mid \cdot))$, where $(\cdot \mid \cdot)$ is the standard inner product on either $(\mathbb{Z}_k)^\ell$ or $(\mathbb{Z}_2)^\ell$. Hence $D_X \cap D_X^\perp$ in $(\mathbb{Z}_k)^\ell$ corresponds to $D_X \cap D_X^\perp$ in $(\mathbb{Z}_2)^\ell$. Therefore, we obtain the following theorem by Propositions 2.5 and 2.6.

THEOREM 8.10. *Let $X \in \mathrm{Irr}(M_{\mathbf{0}})$. Suppose k is even and $D_X \neq 0$.*

(1) *If $k \equiv 0 \pmod 4$, then the irreducible decomposition of the χ_X-twisted M_D-module $M_D \boxtimes_{M_{\mathbf{0}}} X$ is given as*

$$M_D \boxtimes_{M_{\mathbf{0}}} X = \bigoplus_{j=1}^{|D_X|} U^j,$$

where U^j, $1 \leq j \leq |D_X|$, are inequivalent irreducible χ_X-twisted M_D-modules. Furthermore, $U^j \cong \bigoplus_{W \in \mathscr{O}_i} W$ as $M_{\mathbf{0}}$-modules, where $\mathscr{O}_i$ is the D-orbit in $\mathrm{Irr}(M_{\mathbf{0}})$ containing X.

(2) *If $k \equiv 2 \pmod 4$, then the irreducible decomposition of the χ_X-twisted M_D-module $M_D \boxtimes_{M_{\mathbf{0}}} X$ is given as*

$$M_D \boxtimes_{M_{\mathbf{0}}} X = \bigoplus_{j=1}^{|D_X \cap D_X^{\perp}|} (U^j)^{\oplus m},$$

where $m = [D_X : D_X \cap D_X^{\perp}]^{1/2}$, and U^j, $1 \leq j \leq |D_X \cap D_X^{\perp}|$, are inequivalent irreducible χ_X-twisted M_D-modules. Furthermore, $U^j \cong \bigoplus_{W \in \mathscr{O}_i} W^{\oplus m}$ as $M_{\mathbf{0}}$-modules, where $\mathscr{O}_i$ is the D-orbit in $\mathrm{Irr}(M_{\mathbf{0}})$ containing X.

Since any irreducible χ-twisted M_D-module for $\chi \in D^*$ is isomorphic to a direct summand of the χ_X-twisted M_D-module $M_D \boxtimes_{M_{\mathbf{0}}} X$ with $\chi = \chi_X$ for some $X \in \mathrm{Irr}(M_{\mathbf{0}})$, we obtain a classification of all the irreducible χ-twisted M_D-modules for any $\chi \in D^*$ by Theorems 8.9 and 8.10.

As mentioned in Section 2.2, we can write χ_i for χ_X, and D_i for D_X if X belongs to a D-orbit $\mathscr{O}_i$ in $\mathrm{Irr}(M_{\mathbf{0}})$. Let $I(\chi) = \{i \in I \mid \chi_i = \chi\}$. Then $I = \bigcup_{\chi \in D^*} I(\chi)$. The next lemma follows from (2) of Lemma 8.2.

LEMMA 8.11. *$I(\chi) \neq \emptyset$ for any $\chi \in D^*$.*

Theorems 8.9 and 8.10 imply the next theorem.

THEOREM 8.12. *The number of inequivalent irreducible χ-twisted M_D-modules for $\chi \in D^*$ is given as follows.*

$$\begin{aligned} &|I(\chi)| && \text{if } k \text{ is odd},\\ &|I(\chi)_0| + \sum_{i \in I(\chi)_1} |D_i| && \text{if } k \equiv 0 \pmod 4,\\ &|I(\chi)_0| + \sum_{i \in I(\chi)_1} |D_i \cap D_i^{\perp}| && \text{if } k \equiv 2 \pmod 4, \end{aligned}$$

where $I(\chi)_0 = \{i \in I(\chi) \mid D_i = 0\}$ and $I(\chi)_1 = I(\chi) \setminus I(\chi)_0$.

