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Stabilities of rough curvature dimension condition

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Abstract. We study the asymptotic behavior of metric measure spaces satisfying the rough curvature dimension condition. We prove stabilities of the rough curvature dimension condition with respect to the observable distance function and the L^2-transportation distance function.

1. Introduction.

The curvature dimension condition $\mathrm{CD}(K, N)$ for mm-spaces (metric measure spaces) has been introduced by Sturm [**13**], [**14**] and Lott–Villani [**10**]. This is a generalized notion of Ricci curvature bound from below by $K \in \mathbb{R}$ and dimension bound from above by $N \in [1, \infty]$. Since an mm-space satisfying $\mathrm{CD}(K, N)$ is a geodesic space, the notion does not cover the case of discrete spaces. To extend the notion of curvature bounds to discrete spaces, Bonciocat–Sturm [**4**] introduced the rough curvature dimension condition h-$\mathrm{CD}(K, \infty)$ with roughness parameter $h \geq 0$ and constructed the discretization with h-$\mathrm{CD}(K, \infty)$ condition of mm-space satisfying $\mathrm{CD}(K, \infty)$. After that, Bonciocat [**2**], [**3**] introduced the rough curvature dimension condition h-$\mathrm{CD}(K, N)$ with $N \in [1, \infty)$ and proved some rough geometric properties. They also give nice graphs satisfying h-$\mathrm{CD}(K, N)$, which can be embedded isometrically into N-dimensional Riemannian manifolds. Their approach is based on the definition of the curvature dimension condition and removing the connectivity assumptions on geodesics required in the continuous case.

Sturm [**13**] introduced the L^2-transportation distance function $\mathbb{D}$ (or $\mathbb{D}$ distance function) on the set $\mathcal{X}_v$ of isomorphism classes of mm-spaces with finite second moment. This comes from the ideas of the Gromov–Hausdorff distance between two compact metric spaces and the Wasserstein distance between two Borel probability measures. He proved the stability of $\mathrm{CD}(K, N)$ condition with respect to the L^2-transportation distance function. After that, Bonciocat–Sturm proved the stability of h-$\mathrm{CD}(K, N)$ condition with respect to the L^2-transportation distance function in "from discrete to continuous" case, i.e., if a sequence of mm-spaces satisfies h_n-$\mathrm{CD}(K, N)$ with $h_n \to 0$ as $n \to \infty$, then the $\mathbb{D}$-limit mm-space satisfies $\mathrm{CD}(K, N)$.

Gromov [**9**, Chapter 3. $1/2_+$] introduced the observable distance function d_{conc} on the set $\mathcal{X}$ of isomorphism classes of mm-spaces. This comes from the idea of measure

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concentration phenomenon which is stated as that any 1-Lipschitz function on an mm-space is close to a constant function on a Borel set with almost full measure. The observable distance function is defined by the difference between the sets of 1-Lipschitz functions on two mm-spaces. The topology generated by the observable distance function is weaker than the topology generated by the L^2-transportation distance and allows a convergence sequence of Riemannian manifolds to have unbounded dimensions. For example, the sequence $\{S^n\}_{n=1}^{\infty}$ of n-dimensional unit spheres d_{conc}-converges to the one-point mm-space but this $\mathbb{D}$-diverges. Funano–Shioya [**7**] proved the stability of $\mathrm{CD}(K,\infty)$ condition with respect to d_{conc}-convergence in the case when the limit mm-space is proper.

The aim of this paper is to obtain stabilities of the rough curvature dimension condition with respect to the observable distance function and the L^2-transportation distance function in the general case. In particular, our results contain "from discrete to discrete" case. The following are our main results.

THEOREM 1.1. *Let Y, X_n, $n = 1, 2, \dots$ be mm-spaces and let h, h_n, K, K_n, be real numbers with $h, h_n \geq 0$. Assume that X_n satisfies h_n-$\mathrm{CD}(K_n, \infty)$, X_n d_{conc}-converges to Y, and (h_n, K_n) converges to (h, K) as $n \to \infty$. Then we have the following.*

(1) *If $K \geq 0$, then Y satisfies h-$\mathrm{CD}(K,\infty)$.*

(2) *If $K < 0$, then Y satisfies $2h$-$\mathrm{CD}(K,\infty)$.*

THEOREM 1.2. *Let $Y, X_n, n = 1, 2, \dots$ be mm-spaces and let h, h_n, K, K_n, N, N_n, L, L_n be real numbers with $h, h_n \geq 0$, $L, L_n > 0$ and $N, N_n \geq 1$. Assume that X_n satisfies h_n-$\mathrm{CD}(K_n, N_n)$ and $\operatorname{diam} X_n = L_n$, Y is compact, X_n $\mathbb{D}$-converges to Y as $n \to \infty$ and (h_n, K_n, N_n, L_n) converges to (h, K, N, L) satisfying $KL^2 < (N-1)\pi^2$ as $n \to \infty$. Then Y satisfies the rough curvature dimension condition h-$\mathrm{CD}(K, N)$ and $\operatorname{diam} Y \leq L$.*

Note that in Theorem 1.1, we remove the properness assumption of limit mm-space in Funano–Shioya's result. We also find new example of graphs satisfying h-$\mathrm{CD}(0,1)$. This graph cannot be isometrically embedded into any 1-dimensional Riemannian manifold.

THEOREM 1.3. *Denote by (K_n, d_{K_n}) the complete graph of n-vertices equipped with the graph distance. For any Borel probability measure μ on K_n, the mm-space (K_n, d_{K_n}, μ) satisfies h-$\mathrm{CD}(0,1)$ for $h \geq 1/2$.*

2. Observable distance and L^2-transportation distance.

2.1. Observable distance function.

DEFINITION 2.1 (mm-Space). A triple $X = (X, d_X, \mu_X)$ is called an *mm-space* (*metric measure space*) if (X, d_X) is a complete separable metric space and if μ_X is a Borel probability measure on X. We sometimes say that X is an mm-space, in which case the metric and the measure of X are respectively indicated by d_X and μ_X.

DEFINITION 2.2 (mm-Isomorphism). Two mm-spaces X and Y are said to be *mm-isomorphic* to each other if there exists an isometry $f : \operatorname{supp} \mu_X \to \operatorname{supp} \mu_Y$ such that

$f_*\mu_X = \mu_Y$, where $f_*\mu_X$ is the push-forward measure of μ_X by f. Such an f is called an *mm-isomorphism*. Denote by $\mathcal{X}$ the set of mm-isomorphism classes of mm-spaces.

We assume that an mm-space X satisfies $X = \operatorname{supp}\mu_X$ unless otherwise stated.

Let $I := [0,1]$ and let X be an mm-space. A Borel measurable map $\varphi : I \to X$ is called a *parameter of* X if φ satisfies $\varphi_*\mathcal{L} = \mu_X$, where $\mathcal{L}$ denotes the one-dimensional Lebesgue measure on I. Any mm-space has a parameter (see [**12**, Proposition 4.1]). For two Borel measurable functions $f, g : X \to \mathbb{R}$, we define the *Ky Fan distance between* f *and* g by

$$d_{\mathrm{KF}}(f,g) := \inf\{\varepsilon > 0 \mid \mu_X(\{x \in X \mid |\, f(x) - g(x)\,| > \varepsilon\}) \le \varepsilon\}.$$

The distance function d_{KF} is called the *Ky Fan metric* on the set of Borel measurable functions on X. Note that the Ky Fan metric is a metrization of convergence in measure of Borel measurable functions.

Denote by $\mathcal{L}ip_1(X)$ the set of 1-Lipschitz continuous functions on an mm-space X. For any parameter φ of X, we set $\varphi^*\mathcal{L}ip_1(X) := \{f \circ \varphi \mid f \in \mathcal{L}ip_1(X)\}$.

Definition 2.3 (Observable distance function). We define the *observable distance* $d_{\mathrm{conc}}(X, X')$ *between two mm-spaces* X *and* X' by

$$d_{\mathrm{conc}}(X, X') := \inf_{\varphi,\psi} d_{\mathrm{H}}(\varphi^*\mathcal{L}ip_1(X), \psi^*\mathcal{L}ip_1(X')),$$

where $\varphi : I \to X$ and $\psi : I \to X'$ run over all parameters of X and X', respectively, and where d_{H} is the Hausdorff distance with respect to d_{KF}. We say that a sequence of mm-spaces X_n, $n = 1, 2, \dots$, *concentrates* to an mm-space X if X_n d_{conc}-converges to X as $n \to \infty$.

Note that $(\mathcal{X}, d_{\mathrm{conc}})$ is a separable metric space (see [**12**, Theorem 5.13]).

Proposition 2.4 ([**7**, Proposition 3.5, 3.11, Lemma 5.4], [**12**, Lemma 5.27, Corollary 5.35, Proposition 9.31]). *Let* X_n *and* Y *be mm-spaces,* $n = 1, 2, \dots$. *If* X_n *concentrates to* Y *as* $n \to \infty$, *then there exist Borel measurable maps* $p_n : X_n \to Y$, *positive real numbers* ε_n *with* $\varepsilon_n \to 0$ *as* $n \to \infty$ *and Borel subsets* $\tilde{X}_n \subset X_n$ *with* $\mu_{X_n}(\tilde{X}_n) \ge 1 - \varepsilon_n$ *such that*

(1) $d_{\mathrm{H}}(\mathcal{L}ip_1(X_n), p_n^*\mathcal{L}ip_1(Y)) \le \varepsilon_n$,

(2) $(p_n)_*\mu_{X_n}$ *converges weakly to* μ_Y *as* $n \to \infty$,

(3) $d_Y(p_n(x_n), p_n(x_n')) \le d_{X_n}(x_n, x_n') + \varepsilon_n$ *for any* $x_n, x_n' \in \tilde{X}_n$,

(4) $\displaystyle\limsup_{n\to\infty} \sup_{x_n \in X_n \setminus \tilde{X}_n} d_Y(p_n(x_n), y_0) < +\infty$ *for any* $y_0 \in Y$.

