J. Math. Soc. Japan
Vol. 72, No. 3 (2020) pp. 777–794
doi: 10.2969/jmsj/81688168

Dimensions of multi-fan duality algebras

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(Received Nov. 16, 2018)

Abstract. Given an arbitrary non-zero simplicial cycle and a generic vector coloring of its vertices, there is a way to produce a graded Poincare duality algebra associated with these data. The procedure relies on the theory of volume polynomials and multi-fans. The algebras constructed this way include many important examples: cohomology algebras of toric varieties and quasitoric manifolds, and Gorenstein algebras of triangulated homology manifolds, introduced and studied by Novik and Swartz. In all these examples the dimensions of graded components of such duality algebras do not depend on the vector coloring. It was conjectured that the same holds for any simplicial cycle. We disprove this conjecture by showing that the colors of singular points of the cycle may affect the dimensions. However, the colors of nonsingular points are irrelevant. By using bistellar moves we show that the number of distinct dimension vectors arising on a given 3-dimensional pseudomanifold with isolated singularities is a topological invariant. This invariant is trivial on manifolds, but nontrivial on general pseudomanifolds.

1. Introduction.

A multi-fan is a collection of full-dimensional convex cones in an oriented space $V \cong \mathbb{R}^n$, emanating from the origin and equipped with weights [**9**]. In contrast to the ordinary fans, cones of a multi-fan may overlap. The condition of completeness for a multi-fan is as follows: the multi-fan Δ is complete if the weighted sum of its maximal cones is a cycle. This means that for every cone C of codimension one the weights of all maximal cones incident to C sum to zero, being counted with different signs depending on which side of C they lie. A multi-polytope based on a multi-fan Δ is a collection of affine hyperplanes orthogonal to the rays of Δ. The hyperplanes should intersect whenever their corresponding rays are the faces of some cone of Δ. In this work as well as in [**2**] we restrict to the case when all cones of a multi-fan Δ are simplicial. In this case Δ is called a simplicial multi-fan and corresponding multi-polytopes are called simple. For the simplicial case we work with the following definitions.

Definition 1.1 ([**2**], [**9**]). *A complete simplicial multi-fan* is a pair (ω, λ), where

$$\omega = \sum_{I\subset[m],|I|=n} w(I)I \in Z_{n-1}\left(\triangle_{[m]}^{(n-1)};\mathbb{R}\right)$$

2010 *Mathematics Subject Classification.* Primary 05E45; Secondary 05E40, 28A75, 57M25, 52B70, 13H10, 13D40, 52C35, 57N65, 13A02.
Key Words and Phrases. volume polynomial, multi-fan, multi-polytope, invariants of 3-dimensional pseudomanifolds, vector coloring, Poincare duality algebra, Macaulay duality, bistellar moves.
This work is supported by the Russian Science Foundation under grant 18-71-00009.

is a simplicial cycle on m vertices, and $\lambda\colon [m] \to V$ is any function satisfying the condition: the set $\{\lambda(i) \mid i \in I\}$ is a basis of V if $|I| = n$ and $w(I) \neq 0$. In this case λ is called *a characteristic function.* If ω is the fundamental cycle of an $(n-1)$-dimensional oriented simplicial pseudomanifold K, we say that Δ is supported on K.

A simple multi-polytope P is a pair $(\Delta, \{H_1, \ldots, H_m\})$, where Δ is a simplicial multi-fan, and H_i is a hyperplane in V^* orthogonal to $\lambda(i) \in V$, that is

$$H_i = \{x \in V^* \mid \langle x, \lambda(i)\rangle = c_i\}.$$

We say that P is based on Δ. The numbers $c_1, \ldots, c_m \in \mathbb{R}$ are called *the support parameters* of a multi-polytope P.

Every simple multi-polytope based on a given simplicial multi-fan is uniquely characterized by the set of its support parameters.

Many standard notions and facts about convex polytopes and their normal fans are naturally extended to multi-polytopes and multi-fans. In particular, if P is a multi-polytope based on a complete multi-fan, there is a well-defined notion of volume of P, see [**6**]. There is a classical Lawrence's formula to compute the volume of a simple polytope [**8**]. This formula is naturally generalized to simple multi-polytopes [**2**], see Proposition 2.2. Considering volumes of all multi-polytopes based on a fixed multi-fan Δ at once, we get the volume function $V_\Delta\colon \mathbb{R}^m \to \mathbb{R}$ defined on the vector space of support parameters. A tuple $(c_1, \ldots, c_m) \in \mathbb{R}^m$ is mapped by this function to the volume of the multi-polytope which is based on Δ and has support parameters $(c_1, \ldots, c_m)$. As was proved in [**6**], the function V_Δ is a homogeneous polynomial of degree n in the support parameters: $V_\Delta \in \mathbb{R}[c_1, \ldots, c_m]_n$. The polynomial V_Δ is called *the volume polynomial* of a multi-fan Δ.

There is a standard procedure to make a Poincare duality algebra out of any nonzero homogeneous polynomial: Macaulay duality [**10**]. In the case of volume polynomials it was introduced and studied in [**7**], [**14**]. Consider the graded ring of polynomials $\mathcal{D} = \mathbb{R}[\partial_1, \ldots, \partial_m]$ where each symbol ∂_i denotes the partial derivative $\partial/\partial c_i$. For topological reasons we double the degrees by assuming $\deg \partial_i = 2$. Each variable ∂_i acts on $\mathbb{R}[c_1, \ldots, c_m]$ as a differential operator, so we may consider the homogeneous ideal

$$\operatorname{Ann} V_\Delta = \{D \in \mathcal{D} \mid DV_\Delta = 0\}$$

of differential operators annihilating V_Δ. The quotient algebra $\mathcal{A}^*(\Delta) \stackrel{\text{def}}{=} \mathcal{D}/\operatorname{Ann} V_\Delta$ satisfies Poincare duality conditions: its top component $\mathcal{A}^{2n}(\Delta)$ is one-dimensional and the pairing $\mathcal{A}^{2j}(\Delta) \otimes \mathcal{A}^{2n-2j}(\Delta) \xrightarrow{\times} \mathcal{A}^{2n}(\Delta)$ is non-degenerate. We call $\mathcal{A}^*(\Delta)$ the duality algebra of a multi-fan. Let $d_j = \dim \mathcal{A}^{2j}$. Poincare duality implies $d_j = d_{n-j}$.

Multi-fan algebras are important by several reasons. If Δ is an ordinary complete simplicial rational fan, the algebra $\mathcal{A}^*(\Delta)$ is isomorphic to the cohomology algebra of the corresponding toric variety (see [**7**]). Timorin [**14**] gave a purely geometrical proof of Stanley's g-theorem for a polytopal fan Δ by showing that $\mathcal{A}^*(\Delta)$ satisfies Lefschetz property. In [**2**] we proved that whenever the multi-fan Δ is supported on an orientable homology manifold K, the algebra $\mathcal{A}^*(\Delta)$ coincides with the Gorenstein algebra introduced by Novik and Swartz in [**11**]. However, we also showed that every (finite-dimensional commutative) Poincare duality algebra generated in degree 2 is isomorphic to $\mathcal{A}^*(\Delta)$ for some multi-fan Δ. Therefore there exist complete simplicial multi-fans whose algebras do not satisfy Lefschetz property and whose dimensions' sequence $(d_0, d_1, \dots, d_n)$ is not unimodal.