9. Irreducible M_D-modules: Case B.

Let $k \geq 2$, and let D be a $\mathbb{Z}_k$-code of length ℓ satisfying the conditions of Case B in Section 7, that is, k is even, $(\xi|\eta) \in \{0, k/2\}$ for all $\xi, \eta \in D$, and $(\xi|\xi) = k/2$ for some $\xi \in D$. Let D^0 and D^1 be as in Section 7. In this section, we construct all irreducible M_D-modules inside $V_{(\Gamma_{D^0})^\circ}$.

Since D^0 is a $\mathbb{Z}_k$-code of length ℓ in Case A, we can apply the results in Section 8 to the vertex operator algebra M_{D^0}. Let $P \in \mathrm{Irr}(M_{D^0})$. Then P is isomorphic to a direct summand of $M_{D^0} \boxtimes_{M_{\mathbf{0}}} M_{\mu,\nu}$ for some $M_{\mu,\nu} \in \mathrm{Irr}(M_{\mathbf{0}})$. Moreover, there are $\eta \in (\mathbb{Z}_k)^\ell$ and $\delta \in \{0,1\}^\ell$ such that $N(\eta,\delta) \subset (\Gamma_{D^0})^\circ$ and $V_{N(\eta,\delta)}$ contains $M_{\mu,\nu}$ as an $M_{\mathbf{0}}$-submodule.

For simplicity of notation, we identify P with an irreducible direct summand of $M_{D^0} \boxtimes_{M_{\mathbf{0}}} M_{\mu,\nu}$ isomorphic to P. Then P is a submodule of the M_{D^0}-module $M_{D^0} \boxtimes_{M_{\mathbf{0}}} M_{\mu,\nu}$, and the M_D-module $M_D \cdot P$ generated by P is isomorphic to $M_D \boxtimes_{M_{D^0}} P$. Thus $M_D \cdot P = P \oplus Q$ as M_{D^0}-modules, where Q is an irreducible M_{D^0}-module isomorphic to $M_{D^1} \boxtimes_{M_{D^0}} P$. Since $\Gamma_D \subset (\Gamma_D)^\circ \subset (\Gamma_{D^0})^\circ$, and since $M_{\mu,\nu} \subset V_{(\Gamma_{D^0})^\circ}$, we have $M_D \cdot P \subset V_{(\Gamma_{D^0})^\circ}$.

If P and Q are inequivalent as M_{D^0}-modules, then there is a unique M_D-module structure on $P \oplus Q$ which extends the M_{D^0}-module structure. If P and Q are equivalent as M_{D^0}-modules, then $P \oplus Q$ is the direct sum of two inequivalent irreducible M_D-modules, both of which are isomorphic to P as M_{D^0}-modules, see [**33**, Proposition 5.2]. Any irreducible M_D-module is obtained in this way. Therefore, the following theorem holds.

Theorem 9.1. *$V_{(\Gamma_{D^0})^\circ}$ contains any irreducible M_D-module.*

10. Examples.

The vertex operator algebra M_D was previously studied for some small k. The first one is the case $k = 2$, where M^0 is the Virasoro vertex operator algebra $L(1/2, 0)$ of central charge $1/2$, and its simple currents are M^0 and $M^1 = L(1/2, 1/2)$. The next one is the case $k = 3$, where M^0 is $L(4/5, 0) \oplus L(4/5, 3)$, and there are three simple currents. These cases were discussed in [**35**] and [**23**], respectively.

In the case $k = 4$, we have $M^0 = V^+_{\sqrt{6}\mathbb{Z}}$ and $M^2 = V^-_{\sqrt{6}\mathbb{Z}}$. So $M_D = V_{\sqrt{6}\mathbb{Z}}$ for $\ell = 1$ and $D = \{(0), (2)\}$. The case $k = 5$ with $\ell = 2$ and $D = \{(00), (12), (24), (31), (43)\}$, and the case $k = 9$ with $\ell = 1$ and $D = \{(0), (3), (6)\}$ were considered in Sections 3.5 and 3.9 of [**30**], respectively.