We call $\tilde{X}_n$ *the non-exceptional domain of* p_n *for an additive error* ε_n.

Remark 2.5. (1) By the inner regularity of μ_{X_n}, we may assume $\tilde{X}_n$ is a compact set.

(2) The conditions (1) and (2) of Proposition 2.4 imply the d_{conc}-convergence (see [**7**, Proposition 3.5], [**12**, Corollary 5.36]).

Let X be a complete separable metric space. Denote by $\mathcal{P}(X)$ the set of Borel probability measures on X. For two Borel probability measures $\nu_0, \nu_1 \in \mathcal{P}(X)$, we define the *Prokhorov distance* $d_{\mathrm{P}}(\nu_0, \nu_1)$ *between* ν_0 *and* ν_1 by

$$d_{\mathrm{P}}(\nu_0, \nu_1) := \inf\{\varepsilon > 0 \,|\, \nu_0(A) \leq \nu_1(B_\varepsilon(A)) + \varepsilon \text{ for any Borel set } A \subset X\},$$

where $B_\varepsilon(A)$ is an open ε-neighborhood of A. The distance function d_{P} is called the *Prokhorov metric* on $\mathcal{P}(X)$. Note that the Prokhorov metric is a metrization of the weak topology on $\mathcal{P}(X)$.

Proposition 2.6. *Let X_n and Y be mm-spaces, $n = 1, 2, \ldots$. Assume that X_n concentrates to Y as $n \to \infty$. Then we have*

$$\operatorname{diam} Y \leq \liminf_{n\to\infty} \operatorname{diam} X_n.$$

Proof. By Proposition 2.4, there exist Borel measurable maps $p_n : X_n \to Y$, $\varepsilon_n, \varepsilon_n' > 0$ with $\varepsilon_n, \varepsilon_n' \to 0$ and Borel subsets $\tilde{X}_n \subset X_n$ with $\mu_{X_n}(\tilde{X}_n) \geq 1 - \varepsilon_n$ such that $d_{\mathrm{P}}((p_n)_*\mu_{X_n}, \mu_Y) \leq \varepsilon_n'$ and $d_Y(p_n(x_n), p_n(x_n')) \leq d_{X_n}(x_n, x_n') + \varepsilon_n$ for any $x_n, x_n' \in \tilde{X}_n$. Then we have $\mu_Y\left(B_{\varepsilon_n'}\left(\overline{p_n(\tilde{X}_n)}\right)\right) \geq 1 - (\varepsilon_n + \varepsilon_n')$. Let $\{(y_m, y_m')\}_{m=1}^\infty \subset Y^2$ satisfy

$$\lim_{m\to\infty} d_Y(y_m, y_m') = \operatorname{diam} Y.$$

For fixed $m \in \mathbb{N}$, we take sufficiently small $\eta > 0$ satisfying $\min\{\mu_Y(B_\eta(y_m)), \mu_Y(B_\eta(y_m'))\} > \varepsilon_n + \varepsilon_n'$ and then we have $B_\eta(y_m) \cap B_{\varepsilon_n'}\left(\overline{p_n(\tilde{X}_n)}\right) \neq \emptyset$ and $B_\eta(y_m') \cap B_{\varepsilon_n'}\left(\overline{p_n(\tilde{X}_n)}\right) \neq \emptyset$. There exist $\tilde{x}_{nm}, \tilde{x}_{nm}' \in \tilde{X}_n$ such that $d_Y(y_m, p_n(\tilde{x}_{nm})) < \eta + \varepsilon_n'$ and $d_Y(y_m', p_n(\tilde{x}_{nm}')) < \eta + \varepsilon_n'$. Then we obtain

$$\begin{aligned} d_Y(y_m, y_m') &\leq d_Y(y_m, p_n(\tilde{x}_{nm})) + d_Y(p_n(\tilde{x}_{nm}), p_n(\tilde{x}_{nm}')) + d_Y(p_n(\tilde{x}_{nm}'), y_m') \\ &< d_{X_n}(\tilde{x}_{nm}, \tilde{x}_{nm}') + \varepsilon_n + 2(\eta + \varepsilon_n') \\ &\leq \operatorname{diam} X_n + \varepsilon_n + 2(\eta + \varepsilon_n'). \end{aligned}$$

Taking limits of this inequality as $n \to \infty$, $\eta \to 0$, and then $m \to \infty$, we obtain the proposition. $\square$

2.2. L^2-transportation distance function.

Define $\mathcal{X}_v$ by the subset of isomorphism classes of mm-spaces X with

$$\int_X d_X(x, x_0)^2 \, d\mu_X(x) < \infty$$

for some (hence all) $x_0 \in X$.

Definition 2.7 (Coupling). Let $(X_1, d_{X_1}, \mu_{X_1})$ and $(X_2, d_{X_2}, \mu_{X_2})$ be two mm-spaces and $\mathrm{pr}_i : X_1 \times X_2 \to X_i$ be the natural projection $(i = 1, 2)$. A Borel probability measure π on $X_1 \times X_2$ is called a *coupling of* μ_{X_1} *and* μ_{X_2} if π satisfies $(\mathrm{pr}_i)_*\pi = \mu_{X_i}$ $(i = 1, 2)$. Denote by $\Pi(\mu_{X_1}, \mu_{X_2})$ the set of couplings of μ_{X_1} and μ_{X_2}.

Definition 2.8 (L^2-transportation distance function). For $X, Y \in \mathcal{X}_v$, we define the *L^2-transportation distance between X and Y* by

$$\mathbb{D}(X, Y) := \inf_{\hat{d},\, \pi} \left(\int_{X \times Y} \hat{d}(x, y)^2 \, d\pi(x, y) \right)^{1/2},$$

where $\hat{d}$ and π run over all couplings of d_X and d_Y, μ_X and μ_Y respectively. A coupling $\hat{d}$ of d_X and d_Y is a pseudo-metric on the disjoint union $X \sqcup Y$ satisfying $\hat{d}|_{X \times X} = d_X$ and $\hat{d}|_{Y \times Y} = d_Y$.

Remark 2.9. (1) Note that $(\mathcal{X}_v, \mathbb{D})$ is a complete separable length metric space (see [**13**, Theorem 3.6]).

(2) By [**13**, Lemma 3.7] and [**12**, Proposition 5.5], we have $(2^{-1} d_{\mathrm{conc}}(X, Y))^{3/2} \leq \mathbb{D}(X, Y)$ for any $X, Y \in \mathcal{X}_v$. In particular, the $\mathbb{D}$-convergence implies the d_{conc}-convergence.

3. The rough curvature dimension condition.

3.1. Rough Wasserstein distance function and rough curvature dimension condition.

Definition 3.1 (Relative entropy). Let X be a complete separable metric space. For two Borel probability measures μ and ν on X, the *relative entropy* $\mathrm{Ent}(\nu|\mu)$ of ν with respect to μ is defined as follows. If $\nu = \rho \cdot \mu$, then

$$\mathrm{Ent}(\nu|\mu) := \int_X \rho \log \rho \, d\mu,$$

otherwise $\mathrm{Ent}(\nu|\mu) := \infty$.

Lemma 3.2 ([**12**, Lemma 9.15]). *Let $p : X \to Y$ be a Borel measurable map between two complete separable metric spaces, and let μ and ν be two Borel probability measures on X such that ν is absolutely continuous with respect to μ. Then, $p_*\nu$ is absolutely continuous with respect to $p_*\mu$ and we have*

$$\mathrm{Ent}(p_*\nu|p_*\mu) \leq \mathrm{Ent}(\nu|\mu).$$

Lemma 3.3 ([**6**, Lemma 1.4.3 (b)]). *Let X be a complete separable metric space. The relative entropy $\mathrm{Ent}(\cdot|\cdot) : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, \infty]$ is lower semicontinuous with respect to the weak convergence.*

Lemma 3.4 ([**8**, Proposition 4.1]). *Let X be a complete separable metric space and $\{\mu_n\}_{n=1}^\infty$, $\{\nu_n\}_{n=1}^\infty \subset \mathcal{P}(X)$ be two sequences of Borel probability measures. Assume that $\{\mu_n\}_{n=1}^\infty$ is tight and*

$$\sup_{n\in\mathbb{N}} \operatorname{Ent}(\nu_n|\mu_n) < \infty.$$

Then, $\{\nu_n\}_{n=1}^{\infty}$ is also tight.

Definition 3.5 (Rényi entropy). Let X be an mm-space, N a real number with $N \geq 1$, and ν a Borel probability measure on X. The *Rényi entropy* $S_N(\nu|\mu_X)$ of ν with respect to μ_X is defined as follows.

$$S_N(\nu|\mu_X) := -\int_X \rho^{-1/N}\, d\nu,$$

where ρ is the density of the absolutely continuous part ν^c with respect to μ_X in the Lebesgue decomposition $\nu = \nu^c + \nu^s = \rho \cdot \mu_X + \nu^s$.

Lemma 3.6 ([**14**, Lemma 1.1]). *Let X be an mm-space and $N > 1$. The Rényi entropy functional $S_N(\cdot|\mu_X)$ is lower semicontinuous with respect to the weak convergence and satisfies $-1 \leq S_N(\cdot|\mu_X) \leq 0$.*

Definition 3.7 (Rough Wasserstein distance function). Let (X, d_X) be a metric space and h a nonnegative real number. For two Borel probability measures ν_0 and ν_1 on X, we define the *h-rough Wasserstein distance between ν_0 and ν_1* by

$$W_2^{\pm h}(\nu_0,\nu_1) := \inf_{\pi\in\Pi(\nu_0,\nu_1)} \left(\int_{X\times X} (d_X(x_0,x_1) \mp h)_+^2\, d\pi(x_0,x_1)\right)^{1/2}, \tag{3.1}$$

where $(\cdot)_+$ denotes the positive part. We write $W_2(\nu_0,\nu_1) := W_2^0(\nu_0,\nu_1)$ and call it the *Wasserstein distance between ν_0 and ν_1*.

