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Whitehead products in moment-angle complexes

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Abstract. In toric topology, to a simplicial complex K with m vertices, one associates two spaces, the moment-angle complex $\mathcal{Z}_K$ and the Davis–Januszkiewicz space DJ_K. These spaces are connected by a homotopy fibration $\mathcal{Z}_K \to DJ_K \to (\mathbb{C}P^\infty)^m$. In this paper, we show that the map $\mathcal{Z}_K \to DJ_K$ is identified with a wedge of iterated (higher) Whitehead products for a certain class of simplicial complexes K including dual shellable complexes. We will prove the result in a more general setting of polyhedral products.

1. Introduction.

1.1. Moment-angle complex.

In the seminal work on quasitoric manifolds in toric topology [**4**], Davis and Januszkiewicz constructed a certain space from a simple convex polytope (or equivalently, a dual simplicial convex polytope) as a topological analogue of the hyperplane arrangement appearing in the theory of toric varieties so that every quasitoric manifold is obtained as the quotient of the space by a certain torus action. Later on, the construction of this space is generalized to any simplicial complex as follows. Let K be a simplicial complex on the vertex set $[m] = \{1, \dots, m\}$. The *moment-angle complex* $\mathcal{Z}_K$ is defined as the union of subspaces $Z(\sigma) = \{(z_1, \dots, z_m) \in (D^2)^m \mid |z_i| = 1 \text{ for } i \notin \sigma\}$ of $(D^2)^m$ for all $\sigma \in K$, where $D^2 = \{z \in \mathbb{C} \mid |z| \le 1\}$.

The moment-angle complex is now a fundamental object not only as a source of quasitoric manifolds but also as an object connecting toric topology with a broad area of mathematics including algebraic geometry, algebraic topology, combinatorics, commutative algebra, and geometry. In particular, recent development of the study on the homotopy type of $\mathcal{Z}_K$ in connection with combinatorics and commutative algebra is significant [**7**], [**8**], [**10**], [**11**].

1.2. Object of study.

Davis and Januszkiewicz [**4**] also constructed a supplementary space from a simple convex polytope, and it was also generalized to any simplicial complex. The supplementary space associated with a simplicial complex K is called the *Davis–Januszkiewicz space* and denoted by DJ_K, which is defined as the union of subspaces $DJ(\sigma) = \{(x_1, \dots, x_m) \in (\mathbb{C}P^\infty)^m \mid x_i = * \text{ for } i \notin \sigma\}$ of $(\mathbb{C}P^\infty)^m$ for all $\sigma \in K$, where $* \in \mathbb{C}P^\infty$ is a basepoint.

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By definition, there is a natural action of torus $(S^1)^m$ on $\mathcal{Z}_K$, and the Davis–Januszkiewicz space DJ_K is homotopy equivalent to the Borel construction of this torus action. Then in particular, there is a fundamental homotopy fibration

$$\mathcal{Z}_K \xrightarrow{\widetilde{w}} DJ_K \to (\mathbb{C}P^\infty)^m. \tag{1}$$

The object of study in this paper is the fiber inclusion $\widetilde{w}$.

1.3. Problem.

For a finite set V, let Δ^V denote the simplex with vertex set V and $\partial\Delta^V$ be its boundary. When $K = \partial\Delta^{[2]}$, i.e. K consists of two vertices, we have $\mathcal{Z}_K = S^3$ and $DJ_K = \mathbb{C}P^\infty \vee \mathbb{C}P^\infty$ by definition, and the homotopy fibration (1) coincides with Ganea's homotopy fibration

$$S^3 \to \mathbb{C}P^\infty \vee \mathbb{C}P^\infty \to (\mathbb{C}P^\infty)^2.$$

Then in particular, the map $\widetilde{w}$ is the Whitehead product of the bottom cell inclusion $S^2 \to \mathbb{C}P^\infty$ with itself. More generally, when $K = \partial\Delta^{[m]}$ for general m, the map $\widetilde{w}$ is the higher Whitehead product of m-copies of the bottom cell inclusion $S^2 \to \mathbb{C}P^\infty$. Thus the following problem naturally arises.

PROBLEM 1.1. For which simplicial complex is the map $\widetilde{w}$ described by higher Whitehead products?

Grbić and Theriault [**6**] previously studied this problem by introducing a new class of simplicial complexes that they call directed MF-complexes. However, there is a gap in the proof of the main theorem [**6**, Theorem 13.5]. In Step 4 of the proof, it is claimed that since a subset $R' \subset H_*(\bigvee_{\beta\in\mathcal{J}} S^{t_\beta};\mathbb{Z})$ coincides with a subset $R \subset H_*(\bigvee_{\beta\in\mathcal{J}} S^{t_\beta};\mathbb{Q})$ and R generates $H_*(\bigvee_{\beta\in\mathcal{J}} S^{t_\beta};\mathbb{Q})$ over $\mathbb{Q}$, R' generates $H_*(\bigvee_{\beta\in\mathcal{J}} S^{t_\beta};\mathbb{Z})$ over $\mathbb{Z}$, where an ambiguous term "degree one map" used in the proof can only mean an injective integral map. It is impossible to get such an integral generation as long as we use a rational homology calculation without any implication on integral homology as in [**6**].

In this paper, we show that the map $\widetilde{w}$ is identified with a wedge of iterated higher Whitehead products by applying the fat wedge filtration technology for polyhedral products developed in [**11**], which is completely different from the method of Grbić and Theriault [**6**]. The class of simplicial complexes that we consider includes directed MF-complexes, and so our result implies that the main theorem of Grbić and Theriault [**6**] itself is correct.

1.4. Polyhedral product.

In [**2**], Bahri, Bendersky, Cohen and Gitler unified and generalized the construction of $\mathcal{Z}_K$ and DJ_K, and introduced a space called a *polyhedral product*. Polyhedral products enable us to study the homotopy theory of $\mathcal{Z}_K$ and DJ_K with a wide viewpoint and rich homotopy theoretical techniques.

In our case, the map $\widetilde{w}$ can be defined in a more general setting using polyhedral products so that we will study this generalized map $\widetilde{w}$ in what follows. However, in this introduction, we will state our result only in terms of $\mathcal{Z}_K$ and DJ_K for readability.

1.5. Totally fillable complex.

Now we introduce a simplicial complex for which we are going to study the map $\widetilde{w}$. We set notation. Let L be a simplicial complex with vertex set V. Let $|L|$ denote the geometric realization of L. For a non-empty subset $I \subset V$, define the full subcomplex of L on I by $L_I = \{\sigma \in L \,|\, \sigma \subset I\}$. A subset $\sigma \subset V$ is called a minimal non-face of L if it is not a simplex of L and all of its proper subsets are simplices of L. In particular, if we add minimal non-faces to L, then we get a new simplicial complex.

Definition 1.2. A simplicial complex K is called *fillable* if there is a collection of minimal non-faces $\{\sigma_1, \dots, \sigma_r\}$ such that $|K \cup \sigma_1 \cup \cdots \cup \sigma_r|$ is contractible. If any full subcomplex of K is fillable, then it is called *totally fillable*.

Example 1.3. A typical example of totally fillable complexes is a skeleton of a simplex, and a typical example of simplicial complexes which are not fillable is a square graph.

The collection of minimal non-faces $\{\sigma_1, \dots, \sigma_r\}$ in the above definition is called a *filling* of K and denoted by $\mathcal{F}(K)$, where there are possibly several fillings of a fillable complex K. The class of totally fillable complexes includes dual shellable complexes which are especially important in combinatorics, where we refer to Section 2 for the definition of shellable complexes. As mentioned above, we will see that directed MF-complexes that were considered in the previous work [**6**] are dual shellable, and so they are totally fillable.

1.6. Statement of the result.

The key to study the map $\widetilde{w}$ for a totally fillable complex K is the following homotopy decomposition of $\mathcal{Z}_K$ which was obtained in [**11**].