Let $k = 6$ with $\ell = 1$ and $D = \{(0), (3)\}$. Then

$$M_D = M^0 \oplus M^3 \cong L_{\mathrm{NS}}(5/4, 0) \oplus L_{\mathrm{NS}}(5/4, 3),$$

where $L_{\mathrm{NS}}(c, 0)$ is the simple Neveu–Schwarz algebra of central charge c, and $L_{\mathrm{NS}}(c, h)$ is its irreducible highest weight module with highest weight h, see [**3**, Section 4], [**44**]. In fact, let v be an weight $3/2$ element of M^3 such that $v_{(2)}v = (5/6)\mathbf{1}$. Then $L_n = \omega_{(n+1)}$ and $G_{n-1/2} = v_{(n)}$, $n \in \mathbb{Z}$, satisfy the relations for the Neveu–Schwarz algebra of central charge $5/4$. Thus the subalgebra generated by ω and v in V_{Γ_D} is isomorphic to $L_{\mathrm{NS}}(5/4, 0)$. Moreover, the weight 3 primary vector W^3 of M^0 is a highest weight vector for $L_{\mathrm{NS}}(5/4, 0)$.

Let $k = 8$ with $\ell = 1$ and $D = \{(0), (2), (4), (6)\}$. Then

$$M_D = M^0 \oplus M^2 \oplus M^4 \oplus M^6 \cong L_{\mathrm{NS}}(7/10, 0) \otimes L_{\mathrm{NS}}(7/10, 0)$$

is a simple vertex operator superalgebra, where

$$L_{\mathrm{NS}}(7/10, 0) \cong L(7/10, 0) \oplus L(7/10, 3/2).$$

The even part of M_D is

$$M^0 \oplus M^4 \cong \big(L(7/10, 0) \otimes L(7/10, 0)\big) \oplus \big(L(7/10, 3/2) \otimes L(7/10, 3/2)\big),$$

see [**4**, Theorems 4.14, 4.15], [**30**, Section 3.7].

Appendix. Minimal norm of elements in $N(j, \boldsymbol{a})$.

In this appendix, we calculate the minimal norm of elements in the coset $N(j, \boldsymbol{a})$ of N in N° defined in (4.3). Let $\Omega = \{1, 2, \dots, k\}$, and let $\alpha_S = \sum_{p \in S} \alpha_p$ for a subset S of Ω.

Theorem A.1. *Let $\boldsymbol{a} \in \{0, 1\}^k$ and $0 \le j \le k-1$. Set $I = \mathrm{supp}(\boldsymbol{a})$ and $i = \mathrm{wt}(\boldsymbol{a})$.*
(1) *If $j < i$, then*
 (i) $\min\{\langle \mu, \mu \rangle \mid \mu \in N(j, \boldsymbol{a})\} = (ki - (i - 2j)^2)/2k$,
 (ii) *For $\mu \in N(j, \boldsymbol{a})$, the norm $\langle \mu, \mu \rangle$ is minimal if and only if*

$$\mu = \frac{1}{2}\alpha_I - \alpha_J + \frac{2j - i}{2k}\gamma$$

for some $J \subset I$ with $|J| = j$. There are $\binom{i}{j}$ such μ's.
(2) *If $j \ge i$, then*
 (i) $\min\{\langle \mu, \mu \rangle \mid \mu \in N(j, \boldsymbol{a})\} = (k(k - i) - (k + i - 2j)^2)/2k$,
 (ii) *For $\mu \in N(j, \boldsymbol{a})$, the norm $\langle \mu, \mu \rangle$ is minimal if and only if*

$$\mu = \frac{1}{2}\alpha_I - \alpha_J + \frac{2j - i}{2k}\gamma$$

for some $I \subset J \subset \Omega$ with $|J| = j$. There are $\binom{k-i}{j-i}$ such μ's.