Denote by $\mathcal{P}_2(X)$ the set of Borel probability measures μ on X such that

$$\int_X d_X(x,x_0)^2\, d\mu(x) < \infty$$

for some point $x_0 \in X$. If (X, d_X) is a complete separable metric space, then so is $(\mathcal{P}_2(X), W_2)$ (see [**15**, Lemma 6.14]). For an mm-space X, we denote by $\mathcal{P}_2^{ac}(X)$ the subset of $\mathcal{P}_2(X)$ satisfying the absolute continuity with respect to μ_X, and by $\mathcal{P}_2^*(X)$ the subset of measures $\nu \in \mathcal{P}_2(X)$ of $\operatorname{Ent}(\nu|\mu_X) < \infty$.

Lemma 3.8 ([**4**, Remark 3.4], [**15**, Lemma 4.4, Theorem 6.9, Remark 6.12]). *For a complete separable metric space X, we have the following* (1)–(4).

(1) *For $\nu_0, \nu_1 \in \mathcal{P}(X)$, the set $\Pi(\nu_0,\nu_1)$ is compact with respect to the weak topology.*

(2) *There exists a minimizer for the infimum in* (3.1). *We will call it $\pm h$-optimal coupling of ν_0 and ν_1. Denote by $\pm h$-*$\operatorname{Opt}(\nu_0,\nu_1)$ *the set of $\pm h$-optimal couplings of ν_0 and ν_1. If $h = 0$, we omit* 0.

(3) *The topology generated by the Wasserstein distance is stronger than the weak topology. If a metric space X is bounded, then the topology generated by the Wasserstein distance and the weak topology coincide to each other.*

(4) *The Wasserstein distance function is lower semicontinuous with respect to the weak topology, i.e., if* $\{\nu_0^n\}_{n=1}^\infty$ *and* $\{\nu_1^n\}_{n=1}^\infty$ *converge weakly to* ν_0 *and* ν_1*, respectively, we have*

$$W_2(\nu_0,\nu_1) \le \liminf_{n\to\infty} W_2(\nu_0^n,\nu_1^n).$$

LEMMA 3.9 ([**2**, Lemma 1.2.5, 1.2.6], [**4**, Lemma 3.5, 3.6]). *For any* $h,k \ge 0$, $0 \le h_1 \le h_2$ *and any* $\nu_1,\nu_2,\nu_3 \in \mathcal{P}_2(X)$*, we have the following* (1)–(6)*.*

(1) $W_2^{+h}(\nu_1,\nu_2) \le W_2(\nu_1,\nu_2) \le W_2^{+h}(\nu_1,\nu_2) + h$.

(2) $W_2(\nu_1,\nu_2) \le W_2^{-h}(\nu_1,\nu_2) \le W_2(\nu_1,\nu_2) + h$.

(3) $W_2^{-h_1}(\nu_1,\nu_2) \le W_2^{-h_2}(\nu_1,\nu_2)$.

(4) $W_2^{+h_2}(\nu_1,\nu_2) \le W_2^{+h_1}(\nu_1,\nu_2)$.

(5) $W_2^{\pm h\pm k}(\nu_1,\nu_3) \le W_2^{\pm h}(\nu_1,\nu_2) + W_2^{\pm k}(\nu_2,\nu_3)$.

(6) $W_2^{\pm h\mp k}(\nu_1,\nu_2) \le W_2^{\pm h}(\nu_1,\nu_2) + k$.

PROOF. Statements (1)–(4) are proved in [**2**, Lemma 1.2.5, 1.2.6] and [**4**, Lemma 3.5, 3.6]. In this paper, we only prove (5) and (6).

We prove (5). By Lemma 3.8 (2), there exist $\pi_{\pm h} \in \pm h\text{-}\operatorname{Opt}(\nu_1,\nu_2)$ and $\pi_{\pm k} \in \pm k\text{-}\operatorname{Opt}(\nu_2,\nu_3)$. Define a projection $\mathrm{pr}_{i,j} : X^3 \to X^2$, $i,j = 1,2,3$ with $i<j$ by $\mathrm{pr}_{i,j}(x_1,x_2,x_3) := (x_i,x_j)$. By the gluing lemma (see [**15**, Section 1]), there exists a Borel probability measure π on X^3 satisfying $(\mathrm{pr}_{1,2})_*\pi = \pi_{\pm h}$, $(\mathrm{pr}_{2,3})_*\pi = \pi_{\pm k}$, and $\pi_{\pm h\pm k} := (\mathrm{pr}_{1,3})_*\pi \in (\pm h \pm k)\text{-}\operatorname{Opt}(\nu_1,\nu_3)$. By Minkowski's inequality, we obtain

$$\begin{aligned}
&W_2^{\pm h\pm k}(\nu_1,\nu_3)\\
&\quad\le \left(\int_{X\times X\times X} \{(d_X(x_1,x_2)\mp h)_+ + (d_X(x_2,x_3)\mp k)_+\}^2\, d\pi(x_1,x_2,x_3)\right)^{1/2}\\
&\quad\le W_2^{\pm h}(\nu_1,\nu_2) + W_2^{\pm k}(\nu_2,\nu_3).
\end{aligned}$$

We prove (6). By Lemma 3.8 (2), there exists $\pi_{\pm h} \in \pm h\text{-}\operatorname{Opt}(\nu_1,\nu_2)$. By Minkowski's inequality, we obtain

$$\begin{aligned}
W_2^{\pm h\mp k}(\nu_1,\nu_2) &\le \left(\int_{X\times X} \{(d_X(x_1,x_2)\mp h)_+ + k\}^2\, d\pi_{\pm h}(x_1,x_2)\right)^{1/2}\\
&\le W_2^{\pm h}(\nu_1,\nu_2) + k.
\end{aligned}$$

This completes the proof of lemma. □

DEFINITION 3.10 (Rough curvature dimension condition: the case $N=\infty$). Let X be an mm-space, h a nonnegative real number, and K a real number. We say that an mm-space X satisfies the *h-rough curvature dimension condition* h-$\mathrm{CD}(K,\infty)$ if for any $\nu_0,\nu_1 \in \mathcal{P}_2^*(X)$, there exists a family of measures $(\nu_t)_{t\in(0,1)} \subset \mathcal{P}_2(X)$ such that for any $t\in[0,1]$, we have

$$W_2(\nu_t, \nu_i) \le t^{1-i}(1-t)^i W_2(\nu_0, \nu_1) + h, \quad i = 0, 1, \tag{3.2}$$

$$\mathrm{Ent}(\nu_t|\mu_X) \le (1-t)\,\mathrm{Ent}(\nu_0|\mu_X) + t\,\mathrm{Ent}(\nu_1|\mu_X) - \frac{1}{2}Kt(1-t)W_2^{\theta_K h}(\nu_0, \nu_1)^2, \tag{3.3}$$

where $\theta_K = -1$ for $K < 0$ and $\theta_K = 1$ for $K \ge 0$. A map $[0,1] \ni t \mapsto \nu_t \in \mathcal{P}_2(X)$ satisfying (3.2) is called an *h-rough geodesic on* $(\mathcal{P}_2(X), W_2)$.

Lemma 3.11. *Let (X, d_X) be a metric space and $h \ge 0$. If a map $\gamma : [0,1] \to X$ satisfies*

$$(1-t)d_X(\gamma_0, \gamma_t)^2 + t d_Y(\gamma_t, \gamma_1)^2 \le t(1-t)d_X(\gamma_0, \gamma_1)^2 + h^2,$$

then $(\gamma_t)_{t\in[0,1]}$ is an h-rough geodesic on (X, d_X).

Proof. By the triangle inequality,

$$\begin{aligned} h^2 &\ge (1-t)d_X(\gamma_0, \gamma_t)^2 + t d_X(\gamma_t, \gamma_1)^2 - t(1-t)d_X(\gamma_0, \gamma_1)^2 \\ &\ge (1-t)\{d_X(\gamma_0, \gamma_1) - d_X(\gamma_t, \gamma_1)\}^2 + t d_X(\gamma_t, \gamma_1)^2 - t(1-t)d_X(\gamma_0, \gamma_1)^2 \\ &= \{d_X(\gamma_t, \gamma_1) - (1-t)d_X(\gamma_0, \gamma_1)\}^2. \end{aligned}$$

Similarly, we have $d_X(\gamma_t, \gamma_0) \le t d_X(\gamma_0, \gamma_1) + h$. □

For two positive real numbers K, N with $N \ge 1$ and $(t, \theta) \in [0,1] \times \mathbb{R}_{\ge 0}$, we define the function $\tau_{K,N}^{(t)}(\theta)$ by

$$\tau_{K,N}^{(t)}(\theta) := \begin{cases} \infty & \text{if } K\theta^2 \ge (N-1)\pi^2, \\ t^{1/N}\left(\dfrac{\sin\left(t\theta\sqrt{K/(N-1)}\,\right)}{\sin\left(\theta\sqrt{K/(N-1)}\,\right)}\right)^{1-1/N} & \text{if } 0 < K\theta^2 < (N-1)\pi^2, \\ t & \text{if } K\theta^2 = 0 \text{ or} \\ & \text{if } K\theta^2 < 0 \text{ and } N = 1, \\ t^{1/N}\left(\dfrac{\sinh\left(t\theta\sqrt{-K/(N-1)}\,\right)}{\sinh\left(\theta\sqrt{-K/(N-1)}\,\right)}\right)^{1-1/N} & \text{if } K\theta^2 < 0 \text{ and } N > 1. \end{cases}$$

Definition 3.12 (Rough curvature dimension condition: the case $N < \infty$). Let X be an mm-space, h a nonnegative real number, and K, N real numbers with $N \ge 1$. We say that an mm-space X satisfies the *h-rough curvature dimension condition* h-$\mathrm{CD}(K, N)$ if for any two measures $\nu_0 = \rho_0 \cdot \mu_X, \nu_1 = \rho_1 \cdot \mu_X \in \mathcal{P}_2^{ac}(X)$, there exists a $\theta_K h$-optimal coupling π of ν_0 and ν_1 and a family of measures $(\nu_t)_{t\in(0,1)} \subset \mathcal{P}_2(X)$ such that for any $t \in [0,1]$ and any $N' \ge N$, we have

$$W_2(\nu_t, \nu_i) \le t^{1-i}(1-t)^i W_2(\nu_0, \nu_1) + h, \quad i = 0, 1, \tag{3.4}$$

$$\begin{aligned} S_{N'}(\nu_t|\mu_X) \le -\int_{X\times X} \Big\{&\tau_{K,N'}^{(1-t)}((d_X(x_0,x_1) - \theta_K h)_+)\rho_0^{-1/N'}(x_0) \\ &+ \tau_{K,N'}^{(t)}((d_X(x_0,x_1) - \theta_K h)_+)\rho_1^{-1/N'}(x_1)\Big\}\, d\pi(x_0, x_1), \end{aligned} \tag{3.5}$$

where $\theta_K = -1$ for $K < 0$ and $\theta_K = 1$ for $K \geq 0$.