By Definition 1.1 a complete simplicial multi-fan Δ consists of two types of data: a simplicial cycle ω, the combinatorial data, and a characteristic function $\lambda\colon [m] \to V$ encoding the directions of rays of a multi-fan, the geometrical data. In many cases the dimensions of graded components $d_j = \dim \mathcal{A}^{2j}(\Delta)$ depend only on ω, but not on λ. If Δ is supported on a simplicial homology sphere, the dimensions d_j coincide with the h-numbers of this sphere. More generally, if Δ is supported on a simplicial homology manifold, the dimensions d_j coincide with the so called h"-numbers of a manifold, which are the combinatorial invariants of the triangulation (see [**11**]).

We suggested this was a general phenomenon [**2**, Conjecture 1]: Is it true that dimensions $(d_0, d_1, \dots, d_n)$ of a multi-fan algebra $\mathcal{A}^*(\Delta)$ depend only on ω but not on characteristic function λ? If no, is it true for multi-fans supported on pseudomanifolds?

In this paper we answer both questions in negative by providing two counter-examples. First counter-example is manually computed and relies on several simple facts proved previously in our paper [**2**]. However, this counter-example is not a pseudo-manifold so it does not answer the second question.

The second question is more complicated since, in a sense, the simplest interesting example of a pseudomanifold which is not a manifold is the suspended torus. In this case the volume polynomial is a quartic polynomial in at least 9 variables. A script written in GAP [**4**], [**5**] allowed to answer the second question. If K is the suspension over some triangulation of a torus, then there exist two multi-fans supported on K having distinct dimension sequences.

In a theoretical part of this work we explain what makes manifolds so special from the view point of multi-fans. Let ω be a simplicial cycle and $i \in [m]$ be its vertex. The link of the vertex i in a cycle ω may be defined in a natural way. We say that i is a nonsingular vertex of a cycle ω if the link of i is a homology sphere.

Theorem 1. *Dimensions d_j of a multi-fan algebra do not depend on the values of characteristic function at nonsingular vertices.*

In particular, since all vertices of a triangulated manifold are nonsingular, d-vectors of multi-fans supported on manifolds do not depend on characteristic function at all, as was noticed previously. Nevertheless, the calculations show that d-vectors of mutli-fans may crucially depend on the values of λ at singular points.

It is natural to consider the following invariant of a simplicial pseudomanifold K: the number $r(K)$ of distinct dimension vectors of multi-fans supported on K. By the preceding discussion, this invariant equals 1 on homology manifolds, but may be nontrivial in general. Using *bistellar moves* we show that this number is a topological invariant on the class of 3-dimensional pseudomanifolds with isolated singularities, i.e. it does not depend on a triangulation. General properties of this invariant are yet unknown.

We study the following family of examples. Consider an arbitrary link[1] $l\colon \bigsqcup_\alpha S^1_\alpha \hookrightarrow S^3$ and collapse each of its components to a point. We suppose that the resulting pseudomanifold K satisfies $r(K)=1$ if and only if all components of the link are pairwise unlinked. We checked several examples. We have $r(K)=1$ for disjoint union of knots and for Borromean rings. For the Hopf link there holds $r(K)=2$; in this case X is a suspended torus discussed above.

Our considerations suggest that the additive structure of multi-fan algebras over pseudomanifolds may lead to new invariants of 3-pseudomanifolds or, at least, uncover interesting relations between convex geometry and 3-dimensional topology.

2. Preliminaries.

Let $\Psi \in \mathbb{R}[c_1,\dots,c_m]_n$ be an arbitrary non-zero homogeneous polynomial of degree n and let $\mathcal{D}^*/\operatorname{Ann}\Psi$ be the corresponding Poincare duality algebra, i.e. $\mathcal{D}^*=\mathbb{R}[\partial_1,\dots,\partial_m]$, $\partial_i=\partial/\partial c_i$, $\operatorname{Ann}\Psi=\{D\in\mathcal{D}^*\mid D\Psi=0\}$. Let $\operatorname{var}^j(\Psi)\subset\mathbb{R}[c_1,\dots,c_m]_{n-j}$ be the linear span of all partial derivatives of degree j of the polynomial Ψ. The linear map

$$(\mathcal{D}/\operatorname{Ann}\Psi)_{2j}\to\operatorname{var}^j\Psi,\qquad D\mapsto D\Psi \tag{2.1}$$

is an isomorphism by the fundamental theorem on homomorphisms.

Let $\Delta=(\omega,\lambda)$ be a complete simplicial multi-fan on a finite set $M=[m]$ of rays in the space $V\cong\mathbb{R}^n$, where $\omega=\sum_{I\subset M,|I|=n}w(I)I$, $\lambda\colon M\to V$. Consider the simplicial complex K on M whose maximal simplices are all subsets $I\subset M, |I|=n$ such that $w(I)\neq 0$. K is called *the support* of the cycle ω; it has dimension $n-1$. If $J\in K$ is a simplex of any dimension, then we can define the projected multi-fan Δ_J as follows.

CONSTRUCTION 2.1. Consider the link of J in K:

$$\operatorname{link}_K J=\{I\subset[m]\mid I\cap J=\emptyset, I\sqcup J\in K\},$$

and let $M_J\subset M$ be the set of vertices of $\operatorname{link}_K J$. Let V_J be the quotient of the vector space V by the subspace $\langle\lambda(i)\mid i\in J\rangle\cong\mathbb{R}^{|J|}$. Consider the simplicial cycle

$$\omega_J=\sum_{I\in\operatorname{link}_K J,|I|=n-|J|} w(I\sqcup J)I\in Z_{n-1-|J|}(\operatorname{link}_K J;\mathbb{R}).$$

The projected characteristic function $\lambda_J\colon M_J\to V$ is defined by the composition

$$M_J\hookrightarrow M\xrightarrow{\lambda}V\twoheadrightarrow V_J,$$

where the last arrow is the natural projection. The multi-fan $\Delta_J=(\omega_J,\lambda_J)$ is called *the projected multi-fan* of Δ with respect to J.

[1]The term "link" appears in this paper in two different meanings, both being well-established. Here link means "the collection of knots" in S^3, however, the link in a simplicial complex means "the combinatorial boundary of a neighborhood of a vertex". Hopefully, the meaning can be understood from the context.

Let $V_\Delta \in \mathbb{R}[c_1, \ldots, c_m]_n$ be the volume polynomial of Δ. For a subset $J = \{i_1, \ldots, i_j\} \subset [m]$ let ∂_J denote the differential operator $\partial_{i_1} \cdots \partial_{i_j}$. In [**2**, Lemma 1] we proved that $\partial_J V_\Delta$ is zero whenever $J \notin K$; otherwise $\partial_J V_\Delta$ coincides with V_{Δ_J}, the volume polynomial of the projected multi-fan, up to a linear change of variables and up to a constant factor. In particular, we have

$$\dim \operatorname{var}^s \partial_J V_\Delta = \dim \operatorname{var}^s V_{\Delta_J} \text{ for any } s = 0, \ldots, n. \tag{2.2}$$

Let us recall the formula for the volume polynomial.

Proposition 2.2 ([**2**]). *Let $\Delta = (\omega, \lambda)$ be a complete simplicial multi-fan and $v \in V$ be a generic vector. Then*

$$V_\Delta(c_1, \ldots, c_m) = \frac{1}{n!} \sum_{I=\{i_1,\ldots,i_n\} \in K} \frac{w(I)}{|\det \lambda(I)| \prod_{j=1}^n \alpha_{I,j}} (\alpha_{I,1} c_{i_1} + \cdots + \alpha_{I,n} c_{i_n})^n, \tag{2.3}$$

where $\alpha_{I,1}, \ldots, \alpha_{I,n}$ are the coordinates of v in the basis $(\lambda(i_1), \ldots, \lambda(i_n))$, $w(I)$ is the weight of the simplex I, and $\det \lambda(I)$ *is the determinant of the matrix $(\lambda(i_1), \ldots, \lambda(i_n))$.*

The vector v will be called a *polarization vector*. The condition that v is generic simply means that all coefficients $\alpha_{I,i}$ should be nonzero, so that the right hand side of (2.3) makes sense. This is an open condition in V. The volume polynomial does not depend on the choice of a polarization vector.