Proof. Any permutation on $\{\alpha_1, \dots, \alpha_k\}$ induces an isometry on $\mathbb{Q} \otimes_{\mathbb{Z}} L$. The isometry fixes γ and leaves L invariant. Since $\lambda_p = \gamma/2k - \alpha_p/2$ and $2\lambda_p \equiv 2\lambda_k \pmod{N}$, $1 \le p \le k$, we may assume that $I = \{1, \dots, i\}$, that is, $a_p = 1$ for $p \le i$, and $a_p = 0$ for $p \ge i + 1$ in (4.3).

Let $d = (2j - i)/2k$. Since $\alpha_p \equiv \alpha_q \pmod{N}$, $1 \le p, q \le k$, and since any element of N is of the form $c_1\alpha_1 + \cdots + c_k\alpha_k$ for some $c_1, \dots, c_k \in \mathbb{Z}$ with $c_1 + \cdots + c_k = 0$, we see from (4.4) that any element $\mu \in N(j, \boldsymbol{a})$ is of the form

$$\begin{aligned}\mu &= \frac{1}{2}(\alpha_1 + \cdots + \alpha_i) - c_1\alpha_1 - \cdots - c_k\alpha_k + d\gamma \\ &= \sum_{p=1}^{i}(d + 1/2 - c_p)\alpha_p + \sum_{q=i+1}^{k}(d - c_q)\alpha_q\end{aligned}$$

for some $c_1, \ldots, c_k \in \mathbb{Z}$ with $c_1 + \cdots + c_k = j$. Our concern is the minimum of

$$\langle \mu, \mu \rangle/2 = \sum_{p=1}^{i}(d + 1/2 - c_p)^2 + \sum_{q=i+1}^{k}(d - c_q)^2 \tag{A.1}$$

for $c_1, \ldots, c_k \in \mathbb{Z}$ with $c_1 + \cdots + c_k = j$.

We first show the assertion (1). Assume that $0 \le j < i \le k$. Then $-1/2 \le d < 1/2$. If $d = -1/2$, then $i = k$ and $j = 0$. In this case, we have $N(j, \boldsymbol{a}) = N$. Clearly, $\min\{\langle \mu, \mu \rangle \mid \mu \in N\} = 0$, and $\langle \mu, \mu \rangle = 0$ only if $\mu = 0$. Hence the assertion (1) holds in the case $d = -1/2$.

If $d = 0$, then $i = 2j$, and (A.1) reduces to $\langle \mu, \mu \rangle/2 = \sum_{p=1}^{i}(1/2 - c_p)^2 + \sum_{q=i+1}^{k} {c_q}^2$. We see that $(1/2 - c_p)^2$ is $1/4$ if $c_p = 0, 1$, and $9/4$ if $c_p = -1, 2$. Moreover, ${c_q}^2$ is 0 if $c_q = 0$, and 1 if $c_q = \pm 1$. Hence the minimum of $\langle \mu, \mu \rangle/2$ for $c_1, \ldots, c_k \in \mathbb{Z}$ with $c_1 + \cdots + c_k = j$ is attained only when j of $c_1, \ldots, c_i$ are 1, the remaining $i - j$ of $c_1, \ldots, c_i$ are 0, and $c_q = 0$ for $i + 1 \le q \le k$. The minimum of $\langle \mu, \mu \rangle/2$ is $i/4$. Thus the assertion (1) holds in the case $d = 0$.

If $-1/2 < d < 0$, then $0 < d + 1/2 < 1/2$. In this case, $(d + 1/2 - c_p)^2$ belongs to one of the four open intervals $(0, 1/4)$, $(1/4, 1)$, $(1, 9/4)$, or $(9/4, 4)$ according as $c_p = 0$, 1, -1, or 2, respectively. Moreover, $(d - c_q)^2$ belongs to one of the four open intervals $(0, 1/4)$, $(1/4, 1)$, $(1, 9/4)$, or $(9/4, 4)$ according as $c_q = 0$, -1, 1, or -2, respectively. Hence the minimum of (A.1) for $c_1, \ldots, c_k \in \mathbb{Z}$ with $c_1 + \cdots + c_j = j$ is attained only when j of $c_1, \ldots, c_i$ are 1, the remaining $i - j$ of $c_1, \ldots, c_i$ are 0, and $c_q = 0$ for $i + 1 \le q \le k$. The minimum of (A.1) is

$$(d - 1/2)^2 j + (d + 1/2)^2(i - j) + d^2(k - i) = i/4 - (i - 2j)^2/4k.$$

Thus the assertion (1) holds in the case $-1/2 < d < 0$.