We write $\mathrm{CD}(K,N)$ instead of 0-$\mathrm{CD}(K,N)$ and call it the *curvature dimension condition.*

REMARK 3.13. (1) On the definition of rough curvature dimension condition, the reference measure μ_X is not necessary probability measure. In Example 3.16, we consider mm-spaces satisfying the rough curvature dimension condition with infinite measures.

(2) By the continuity of the Rényi entropy $S_\cdot(\nu|\mu_X) : [1,\infty) \to \mathbb{R}$ and Fatou's lemma, it suffices to check the case $N' > N$ in Definition 3.12.

For $\nu_0, \nu_1 \in \mathcal{P}_2^{ac}(X)$ and a coupling π of ν_0 and ν_1, we define

$$\begin{aligned}
T^{(1-t),0}_{h,K,N'}(\pi|\mu_X) &:= -\int_{X\times X} \tau^{(1-t)}_{K,N'}((d_X(x_0,x_1) - \theta_K h)_+)\rho_0^{-1/N'}(x_0)\,d\pi(x_0,x_1),\\
T^{(t),1}_{h,K,N'}(\pi|\mu_X) &:= -\int_{X\times X} \tau^{(t)}_{K,N'}((d_X(x_0,x_1) - \theta_K h)_+)\rho_1^{-1/N'}(x_1)\,d\pi(x_0,x_1),\\
T^{(t)}_{h,K,N'}(\pi|\mu_X) &:= T^{(1-t),0}_{h,K,N'}(\pi|\mu_X) + T^{(t),1}_{h,K,N'}(\pi|\mu_X).
\end{aligned}$$

THEOREM 3.14 ([**11**, Theorem 1.1], [**13**, Theorem 4.9], [**14**, Theorem 1.7], [**10**, Theorem 7.3]). *Let M be a complete Riemannian manifold and K a real number, and $N \in [1,\infty]$. Then M satisfies* $\mathrm{CD}(K,N)$ *if and only if* $\mathrm{Ric}_M \geq K$ *and* $\dim M \leq N$, *where* Ric_M *denotes the Ricci curvature of M.*

LEMMA 3.15 ([**2**, Proposition 2.2.7], [**3**, Proposition 3.7]). *Let h, K, N be real numbers with $h \geq 0$ and $N \geq 1$. If an mm-space X satisfies the rough curvature dimension condition h-*$\mathrm{CD}(K,N)$*, then X satisfies h-*$\mathrm{CD}(K,\infty)$.

EXAMPLE 3.16 ([**4**, Example 3.2, 4.2, 4.4], [**2**, Subsection 2.5], [**3**, Section 6]).

(1) The space $\mathbb{Z}^n \subset \mathbb{R}^n$ equipped with the l_1-norm $\|\cdot\|_1$ and the counting measure $\mu_{\mathbb{Z}^n}$ satisfies h-$\mathrm{CD}(0,n)$ for $h \geq 2n$.

(2) The n-dimensional grid $\mathbb{G}^n$ having $\mathbb{Z}^n$ as the set of vertices, equipped with the graph distance (l_1-norm) and the 1-dimensional Lebesgue measure on edges, satisfies h-$\mathrm{CD}(0,n)$ for $h \geq 2(n+1)$.

(3) Let $\mathbb{G}(l,n,r)$ be a homogeneous planar graph and $\mu_{\mathbb{G}}$ be the uniform measure on the set of edges. We assume that vertices have constant degree $l \geq 3$, faces are bounded by polygons with $n \geq 3$ edges, and edges have the same length $r > 0$. Denote $\mathbb{V}(l,n,r)$ the set of vertices of $\mathbb{G}(l,n,r)$ equipped with the counting measure $\mu_{\mathbb{V}}$. $\mathbb{G}(l,n,r)$ and $\mathbb{V}(l,n,r)$ are embedded into the 2-dimensional Riemannian manifold $(M_K^2, d_{M_K^2})$ with constant sectional curvature $K = K(l,n,r)$, where K is defined by

$$K = K(l,n,r) := \begin{cases} -\dfrac{1}{r^2}\left[\operatorname{arccosh}\left(\dfrac{2\cos^2(\pi/n)}{\sin^2(\pi/l)} - 1\right)\right]^2 & \text{if } \dfrac{1}{l} + \dfrac{1}{n} < \dfrac{1}{2}, \\ 0 & \text{if } \dfrac{1}{l} + \dfrac{1}{n} = \dfrac{1}{2}, \\ \dfrac{1}{r^2}\left[\arccos\left(\dfrac{2\cos^2(\pi/n)}{\sin^2(\pi/l)} - 1\right)\right]^2 & \text{if } \dfrac{1}{l} + \dfrac{1}{n} > \dfrac{1}{2}. \end{cases}$$

Then two mm-spaces $(\mathbb{G}(l,n,r), d_{M_K^2}, \mu_{\mathbb{G}})$ and $(\mathbb{V}(l,n,r), d_{M_K^2}, \mu_{\mathbb{V}})$ satisfy h-$\mathrm{CD}(K,2)$ for $h \geq r \cdot C(l,n)$, where

$$C(l,n) := 4\operatorname{arcsinh}\left(\frac{1}{\sin(\pi/n)}\sqrt{\frac{\cos^2(\pi/n)}{\sin^2(\pi/l)} - 1}\right)\left(\operatorname{arccosh}\left(\frac{2\cos^2(\pi/n)}{\sin^2(\pi/l)} - 1\right)\right)^{-1}.$$

Proof of Theorem 1.3. Put $\mu = \sum_{i=1}^n m_i\delta_i$, where δ_i is the Dirac measure at $i \in K_n$. Take $\nu_0 = \sum_{i=1}^n a_i\delta_i, \nu_1 = \sum_{j=1}^n b_j\delta_j \in \mathcal{P}(K_n)$. For any $0 \leq h < 1$, we first prove

$$W_2^{+h}(\nu_0,\nu_1)^2 = (1-h)^2\sum_{i\in A}(a_i - b_i) = (1-h)^2\sum_{i\in A^c}(b_i - a_i) = \frac{(1-h)^2}{2}\sum_{i=1}^n|a_i - b_i|, \quad (3.6)$$

where $A := \{i \in K_n \,|\, a_i \geq b_i\}$. We may assume $A = \{1,2,\ldots,k\}$ with $k < n$. Note that $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 1$ and $\sum_{i=1}^n |a_i - b_i| = \sum_{i\in A}(a_i - b_i) + \sum_{i\in A^c}(-a_i + b_i)$ imply the second and the third equality. We check the first equality. By the Kantorovich duality (see [**15**, Theorem 5.10]),

$$\begin{aligned} &W_2^{+h}(\nu_0,\nu_1)^2 \\ &= \sup\left\{\sum_{i=1}^n a_i\varphi(i) + \sum_{i=1}^n b_i\psi(i)\ \middle|\ \varphi \in L^1(\nu_0),\, \psi \in L^1(\nu_1),\, \varphi(i) + \psi(j) \leq (d_{K_n}(i,j) - h)_+^2\right\}. \end{aligned}$$

Choose functions φ and ψ by

$$\psi(i) := \begin{cases} (1-h)^2 & \text{if } i \in A, \\ 0 & \text{if } i \in A^c, \end{cases} \qquad \psi(j) := \begin{cases} -(1-h)^2 & \text{if } j \in A, \\ 0 & \text{if } j \in A^c. \end{cases}$$

Then we have

$$(1-h)^2\sum_{i\in A}(a_i - b_i) \leq W_2^{+h}(\nu_0,\nu_1)^2.$$

On the other hand, we construct a coupling $\pi = \sum_{i,j=1}^n w_{ij}\delta_{(i,j)}$ of ν_0 and ν_1 as follows.

$$w_{ij} := \begin{cases} b_i & \text{if } \ i = j, \ 1 \le i \le k, \\ a_i & \text{if } \ i = j, \ k+1 \le i \le n, \\ \left\{ \sum_{l=1}^{k} (a_l - b_l) \right\}^{-1} (a_i - b_i)(b_j - a_j) & \text{if } \ i \ne j, \ 1 \le i \le k, \ k+1 \le j \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} W_2^{+h}(\nu_0, \nu_1)^2 &\le \sum_{i \ne j} (1-h)^2 w_{ij} \\ &= (1-h)^2 \left\{ \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} \min\{a_i, b_i\} \right\} \\ &= (1-h)^2 \sum_{i \in A} (a_i - b_i). \end{aligned}$$

Thus we obtain (3.6).