Remark 2.3. It can be seen from this formula that whenever the operator $A \in \mathrm{GL}(V)$ acts on all values of characteristic function simultaneously, the volume polynomial does not change up to constant factor. Indeed, if we take Av as a polarization vector for the multi-fan $A\Delta = (\omega, A\lambda)$, all the coefficients $\alpha_{I,i}$ remain unchanged, and all the determinants $\det \lambda_I$ are multiplied by the same factor $\det A$. Therefore, $\mathcal{A}^*(A\Delta) \cong \mathcal{A}^*(\Delta)$.

We finish this section with another small remark.

Remark 2.4. Let us fix a finite set $M = [m]$ and a generic map $\lambda \colon [m] \to V \cong \mathbb{R}^n$. All n-dimensional multi-fans on $[m]$ having λ as a characteristic function form a vector space (essentially, this is just a certain subspace of the space of all $(n-1)$-cycles on $[m]$ vertices). Therefore, one can form sums and differences of multi-fans, provided that they have the same vertex sets and characteristic functions. Volume polynomial is additive with respect to this operation, which easily follows from its formula (2.3).

It may happen that the underlying cycle of a multi-fan does not contain some vertices from $[m]$. We call such vertices ghost vertices. The polynomial V_Δ does not actually depend on the variable c_i for any ghost vertex $i \in M$. In this case $\partial_i = 0$ in $\mathcal{A}^*(\Delta)$.

3. Characteristic function at nonsingular points.

In this section we prove Theorem 1.

Definition 3.1. A simplicial cycle $\omega \in Z_{n-1}(\triangle_{[m]}^{n-1}; \mathbb{R})$ is called *rigid* if the dimensions $d_j = \dim \mathcal{A}^{2j}(\Delta)$ of all multi-fans $\Delta = (\omega, \lambda)$ do not depend on λ.

As was mentioned in the introduction, the fundamental cycle of any oriented homology manifold K is rigid since $d_j = h''_j(K)$; the necessary definitions and the proof of this statement can be found in our previous paper [**2**]. In particular, every homology sphere K is rigid and $d_j = h_j(K)$. If ω is rigid, we denote the dimension $\dim \mathcal{A}^{2j}$ by $d_j(\omega)$.

CONSTRUCTION 3.2. Let $\omega' = \sum_{|I'|=n'} w'(I')I'$, $\omega'' = \sum_{|I''|=n''} w''(I'')I''$ be two simplicial cycles on disjoint vertex sets M', M''. Define the join $\omega' * \omega''$ as a simplicial cycle on $M' \sqcup M''$ in a natural way:

$$\omega' * \omega'' = \sum_{\substack{I' \in M', |I'|=n' \\ I'' \in M'', |I''|=n''}} \omega'(I')\omega''(I'')I' \sqcup I'' \in Z_{n'+n''-1}\left(\triangle^{n'+n''-1}_{M' \sqcup M''}; \mathbb{R}\right).$$

Let us define the join of two multi-fans. Let $\Delta' = (\omega', \lambda')$ and $\Delta'' = (\omega'', \lambda'')$ be multi-fans in the spaces V' and V'' with the ray-sets M' and M'' respectively. Consider the multi-fan $\Delta' * \Delta'' = (\omega' * \omega'', \lambda' * \lambda'')$, where $\lambda' * \lambda'' \colon M' \sqcup M'' \to V' \oplus V''$ is given by

$$\lambda' * \lambda''(i) = \begin{cases} (\lambda'(i), 0), & \text{if } \ i \in M', \\ (0, \lambda''(i)), & \text{if } \ i \in M''. \end{cases} \tag{3.1}$$

There holds $V_{\Delta' * \Delta''} = V_{\Delta'} \cdot V_{\Delta''}$. This formula can be deduced either from the exact formula of the volume polynomial, or geometrically, by noticing that every multi-polytope based on Δ is just the cartesian product of a multi-polytope based on Δ' and a multi-polytope based on Δ'', so the volumes are multiplied. The polynomials $V_{\Delta'}$ and $V_{\Delta''}$ have distinct sets of variables, which implies

$$\mathcal{A}^*(\Delta' * \Delta'') \cong \mathcal{A}^*(\Delta') \otimes \mathcal{A}^*(\Delta''). \tag{3.2}$$

Therefore,

$$\operatorname{Hilb}(\mathcal{A}^*(\Delta' * \Delta''); t) = \operatorname{Hilb}(\mathcal{A}^*(\Delta'); t) \cdot \operatorname{Hilb}(\mathcal{A}^*(\Delta''); t). \tag{3.3}$$

Let S^0 denote the simplicial complex consisting of two disjoint vertices x and y. Abusing the notation, we adopt the same symbol S^0 for its underlying simplicial cycle, which lies in $Z_0(S^0; \mathbb{R})$ (this is just the formal difference of two vertices). The join of a cycle ω with S^0 is called a *suspension* and is denoted by $\Sigma\omega$.

A multi-fan Δ is called *suspension-shaped* if its underlying simplicial cycle is isomorphic to $\Sigma\omega$ for some cycle ω. Note that the algebra of a suspension-shaped multi-fan contains two marked elements $\partial_x, \partial_y \in \mathcal{A}^*(\Delta)$ corresponding to the apices of the suspension. Since the 1-simplex $\{x, y\}$ does not lie in the support of $\Sigma\omega$, there holds $\partial_x\partial_y = 0$ in $\mathcal{A}^*(\Delta)$. Let us consider two operators

$$\times\partial_x, \times\partial_y \colon \mathcal{A}^*(\Delta) \to \mathcal{A}^{*+2}(\Delta),$$

acting on the multi-fan algebra of a suspension-shaped multi-fan. There holds $\operatorname{Im}(\times\partial_y) \subseteq \operatorname{Ker}(\times\partial_x)$ according to the relation.

Definition 3.3. A suspension-shaped multi-fan Δ is called *editable*, if

$$\mathrm{Im}(\times\partial_y) = \mathrm{Ker}(\times\partial_x).$$

Our next goal is to prove that suspensions over homology spheres are editable, see Corollary 3.7. Several technical lemmas are needed for the proof.

Lemma 3.4. *Let $\mathcal{A}^*$ be a Poincare duality algebra of formal dimension $2n$ and $\mathcal{I} \subset \mathcal{A}^*$ be a graded ideal. Let $\mathrm{Ann}\,\mathcal{I} := \{a \in \mathcal{A}^* \mid a\mathcal{I} = 0\}$ and let $\mathcal{I}^\perp = \bigoplus_j (\mathcal{I}^\perp)^{2j}$ denote the component-wise orthogonal complement*:

$$(\mathcal{I}^\perp)^{2j} = \{a \in \mathcal{A}^{2j} \mid a\mathcal{I}^{2n-2j} = 0\}.$$

Then $\mathrm{Ann}\,\mathcal{I} = \mathcal{I}^\perp$.