Theorem 1.4. *Let K be a totally fillable complex on the vertex set $[m]$ with fillings $\mathcal{F}(K_I)$ for $\emptyset \neq I \subset [m]$. Then there is a homotopy equivalence*

$$\mathcal{Z}_K \simeq \bigvee_{\emptyset \neq I \subset [m]} \bigvee_{\sigma \in \mathcal{F}(K_I)} S^{|\sigma|+|I|-1}.$$

Let $\widetilde{a}_i \colon S^2 \to DJ_K$ be the inclusion of the bottom cell of the i-th $\mathbb{C}P^\infty$ in DJ_K. For $\sigma \subset [m]$ with $|\sigma| \geq 2$, let $\widetilde{w}_\sigma$ be the higher Whitehead product of $\widetilde{a}_i$ for $i \in \sigma$ if it is defined, where we refer to Section 3 for the definition of higher Whitehead products. Now we state our result.

Theorem 1.5. *Let K be a totally fillable complex on the vertex set $[m]$ with fillings $\mathcal{F}(K_I)$ for $\emptyset \neq I \subset [m]$. The equivalence of Theorem 1.4 can be chosen so that the composite*

$$S^{|\sigma|+|I|-1} \to \bigvee_{\emptyset \neq I \subset [m]} \bigvee_{\sigma \in \mathcal{F}(K_I)} S^{|\sigma|+|I|-1} \simeq \mathcal{Z}_K \xrightarrow{\widetilde{w}} DJ_K$$

is the iterated Whitehead product

$$[\cdots[[\widetilde{w}_\sigma,\widetilde{a}_{i_1}],\widetilde{a}_{i_2}],\dots,\widetilde{a}_{i_k}],$$

where $i_1,\dots,i_k$ *is a certain ordering of* $I-\sigma$*.*

Remark 1.6. The equivalence in Theorem 1.4 and iterated Whitehead products in Theorem 1.5 depend on the choice of fillings $\mathcal{F}(K_I)$ for all $\emptyset\ne I\subset[m]$.

Remark 1.7. Recently, Abramyan [**1**] showed that in general, $\widetilde{w}$ is not necessarily a wedge of iterated Whitehead products even if $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres.

Throughout this paper, we assume that spaces have non-degenerate basepoints.

2. Fillable complex.

Throughout this paper, let K be a simplicial complex on the vertex set $[m]$. We will assume that a totally fillable complex K is given specific fillings $\mathcal{F}(K_I)$ of K_I for all $\emptyset\ne I\subset[m]$ unless otherwise is specified.

2.1. Deletable complex.

In [**11**], it is proved that dual shellable complexes are totally fillable. The proof there actually shows that dual shellable complexes are in a certain subclass of totally fillable complexes, which we introduce here. A simplicial complex K is called *deletable* if there are facets $\sigma_1,\dots,\sigma_r$ such that $K-\{\sigma_1,\dots,\sigma_r\}$ is collapsible, where r can be 0, i.e. K itself can be collapsible. K is called *totally deletable* if K itself and $\mathrm{lk}_{K_V}(v)$ are deletable for any $\emptyset\ne V\subset[m]$ and $v\in V$, where $\mathrm{lk}_L(w)=\{\sigma\in L\,|\,w\notin\sigma,\,\sigma\cup w\in L\}$ is the link of a vertex w of a simplicial complex L.

Let L be a simplicial complex with ground set S, where the ground set is a superset of the vertex set and possibly they are different. The Alexander dual of L with respect to S, denoted $L^\vee$, is the simplicial complex consisting of $\sigma\subset S$ such that $S-\sigma$ is not a simplex of L. If we do not specify the ground set, then the Alexander dual will be taken over the vertex set. The following dictionary is useful, which is proved in [**11**]. For a vertex v of L, let $\mathrm{dl}_L(v)=\{\sigma\in L\,|\,\sigma\subset S-\{v\}\}$ be the deletion of v.

Proposition 2.1. *Let* L *be a simplicial complex with ground set* S*.*

1. $(L^\vee)^\vee=L$*, where the Alexander duals are taken over* S*.*
2. $\sigma\subset S$ *is a facet of* L *if and only if* $\sigma^\vee$ *is a minimal non-face of* $L^\vee$*, where* $\sigma^\vee=S-\sigma$*.*
3. *For any* $v\in V$*,* $\mathrm{dl}_L(v)^\vee=\mathrm{lk}_{L^\vee}(v)$*, where the Alexander duals of* $\mathrm{dl}_L(v)$ *and* L *are taken over* $S-\{v\}$ *and* S*, respectively.*

The following is proved in [**11**].

Proposition 2.2. *If* K *is collapsible, then* $|K^\vee|$ *is contractible.*

Then by Proposition 2.1 one gets:

Corollary 2.3. *Dual (totally) deletable complexes are (totally) fillable.*

2.2. Shellable complex.

Recall that K is called *shellable* if there is an ordering of facets $\sigma_1, \dots, \sigma_k$ of K, called a *shelling*, such that $\langle \sigma_1, \dots, \sigma_{i-1} \rangle \cap \langle \sigma_i \rangle$ is pure and $(|\sigma_i| - 2)$-dimensional for $i = 2, \dots, k$, where $\langle \tau_1, \dots, \tau_r \rangle$ means the simplicial complex generated by simplices $\tau_1, \dots, \tau_r$ and a simplicial complex is called pure if its facets have the same dimension. Shellable complexes were originally introduced as a combinatorial criterion for Cohen–Macaulayness and are now one of the most important classes of simplicial complexes in combinatorics.

Example 2.4. Any skeleton of a simplex is a shellable complex, where any ordering of its facets is a shelling.

As is seen in [**3**], [**11**], if K is shellable, then K is deletable and the link of any of its vertices is shellable. Then by Proposition 2.1 we get the following.

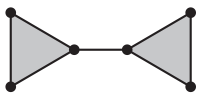
Proposition 2.5. *Shellable complexes are totally deletable.*

By Corollary 2.3, we obtain:

Corollary 2.6. *Dual shellable complexes are totally fillable.*

Example 2.7. Any skeleton of a simplex is shellable as in Example 2.4, and its Alexander dual is again a skeleton of a simplex which is obviously totally fillable.

Example 2.8. Let K be the following simplicial complex with six vertices.



Then K is collapsible, so it is deletable. Moreover, for any vertex v, $\mathrm{lk}_K(v)$ is either the interval graph or the disjoint union of the interval graph and one point. Then $\mathrm{lk}_K(v)$ is shellable for any vertex v, implying that K is totally deletable. However, we see that K itself is not shellable by looking at the middle edge which is a facet. So the class of deletable complexes is strictly larger than that of shellable complexes.

2.3. Directed MF-complex.

In the previous work [**6**], Grbić and Theriault introduced a simplicial complex called a directed MF-complex and studied the map $\widetilde{w}$ for a directed MF-complex K. A simplicial complex K is called a directed MF-complex if there is a filtration of subcomplexes $\emptyset = K_1 \subset K_2 \subset \cdots \subset K_r = K$ such that for $i = 1, \dots, r$, $K_i = K_{i-1} \cup \partial\Delta^{n_i}$, and $K_{i-1} \cap \partial\Delta^{n_i}$ is a common face of K_{i-1} and Δ^{n_i}.

Example 2.9. The k-skeleton of an n-dimensional simplex for $k > 0$ is a directed MF-complex if and only if $k = n - 1$.

We shall show that directed MF-complexes are dual shellable. For this, we will use the following lemma.

Lemma 2.10. *Suppose that there is an ordering of minimal non-faces* $\sigma_1 < \cdots < \sigma_r$ *of* K *such that for any* $i < j$*, there is* $k < j$ *satisfying that* $\sigma_k \cup \sigma_j \subset \sigma_i \cup \sigma_j$ *and* $|\sigma_k \cup \sigma_j| = |\sigma_j| + 1$*. Then the ordering of facets* $\sigma_1^\vee < \cdots < \sigma_r^\vee$ *of* $K^\vee$ *is a shelling.*

Proof. The assumption is equivalent to that for any $i < j$, there is $k < j$ satisfying that $\sigma_k^\vee \cap \sigma_j^\vee \supset \sigma_i^\vee \cap \sigma_j^\vee$ and $\sigma_k^\vee \cap \sigma_j^\vee$ is $(m - |\sigma_j| - 2)$-dimensional. Then we get that for $j \geq 2$, $\langle \sigma_1^\vee, \ldots, \sigma_{j-1}^\vee \rangle \cap \langle \sigma_j^\vee \rangle$ is pure and $(m - |\sigma_j| - 2)$-dimensional, completing the proof. □

Proposition 2.11. *Directed MF-complexes are dual shellable.*

Proof. Let K be a directed MF-complex. Then there is a filtration $\emptyset = K_0 \subset K_1 \subset \cdots \subset K_r = K$ such that $K_i = K_{i-1} \cup \partial\Delta^{\sigma_i}$ and $K_{i-1} \cap \partial\Delta^{\sigma_i}$ is a common face of K_{i-1} and Δ^{σ_i}. The filtration induces an ordering $\sigma_1 < \cdots < \sigma_r$. We consider an ordering of the vertex set induced by this ordering of simplices with $v < w$ whenever $v \in \sigma_i$ and $w \in \sigma_{i+1}$.