Put

$$\nu_t := (1-t)\nu_0 + t\nu_1 = \sum_{i=1}^{n} \{(1-t)a_i + tb_i\}\delta_i.$$

By (3.6),

$$\begin{aligned} &(1-t)W_2(\nu_0, \nu_t)^2 + tW_2(\nu_t, \nu_1)^2 - t(1-t)W_2(\nu_0, \nu_1)^2 \\ &\quad = \frac{1-t}{2} \sum_{i=1}^{n} |a_i - (1-t)a_i - tb_i| + \frac{t}{2} \sum_{i=1}^{n} |(1-t)a_i + tb_i - b_i| - \frac{t(1-t)}{2} \sum_{i=1}^{n} |a_i - b_i| \\ &\quad = \frac{t(1-t)}{2} \sum_{i=1}^{n} |a_i - b_i| \\ &\quad \le \frac{1}{4}. \end{aligned}$$

By Lemma 3.11, $(\nu_t)_{t \in [0,1]}$ is an h-rough geodesic for $h \ge 1/2$. By Jensen's inequality and the convexity of $f(s) = -s^{1-1/N}$ with $N > 1$,

$$\begin{aligned} S_N(\nu_t|\mu) &= - \sum_{i \in \operatorname{supp} \mu} \left\{ \frac{(1-t)a_i + tb_i}{m_i} \right\}^{1-1/N} m_i \\ &\le -(1-t) \sum_{i \in \operatorname{supp} \mu} \left(\frac{a_i}{m_i} \right)^{1-1/N} m_i - t \sum_{i \in \operatorname{supp} \mu} \left(\frac{b_i}{m_i} \right)^{1-1/N} m_i \\ &= (1-t)S_N(\nu_0|\mu) + tS_N(\nu_1|\mu). \end{aligned}$$

Therefore (K_n, d_{K_n}, μ) satisfies h-CD$(0,1)$ for $h \ge 1/2$. $\square$

Remark 3.17. We do not know that the lower curvature bound of (K_n, d_{K_n}, μ)

is sharp. For sufficiently small $\varepsilon > 0$, we put $\nu_0^\varepsilon := (2^{-1}+\varepsilon)\delta_1 + (2^{-1}-\varepsilon)\delta_2$ and $\nu_1^\varepsilon := (2^{-1}-\varepsilon)\delta_1 + (2^{-1}+\varepsilon)\delta_2$. $\nu_t^\varepsilon := (1-t)\nu_0 + t\nu_1$ is an h-rough geodesic for $h \geq 1/2$. We assume $(\nu_t^\varepsilon)_{t\in[0,1]}$ satisfy (3.3) for $K > 0$. Taking the limit as $\varepsilon \to 0$, this leads the contradiction. Unfortunately, we do not know whether for any other h-rough geodesic (3.3) is satisfied or not.

The following is an example and a corollary of Theorem 1.2 and 1.3.

Example 3.18. Let $i \in \mathbb{N} \cup \{\infty\}$ and $k, n \in \mathbb{N}$ with $k < n$. Define a probability measure on K_n by

$$\mu_{n,k}^i := \sum_{l=1}^{k} \frac{i}{k(i-1)+n}\delta_l + \sum_{l=k+1}^{n} \frac{1}{k(i-1)+n}\delta_l.$$

For each i, k, n, the mm-space $K_{n,k}^i := (K_n, d_{K_n}, \mu_{n,k}^i)$ satisfies h-CD$(0,1)$ for $h \geq 1/2$. The sequence $\{K_{n,k}^i\}_{i=1}^\infty$ $\mathbb{D}$-converges to $K_{n,k}^\infty$, which is isomorphic to K_k. Indeed, by (3.6),

$$\begin{aligned}\mathbb{D}(K_{n,k}^i, K_{n,k}^\infty) &\leq W_2(\mu_{n,k}^i, \mu_{n,k}^\infty) \\ &= \sqrt{\sum_{l=1}^{k}\left|\frac{i}{k(i-1)+n} - \frac{1}{k}\right| + \sum_{l=k+1}^{n}\frac{1}{k(i-1)+n}} \\ &\to 0,\end{aligned}$$

as $i \to \infty$.

4. Proof of Theorem 1.1.

For an mm-space X, we denote by $\mathcal{P}^{cb}(X)$ the set of Borel probability measures ν on X with compact support that are absolutely continuous with respect to μ_X and their density functions are essentially bounded on X. Note that $\mathcal{P}^{cb}(X)$ is a dense subset in $(\mathcal{P}_2(X), W_2)$.

Lemma 4.1 ([**12**, Lemma 9.20]). *Let X be an mm-space and $\nu \in \mathcal{P}_2^*(X)$. Then, for any $\varepsilon > 0$, there exists $\tilde{\nu} \in \mathcal{P}^{cb}(X)$ such that*

$$W_2(\tilde{\nu}, \nu) < \varepsilon \quad \textit{and} \quad |\operatorname{Ent}(\tilde{\nu}|\mu_X) - \operatorname{Ent}(\nu|\mu_X)| < \varepsilon.$$

Lemma 4.2. *Let X be an mm-space, h a nonnegative real number, and K a real number. If we assume that any $\nu_0, \nu_1 \in \mathcal{P}^{cb}(X)$ satisfy the conditions in the definition of h-CD(K,∞), then X satisfies h-CD(K,∞).*

Proof. Lemma 3.9 (5) and Lemma 4.1 together imply the lemma. □

For a Borel subset B of an mm-space X with positive measure, we define a Borel probability measure μ_B by

$$\mu_B := \frac{\mu_X|_B}{\mu_X(B)}.$$

Lemma 4.3 ([**7**, Lemma 3.13], [**12**, Lemma 9.33]). *Let X_n and Y be mm-spaces, $n = 1, 2, \ldots$. Assume that a sequence of Borel measurable maps $p_n : X_n \to Y$ and a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive real numbers with $\varepsilon_n \to 0$ satisfy (1)–(3) of Proposition 2.4. For a real number $\delta > 0$, we give two Borel subsets $B_0, B_1 \subset Y$ such that*

$$\operatorname{diam} B_i \le \delta, \quad \mu_Y(B_i) > 0, \quad \text{and} \quad \mu_Y(\partial B_i) = 0$$

for $i = 0, 1$, and set

$$\tilde{B}_i := p_n^{-1}(B_i) \cap \tilde{X}_n \subset X_n,$$

where $\tilde{X}_n$ is a non-exceptional domain of p_n. Then, there exist Borel probability measures $\tilde{\mu}_0^n$, $\tilde{\mu}_1^n$ on X_n and couplings $\tilde{\pi}_n$ between $\tilde{\mu}_0^n$ and $\tilde{\mu}_1^n$, $n = 1, 2, \ldots$, such that, for every sufficiently large natural number n,

(1) $\tilde{\mu}_i^n \le (1 + O(\delta^{1/2}))\mu_{\tilde{B}_i}$ $(i = 0, 1)$, *where $O(\cdot)$ is a Landau symbol,*

(2) $d_{X_n}(x_0, x_1) \ge d_Y(B_0, B_1) - \varepsilon_n$ *for any $x_i \in \tilde{B}_i$, $i = 0, 1$,*

(3) $\operatorname{supp} \tilde{\pi}^n \subset \{(x_n, x_n') \in X_n^2 \mid d_{X_n}(x_n, x_n') \le d_Y(B_0, B_1) + \delta^{1/2}\}$,

(4) $-\varepsilon_n \le W_2^{\pm h}(\tilde{\mu}_0^n, \tilde{\mu}_1^n) - (d_Y(B_0, B_1) \mp h)_+ \le \delta^{1/2}$ *for any nonnegative real number h.*

Proof. Existence of $\tilde{\mu}_0^n, \tilde{\mu}_1^n$ and statements (1)–(3) are proved in [**12**, Lemma 9.33]. We only prove that (1)–(3) imply (4). By (2), we have

$$(d_Y(B_0, B_1) \mp h)_+ \le (d_{X_n}(x_0, x_1) \mp h)_+ + \varepsilon_n$$

for any $x_i \in \tilde{B}_i$, $i = 0, 1$. Let $\pi \in h\text{-}\operatorname{Opt}(\tilde{\mu}_0^n, \tilde{\mu}_1^n)$. By (1), we have $\operatorname{supp} \pi \subset \tilde{B}_0 \times \tilde{B}_1$. Then, Minkowski's inequality and the above inequality imply

$$\begin{aligned} (d_Y(B_0, B_1) \mp h)_+ &\le \left(\int_{X_n \times X_n} \{(d_{X_n}(x_n, x_n') \mp h)_+ + \varepsilon_n\}^2 \, d\pi(x_n, x_n') \right)^{1/2} \\ &\le W_2^{\pm h}(\tilde{\mu}_0^n, \tilde{\mu}_1^n) + \varepsilon_n. \end{aligned}$$

By (3), we have

$$\operatorname{supp} \tilde{\pi}^n \subset \left\{ (x_n, x_n') \in X_n^2 \,\middle|\, (d_{X_n}(x_n, x_n') \mp h)_+ \le (d_Y(B_0, B_1) \mp h)_+ + \delta^{1/2} \right\}.$$

Then, we obtain

$$\begin{aligned} W_2^{\pm h}(\tilde{\mu}_0^n, \tilde{\mu}_1^n) &\le \left(\int_{X_n \times X_n} (d_{X_n}(x_n, x_n') \mp h)_+^2 \, d\tilde{\pi}^n(x_n, x_n') \right)^{1/2} \\ &= (d_Y(B_0, B_1) \mp h)_+ + \delta^{1/2}. \end{aligned}$$