Proof. The subspaces $\mathrm{Ann}\,\mathcal{I}$ and $\mathcal{I}^\perp$ are defined similarly so at first glance the statement may seem a tautology. Notice, however, that $\mathrm{Ann}\,\mathcal{I}$ consists of elements annihilated by the whole ideal $\mathcal{I}$, while $\mathcal{I}^\perp$ is annihilated only by the elements of the complementary degree. Nevertheless, these conditions are equivalent in a Poincare duality algebra. If the element $a \in \mathcal{A}^{2j}$ is not annihilated by some $b \in \mathcal{A}^{2k}$, then it is possible to choose $c \in \mathcal{A}^{2n-2j-2k}$ such that $abc \neq 0$, so that we have an element bc of degree complementary to a, which does not annihilate a as well. □

The ideals $\mathcal{I}$ and $\mathrm{Ann}\,\mathcal{I} = \mathcal{I}^\perp$ are called *orthogonal*. If Δ is a suspension-shaped multi-fan, then the ideal $\mathrm{Im}(\times\partial_x)$ (which is just the principal ideal of $\mathcal{A}^*(\Delta)$ generated by ∂_x) is orthogonal to $\mathrm{Ker}(\times\partial_x)$ by definition.

Lemma 3.5. $\mathrm{Hilb}(\mathrm{Im}(\times\partial_x); t) = t^2\,\mathrm{Hilb}(\mathcal{A}^*(\Delta_x); t)$.

Proof. Recall from Section 2 that there is an isomorphism $\mathcal{A}^{2j}(\Delta) \to \mathrm{var}^j\, V_\Delta$ which sends D to DV_Δ. We have

$$\dim \mathrm{Im}(\times\partial_x)_{2j} = \dim\{DV_\Delta \mid D \in \mathrm{Im}(\times\partial_x)_{2j}\}.$$

The latter space may be identified with

$$\{D\partial_x V_\Delta \mid D \in \mathcal{D}_{2j-2}\} = \mathrm{var}^{j-1}(\partial_x V_\Delta).$$

Equation (2.2) implies that $\dim \mathrm{var}^{j-1}(\partial_x V_\Delta) = \dim \mathrm{var}^{j-1} V_{\Delta_x} = \dim \mathcal{A}^{2j-2}(V_{\Delta_x})$. This finishes the proof. □

Lemma 3.6. *Suppose that a simplicial $(n-2)$-cycle ω is rigid and its suspension $\Sigma\omega$ is rigid. Then any suspension-shaped multi-fan Δ on $\Sigma\omega$ is editable.*

Proof. For any multi-fan Δ' based on S^0 we have $\mathrm{Hilb}(\Delta'; t) = 1 + t^2$, since S^0 is a sphere and its h-numbers are $(1, 1)$. Since ω, S^0, and $\Sigma\omega$ are rigid, the dimension vectors of the corresponding multi-fans do not depend on characteristic functions, so that Hilbert polynomials can be computed for a particular joined characteristic function, defined by (3.1). In this case, formula (3.3) implies

$$\mathrm{Hilb}(\mathcal{A}^*(\Delta);t) = (1+t^2)\sum_{j=0}^{n-1} d_j(\omega)t^{2j}.$$

Lemma 3.5 asserts that $\mathrm{Hilb}(\mathrm{Im}(\times\partial_x);t) = t^2\,\mathrm{Hilb}(\mathcal{A}^*(\Delta_x);t)$. Since Δ_x is a multi-fan based on ω, there holds $\mathrm{Hilb}(\mathrm{Im}(\times\partial_x);t) = t^2\sum_{j=0}^{n-1} d_j(\omega)t^{2j}$. Similarly, $\mathrm{Hilb}(\mathrm{Im}(\times\partial_y);t) = t^2\sum_{j=0}^{n-1} d_j(\omega)t^{2j}$. Using Lemma 3.4, we may find the dimensions of the orthogonal complement $\mathrm{Ker}(\times\partial_x) = \mathrm{Im}(\times\partial_x)^{\perp}$ in each degree:

$$\dim \mathrm{Ker}(\times\partial_x)_{2j} = \dim \mathcal{A}^{2j}(\Delta) - \dim \mathrm{Im}(\times\partial_x)_{2n-2j}.$$

This implies

$$\mathrm{Hilb}(\mathrm{Ker}(\times\partial_x);t) = t^2\,\mathrm{Hilb}(\mathcal{A}^*(\Delta_x);t) = \mathrm{Hilb}(\mathrm{Im}(\times\partial_y);t).$$

Since $\mathrm{Im}(\times\partial_y)$ lies in $\mathrm{Ker}(\times\partial_x)$, and these vector spaces have equal dimensions, they coincide. □

Corollary 3.7. *Let K be a homology $(n-2)$-sphere (or its underlying simplicial cycle). Then any suspension-shaped multi-fan Δ on ΣK is editable.*

Proof. Suspension over a homology sphere is again a homology sphere. Thus both K and ΣK are rigid and Lemma 3.6 applies. □

Remark 3.8. The arguments in the proof of Lemma 3.6 show that in general, if Δ is a suspension-shaped multi-fan with suspension points x, y, there holds

$$\mathrm{Hilb}(\mathcal{A}^*(\Delta);t) \geq \mathrm{Hilb}(\mathcal{A}^*(\Delta_x);t) + t^2\,\mathrm{Hilb}(\mathcal{A}^*(\Delta_y);t),$$

where the inequality of polynomials is understood coefficient-wise.

The next construction shows how suspension-shaped multi-fans arise when the value of characteristic function is changed at a single point.

Construction 3.9. Let $\Delta' = (\omega, \lambda')$, $\Delta'' = (\omega, \lambda'')$ be multi-fans based on the same simplicial cycle ω, and assume that $\lambda'(i) = \lambda''(i)$ for all $i \in [m]$ except $i = k$, where k is a fixed vertex. It is convenient to take two copies x, y of k and consider Δ' and Δ'' as the multi-fans on the same set $M := ([m]\setminus\{k\})\sqcup\{x,y\}$ having the same characteristic function λ:

$$\lambda(i) \stackrel{\text{def}}{=} \begin{cases} \lambda'(i) = \lambda''(i), & \text{if } \ i \in [m]\setminus\{k\},\\ \lambda'(k), & \text{if } \ i = x,\\ \lambda''(k), & \text{if } \ i = y.\end{cases}$$

The underlying cycles of Δ' and Δ'' are isomorphic, however they are different as the elements of $Z_{n-1}(\triangle_M^{n-1};\mathbb{R})$. The cycle ω' passes through x and has ghost vertex y, while the cycle ω'' passes through y and has ghost vertex x. Since Δ', Δ'' have the same characteristic function, their difference is well-defined:

$$T = \Delta'' - \Delta'.$$

It is easy to observe that T is a suspension-shaped multi-fan with the suspension points x and y; its underlying simplicial cycle has the form $\Sigma\omega_x = \Sigma\omega_y$ (recall that ω_x, ω_y are the projected simplicial cycles with respect to vertices x and y respectively).

THEOREM 2. *Let Δ', Δ'' be as above. If the projected $(n-2)$-cycle ω_k is rigid and $T = \Delta'' - \Delta'$ is an editable suspension-shaped multi-fan, then*

$$\operatorname{Hilb}(\mathcal{A}^*(\Delta'); t) = \operatorname{Hilb}(\mathcal{A}^*(\Delta''); t).$$

PROOF. Consider the ring of differential operators $\mathcal{D}^* = \mathbb{R}[\partial_i \mid i \in M]$ and the principal ideals (∂_x), (∂_y) of this ring. Note that $(\partial_y) \subset \operatorname{Ann} V_{\Delta'}$ since y is a ghost vertex of Δ', and similarly $(\partial_x) \subset \operatorname{Ann} V_{\Delta''}$.