Let I be the set of all 1-dimensional minimal non-faces of K and put $\{\tau_1, \ldots, \tau_s\} = \{\sigma_1, \ldots, \sigma_r\} - I$, where $\tau_1 < \cdots < \tau_s$. Then all minimal non-faces of dimension > 1 are included in $\{\tau_1, \ldots, \tau_s\}$. Consider the lexicographic ordering on I such that $\{k, l\} < \{k', l'\} \in I$ if $k < k'$ or $k = k', l < l'$. Then the ordering $I < \tau_1 < \cdots < \tau_s$ satisfies the condition of Lemma 2.10, where $I \sqcup \{\tau_1, \ldots, \tau_s\}$ is the set of all minimal non-faces of K. Thus the proof is done. □

Summarizing, we have obtained the implications:

$$\text{directed MF} \quad \Rightarrow \quad \text{dual shellable} \quad \Rightarrow \quad \text{dual totally deletable} \quad \Rightarrow \quad \text{totally fillable.}$$

2.4. Homotopy type.

It is observed in [**11**] that if K is fillable, then $|\Sigma K|$ is homotopy equivalent to a wedge of spheres. Here we consider the naturality of this homotopy equivalence which will be used later. For a fillable complex K, we put $\overline{K} = K \bigcup_{\sigma \in \mathcal{F}(K)} \sigma$, where $\mathcal{F}(K)$ is defined in Section 1.

Proposition 2.12. *If* K *is fillable with filling* $\mathcal{F}(K)$*, then there is a homotopy equivalence*

$$|\Sigma K| \simeq \bigvee_{\sigma \in \mathcal{F}(K)} S^{|\sigma|-1}$$

such that for a fillable subcomplex L *of* K *with filling* $\mathcal{F}(L)$ *satisfying* $\mathcal{F}(L) \subset \mathcal{F}(K) \cup K$*, the square diagram*

$$\begin{array}{ccc} |\Sigma L| & \xrightarrow{\simeq} & \bigvee_{\tau \in \mathcal{F}(L)} S^{|\tau|-1} \\ \downarrow & & \downarrow \widetilde{g} \\ |\Sigma K| & \xrightarrow{\simeq} & \bigvee_{\sigma \in \mathcal{F}(K)} S^{|\sigma|-1} \end{array}$$

commutes, where $\widetilde{g}|_{S^{|\tau|-1}}$ *is the inclusion for* $\tau \in \mathcal{F}(K) \cap \mathcal{F}(L)$ *and the constant map otherwise.*

PROOF. Since $|\overline{K}|$ is contractible, there is a homotopy equivalence $|CK| \to |\overline{K}|$ which restricts to the identity map of $|K|$. Then we get the desired homotopy equivalence by pinching $|K|$ to a point. The assumption on L is equivalent to that $\overline{L}$ is a subcomplex of $\overline{K}$, so one gets the commutative square in the statement. □

2.5. Contraction ordering.

We define a contraction ordering of vertices of a fillable complex. Let V be a finite set and $S \subset V$ be a subset with $|S| \geq 2$. Let L be a simplicial complex with vertex set V obtained by attaching trees $T_1, \ldots, T_k$ to $\partial\Delta^S$ by their roots. Let V_i be the vertex set of T_i and $r_i \in V_i \cap S$ be the root of T_i. Then one has $V = S \sqcup (V_1 - r_1) \sqcup \cdots \sqcup (V_k - r_k)$. An ordering $v_1 < \cdots < v_n$ of $V_i - r_i$ is called a *local contraction ordering* if the full subcomplex $(T_i)_{V_i - \{v_j, \ldots, v_n\}}$ is connected for any $j = 1, \ldots, n$. An ordering of $V - S$ is called a *contraction ordering* if it is the union of local contraction orderings of $V_i - r_i$. Note that a deformation retract of $|L|$ onto $|\partial\Delta^S|$ is given by a contraction ordering.

For a finite set V and its non-empty subset S, let $\Delta(V, S)$ be the simplicial complex which is the disjoint union of $\partial\Delta^S$ and vertices $v \in V - S$.

PROPOSITION 2.13. *If K is fillable and $\sigma \in \mathcal{F}(K)$, then there are trees $T_1, \ldots, T_k$ such that there is a subcomplex of $\overline{K}$ with vertex set $[m]$ obtained by attaching $T_1, \ldots, T_k$ to $\partial\Delta^\sigma$ by their roots.*

PROOF. Choose any maximal tree of $\overline{K}$. Then since $\overline{K}$ is connected, the vertex set of T is $[m]$. If we remove all edges of $\partial\Delta^\sigma$ from T, then we get a collection of trees which gives a desired subcomplex. □

Then we can define a contraction ordering of $[m] - \sigma$ for a fillable complex K and $\sigma \in \mathcal{F}(K)$.

3. Polyhedral products and the map w.

3.1. Polyhedral product.

Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be a collection of pairs of spaces. The polyhedral product of $(\underline{X}, \underline{A})$ associated with K is defined in [**2**] as

$$Z(K; (\underline{X}, \underline{A})) = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \subset \prod_{i=1}^m X_i,$$

where $(\underline{X}, \underline{A})^\sigma = Y_1 \times \cdots \times Y_m$ such that $Y_i = X_i$ for $i \in \sigma$ and $Y_i = A_i$ for $i \notin \sigma$. The most fundamental property of polyhedral products, first observed in [**5**], is the following which we will use implicitly, where we omit the proof because it is obvious. For $\emptyset \neq I \subset [m]$, let $(\underline{X}, \underline{A})_I = \{(X_i, A_i)\}_{i \in I}$.

PROPOSITION 3.1. *For $\emptyset \neq I \subset [m]$, $Z(K_I; (\underline{X}, \underline{A})_I)$ is a retract of $Z(K; (\underline{X}, \underline{A}))$.*

If all (X_i, A_i) are (D^2, S^1) (resp. $(\mathbb{C}P^\infty, *)$), the resulting polyhedral product is the moment-angle complex $\mathcal{Z}_K$ (resp. DJ_K). Hereafter, let $\underline{X} = \{X_i\}_{i=1}^m$ be a collection of pointed spaces. We will generalize the map $\widetilde{w}\colon \mathcal{Z}_K \to DJ_K$ to the polyhedral products

$$\mathcal{Z}_K(\underline{X}) = Z(K; (C\underline{X}, \underline{X})) \quad \text{and} \quad DJ_K(\underline{X}) = Z(K; (\underline{X}, *))$$

which are generalization of $\mathcal{Z}_K$ and DJ_K, respectively, where $(C\underline{X}, \underline{X}) = \{(CX_i, X_i)\}_{i=1}^m$ and $(\underline{X}, *) = \{(X_i, *)\}_{i=1}^m$. Here we remark that the same notation $\mathcal{Z}_K(\underline{X})$ is used in [**6**] to express a different polyhedral product $Z(K; (C\Omega\underline{X}, \Omega\underline{X}))$, where $\Omega\underline{X} = \{\Omega X_i\}_{i=1}^m$.

3.2. Decomposition of the map $\widetilde{w}$.

As in [**11**], there is a homotopy fibration

$$\mathcal{Z}_K(\Omega\underline{X}) \xrightarrow{\widetilde{w}} DJ_K(\underline{X}) \to \prod_{i=1}^m X_i \tag{2}$$

which specializes to the homotopy fibration (1). We decompose the map $\widetilde{w}$ to clarify the point of our study.