This completes the proof of (4). $\square$

Proof of Theorem 1.1. We take any $\nu_0, \nu_1 \in \mathcal{P}^{cb}(Y)$ and fix them. For any natural number m, there are finite disjoint Borel subsets $B_j \subset Y$, $j = 1, 2, \ldots, J$, such that $\bigcup_{j=1}^{J} \overline{B_j} = \operatorname{supp} \nu_0 \cup \operatorname{supp} \nu_1$, $\operatorname{diam} B_j \le m^{-1}$, $\mu_Y(B_j) > 0$, and $\mu_Y(\partial B_j) = 0$ for any j. For each $(j,k) \in \{1, \ldots, J\}^2$, we apply Lemma 4.3 to B_j and B_k and obtain Borel probability measures $\tilde{\xi}_{jk}^{mn} \in \mathcal{P}^{cb}(X_n)$, $n = 1, 2, \ldots$, such that

$$\tilde{\xi}_{jk}^{mn} \le (1 + \theta(m^{-1}))\mu_{\tilde{B}_j}, \tag{4.1}$$

$$\left| W_2^{\theta_{K_n} h_n}(\tilde{\xi}_{jk}^{mn}, \tilde{\xi}_{kj}^{mn}) - (d_Y(B_j, B_k) - \theta_{K_n} h_n)_+ \right| \le \theta(m^{-1}), \tag{4.2}$$

for any sufficiently large natural number n. Here, $\theta(\cdot)$ is a function with $\theta(\delta) \to 0$ as $\delta \to 0$. By the diagonal argument, we may assume that $(p_n)_* \tilde{\xi}_{jk}^{mn}$ converges weakly to a Borel probability measure $\tilde{\xi}_{jk}^{m} \in \mathcal{P}^{cb}(Y)$ as $n \to \infty$ for each $(j,k,m) \in \{1, \ldots, J\}^2 \times \mathbb{N}$. Take a coupling π of ν_0 and ν_1 as follows. If $K \ge 0$, the measure π is an optimal coupling for $W_2(\nu_0, \nu_1)$. If $K < 0$, the measure π is an optimal coupling for $W_2^{\theta_K h}(\nu_0, \nu_1)$. We define

$$\begin{aligned}
w_{jk} &:= \pi(B_j \times B_k), \\
\tilde{\nu}_0^{mn} &:= \sum_{j,k=1}^{J} w_{jk} \tilde{\xi}_{jk}^{mn}, & \tilde{\nu}_1^{mn} &:= \sum_{j,k=1}^{J} w_{jk} \tilde{\xi}_{kj}^{mn} \quad \in \mathcal{P}^{cb}(X_n), \\
\tilde{\nu}_0^{m} &:= \sum_{j,k=1}^{J} w_{jk} \tilde{\xi}_{jk}^{m}, & \tilde{\nu}_1^{m} &:= \sum_{j,k=1}^{J} w_{jk} \tilde{\xi}_{kj}^{m} \quad \in \mathcal{P}^{cb}(Y).
\end{aligned}$$

Then, $(p_n)_* \tilde{\nu}_0^{mn}$ and $(p_n)_* \tilde{\nu}_1^{mn}$ converge weakly to $\tilde{\nu}_0^{m}$ and $\tilde{\nu}_1^{m}$, respectively, as $n \to \infty$. $\tilde{\nu}_0^{m}$ and $\tilde{\nu}_1^{m}$ converge weakly to ν_0 and ν_1, respectively, as $m \to \infty$. Moreover, $W_2((p_n)_* \tilde{\nu}_0^{mn}, \nu_0)$, $W_2((p_n)_* \tilde{\nu}_1^{mn}, \nu_1) \to 0$ as $n \to \infty$ and then $m \to \infty$. The condition h_n-CD(K_n, ∞) implies that, for any $t \in (0,1)$, there is $\tilde{\nu}_t^{mn} \in \mathcal{P}_2(X_n)$ such that

$$W_2(\tilde{\nu}_t^{mn}, \tilde{\nu}_i^{mn}) \le t^{1-i}(1-t)^i W_2(\tilde{\nu}_0^{mn}, \tilde{\nu}_1^{mn}) + h_n, \quad i = 0, 1, \tag{4.3}$$

$$\begin{aligned}
\operatorname{Ent}(\tilde{\nu}_t^{mn} | \mu_{X_n}) &\le (1-t) \operatorname{Ent}(\tilde{\nu}_0^{mn} | \mu_{X_n}) + t \operatorname{Ent}(\tilde{\nu}_1^{mn} | \mu_{X_n}) \\
&\quad - \frac{1}{2} K_n t(1-t) W_2^{\theta_{K_n} h_n}(\tilde{\nu}_0^{mn}, \tilde{\nu}_1^{mn})^2.
\end{aligned} \tag{4.4}$$

Let $\tilde{\pi}$ be an optimal coupling of $W_2^{\theta_{K_n} h_n}(\tilde{\nu}_0^{mn}, \tilde{\nu}_1^{mn})$. Then, $(p_n \times p_n)_* \tilde{\pi}$ is a coupling of $(p_n)_* \tilde{\nu}_0^{mn}$ and $(p_n)_* \tilde{\nu}_1^{mn}$. Proposition 2.4 (3), $\operatorname{supp} \tilde{\nu}_i^{mn} \subset \tilde{X}_n$ $(i = 0, 1)$ together imply

$$\begin{aligned}
&W_2^{\theta_K h}((p_n)_* \tilde{\nu}_0^{mn}, (p_n)_* \tilde{\nu}_1^{mn})^2 \\
&\quad \le \int_{Y \times Y} (d_Y(y, y') - \theta_K h)_+^2 \, d(p_n \times p_n)_* \tilde{\pi}(y, y') \\
&\quad \le \int_{X_n \times X_n} \{(d_{X_n}(x_n, x_n') - \theta_{K_n} h_n)_+ + |\theta_{K_n} h_n - \theta_K h| + \varepsilon_n\}^2 \, d\tilde{\pi}(x_n, x_n') \\
&\quad \le (W_2^{\theta_{K_n} h_n}(\tilde{\nu}_0^{mn}, \tilde{\nu}_1^{mn}) + |\theta_{K_n} h_n - \theta_K h| + \varepsilon_n)^2.
\end{aligned}$$

Since $(p_n)_*\tilde{\nu}_0^{mn}$ and $(p_n)_*\tilde{\nu}_1^{mn}$ W_2-converge to ν_0 and ν_1, respectively, this inequality and Lemma 3.9 (5) together imply

$$W_2^{\theta_K h}(\nu_0,\nu_1) \leq \liminf_{m\to\infty}\ \liminf_{n\to\infty}\left(W_2^{\theta_{K_n} h_n}(\tilde{\nu}_0^{mn},\tilde{\nu}_1^{mn}) + |\theta_{K_n} h_n - \theta_K h|\right). \tag{4.5}$$

Let $\tilde{\pi}_t$ be an optimal coupling for $W_2(\tilde{\nu}_t^{mn},\tilde{\nu}_i^{mn})$. By Proposition 2.4 (4), $\tilde{\nu}_i^{mn}(p_n^{-1}(\operatorname{supp}\nu_0 \cup \operatorname{supp}\nu_1)) = 1$ and the compactness of $\operatorname{supp}\nu_0 \cup \operatorname{supp}\nu_1$, there exists a constant $D > 0$ such that $d_Y(p_n(x_n), p_n(x_n')) \leq D$ for $\tilde{\pi}_t|_{(X_n\setminus \tilde{X}_n)\times X_n}$-a.e. $(x_n, x_n') \in X_n^2$. This together with Proposition 2.4 (3) and Minkowski's inequality imply

$$\begin{aligned}
&W_2((p_n)_*\tilde{\nu}_t^{mn},(p_n)_*\tilde{\nu}_i^{mn})^2\\
&\quad\leq \int_{Y\times Y} d_Y(y,y')^2\, d(p_n\times p_n)_*\tilde{\pi}_t(y,y')\\
&\quad\leq \int_{X_n\times X_n} \{d_{X_n}(x_n,x_n') + \varepsilon_n\}^2\, d\tilde{\pi}_t(x_n,x_n')\\
&\qquad + \int_{(X_n\setminus\tilde{X}_n)\times\tilde{X}_n} d_Y(p_n(x_n),p_n(x_n'))^2\, d\tilde{\pi}_t(x_n,x_n')\\
&\quad\leq (W_2(\tilde{\nu}_t^{mn},\tilde{\nu}_i^{mn}) + \varepsilon_n)^2 + D^2\tilde{\nu}_t^{mn}(X_n\setminus\tilde{X}_n).
\end{aligned} \tag{4.6}$$

Note that we can prove

$$\lim_{n\to\infty}\tilde{\nu}_t^{mn}(X_n\setminus\tilde{X}_n) = 0$$

as in [**7**, Lemma 3.15] and [**12**, Lemma 9.34].

<u>K-Convexity : the case of $K \geq 0$.</u>

In this case,

$$W_2(\nu_0,\nu_1) = \lim_{m\to\infty}\liminf_{n\to\infty} W_2(\tilde{\nu}_0^{mn},\tilde{\nu}_1^{mn}) = \lim_{m\to\infty}\limsup_{n\to\infty} W_2(\tilde{\nu}_0^{mn},\tilde{\nu}_1^{mn}), \tag{4.7}$$

$$\operatorname{Ent}(\nu_i|\mu_Y) \geq \limsup_{m\to\infty}\ \limsup_{n\to\infty}\operatorname{Ent}(\tilde{\nu}_i^{mn}|\mu_{X_n}),\quad i=0,1, \tag{4.8}$$

are proved in the proof of [**7**, Lemma 3.15] and [**12**, Lemma 9.34]. If $K_n \to 0$, Lemma 3.9 (2) implies

$$\lim_{n\to\infty} K_n W_2^{\theta_{K_n} h_n}(\tilde{\nu}_0^{mn},\tilde{\nu}_1^{mn}) \leq \lim_{n\to\infty} K_n(W_2(\tilde{\nu}_0^{mn},\tilde{\nu}_1^{mn}) + h_n) = 0. \tag{4.9}$$

Thus Lemma 3.2, (4.3), (4.4), (4.6), (4.5), (4.7), (4.8), and (4.9) together imply

$$\limsup_{m\to\infty}\ \limsup_{n\to\infty} W_2((p_n)_*\tilde{\nu}_t^{mn},(p_n)_*\tilde{\nu}_i^{mn}) \leq t^{1-i}(1-t)^i W_2(\nu_0,\nu_1) + h,\quad i=0,1, \tag{4.10}$$

$$\begin{aligned}
\limsup_{m\to\infty}\ \limsup_{n\to\infty}\operatorname{Ent}((p_n)_*\tilde{\nu}_t^{mn}|(p_n)_*\mu_{X_n}) &\leq (1-t)\operatorname{Ent}(\nu_0|\mu_Y) + t\operatorname{Ent}(\nu_1|\mu_Y)\\
&\quad - \frac{1}{2}Kt(1-t)W_2^{\theta_K h}(\nu_0,\nu_1)^2.
\end{aligned} \tag{4.11}$$

K-Convexity : the case of $K < 0$.