LEMMA 3.10. $\operatorname{Ann} V_{\Delta'} + (\partial_x) = \operatorname{Ann} V_{\Delta''} + (\partial_y)$.

PROOF. It is sufficient to prove that $\operatorname{Ann} V_{\Delta'} \subset V_{\Delta''} + (\partial_y)$: the symmetry of the statement would imply $\operatorname{Ann} V_{\Delta''} \subset V_{\Delta'} + (\partial_x)$.

First note that $V_{\Delta''} = V_{\Delta'} + V_T$. Hence $\partial_x V_{\Delta'} + \partial_x V_T = \partial_x V_{\Delta''} = 0$. Consider $D \in \operatorname{Ann} V_{\Delta'}$. Then $\partial_x D V_T = -D\partial_x V_{\Delta'} = 0$. Therefore the class of D in the algebra $\mathcal{A}^*(T)$ lies in the kernel of $\times\partial_x \colon \mathcal{A}^*(T) \to \mathcal{A}^{*+2}(T)$. By assumption, T is editable, which implies $D \in \operatorname{Ann} V_T + (\partial_y)$. Thus $D = D_1 + D_2$ where $D_1 \in \operatorname{Ann} V_T$ and $D_2 \in (\partial_y)$.

Applying $D_1 = D - D_2$ to the polynomial $V_{\Delta''} = V_T + V_{\Delta'}$ we get

$$D_1 V_{\Delta''} = D_1(V_T) + (D - D_2)V_{\Delta'} = 0,$$

since $D \in \operatorname{Ann} V_{\Delta'}$ by assumption and $D_2 \in (\partial_y) \subset \operatorname{Ann} V_{\Delta'}$. Thus $D = D_1 + D_2$ where $D_1 \in \operatorname{Ann} V_{\Delta''}$ and $D_2 \in (\partial_y)$ which proves the claim. ☐

Now consider the diagram of inclusions of ideals:

$$\begin{tikzcd}
\operatorname{Ann} V_{\Delta'} \arrow[r, hook] & \operatorname{Ann} V_{\Delta'} + (\partial_x) = \operatorname{Ann} V_{\Delta''} + (\partial_y) & \operatorname{Ann} V_{\Delta''} \arrow[l, hook'] \\
\operatorname{Ann} V_{\Delta'} \cap (\partial_x) \arrow[u, hook] \arrow[r, hook] & (\partial_x, \partial_y) \arrow[u, hook] & \operatorname{Ann} V_{\Delta''} \cap (\partial_y) \arrow[u, hook] \arrow[l, hook']
\end{tikzcd} \tag{3.4}$$

LEMMA 3.11. $\operatorname{Hilb}(\operatorname{Ann} V_{\Delta'} \cap (\partial_x); t) = \operatorname{Hilb}(\operatorname{Ann} V_{\Delta''} \cap (\partial_y); t)$.

PROOF. We have $\operatorname{Ann} V_{\Delta'} \cap (\partial_x) = \{\partial_x D \in \mathcal{D} \mid \partial_x D V_{\Delta'} = 0\}$. Up to a shift of grading, this vector space is isomorphic to $\operatorname{Ann}(\partial_x V_{\Delta'})$. There holds $\operatorname{Hilb}(\operatorname{Ann}(\partial_x V_{\Delta'}); t) = \operatorname{Hilb}(\mathcal{D}; t) - \operatorname{Hilb} \mathcal{A}^*(\partial_x V_{\Delta'})$. Note that $\dim \mathcal{A}^{2j}(\partial_x V_{\Delta'}) = \dim \operatorname{var}^j(\partial_x V_{\Delta'})$ and the latter space has the same dimension as $\dim \operatorname{var}^j(V_{\Delta'_x}) = \dim \mathcal{A}^{2j}(\Delta'_x)$ according to equation (2.2). Both projected multi-fans Δ'_x and Δ''_y are based on the same projected cycle ω_k. By the assumption of the theorem, ω_k is rigid, thus $\dim \mathcal{A}^{2j}(\Delta'_x) = \dim \mathcal{A}^{2j}(\Delta''_y)$. This proves the lemma. ☐

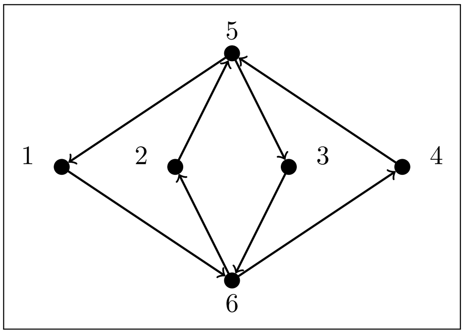

Figure 1. Graph Γ.

It remains to notice, that given a pull-back square of graded vector spaces, that is a pair of subspaces $A, B \subset C$ together with their intersection $A \cap B$, there holds an inclusion-exclusion formula

$$\mathrm{Hilb}(A \cap B; t) + \mathrm{Hilb}(C; t) = \mathrm{Hilb}(A; t) + \mathrm{Hilb}(B; t)$$

so that the Hilbert functions of any three spaces among $A \cap B, A, B, C$ determines the fourth. Both squares in the diagram (3.4) are the pull-back squares. Therefore, Lemma 3.11 implies $\mathrm{Hilb}(\mathrm{Ann}\, V_{\Delta'}; t) = \mathrm{Hilb}(\mathrm{Ann}\, V_{\Delta''}; t)$. This proves the theorem, since $\mathrm{Hilb}(\mathcal{A}(\Delta'); t) = \mathrm{Hilb}(\mathcal{D}^*; t) - \mathrm{Hilb}(\mathrm{Ann}\, V_{\Delta'}; t)$, and the same for Δ''. $\square$

Corollary 3.7 and Theorem 2 imply Theorem 1.

4. Singular examples.

PROPOSITION 4.1. *There exist two complete simplicial multi-fans Δ_1, Δ_2 having the same underlying simplicial cycle but different dimension vectors.*

PROOF. Let $m = 6$ and $n = 2$. Consider the oriented graph Γ on Figure 1. Let $\omega_\Gamma \in C_1(\triangle_{[6]}; \mathbb{Z})$ be the simplicial cycle which is the sum of all oriented edges of Γ with weights 1.

We define two complete simplicial multi-fans Δ_1, Δ_2 with underlying cycle ω_Γ. To do this, we need to specify the values of characteristic functions $\lambda_{1,2} \colon [6] \to \mathbb{R}^2$.

(1) Define the function λ_1 so that its values in the vertices $1, 2, 3, 4$ lie on x-axis and the values in vertices $5, 6$ lie in y-axis. For example, take $\lambda_1(1) = \lambda_1(2) = \lambda_1(3) = \lambda_1(4) = e_1$, $\lambda_1(5) = \lambda_1(6) = e_2$, where e_1, e_2 is the basis of $\mathbb{R}^2$. In this case the multi-fan $\Delta_1 = (w_\Gamma, \lambda_1)$ is the join of two 1-dimensional multi-fans Δ' and Δ'' shown on Figure 2. Equation (3.2) implies that $\mathcal{A}^*(\Delta_1) = \mathcal{A}^*(\Delta') \otimes \mathcal{A}^*(\Delta'')$. Algebra of any 1-dimensional multi-fan has Hilbert function $1 + t^2$ due to Poincare duality. Finally, $\mathrm{Hilb}(\mathcal{A}^*(\Delta_1); t) = (1 + t^2)(1 + t^2) = 1 + 2t^2 + t^4$.