Let $\Omega X_i \to PX_i \to X_i$ be the path-loop fibration. Then for each i, there is a pair of fibrations $(PX_i, \Omega X_i) \to (X_i^{[0,1]}, PX_i) \to (X_i, X_i)$, where the second map is the evaluation at 1, and as in [**5**], [**11**], this induces a homotopy fibration

$$Z(K; (P\underline{X}, \Omega\underline{X})) \to Z(K; (\underline{X}^{[0,1]}, P\underline{X})) \to \prod_{i=1}^m X_i. \tag{3}$$

The maps $C\Omega X_i \to PX_i$, $(s, l) \mapsto [t \mapsto l((1-s)t)]$ and the evaluations $X_i^{[0,1]} \to X_i$ at 0 induce homotopy equivalences $Z(K; (P\underline{X}, \Omega\underline{X})) \simeq \mathcal{Z}_K(\Omega\underline{X})$ and $Z(K; (\underline{X}^{[0,1]}, P\underline{X})) \simeq DJ_K(\underline{X})$. Then by applying these homotopy equivalences to (3), one gets the homotopy fibration (2). Hence one gets the following. Let $w\colon \mathcal{Z}_K(\underline{X}) \to DJ_K(\Sigma\underline{X})$ be the map induced by the maps of pairs $(CX_i, X_i) \to (\Sigma X_i, *)$, where $\Sigma\underline{X} = \{\Sigma X_i\}_{i=1}^m$ and $CX_i \to \Sigma X_i$ is the pinch map.

Proposition 3.2. *The map $\widetilde{w}\colon \mathcal{Z}_K(\Omega\underline{X}) \to DJ(\underline{X})$ is the composite of maps*

$$\mathcal{Z}_K(\Omega\underline{X}) \xrightarrow{w} DJ_K(\Sigma\Omega\underline{X}) \to DJ_K(\underline{X}),$$

where the second map is induced from the evaluation maps $\Sigma\Omega X_i \to X_i$.

Thus we study the map w and apply its properties to understand the map $\widetilde{w}$. By definition, the map w has the following naturality.

Proposition 3.3. *For a subcomplex L of K on the same vertex set $[m]$, the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{Z}_L(\underline{X}) & \xrightarrow{w} & DJ_L(\Sigma\underline{X}) \\ \downarrow & & \downarrow \\ \mathcal{Z}_K(\underline{X}) & \xrightarrow{w} & DJ_K(\Sigma\underline{X}) \end{array}$$

3.3. Higher Whitehead product.

Suppose that K consists only of two vertices, where $m = 2$. Then we have $\mathcal{Z}_K(\underline{X}) = X_1 * X_2$ and $DJ_K(\Sigma\underline{X}) = \Sigma X_1 \vee \Sigma X_2$ so that the map $w\colon \mathcal{Z}_K(\underline{X}) \to DJ_K(\Sigma\underline{X})$ is by definition the (generalized) Whitehead product of the inclusions $\Sigma X_i \to DJ_K(\Sigma\underline{X})$ for $i = 1, 2$, where $X * Y$ means the join of spaces X and Y.

Suppose next that $K = \partial\Delta^{[m]}$ for general m. Then we have $\mathcal{Z}_K(\underline{X}) = X_1 * \cdots * X_m$ and $DJ_K(\Sigma\underline{X})$ is the fat wedge of ΣX_i, which is the subspace of $\prod_{i=1}^m \Sigma X_i$ consisting of points $(x_1, \ldots, x_m)$, where at least one x_i is the basepoint. Porter [**15**] defined the universal higher Whitehead product of the inclusions $a_i\colon \Sigma X_i \to DJ_K(\Sigma\underline{X})$ for $i = 1, \ldots, m$ by the map $w\colon \mathcal{Z}_K(\underline{X}) \to DJ_K(\Sigma\underline{X})$ in this special case that K is the boundary of $\Delta^{[m]}$.

We finally consider general K. Suppose that $\sigma \subset [m]$ is a minimal non-face of K. Then there is the inclusion $DJ_{\partial\Delta^\sigma}(\Sigma\underline{X}_\sigma) \to DJ_K(\Sigma\underline{X})$, where $\underline{X}_\sigma = \{X_i\}_{i\in\sigma}$. Let $a_i\colon \Sigma X_i \to DJ_K(\Sigma\underline{X})$ be the inclusion for $i = 1, \ldots, m$. Then the higher Whitehead product of the inclusions a_i for $i \in \sigma$ is defined as the composite $\mathcal{Z}_K(\underline{X}_\sigma) \xrightarrow{w} DJ_{\partial\Delta^\sigma}(\Sigma\underline{X}_\sigma) \to DJ_K(\Sigma\underline{X})$, which we write w_σ.

4. Fat wedge filtration.

4.1. Definition.

For a collection of pointed spaces $\underline{Y} = \{Y_i\}_{i=1}^m$, let $T^i(\underline{Y})$ be the subspace of $\prod_{i=1}^m Y_i$ consisting of points $(y_1, \ldots, y_m)$ such that at least $m - i$ of y_j are the basepoints, where $T^i(\underline{Y})$ are called the *generalized fat wedge* of Y_i. Put $\mathcal{Z}_K^i(\underline{X}) = \mathcal{Z}_K(\underline{X}) \cap T^i(C\underline{X})$. Then there is a filtration

$$* = \mathcal{Z}_K^0(\underline{X}) \subset \mathcal{Z}_K^1(\underline{X}) \subset \cdots \subset \mathcal{Z}_K^m(\underline{X}) = \mathcal{Z}_K(\underline{X})$$

which we call the *fat wedge filtration* of $\mathcal{Z}_K(\underline{X})$. The fat wedge filtration of $\mathcal{Z}_K(\underline{X})$ is studied in [**11**]: the fat wedge filtration connects the homotopy type of $\mathcal{Z}_K(\underline{X})$ and the combinatorics of a simplicial complex K, and produces application of homotopical technique to combinatorics.

4.2. Cone decomposition.

In [**11**], it is shown that if all X_i are suspensions, then the fat wedge filtration of $\mathcal{Z}_K(\underline{X})$ is a cone decomposition with explicitly described attaching maps. We recall this result here. Let $\mathbb{R}\mathcal{Z}_K$ be the polyhedral product $\mathcal{Z}_K(\underline{X})$ such that all X_i are S^0, which we call the *real moment-angle complex*. We first recall from [**11**] properties of the fat wedge filtration of $\mathbb{R}\mathcal{Z}_K$. We denote the i-th filter of the fat wedge filtration of $\mathbb{R}\mathcal{Z}_K$ by $\mathbb{R}\mathcal{Z}_K^i$.

THEOREM 4.1. *For any* $\emptyset \neq I \subset [m]$, *there is a map* $\varphi_{K_I}\colon |K_I| \to \mathbb{R}\mathcal{Z}_{K_I}^{|I|-1}$ *satisfying the following properties*:

1. $\mathbb{R}\mathcal{Z}_K^i$ *is obtained from* $\mathbb{R}\mathcal{Z}_K^{i-1}$ *by attaching cones by* φ_{K_I} *for* $|I| = i$ *so that*

$$\mathbb{R}\mathcal{Z}_K^i = \mathbb{R}\mathcal{Z}_K^{i-1} \bigcup_{I\subset[m],\,|I|=i} C|K_I|.$$

2. *If* L *is a subcomplex of* K, *then the following diagram commutes, where the vertical arrows are the inclusions.*

$$\begin{array}{ccc} |L_I| & \xrightarrow{\varphi_{L_I}} & \mathbb{R}Z_{L_I}^{|I|-1} \\ \downarrow & & \downarrow \\ |K_I| & \xrightarrow{\varphi_{K_I}} & \mathbb{R}Z_{K_I}^{|I|-1} \end{array}$$

3. *Let* $\widehat{K}_I$ *be the simplicial complex obtained from* K_I *by adding all of its minimal non-faces. Then* φ_{K_I} *factors through the inclusion* $|K_I| \to |\widehat{K}_I|$.