The limit inequality (4.8) for this case is obtained in the same way as in [**7**, Lemma 3.15] and [**12**, Lemma 9.34]. Let $\tilde{\pi}_{jk}$ be an optimal coupling of $W_2^{\theta_{K_n} h_n}(\tilde{\xi}_{jk}^{mn}, \tilde{\xi}_{kj}^{mn})$. Define the coupling $\tilde{\pi}'$ of $\tilde{\nu}_0^{mn}$ and $\tilde{\nu}_1^{mn}$ by

$$\tilde{\pi}' := \sum_{j,k=1}^{J} w_{jk} \tilde{\pi}_{jk} \in \mathcal{P}(X_n \times X_n).$$

For sufficiently large n, (4.2), Minkowski's inequality, and Lemma 3.9 (5)–(6) together imply

$$\begin{aligned}
W_2^{\theta_{K_n} h_n}(\tilde{\nu}_0^{mn}, \tilde{\nu}_1^{mn})^2 &\le \sum_{j,k=1}^{J} w_{jk} W_2^{\theta_{K_n} h_n}(\tilde{\xi}_{jk}^{mn}, \tilde{\xi}_{kj}^{mn})^2 \\
&\le \sum_{j,k=1}^{J} w_{jk} \{(d_Y(B_j, B_k) - \theta_{K_n} h_n)_+ + \theta(m^{-1})\}^2 \\
&= \sum_{j,k=1}^{J} \int_{B_j \times B_k} \{(d_Y(B_j, B_k) - \theta_{K_n} h_n)_+ + \theta(m^{-1})\}^2 \, d\pi(y, y') \\
&\le \int_{Y \times Y} \{(d_Y(y, y') - \theta_{K_n} h_n)_+ + \theta(m^{-1})\}^2 \, d\pi(y, y') \\
&\le \left(W_2^{\theta_K h}(\nu_0, \nu_1) + |\theta_{K_n} h_n - \theta_K h| + \theta(m^{-1})\right)^2.
\end{aligned}$$

Thus

$$\limsup_{m \to \infty} \limsup_{n \to \infty} W_2^{\theta_{K_n} h_n}(\tilde{\nu}_0^{mn}, \tilde{\nu}_1^{mn}) \le W_2^{\theta_K h}(\nu_0, \nu_1), \tag{4.12}$$

and this limit inequality and (4.8) together lead to the limit inequality (4.11) for $K < 0$. For sufficiently large n and $i = 0, 1$, by (4.3) and Lemma 3.9,

$$\begin{aligned}
W_2(\tilde{\nu}_t^{mn}, \tilde{\nu}_i^{mn}) &\le t^{1-i}(1-t)^i W_2(\tilde{\nu}_0^{mn}, \tilde{\nu}_1^{mn}) + h_n \\
&\le t^{1-i}(1-t)^i W_2^{\theta_{K_n} h_n}(\tilde{\nu}_0^{mn}, \tilde{\nu}_1^{mn}) + h_n.
\end{aligned}$$

Thus this inequality, (4.12), and Lemma 3.9 (2) together imply

$$\begin{aligned}
\limsup_{m \to \infty} \limsup_{n \to \infty} W_2(\tilde{\nu}_t^{mn}, \tilde{\nu}_i^{mn}) &\le t^{1-i}(1-t)^i W_2^{\theta_K h}(\nu_0, \nu_1) + h \\
&\le t^{1-i}(1-t)^i W_2(\nu_0, \nu_1) + 2h. \qquad (4.13)
\end{aligned}$$

Existence of h-rough geodesic.

We prove the existence of h-rough geodesic $(\nu_t)_{t \in [0,1]}$ between ν_0 and ν_1. By the limit inequality (4.11) for each K, there exists a subsequence $\{(m_k, n_k)\}_{k=1}^{\infty} \subset \mathbb{N} \times \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \operatorname{Ent}\left((p_{n_k})_*\tilde{\nu}_t^{m_k n_k}\big|(p_{n_k})_*\mu_{X_{n_k}}\right) < \infty.$$

Since the sequence $\{(p_{n_k})_*\mu_{X_{n_k}}\}_{k=1}^{\infty}$ is tight, Lemma 3.4 implies that $\{(p_{n_k})_*\tilde{\nu}_t^{m_k n_k}\}_{k=1}^{\infty}$ is also tight. We denote its weak convergence limit by ν_t. By Lemma 3.3, we have

$$\operatorname{Ent}(\nu_t|\mu_Y) \leq \liminf_{k\to\infty} \operatorname{Ent}\left((p_{n_k})_*\tilde{\nu}_t^{m_k n_k}\big|(p_{n_k})_*\mu_{X_{n_k}}\right). \tag{4.14}$$

Let π_t^k be an optimal coupling of $W_2((p_{n_k})_*\tilde{\nu}_t^{m_k n_k}, (p_{n_k})_*\tilde{\nu}_i^{m_k n_k})$, $i = 0, 1$. Since $\{(p_{n_k})_*\tilde{\nu}_t^{m_k n_k}\}_{k=1}^{\infty}$ and $\{(p_{n_k})_*\tilde{\nu}_i^{m_k n_k}\}_{k=1}^{\infty}$ are both tight, $\{\pi_t^k\}_{k=1}^{\infty}$ is also tight. We denote its weak convergence limit by π_t. This is a coupling of ν_t and ν_i. Then, we obtain

$$\begin{aligned} W_2(\nu_t, \nu_i)^2 &\leq \int_{Y\times Y} d_Y(y, y')^2\, d\pi_t(y, y') \\ &\leq \liminf_{k\to\infty} \int_{Y\times Y} d_Y(y, y')^2\, d\pi_t^k(y, y') \\ &= \liminf_{k\to\infty} W_2\left((p_{n_k})_*\tilde{\nu}_t^{m_k n_k}, (p_{n_k})_*\tilde{\nu}_i^{m_k n_k}\right)^2. \end{aligned} \tag{4.15}$$

Combining (4.10), (4.11), (4.13), (4.14), and (4.15), we obtain the conclusion. $\square$

5. Proof of Theorem 1.2.

Let π be a coupling of μ_X and μ_Y, and let $\hat{d}$ be a coupling of d_X and d_Y. Let ξ and ξ' be the disintegrations of π with respect to μ_X and μ_Y respectively, i.e., $d\pi(x, y) = d\xi_x(y)\, d\mu_X(x) = d\xi'_y(x)\, d\mu_Y(y)$. Recall that ξ defines a map $\tilde{\xi} : \mathcal{P}_2^{ac}(Y) \to \mathcal{P}_2^{ac}(X)$, which was constructed in [**13**, Section 4.5]. For $\nu = \rho'\mu_Y \in \mathcal{P}_2^{ac}(Y)$, we define $\tilde{\xi}(\nu) = \rho\mu_X \in \mathcal{P}_2^{ac}(X)$ by

$$\rho(x) := \int_Y \rho'(y)\, d\xi_x(y).$$

In the same way, we also define a map $\tilde{\xi}' : \mathcal{P}_2^{ac}(X) \to \mathcal{P}_2^{ac}(Y)$ using the disintegration ξ'. Denote by $\hat{L}$ the μ_X-essential supremum of the map

$$x \mapsto \left(\int_Y \hat{d}(x, y)^2\, d\xi_x(y)\right)^{1/2}.$$

Lemma 5.1 ([**13**, Lemma 4.19]). *Let $X, Y \in \mathbb{X}_v$ with $\mathbb{D}(X, Y) < 1$. $\tilde{\xi}'$ and $\hat{L}$ are defined as above. For any $\nu \in \mathcal{P}_2^{ac}(X)$, we have following two properties.*

(1) $\operatorname{Ent}(\tilde{\xi}'(\nu)|\mu_Y) \leq \operatorname{Ent}(\nu|\mu_X)$.

(2) $W_2(\nu, \tilde{\xi}'(\nu))^2 \leq \dfrac{2 + \hat{L}^2 \operatorname{Ent}(\nu|\mu_X)}{-\log \mathbb{D}(X, Y)}$, *where W_2 is the Wasserstein distance on $\mathcal{P}_2(X \sqcup Y, \hat{d})$.*

Lemma 5.2 ([**2**, Lemma 2.4.2], [**3**, Lemma 5.2]). *Let X be an mm-space and $\nu_0, \nu_1 \in \mathcal{P}_2^{ac}(X)$. Assume that a sequence $\{\pi^n\}_{n=1}^{\infty}$ of couplings of ν_0 and ν_1 converges to a coupling π^∞ weakly. Then we have*

$$\limsup_{n\to\infty} T_{h,K,N}^{(t),i}(\pi^n|\mu_X) \le T_{h,K,N}^{(t),i}(\pi^\infty|\mu_X). \tag{5.1}$$

Proof of Theorem 1.2. By Remark 2.9 and Proposition 2.6, the limit space Y has $\operatorname{diam} Y \le L$.