(2) Let us define the function $\lambda_2 \colon [6] \to \mathbb{R}^2$. Set the values $\lambda_2(1), \lambda_2(2), \lambda_2(3), \lambda_2(4)$ arbitrarily (for example we may set them equal to e_1). Now choose $\lambda_2(5)$ and $\lambda_2(6)$ so that they are linearly independent. Then $\Delta_2 = (w_\Gamma, \lambda_2)$ may be represented as a connected

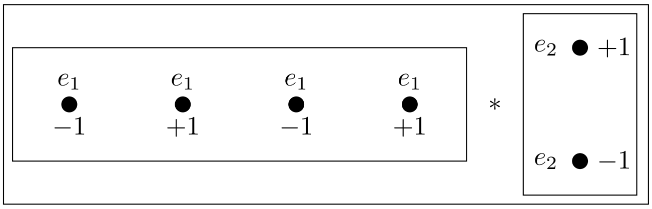

Figure 2. The join of multi-fans Δ' and Δ''.

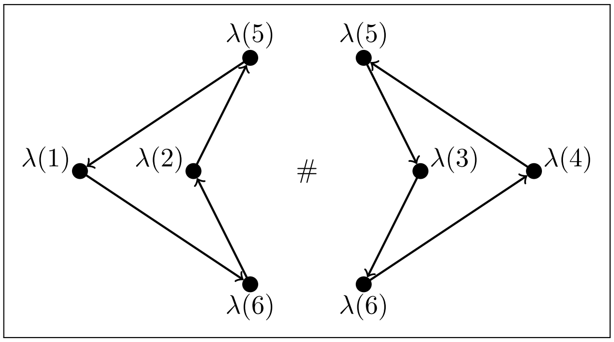

Figure 3. Connected sum of multi-fans $\dot{\Delta}$ and $\ddot{\Delta}$.

sum of 2-dimensional multi-fans $\dot{\Delta}$, $\ddot{\Delta}$ depicted in Figure 3 (The definition of connected sum is given in [**2**]. There we do not require that the set along which the connected sum is taken is a simplex: it is only required that the values of characteristic function on this set are linearly independent.) By [**2**, Proposition 11.3] we have $\mathrm{Hilb}(\Delta_2; t) = \mathrm{Hilb}(\dot{\Delta}; t) + \mathrm{Hilb}(\ddot{\Delta}; t) - (1 + t^4)$. Multi-fans $\dot{\Delta}, \ddot{\Delta}$ are supported on spheres, therefore dimensions of their algebras are the h-vectors. These are $(1, 2, 1)$ in both cases. Thus $\mathrm{Hilb}(\Delta_2; t) = 1 + 4t^2 + t^4$. $\square$

Remark 4.2. The vertices $1, 2, 3, 4$ of Γ are nonsingular vertices. Hence the Hilbert function of $\mathcal{A}^*(\Delta)$ does not depend on the values of λ in these vertices by Theorem 1. Proposition 4.1 implies the following alternative: the dimensions-vector of a multi-fan on Γ is either $(1, 2, 1)$ (if the values $\lambda(5)$, $\lambda(6)$ are collinear) or $(1, 4, 1)$ (if $\lambda(5)$, $\lambda(6)$ are linearly independent).

Next we construct an example similar to Proposition 4.1 but based on a pseudomanifold. At first we specify, what is meant by a simplicial pseudomanifold.

Definition 4.3. A simplicial pseudomanifold of dimension is defined inductively

- A simplicial pseudomanifold of dimension 0 is a pair of points, i.e. the complex S^0.
- A simplicial pseudomanifold of dimension 1 is a cycle graph.
- A simplicial pseudomanifold of dimension $k \geq 2$ is a simplicial complex K such that for every vertex i of K the complex $\mathrm{link}_K\{i\}$ is a connected simplicial pseudomanifold of dimension $k - 1$.

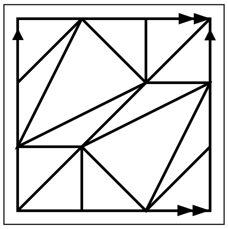

Figure 4. The minimal triangulation of a torus.

This definition implies (but is not equivalent to) the condition that each codimension one simplex of K lies in exactly two maximal simplices of K, which is often taken as the definition of the pseudomanifold. However, we will work with the definition above; so the objects like a 2-sphere pinched at two points, are not considered as pseudomanifolds. Note that the class of simplicial pseudomanifolds of dimension 2 coincides with the class of triangulated surfaces.

PROPOSITION 4.4. *There exist two complete simplicial multi-fans Δ_1, Δ_2 which are supported on the same pseudomanifold but have different dimensions of their multi-fan algebras.*

PROOF. Let L be the minimal triangulation of a 2-torus shown on Figure 4. Using general formulas for h"-numbers, one can show that h"-numbers of L are $(1, 4, 4, 1)$. Let Δ' be any multi-fan supported on L. Since L is a manifold, we have $\mathrm{Hilb}(\mathcal{A}^*(\Delta'); t) = 1 + 4t^2 + 4t^4 + t^6$.

Consider the suspension $K = \Sigma L$, this simplicial complex is obviously a pseudomanifold. We claim that there exist two multi-fans supported on K with different d-vectors of multi-fan algebras

(1) Let Δ'' be any multi-fan supported on S^0. Then we have $\mathrm{Hilb}(\mathcal{A}^*(\Delta''); t) = 1+t^2$. Therefore,

$$\mathrm{Hilb}(\mathcal{A}^*(\Delta' * \Delta''); t) = (1 + 4t^2 + 4t^4 + t^3)(1 + t^2),$$

so the dimension vector of multi-fan $\Delta' * \Delta''$ supported on K is $(1, 5, 8, 5, 1)$. This multi-fan has the property that the values of its characteristic function in suspension points are collinear.

(2) Now we take another multi-fan supported on K and set the values of characteristic function in suspension points to be non-collinear. In this case there is no way to compute dimensions by hands. The computations were implemented in GAP [**5**] with the use of the package simpcomp [**4**]. The script for the computation of dimension vectors is available at [**1**]. It outputs the d-vector $(1, 5, 12, 5, 1)$ whenever the values of λ at suspension points (last two values of the list) are non-collinear. $\square$

In fact, there are exactly two alternatives for the dimension vector of the suspension over a torus.

Proposition 4.5. *Let N be a triangulation of a 2-torus with $m-2$ vertices, and $K = \Sigma N$ its suspension with additional suspension points x, y. For a multi-fan $\Delta = ([K], \lambda)$ supported on the pseudomanifold K there are two alternatives*:

1. *d-vector equals* $(1, m-4, 2m-10, m-4, 1)$ *if the vectors $\lambda(x), \lambda(y)$ are collinear*;
2. *d-vector equals* $(1, m-4, 2m-6, m-4, 1)$ *if the vectors $\lambda(x), \lambda(y)$ are non-collinear.*

Proof. At first note that all vertices of N, that is $M \setminus \{x, y\}$, are nonsingular in the pseudomanifold K. Therefore, the d-vector of a multi-fan does not depend on the values of characteristic function at these vertices according to Theorem 1.

(1) Let $M = [m]$ denote the vertex set of K. Let $\lambda(x), \lambda(y) \in V \cong \mathbb{R}^4$ be collinear. We may assume that all values $\{\lambda(i) \mid i \in M \setminus \{x, y\}\}$ lie in the 3-plane transversal to $\langle \lambda(x) \rangle$. Then Δ is just the join of two multi-fans: one supported on N and another supported on S^0. Formula (3.3) implies

$$\mathrm{Hilb}(\mathcal{A}^*(\Delta); t) = (h''_0(N) + h''_1(N)t^2 + h''_2(N)t^4 + h''_3(N)t^6)(1 + t^2).$$

h''-numbers of any 2-surface are easily computed: they are symmetric, $h''_0 = 1$, and $h''_1 = m - 3$. Therefore,

$$\begin{aligned}\mathrm{Hilb}(\mathcal{A}^*(\Delta); t) &= (1 + (m-5)t^2 + (m-5)t^4 + t^6)(1 + t^2) \\ &= 1 + (m-4)t^2 + (2m-10)t^4 + (m-4)t^6 + t^8.\end{aligned}$$

This proves the first case.