The fat wedge filtration of $\mathcal{Z}_K(\underline{X})$ is not a cone decomposition in general unlikely to $\mathbb{R}\mathcal{Z}_K$ in Theorem 4.1. However, as mentioned above, it is indeed a cone decomposition whenever all X_i are suspensions. This is proved in [**11**] only for the moment-angle complex $\mathcal{Z}_K$, but it can be proved in the general case by the same construction using higher Whitehead product. We demonstrate it here. Define a map $\widetilde{\Phi}\colon I^m \times \prod_{i=1}^m X_i \to \prod_{i=1}^m CX_i$ by $\widetilde{\Phi}(t_1,\dots,t_m,x_1,\dots,x_m) = ((t_1,x_1),\dots,(t_m,x_m))$. Then $\widetilde{\Phi}$ restricts to a map $\Phi\colon \mathbb{R}\mathcal{Z}_K \times \prod_{i=1}^m X_i \to \mathcal{Z}_K(\underline{X})$ such that

$$\Phi^{-1}(\mathcal{Z}_K^{m-1}(\underline{X})) = (\mathbb{R}\mathcal{Z}_K \times T^{m-1}(\underline{X})) \cup \left(\mathbb{R}\mathcal{Z}_K^{m-1} \times \prod_{i=1}^m X_i\right).$$

If $\underline{X} = \Sigma\underline{Y}$ for $\underline{Y} = \{Y_i\}_{i=1}^m$, then there is the higher Whitehead product $\omega\colon Y^{*[m]} \to T^{m-1}(\underline{X})$, where $Y^{*[m]} = Y_1 * \cdots * Y_m$. Now we define the map $\overline{\varphi}_K\colon |K| * Y^{*[m]} \to \mathcal{Z}_K^{m-1}(\underline{X})$ by the composite

$$\begin{aligned} &|K| * Y^{*[m]} = (C|K| \times Y^{*[m]}) \cup (|K| \times C(Y^{*[m]})) \\ &\qquad \xrightarrow{(C\varphi_K\times\omega)\cup(\varphi_K\times C\omega)} (\mathbb{R}\mathcal{Z}_K \times T^{m-1}(\underline{X})) \cup \left(\mathbb{R}\mathcal{Z}_K^{m-1} \times \prod_{i=1}^m X_i\right) \xrightarrow{\Phi} \mathcal{Z}_K^{m-1}(\underline{X}). \end{aligned}$$

THEOREM 4.2. *If* $\underline{X} = \Sigma\underline{Y}$, *then the fat wedge filtration of* $\mathcal{Z}_K(\underline{X})$ *is a cone decomposition such that*

$$\mathcal{Z}_K^i(\underline{X}) = \mathcal{Z}_K^{i-1}(\underline{X}) \bigcup_{I\subset[m],\,|I|=i} C(|K_I| * Y^{*I}),$$

where the attaching maps are $\overline{\varphi}_{K_I}$.

It is shown in [**11**] that if $\varphi_{K_I} \simeq *$ for any I, then $\overline{\varphi}_{K_I} \simeq *$ for any I as a consequence of a more general result, where φ_{K_I} is as in Theorem 4.1. We will prove this fact by a more direct argument, which enables us to consider the naturality among null homotopies.

Proposition 4.3. *If φ_K is null homotopic, then so is $\overline{\varphi}_K$. Moreover, if a null homotopy of φ_K restricts to that of φ_L for a subcomplex $L \subset K$, then we may choose a null homotopy of $\overline{\varphi}_K$ such that it restricts to that of $\overline{\varphi}_L$.*

Proof. Suppose that $\varphi_K \simeq *$ and we fix a null homotopy. Then the map $(C\varphi_K \times \omega) \cup (\varphi_K \times C\omega)$ in the definition of $\overline{\varphi}_K$ is homotopic to the composite

$$(C|K| \times Y^{*[m]}) \cup (|K| \times C(Y^{*[m]})) \to (|\Sigma K| \times Y^{*[m]}) \cup (* \times C(Y^{*[m]}))$$
$$\xrightarrow{(f\times\omega)\cup(*\times C\omega)} (\mathbb{R}\mathcal{Z}_K \times T^{m-1}(\underline{X})) \cup \left(\mathbb{R}\mathcal{Z}_K^{m-1} \times \prod_{i=1}^m X_i\right)$$

for a map $f\colon |\Sigma K| \to \mathbb{R}\mathcal{Z}_K$ defined by gluing $C\varphi_K$ and the null homotopy of φ_K. Then for $\Phi(\mathbb{R}\mathcal{Z}_K \vee \prod_{i=1}^m X_i) = *$, the map $\overline{\varphi}_K$ factors through the map $f \wedge \omega\colon |\Sigma K| \wedge Y^{*[m]} \to \mathbb{R}\mathcal{Z}_K \wedge T^{m-1}(\underline{X})$. Note that $f \wedge \omega = (f \wedge 1_{Y^{*[m]}}) \circ (1_{|\Sigma K|} \wedge \omega) = (f \wedge 1_{Y^{*[m]}}) \circ \Sigma(1_{|K|} \wedge \omega)$. For $\Sigma\omega \simeq *$, one gets $\Sigma(1_{|K|} \wedge \omega) \simeq *$ so that $\overline{\varphi}_K \simeq *$ as desired. The naturality of null homotopies is obvious by the above deformation of maps. □

4.3. Homotopy decomposition.

We apply Theorem 4.2 to obtain a homotopy decomposition of $\mathcal{Z}_K(\underline{X})$ together with its naturality. To this end, we will use the following simple lemma, where the proof is easy and omitted.

Lemma 4.4. *If a map $\varphi\colon A \to X$ is null homotopic, then there is a homotopy equivalence*

$$\epsilon_\varphi\colon X \vee \Sigma A \xrightarrow{\simeq} X \cup_\varphi CA$$

which is natural with respect to φ and its null homotopy.

By Theorem 4.2, Proposition 4.3 and Lemma 4.4, one gets:

Corollary 4.5. *Suppose that $\underline{X} = \Sigma\underline{Y}$. If $\varphi_{K_I} \simeq *$ for any $\emptyset \neq I \subset [m]$, then there is a homotopy equivalence*

$$\epsilon_K\colon \mathcal{Z}_K(\underline{X}) \xrightarrow{\simeq} \bigvee_{\emptyset\neq I\subset[m]} |\Sigma K_I| \wedge \widehat{X}^I,$$

where $\widehat{X}^I = \bigwedge_{i\in I} X_i$. Moreover, if L is a subcomplex of K with vertex set $[m]$ such that a null homotopy of φ_{K_I} restricts to that of φ_{L_I} for any $\emptyset \neq I \subset [m]$, up to homotopy, then there is a homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_L(\underline{X}) & \xrightarrow{\epsilon_L} & \bigvee_{\emptyset \ne I \subset [m]} |\Sigma L_I| \wedge \widehat{X}^I \\ \downarrow & & \downarrow \\ \mathcal{Z}_K(\underline{X}) & \xrightarrow{\epsilon_K} & \bigvee_{\emptyset \ne I \subset [m]} |\Sigma K_I| \wedge \widehat{X}^I, \end{array}$$

where the vertical arrows are inclusions.

5. Main theorem and its proof.

5.1. Main theorem.

We first show the homotopy decomposition of $\mathcal{Z}_K(\underline{X})$ for a totally fillable complex K. By Theorem 4.1, we have:

Lemma 5.1. *If K is totally fillable, then $\varphi_{K_I} \simeq *$ for any $\emptyset \ne I \subset [m]$.*

For a totally fillable complex K, we put

$$W_K(\underline{X}) = \bigvee_{\emptyset \ne I \subset [m]} \bigvee_{\sigma \in \mathcal{F}(K_I)} \Sigma^{|\sigma|-1} \widehat{X}^I.$$

Then by Proposition 2.12, Corollary 4.5 and Lemma 5.1, one gets the following homotopy decomposition which specializes to Theorem 1.4 by putting $X_i = S^1$ for all i.