If $0 < d < 1/2$, then $1/2 < d + 1/2 < 1$. In this case, $(d + 1/2 - c_p)^2$ belongs to one of the four open intervals $(0, 1/4)$, $(1/4, 1)$, $(1, 9/4)$, or $(9/4, 4)$ according as $c_p = 1$, 0, 2, or -1, respectively. Moreover, $(d - c_q)^2$ belongs to one of the four open intervals $(0, 1/4)$, $(1/4, 1)$, $(1, 9/4)$, or $(9/4, 4)$ according as $c_q = 0$, 1, -1, or 2, respectively. Hence the minimum of (A.1) for $c_1, \ldots, c_k \in \mathbb{Z}$ with $c_1 + \cdots + c_k = j$ is attained only when j of $c_1, \ldots, c_i$ are 1, the remaining $i - j$ of $c_1, \ldots, c_i$ are 0, and $c_q = 0$ for $i + 1 \le q \le k$. Thus the assertion (1) holds in the case $0 < d < 1/2$. We have shown that (1) holds for all $0 \le j < i \le k$.

Next, we show the assertion (2). Assume that $j \ge i$. We use Lemma 4.3. Let $a'_p = 1 - a_p$, $1 \le p \le k$, $\boldsymbol{a}' = (a'_1, \ldots, a'_k)$, and $I' = \operatorname{supp}(\boldsymbol{a}')$. Then $I \cup I' = \Omega$ and $I \cap I' = \emptyset$. Let $i' = \operatorname{wt}(\boldsymbol{a}')$ and $j' = j - i$. Then $i' = k - i$ and $0 \le j' < i' \le k$. The assertion (1) for $N(j', \boldsymbol{a}')$ implies that

(i)$'$ $\min\{\langle \mu, \mu \rangle \mid \mu \in N(j', \boldsymbol{a}')\} = (ki' - (i' - 2j')^2)/2k$, (ii)′ For $\mu \in N(j', \boldsymbol{a}')$, the norm $\langle \mu, \mu \rangle$ is minimal if and only if

$$\mu = \frac{1}{2}\alpha_{I'} - \alpha_{J'} + \frac{2j' - i'}{2k}\gamma \tag{A.2}$$

for some $J' \subset I'$ with $|J'| = j'$. There are $\binom{i'}{j'}$ such μ's.

Since $\alpha_{I'} = \gamma - \alpha_I$, and since $2j' - i' = 2j - i - k$, the element μ of (A.2) is equal to

$$\mu = -\frac{1}{2}\alpha_I - \alpha_{J'} + \frac{2j - i}{2k}\gamma.$$

The set $\{J \subset \Omega \mid I \subset J, |J| = j\}$ is in one-to-one correspondence with the set $\{J' \subset \Omega - I \mid |J'| = j - i\}$ by $J \mapsto J - I$ and $J' \mapsto J' \cup I$. Let $J = J' \cup I$. Then $\alpha_J = \alpha_{J'} + \alpha_I$, as $J' \cap I = \emptyset$. Thus the assertion (2) holds. □

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Tomoyuki Arakawa
Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan
E-mail: arakawa@kurims.kyoto-u.ac.jp

Hiromichi Yamada
Department of Mathematics
Hitotsubashi University
Kunitachi, Tokyo 186-8601, Japan
E-mail: yamada.h@r.hit-u.ac.jp

Hiroshi Yamauchi
Department of Mathematics
Tokyo Woman's Christian University
Suginami, Tokyo 167-8585, Japan
E-mail: yamauchi@lab.twcu.ac.jp