(2) Now suppose that $\lambda(x), \lambda(y)$ are non-collinear. At first, we will show, that proposition holds for the minimal triangulation L of a torus.

Indeed, the statement holds for some particular choice of characteristic function using GAP. In this case we have $m = 9$ and the d-vector is $(1, 5, 12, 5, 1)$. However any two non-collinear pairs of vectors (values in suspension points) may be translated into each other by an element $A \in \mathrm{GL}(V)$. Therefore the d-vector is $(1, 5, 12, 5, 1)$ for any characteristic function on ΣL, according to Remark 2.3.

Let us prove the statement for an arbitrary triangulation of a torus. Any triangulation N is connected to the minimal one by a sequence of bistellar moves according to Pachner's theorem [**13**]. Therefore the corresponding sequence of "suspended bistellar moves" joins $K = \Sigma N$ with ΣL. There are 3 types of bistellar moves in dimension 2, shown on the top of Figure 5. Suspended bistellar moves are shown below.

Each suspended move is decomposed as a sequence of 3-dimensional bistellar moves. In [**2**, Theorem 10] we proved that under bistellar moves (otherwise called *flips*) the dimension vector of multi-fan algebra changes in exactly the same way as the h-vector of simplicial complex does. Let us make all the computations.

The suspended move $\Sigma(0,2)$ is the same as adding new vertex d in the tetrahedron $\{a, b, c, y\}$ (this is $(0,3)$-move) followed by the $(1,2)$-move applied to adjacent tetrahedra $\{a, b, c, d\}$ and $\{a, b, c, x\}$. The $(0,3)$-move adds $(0, 1, 1, 1, 0)$ to the dimension vector, and $(1,2)$-move adds $(0, 0, 1, 0, 0)$. Therefore $\Sigma(0,2)$ increases $d_1 = d_3$ by 1 and d_2 by 2. Similarly, the inverse move $\Sigma(2,0)$ subtracts $(0, 1, 2, 1, 0)$ from the d-vector.

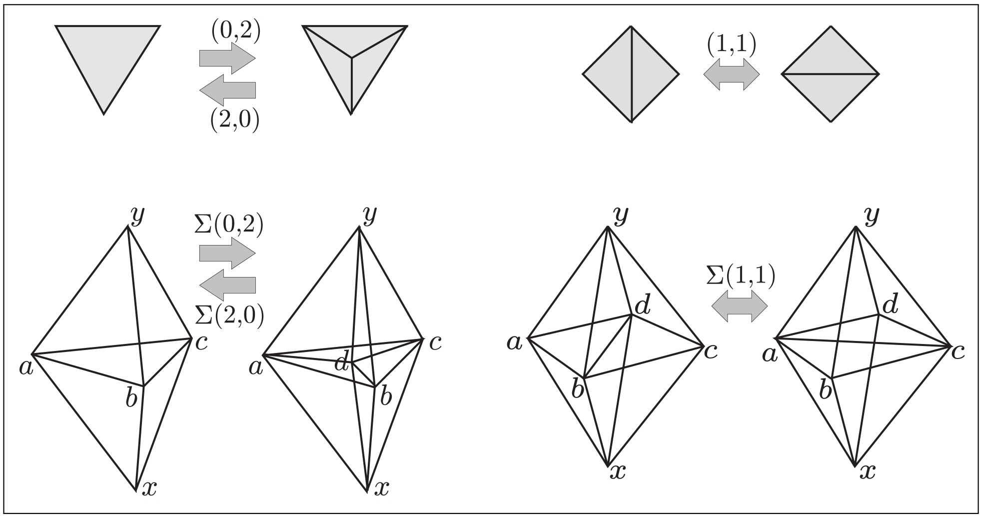

Figure 5. Bistellar moves and their suspensions.

The suspended move $\Sigma(1,1)$ is equivalent to the application of $(1,2)$-move to adjacent tetrahedra $\{a,b,d,y\}$ and $\{b,c,d,y\}$ followed by the application of $(2,1)$-move to tetrahedra $\{a,b,c,d\}$,$\{a,b,d,x\}$,$\{b,d,c,x\}$. First move adds $(0,0,1,0,0)$ to the d-vector, and the second subtracts the same value. So far, the suspended $(1,1)$-move does not change the d-vector.

In all three cases the d-vector changes in the same way, as the expression $(1,m-4,2m-6,m-4,1)$. Since the d-vector is equal to this expression for the minimal triangulation of a torus, the same holds for any triangulation of a 2-torus. $\square$

REMARK 4.6. Propositions 4.4, 4.5 show that suspension K over a 2-torus is not rigid. A suspension-shaped multi-fan supported on K is editable if and only if the values of its characteristic function in the suspension points are collinear.

Propositions show that multi-fans supported on a suspension of a fixed triangulation of a torus may have two different values of d-vector. The technique used in the proof shows that d-vectors of multi-fans supported on suspended orientable surfaces of any genus $g \geq 2$ can take no more than 2 values, depending on whether the values of λ in suspension points are collinear or not. The results of calculations performed in GAP support the following claim.

CLAIM 4.7. *Let N be a triangulated surface of genus g with $m-2$ vertices, and $K = \Sigma N$ its suspension with additional suspension points x, y. For a multi-fan $\Delta = ([K], \lambda)$ supported on the pseudomanifold K there are two alternatives*:

1. *d-vector equals $(1, m-4, 2m-10, m-4, 1)$ if the vectors $\lambda(x), \lambda(y)$ are collinear*;
2. *d-vector equals $(1, m-4, 2m-10+4g, m-4, 1)$ if the vectors $\lambda(x), \lambda(y)$ are non-collinear.*

So far, the gap between possible values of d-vector depends on the links of singular points. The claim was checked for $g \leq 10$ however we cannot explain this result rigorously.

5. 3-dimensional pseudomanifolds.

Let X be any triangulated 3-dimensional closed oriented pseudomanifold with isolated singularities. By this we mean that X is a pure 3-dimensional simplicial complex such that links of all its vertices are orientable surfaces and X has a fundamental cycle. We also assume that there are no two singular points connected by an edge.

Consider the following number

$$r(X) = \text{number of distinct d-vectors of multi-fans supported on } X.$$

PROPOSITION 5.1. *$r(X)$ is a topological invariant, that is $X_1 \cong X_2$ implies $r(X_1) = r(X_2)$.*

PROOF. Any two triangulations of a given topological pseudomanifold with isolated singularities are connected by a sequence of bistellar moves performed outside singularities, as shown in [**3**, Theorem 4.6]. However, each bistellar move have the same effect on all d-vectors, so the number of possible d-vectors coincides for all triangulations. □

EXAMPLE 5.2. As was proved earlier, $r(X) = 1$ for all manifolds. If the pseudomanifold X has a unique singular point, we still have $r(X) = 1$ since the value of λ at the singular point (the only value that matters according to Theorem 1) can be made arbitrary by a linear transformation of an ambient space. If X is a connected sum of several pseudomanifolds X_i with $r(X_i) = 1$, then $r(X)$ also equals 1. Indeed, in [**2**] we showed that $d_j(\Delta_1 \# \Delta_2) = d_j(\Delta_1) + d_j(\Delta_2)$ for $j \neq 0, n$, therefore there is only one possibility for the d-vector of connected sum, whenever this is true for the summands. Hence there exist pseudomanifolds X with any number of singular points having $r(X) = 1$. These examples are consistent with the result of [**12**, Theorem 4.7], which asserts that $\mathrm{Hilb}(\Bbbk[K]/\Theta; t)$ does not depend on the linear system of parameters Θ in the Stanley–Reisner ring $\Bbbk[K]$ for a certain class of pseudomanifolds with isolated singularites.