Theorem 5.2. *If K is totally fillable and $\underline{X} = \Sigma \underline{Y}$, then there is a homotopy equivalence*

$$\epsilon_K \colon \mathcal{Z}_K(\underline{X}) \xrightarrow{\simeq} W_K(\underline{X}).$$

Remark 5.3. As is seen in [**11**], the assumption $\underline{X} = \Sigma \underline{Y}$ in Theorem 5.2 is redundant to get the decomposition. But under this assumption, we can construct ϵ_K explicitly as above, which gives us its naturality that will be used to prove the main theorem.

Remark 5.4. The homotopy equivalence ϵ_K depends on the choice of $\mathcal{F}(K_I)$ and contraction ordering of $I - \sigma$ for $\sigma \in \mathcal{F}(K_I)$.

Now we state the main theorem. For a totally fillable complex K, we fix a contraction ordering of $I - \sigma$ for each $\sigma \in \mathcal{F}(K_I)$ and $\emptyset \ne I \subset [m]$. Let $a_i \colon X_i \to DJ_K(\underline{X})$ be the inclusion for $i = 1, \ldots, m$ as above. Now we state the main theorem.

Theorem 5.5. *Suppose that $\underline{X} = \Sigma \underline{Y}$ and K is a totally fillable complex. Then for $\sigma \in \mathcal{F}(K_I)$, the composite*

$$\Sigma^{|\sigma|-1} \widehat{X}^I \to W_K(\underline{X}) \xrightarrow{\epsilon_K^{-1}} \mathcal{Z}_K(\underline{X}) \xrightarrow{w} DJ_K(\Sigma \underline{X})$$

is the iterated Whitehead product

$$[[\cdots[w_\sigma, a_{i_1}],\cdots], a_{i_k}]$$

up to permutation of the smash factors of $\Sigma^{|\sigma|-1}\widehat{X}^I$, *where* $i_1 < \cdots < i_k$ *is a contraction ordering of* $I-\sigma$ *and* w_σ *is the higher Whitehead product defined in Section* 3.

Remark 5.6. As in Remark 5.4, a different choice of $\mathcal{F}(K_I)$ and contraction ordering may produce a different equivalence ϵ_K so that the appearing Whitehead products may change.

Let $\widetilde{a}_i\colon S^2 \to DJ_K$ be the inclusion of the bottom cell of the i-th $\mathbb{C}P^\infty \to DJ_K$. For a minimal non-face σ of K, let $\widetilde{w}_\sigma$ be the composite

$$\mathcal{Z}_{\partial\Delta^\sigma} \xrightarrow{w_\sigma} DJ_K(S^2) \to DJ_K,$$

where the second arrow is induced from the bottom cell inclusion $S^2 \to \mathbb{C}P^\infty$. Then $\widetilde{w}_\sigma$ is the higher Whitehead product of $\widetilde{a}_i$ for $i \in \sigma$. The following is immediate from Theorem 5.5 and the naturality of (higher) Whitehead products.

Corollary 5.7. *If* K *is a totally fillable complex, then for* $\sigma \in \mathcal{F}(K_I)$, *the composite*

$$S^{|\sigma|+|I|-1} \to \bigvee_{\emptyset\neq I\subset[m]}\ \bigvee_{\sigma\in\mathcal{F}(K_I)} S^{|\sigma|+|I|-1} \xrightarrow{\epsilon_K^{-1}} \mathcal{Z}_K \xrightarrow{\widetilde{w}} DJ_K$$

is the iterated Whitehead product

$$[[\cdots[\widetilde{w}_\sigma, \widetilde{a}_{i_1}],\cdots], \widetilde{a}_{i_k}],$$

where $i_1 < \cdots < i_k$ *is a contraction ordering of* $I-\sigma$.

5.2. Proof of Theorem 5.5.

Let ϵ_K be the homotopy equivalence of Theorem 5.2. The following naturality of ϵ_K is obvious by its construction.

Corollary 5.8. *The homotopy equivalence* ϵ_K *retracts to* ϵ_{K_I} *for any* $\emptyset \neq I \subset [m]$.

Corollary 5.9. *Suppose that* K *is totally fillable and* $\underline{X} = \Sigma\underline{Y}$. *The homotopy equivalence* ϵ_K *satisfies a homotopy commutative diagram*

$$\begin{array}{ccc} \mathcal{Z}_{\Delta([m],\sigma)}(\underline{X}) & \longrightarrow & \mathcal{Z}_K(\underline{X}) \\ \downarrow{\scriptstyle \epsilon_{\Delta([m],\sigma)}} & & \downarrow{\scriptstyle \epsilon_K} \\ W_{\Delta([m],\sigma)}(\underline{X}) & \xrightarrow{g} & W_K(\underline{X}) \end{array}$$

for $\sigma \in \mathcal{F}(K)$, *where* g *restricts to the identity map of* $\Sigma^{|\sigma|-1}\widehat{X}^{[m]}$.

Proof. The null homotopy of φ_{K_I} is given by the contraction of $|\overline{K_I}|$ which restricts to a contraction of $|\overline{\Delta([m],\sigma)_I}|$ given by a contraction ordering. Then the corollary follows from Corollary 4.5. □

For $k < m$, put $\widetilde{Z}(k) = (\mathcal{Z}_{\Delta([m-1],[k])}(\underline{X}_{[m-1]}) \times X_m) \cup (* \times CX_m)$ and $\widetilde{Z}^i(k) = \widetilde{Z}(k) \cap T^i(C\underline{X})$. Then the following is clear from the definition of $\overline{\varphi}_K$.

Proposition 5.10. *If $\underline{X} = \Sigma\underline{Y}$, then for each $I \subset [m]$ with $I \neq \emptyset, \{m\}$, the map $\overline{\varphi}_{\Delta([m],[k])_I}$ restricts to a map $\widetilde{\varphi}_I \colon |\Delta([m-1],[k])_I| * Y^{*I} \to \widetilde{Z}^{|I|-1}(k)$ such that*

$$\widetilde{Z}^i(k) = \widetilde{Z}^{i-1}(k) \bigcup_{I \subset [m],\, |I|=i} C(|\Delta([m-1],[k])_I| * Y^{*I}),$$

where the attaching maps are $\widetilde{\varphi}_I$.

As mentioned above, $\Delta([m],[k])$ is totally fillable. Put $\mathcal{F}(\Delta([m-1],[k])_I) = \mathcal{F}(\Delta([m],k)_I)$ for any $\emptyset \neq I \subset [m-1]$. Then any null homotopy of $\overline{\varphi}_{\Delta([m],[k])_I}$ given by a contraction ordering induces a null homotopy of $\widetilde{\varphi}_I$. Put

$$\widetilde{W}(k) = \bigvee_{\emptyset \neq I \subset [m-1]} \bigvee_{\sigma \in \mathcal{F}(\Delta([m],[k])_I)} (\Sigma^{|\sigma|-1}\widehat{X}^I \vee \Sigma^{|\sigma|-1}\widehat{X}^{I \cup \{m\}}).$$

Then by Proposition 5.10, one gets:

Corollary 5.11. *If $\underline{X} = \Sigma\underline{Y}$, then any null homotopy of $\overline{\varphi}_{\Delta([m],[k])_I}$ given by a contraction ordering induces a homotopy equivalence $\widetilde{\epsilon} \colon \widetilde{Z}(k) \to \widetilde{W}(k)$ satisfying a homotopy commutative diagram*

$$\begin{array}{ccc} \widetilde{Z}(k) & \longrightarrow & \mathcal{Z}_{\Delta([m],[k])}(\underline{X}) \\ \downarrow{\scriptstyle \widetilde{\epsilon}} & & \downarrow{\scriptstyle \epsilon_{\Delta([m],[k])}} \\ \widetilde{W}(k) & \longrightarrow & W_{\Delta([m],[k])}(\underline{X}), \end{array}$$

where the horizontal arrows are inclusions.

We will use the following lemma to show the naturality of the homotopy equivalence $\widetilde{\epsilon}$. To state the lemma, we set notation. Let A, B be pointed spaces. Put $A \rtimes B = (A \times B)/(* \times B)$. Let $\pi \colon CB \to \Sigma B$, $\bar{q} \colon A * B/CB \to \Sigma A \wedge B$, $r \colon A * B \to A * B/CB$, and $p' \colon A \rtimes \Sigma B \to A \wedge \Sigma B$ be the obvious projections. Then q and $\bar{q}$ are homotopy equivalences, and $q = \bar{q} \circ r$. Put $p = q^{-1} \circ p'$.