However, $r(X) = 2$ for the suspended torus (and conjecturally for all suspended surfaces by Claim 4.7). By taking connected sums of suspended tori, we can construct pseudomanifolds X with arbitrarily large $r(X)$.

EXAMPLE 5.3. Consider two 3-pseudomanifolds: X_1 is the suspended 2-torus and X_2 is the connected sum of two copies of the space Y, where Y is the quotient of the solid torus by its boundary. We have $r(X_1) = 2$ by Proposition 4.5 and $r(X_2) = 1$ by the previous example, since the summand Y have only one singularity. The spaces X_1 and X_2 are different, but this difference is not easy to see. The cohomology rings are isomorphic: in both cases cohomology is torsion-free, and the Betti numbers are $(1, 0, 2, 1)$. The multiplication is trivial by dimensional reasons. Both spaces have exactly two singular points with toric links. The difference may be seen by cutting singular points

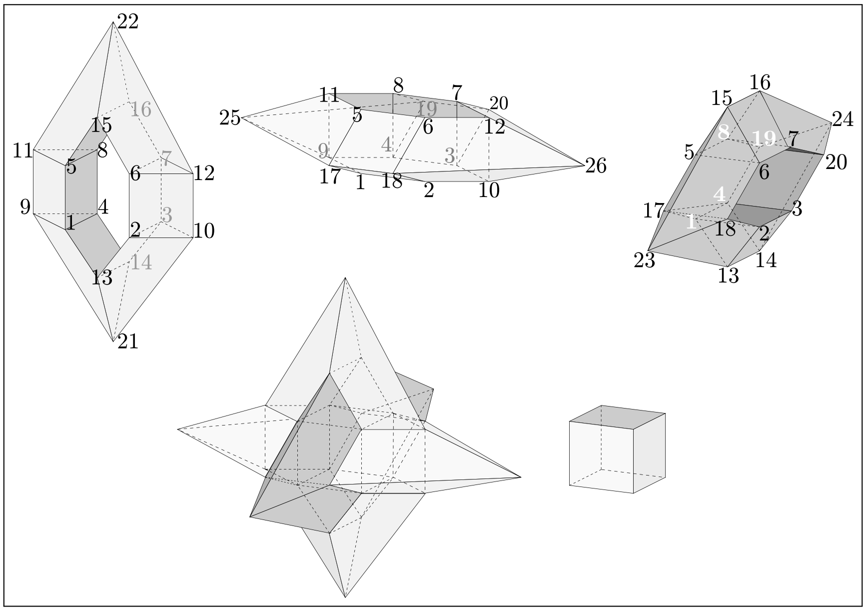

Figure 6. Parts of construction of collapsed Borromean rings.

and noticing that the first space becomes S^3 minus Hopf link, while the second space becomes S^3 minus two unlinked circles. It is interesting that invariant r can sense such knot-theoretical distinctions.

The invariant $r(X)$ somehow measures the complexity of spatial relationships between singular points. It would be interesting to describe this number in a more formal and computable way or at least find out when $r(X) = 1$. The next question is motivated by Example 5.3.

PROBLEM 5.4. Let $l\colon \bigsqcup_\alpha S^1_\alpha \hookrightarrow S^3$ be a link, and X be a pseudomanifold obtained by collapsing each component of l to a point. Is it true that $r(X) = 1$ if and only if each two circles of the link are unlinked?

Example 5.3 shows that pairwise linking numbers affect $r(X)$.

PROPOSITION 5.5. *Let $l\colon \bigsqcup_{\alpha=1,2,3} S^1_\alpha \hookrightarrow S^3$ be the Borromean rings and X be a pseudomanifold obtained by collapsing each component of l to a point. Then $r(X) = 1$.*

PROOF. Figure 6 shows how to triangulate the space X. We specialize three tori in $\mathbb{R}^3$ linked together like Borromean rings (see left part of Figure 7), triangulate the remaining space, and put a cone over each torus. This triangulation is further used perform computations in GAP [**1**].

In order to define multi-fans over this triangulation, we need to specify the values of characteristic function. In our implementation, the singular vertices, which are the apices of the cones over tori, are labeled by numbers $29, 30, 31$. The symmetry group of

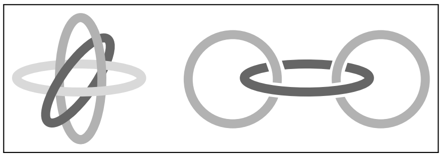

Figure 7. Two links.

X acts transitively on the set of singular vertices, therefore, up to linear transformation of the ambient vector space V we have the following possibilities:

1. $\lambda(29)$, $\lambda(30)$ and $\lambda(31)$ are linearly independent. Without loss of generality, $\lambda(29) = (1,0,0,0)$, $\lambda(30) = (0,1,0,0)$, $\lambda(31) = (0,0,1,0)$.

2. $\lambda(29)$, $\lambda(30)$ and $\lambda(31)$ lie in one 2-space, but any two of them are non-collinear. Without loss of generality $\lambda(29) = (1,0,0,0)$, $\lambda(30) = (0,1,0,0)$, $\lambda(31) = (1,1,0,0)$.

3. $\lambda(29)$ is collinear to $\lambda(30)$ and non-collinear to $\lambda(31)$. Without loss of generality $\lambda(29) = (1,0,0,0)$, $\lambda(30) = (1,0,0,0)$, $\lambda(31) = (0,1,0,0)$.

4. $\lambda(A)$, $\lambda(B)$, and $\lambda(C)$ are collinear. Without loss of generality $\lambda(29) = \lambda(30) = \lambda(31) = (1,0,0,0)$.

These cases are checked in GAP, and the resulting d-vector is $(1,27,100,27,1)$ all the time. Therefore, for the collapsed Borromean rings we have $r(X) = 1$. □

REMARK 5.6. We checked another example shown on the right part of Figure 7. After collapsing each component of this link we obtain a pseudomanifold with 3 singular points. We considered a special triangulation of this space, constructed similarly to the case of Borromean rings. This triangulation has 18 vertices with singular vertices labeled 16 (corresponds to middle circle), 17, and 18. Calculations show that there are 3 alternatives:

1. if $\lambda(16)$, $\lambda(17)$, $\lambda(18)$ are collinear, then d-vector is $(1,14,34,14,1)$;

2. if $\lambda(16)$, $\lambda(17)$, $\lambda(18)$ span 2-dimensional space, then d-vector is $(1,14,38,14,1)$;

3. if $\lambda(16)$, $\lambda(17)$, $\lambda(18)$ are linearly independent, then d-vector is $(1,14,40,14,1)$.

Surprisingly, the relation between the values of characteristic function at singular points corresponding to unlinked circles (vertices labeled by $17, 18$) affect the answer. Indeed, when $\lambda(17), \lambda(18), \lambda(16)$ are all in general position, the answer differs from the case when $\lambda(17) = \lambda(18)$, and $\lambda(16)$ is in general position. This is another strange phenomenon to be explained.

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