Lemma 5.12. *Given a map $f \colon (CA, A) \to (V, W)$, there is a homotopy commutative diagram*

$$\begin{array}{ccc} A \rtimes \Sigma B & \xrightarrow{f \rtimes 1} & W \rtimes \Sigma B \\ \downarrow{\scriptstyle p} & & \downarrow \\ A * B & \xrightarrow{\bar{f}} & (V \rtimes *) \cup (W \rtimes \Sigma B), \end{array}$$

where $\bar{f}$ *is the composite* $A * B \xrightarrow{f \times \pi} (V \times *) \cup (W \times \Sigma Z) \xrightarrow{\text{proj}} (V \rtimes *) \cup (W \rtimes \Sigma Z)$.

Proof. Let $c_t \colon CA \to CA$ be a contraction. Since $(A * B, (CA \times B) \cup (* \times CB))$ is an NDR-pair [**14**], $(A * B/CB, CA \rtimes B)$ is an NDR-pair too. Then by applying the homotopy extension property, we get an extension $h_t \colon A * B/CB \to A * B/CB$ of a contraction $c_t \rtimes 1 \colon CA \rtimes B \to CA \rtimes B$ such that h_0 is the identity map of $A * B/CB$. Thus there is a homotopy commutative diagram

$$\begin{array}{ccc} A \rtimes CB & \xrightarrow{f \rtimes \pi} & W \rtimes \Sigma B \\ \downarrow{\scriptstyle h_1 \circ j} & & \downarrow \\ A * B/CB & \xrightarrow{f \rtimes \pi} & (V \rtimes *) \cup (W \rtimes \Sigma B) \end{array}$$

such that a commuting homotopy is $(f \rtimes \pi) \circ h_t \circ j$, where $j \colon A \rtimes CB \to A * B/CB$ is the inclusion. By definition, the map h_1 decomposes as

$$A * B/CB \xrightarrow{\text{proj}} A * B/(CA \times B) \cup (* \times CB) = A \wedge \Sigma B \xrightarrow{r'} A * B/CB. \tag{4}$$

for some map r'. Then $h_1 \circ j$ decomposes as

$$A \rtimes CB \xrightarrow{\text{proj}} A \rtimes \Sigma B \xrightarrow{p'} A \wedge \Sigma B \xrightarrow{r'} A * B/CB.$$

Since the first arrow of (4) is homotopic to $\bar{q}$ and h_1 is homotopic to the identity map, r' is homotopic to $\bar{q}^{-1}$.

On the other hand, since $(f \rtimes \pi) \circ h_t \circ j(a, b) = (c_t(a), *)$ for $(a, b) \in A \rtimes B \subset A \rtimes CB$, the above homotopy commutative diagram induces a homotopy commutative diagram.

$$\begin{array}{ccc} A \rtimes \Sigma B & \xrightarrow{f \rtimes 1} & W \rtimes \Sigma B \\ \downarrow{\scriptstyle r' \circ p'} & & \downarrow \\ A * B/CB & \xrightarrow{f \rtimes \pi} & (V \rtimes *) \cup (W \rtimes \Sigma B). \end{array}$$

Since $r^{-1} \circ r' \circ p' \simeq r^{-1} \circ \bar{q}^{-1} \circ p' = q^{-1} \circ p' = p$ and $r \circ (f \rtimes \pi) = \bar{f}$, the proof is done. $\square$

For $k \geq 2$, let $\hat{q}$ be the composite of maps

$$\Sigma^{k-1} \widehat{X}^{[m]} \xrightarrow{q^{-1}} (\Sigma^{k-2} \widehat{X}^{[m-1]}) * X_m \xrightarrow{\text{proj}} (\Sigma^{k-1} \widehat{X}^{[m-1]} \times X_m) \cup (* \times CX_m).$$

Proposition 5.13. *If* $\underline{X} = \Sigma \underline{Y}$, *then the homotopy equivalence* $\widetilde{\epsilon}$ *of Corollary* 5.11 *satisfies a homotopy commutative diagram*

$$\begin{CD}
\Sigma^{k-1}\widehat{X}^{[m]} @>{\hat{q}}>> (\Sigma^{k-1}\widehat{X}^{[m-1]} \times X_m) \cup (* \times CX_m) \\
@VVV @VVV \\
\widetilde{W}(k) @. (W_{\Delta([m-1],[k])}(\underline{X}_{[m-1]}) \times X_m) \cup (* \times CX_m) \\
@V{\widetilde{\epsilon}^{-1}}VV @VV{\epsilon^{-1}_{\Delta([m-1],[k])} \times 1}V \\
\widetilde{Z}(k) @= \widetilde{Z}(k),
\end{CD}$$

where $k \geq 2$ and the upper vertical arrows are inclusions.

Proof. Let $w_k \colon Y^{*[k]} \to T^{k-1}(\Sigma \underline{Y}_{[k]})$ denote the higher Whitehead product. Then $w_k = w_{k-1} \times \pi$ for the projection $\pi \colon CY_k \to \Sigma Y_k$. Then by Lemma 5.12, we get a homotopy commutative diagram

$$\begin{CD}
Y^{*[m-1]} \rtimes \Sigma Y_m @>{w_{m-1} \rtimes 1}>> T^{m-2}(\Sigma \underline{Y}_{[m-1]}) \times \Sigma Y_m \\
@V{p}VV @VVV \\
Y^{*[m]} @>{\bar{w}_m}>> T^{m-1}(\Sigma \underline{Y})/\Sigma Y_m,
\end{CD}$$

where $\bar{w}_k$ is the composite of w_k and the projection $T^{k-1}(\Sigma \underline{Y}_{[k]}) \to T^{k-1}(\Sigma \underline{Y}_{[k]})/\Sigma Y_k$.

Put $L = \Delta([m-1],[k])$. Then by the definition of $\overline{\varphi}_L$, one gets a homotopy commutative diagram

$$\begin{CD}
(|L| * Y^{*[m-1]}) \rtimes \Sigma Y_m @>{\overline{\varphi}_L \rtimes 1}>> \mathcal{Z}_L^{m-2}(\underline{X}_{[m-1]}) \rtimes X_m \\
@V{p_1}VV @VVV \\
|L| * Y^{*[m]} @>{\tilde{r} \circ \widetilde{\varphi}}>> \widetilde{Z}^{m-1}(k)/CX_m,
\end{CD}$$

where p_1 is induced from p and $\tilde{r} \colon \widetilde{Z}^{m-1}(k) \to \widetilde{Z}^{m-1}(k)/CX_m$ is the projection which is a homotopy equivalence, and the right vertical arrow is the inclusion. Thus by the definitions of $\widetilde{\epsilon}$ and ϵ_L, one obtains a homotopy commutative diagram

$$\begin{CD}
\Sigma^{k-1}\widehat{X}^{[m-1]} \rtimes X_m @>>> W_L(\underline{X}_{[m-1]}) \rtimes X_m @>{\epsilon_L^{-1} \rtimes 1}>> \widetilde{Z}(k)/CX_m \\
@V{p_2}VV @. @VV{\tilde{r}^{-1}}V \\
\Sigma^{k-1}\widehat{X}^{[m]} @>>> \widetilde{W}(k) @>{\widetilde{\epsilon}^{-1}}>> \widetilde{Z}(k),
\end{CD}$$

where p_2 is induced from p. Since $p_2 \circ \hat{q} \simeq 1$, the proof is completed. □

Lemma 5.14. *If $\underline{X} = \Sigma \underline{Y}$ and $2 \leq k < m$, then the composite*

$$\Sigma^{k-1}\widehat{X}^{[m]} \to W_M(\underline{X}) \xrightarrow{\epsilon_M^{-1}} \mathcal{Z}_M(\underline{X}) \xrightarrow{w} DJ_M(\Sigma \underline{X})$$

is the iterated Whitehead product $[[\cdots[w_{[k]}, a_{k+1}], \cdots], a_m]$*, where* $M = \Delta([m],[k])$*.*

Proof. Put $L = \Delta([m-1],[k])$. By Proposition 5.13, we see that there is a homotopy commutative diagram

$$\begin{array}{ccccc}
\Sigma^{k-1}\widehat{X}^{[m]} & = & \Sigma^{k-1}\widehat{X}^{[m]} & = & \Sigma^{k-1}\widehat{X}^{[m]} \\
\downarrow{\scriptstyle \overline{w}} & & \downarrow{\scriptstyle \hat{q}} & & \downarrow \\
\Sigma^{k-1}\widehat{X}^{[m-1]} \vee \Sigma X_m & \xleftarrow{\text{proj}} & (\Sigma^{k-1}\widehat{X}^{[m-1]} \times X_m) \cup (* \times CX_m) & & \\
\downarrow & & \downarrow & & \\
W_L(\underline{X}_{[m-1]}) \vee \Sigma X_m & \xleftarrow{\text{proj}} & (W_L(\underline{X}_{[m-1]}) \times X_m) \cup (* \times CX_m) & & \widetilde{W}(k) \\
\downarrow{\scriptstyle \epsilon_L^{-1}\vee 1} & & \downarrow{\scriptstyle \epsilon_L^{-1}\times 1} & & \downarrow{\scriptstyle \tilde{\epsilon}^{-1}} \\
\mathcal{Z}_L(\underline{X}_{[m-1]}) \vee \Sigma X_m & \xleftarrow{\text{proj}} & \widetilde{Z}(k) & = & \widetilde{Z}(k) \\
\downarrow{\scriptstyle w\vee 1} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} \\
DJ_L(\Sigma\underline{X}_{[m-1]}) \vee \Sigma X_m & = & DJ_L(\Sigma\underline{X}_{[m-1]}) \vee \Sigma X_m & = & DJ_L(\Sigma\underline{X}_{[m-1]}) \vee \Sigma X_m,
\end{array}$$

where $\overline{w}$ is the Whitehead product of the identity maps of $\Sigma^{k-1}\widehat{X}^{[m-1]}$ and ΣX_m. On the other hand, by Corollaries 5.9 and 5.11 there is a homotopy commutative diagram

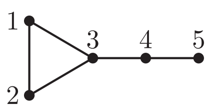
$$\begin{array}{ccc}
\widetilde{W}(k) & \longrightarrow & W_M(\underline{X}) \\
\downarrow{\scriptstyle \tilde{\epsilon}^{-1}} & & \downarrow{\scriptstyle \epsilon_M^{-1}} \\
\widetilde{Z}(k) & \longrightarrow & \mathcal{Z}_M(\underline{X}) \\
\downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} \\
DJ_L(\Sigma\underline{X}_{[m-1]}) \vee \Sigma X_m & = & DJ_M(\Sigma\underline{X}).
\end{array}$$

Then by juxtaposing the above two diagrams, one gets that the composite in the statement is the Whitehead product of the identity map of ΣX_m and $w \circ \epsilon_L$. Thus the proof is completed by induction on m. □

Proof of Theorem 5.5. The proof is done by Theorem 5.2, Corollary 5.9 and Lemma 5.14, where the induction in the proof of Lemma 5.14 is done by a contraction ordering. □

6. Example.

Let K be the following 1-dimensional simplicial complex with five vertices.



We explain how to apply Corollary 5.7 to this simplicial complex K. We first have to show that K is a totally fillable complex, so we check that all non-contractible full subcomplexes of K are fillable. Non-contractible full subcomplexes of K are K itself and K_I for

$$I=\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{3,5\},\{1,2,3\},\{1,2,4\},\{1,2,5\},\\ \{1,3,5\},\{2,3,5\},\{1,4,5\},\{2,4,5\},\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},$$

where

$$K_{\{i,j\}}=\partial\Delta^{\{i,j\}},\quad K_{\{1,2,3\}}=\partial\Delta^{\{1,2,3\}},\quad K_{\{p,q,r\}}=\Delta^{\{p,q\}}\sqcup\{r\}$$

for $(i,j)=(1,4),(1,5),(2,4),(2,5),(3,5)$, $(p,q,r)=(1,2,4),(1,2,5),(1,3,5),(2,3,5),(4,5,1),(4,5,2)$ and

$$K_{\{1,2,3,4\}}=\partial\Delta^{\{1,2,3\}}\cup\Delta^{\{3,4\}},\ K_{\{1,2,3,5\}}=\partial\Delta^{\{1,2,3\}}\sqcup\{5\},\ K_{\{1,2,4,5\}}=\Delta^{\{1,2\}}\sqcup\Delta^{\{4,5\}}.$$

Then we see that these full subcomplexes are fillable, so K is totally fillable as desired.

Next we choose fillings of these K_I. Each of $K_{\{i,j\}}$ for $(i,j)=(1,4),(1,5),(2,4),(2,5),(3,5)$ and K_I for $I=\{1,2,3\},\{1,2,3,4\},\{1,2,3,4,5\}$, has the unique filling such that

$$\mathcal{F}(K_{\{i,j\}})=\{ij\},\quad \mathcal{F}(K_I)=\{123\}.$$

In the remaining cases, there are several choices of fillings, and here we choose

$$\mathcal{F}(K_{\{1,2,3,5\}})=\{123,35\},\quad \mathcal{F}(K_{\{1,2,4,5\}})=\{24\},\quad \mathcal{F}(K_{\{p,q,r\}})=\{qr\}$$

for $(p,q,r)=(1,2,4),(1,2,5),(1,3,5),(2,3,5),(4,5,1),(4,5,2)$.

Next we choose contraction ordering. This is needed for $35\in\mathcal{F}(K_{\{1,2,3,5\}})$, $24\in\mathcal{F}(K_{\{1,2,4,5\}})$, $123\in\mathcal{F}(K)$. For $35\in\mathcal{F}(K_{\{1,2,3,5\}})$, there are two contraction ordering $1<2$ and $2<1$, and we choose $1<2$. For $24\in\mathcal{F}(K_{\{1,2,4,5\}})$, there are also two contraction ordering $1<5$ and $5<1$, and we choose $1<5$. For $123\in\mathcal{F}(K)$, there is only one contraction ordering $4<5$.

With this choice of fillings and contraction ordering, we get a homotopy equivalence

$$\epsilon_K\colon S^3_{1,4}\vee S^3_{1,5}\vee S^3_{2,4}\vee S^3_{2,5}\vee S^3_{3,5}\vee S^4_{1,2,4}\vee S^4_{1,2,5}\vee S^4_{1,3,5}\vee S^4_{2,3,5}\vee S^4_{4,5,1}\vee\\ S^4_{4,5,2}\vee S^5_{1,2,3}\vee S^6_{1,2,3,4}\vee S^6_{1,2,3,5}\vee S^5_{1,2,3,5}\vee S^5_{1,2,4,5}\vee S^7_{1,2,3,4,5}\simeq\mathcal{Z}_K,$$

where the indices of spheres indicate the corresponding full subcomplexes. Then through this homotopy equivalence, we obtain

$$w|_{S^3_{i,j}}=[\widetilde{a}_i,\widetilde{a}_j]\qquad w|_{S^4_{p,q,r}}=[[\widetilde{a}_q,\widetilde{a}_r],\widetilde{a}_p]\qquad w|_{S^5_{1,2,3}}=\widetilde{w}_{1,2,3}$$
$$w|_{S^6_{1,2,3,5}}=[\widetilde{w}_{1,2,3},\widetilde{a}_5]\qquad w|_{S^5_{1,2,3,5}}=[[[\widetilde{a}_3,\widetilde{a}_5],\widetilde{a}_1],\widetilde{a}_2]\qquad w|_{S^5_{1,2,4,5}}=[[[\widetilde{a}_2,\widetilde{a}_4],\widetilde{a}_1],\widetilde{a}_5]$$
$$w|_{S^7_{1,2,3,4,5}}=[[\widetilde{w}_{1,2,3},\widetilde{a}_4],\widetilde{a}_5]$$

for $(i,j) = (1,4),(1,5),(2,4),(2,5),(3,5)$ and $(p,q,r) = (1,2,4),(1,2,5),(1,3,5),(2,3,5),(4,5,1),(4,5,2)$.

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