Define $\tilde{L}, C > 0$ by

$$\tilde{L} := \sup_{n\in\mathbb{N}} \operatorname{diam} X_n + \sup_{n\in\mathbb{N}} h_n, \quad C := \sup_{t',\, K',\, N',\, \theta} \left| \frac{\partial}{\partial\theta} \tau_{K',N'}^{t'}(\theta) \right|,$$

where t', K', N', and θ run over $t' \in [0,1]$, $K' \le \sup_{n\in\mathbb{N}} K_n$, $N' \ge \inf_{n\in\mathbb{N}} N_n$, and $\theta \le \tilde{L}$.

We first consider two Borel probability measures on Y with finite densities. Take any $\varepsilon > 0$ with $L\sqrt{(K+\varepsilon)/(N-1)} < \pi$ and any $\nu_0 = \rho_0\mu_Y, \nu_1 = \rho_1\mu_Y \in \mathcal{P}_2^{ac}(Y)$ with $\|\rho_i\|_\infty \le r$ $(i = 0,1)$ for some $r \ge 1$. Set

$$R = R(r) := r\log r + \frac{1}{8}\sup_{n\in\mathbb{N}} |K_n|\tilde{L}^2.$$

By the assumption, for sufficiently large n, we can find a coupling $\hat{d}_n$ of d_{X_n} and d_Y, and a coupling $\hat{\pi}_n$ of μ_{X_n} and μ_Y such that

$$\frac{1}{2}\left(\int_{X_n\times Y} \hat{d}_n^2 \, d\hat{\pi}_n\right)^{1/2} \le \mathbb{D}(X_n, Y) \le \min\left\{\frac{\varepsilon}{2}, \exp\left(-\frac{2+4\tilde{L}^2R}{\varepsilon^2}\right)\right\}. \tag{5.2}$$

This leads to

$$\hat{\pi}_n(\{(x,y) \in X_n \times Y \mid \hat{d}_n(x,y) \le \sqrt{\varepsilon}\}) \ge 1 - \varepsilon. \tag{5.3}$$

Let ξ^n and $\bar{\xi}^n$ be disintegrations of $\hat{\pi}_n$ with respect to μ_{X_n} and μ_Y respectively, i.e.,

$$d\hat{\pi}_n(x,y) = d\xi_x^n(y) d\mu_{X_n}(x) = d\bar{\xi}_y^n(x)\, d\mu_Y(y).$$

We set

$$\mu_i^n := \sigma_i^n \mu_{X_n}, \quad \sigma_i^n(x) := \int_Y \rho_i(y)\, d\xi_x^n(y), \ i = 0,1.$$

By Jensen's inequality, Lemma 5.1, (5.2) and $\|\rho_i\|_\infty \le r$, for $i = 0,1$ and $N' > 1$, we have

$$S_{N'}(\nu_i^n|\mu_{X_n}) \le S_{N'}(\mu_i^n|\mu_Y), \tag{5.4}$$

$$\operatorname{Ent}(\mu_i^n|\mu_{X_n}) \le \operatorname{Ent}(\nu_i|\mu_Y) \le r\log r, \tag{5.5}$$

$$W_2(\mu_i^n, \nu_i)^2 \le \frac{2 + \tilde{L}^2\operatorname{Ent}(\nu_i|\mu_Y)}{-\log \mathbb{D}(X_n, Y)} \le \varepsilon^2. \tag{5.6}$$

On the other hand, since X_n satisfies the rough curvature dimension condition h_n-$\mathrm{CD}(K_n, N_n)$, for two measures $\mu_0^n, \mu_1^n \in \mathcal{P}_2^{ac}(X_n)$, there exists a coupling $\pi_n \in (\theta_{K_n} h_n)$-$\mathrm{Opt}(\mu_0^n, \mu_1^n)$ such that for each $t \in [0,1]$, there exists a measure $\mu_t^n = \sigma_t^n \mu_{X_n} \in \mathcal{P}_2^{ac}(X_n)$ such that for any $N' > N_n$, the following two conditions hold;

$$W_2(\mu_t^n, \mu_i^n) \le t^{1-i}(1-t)^i W_2(\mu_0^n, \mu_1^n) + h_n, \quad i = 0, 1, \tag{5.7}$$

$$S_{N'}(\mu_t^n | \mu_{X_n}) \le T^{(t)}_{h_n, K_n, N'}(\pi_n | \mu_{X_n}). \tag{5.8}$$

Put

$$\nu_t^n := \rho_t^n \mu_Y, \quad \rho_t^n(y) := \int_{X_n} \sigma_t^n(x)\, d\bar{\xi}_y^n(x).$$

Note that ν_t^n and π_n depend on (r, ε). By Lemma 3.15 and Lemma 5.1, we get

$$\begin{aligned}
\mathrm{Ent}(\nu_t^n | \mu_Y) &\le \mathrm{Ent}(\mu_t^n | \mu_{X_n}) \\
&\le (1-t)\mathrm{Ent}(\mu_0^n | \mu_{X_n}) + t\mathrm{Ent}(\mu_1^n | \mu_{X_n}) - \frac{1}{2} K_n t(1-t) W_2^{\theta_{K_n} h_n}(\mu_0^n, \mu_1^n)^2 \\
&\le r \log r + \frac{1}{8} \sup_{n \in \mathbb{N}} |K_n| \tilde{L}^2 \\
&= R,
\end{aligned} \tag{5.9}$$

and

$$W_2(\nu_t^n, \mu_t^n)^2 \le \frac{2 + \tilde{L}^2 \mathrm{Ent}(\mu_t^n | \mu_{X_n})}{-\log \mathbb{D}(X_n, Y)} \le \varepsilon^2. \tag{5.10}$$

Thus, (5.6), (5.7) and (5.10) imply

$$\begin{aligned}
W_2(\nu_t^n, \nu_i) &\le W_2(\mu_t^n, \mu_i^n) + 2\varepsilon \\
&\le t^{1-i}(1-t)^i W_2(\mu_0^n, \mu_1^n) + h_n + 2\varepsilon \\
&\le t^{1-i}(1-t)^i W_2(\nu_0, \nu_1) + h_n + 4\varepsilon.
\end{aligned} \tag{5.11}$$

By Jensen's inequality,

$$\begin{aligned}
S_{N'}(\nu_t^n | \mu_Y) &= -\int_Y (\rho_t^n(y))^{1-1/N'}\, d\mu_Y(y) \\
&\le -\int_Y \int_{X_n} (\sigma_t^n(x))^{1-1/N'}\, d\bar{\xi}_y^n(x) d\mu_Y(y) \\
&= S_{N'}(\mu_t^n | \mu_{X_n}).
\end{aligned} \tag{5.12}$$

Define a probability measure $\bar{\pi}_n \in \mathcal{P}(Y^2)$ as

$$d\bar{\pi}_n(y, y') := \int_{X_n \times X_n} \frac{\rho_0(y)\rho_1(y')}{\sigma_0^n(x)\sigma_1^n(x')}\, d\xi_{x'}^n(y') d\xi_x^n(y) d\pi_n(x, x').$$

We check $\bar{\pi}_n \in \Pi(\nu_0, \nu_1)$. For any Borel subset $A \subset Y$, we have

$$
\begin{aligned}
\int_{A\times Y} d\bar{\pi}_n(y,y') &= \int_{X_n^2}\int_{A\times Y} \frac{\rho_0(y)\rho_1(y')}{\sigma_0^n(x)\sigma_1^n(x')}\, d\xi^n_{x'}(y')d\xi^n_x(y)d\pi_n(x,x') \\
&= \int_{X_n}\int_A \frac{\rho_0(y)}{\sigma_0^n(x)}\, d\xi^n_x(y)d\mu_0^n(x) \\
&= \int_{X_n\times A} \rho_0(y)\, d\hat{\pi}_n(x,y) \\
&= \int_A d\nu_0(y).
\end{aligned}
$$

Similarly, $\int_{Y\times A}\ d\bar{\pi}_n(y,y') = \int_A d\nu_1(y')$.

Claim 5.3. *We assume that $|\theta_{K_n}h_n - \theta_K h| \to 0$ as $n\to\infty$. There exist a coupling $\pi^{r,\varepsilon}\in \Pi(\nu_0,\nu_1)$ and $(\nu_t^{r,\varepsilon})_{t\in(0,1)} \subset \mathcal{P}_2^{ac}(Y)$ such that*

(1) $W_2(\nu_t^{r,\varepsilon},\nu_i) \le t^{1-i}(1-t)^i W_2(\nu_0,\nu_1) + h + 4\varepsilon, \quad i=0,1,$

(2) *for any $N' > N+\varepsilon$,*

$$
S_{N'}(\nu_t^{r,\varepsilon}|\mu_Y) \le T^{(t)}_{h,K,N'}(\pi^{r,\varepsilon}|\mu_Y) + 4Cr^{1-1/N'}\max\{\varepsilon, (2\tilde{L}+\sqrt{\varepsilon})^{2/N'-1}\varepsilon^{2-2/N'}\},
$$

(3) $\displaystyle \left(\int_{Y\times Y} (d_Y(y,y') - \theta_K h)_+^2\, d\pi^{r,\varepsilon}(y,y')\right)^{1/2} \le W_2^{\theta_K h}(\nu_0,\nu_1) + 2\varepsilon(1+\sqrt{r}).$

Proof. Take $N' > N+\varepsilon$. We may assume $N' > N_n$ and $|K_n - K| < \varepsilon$ for sufficiently large n. By the fundamental theorem of calculus,

$$
\begin{aligned}
&\tau^{(1-t)}_{K_n,N'}((d_Y(y,y') - \theta_K h)_+) \\
&\le \tau^{(1-t)}_{K_n,N'}((d_{X_n}(x,x') - \theta_{K_n}h_n)_+) + C|(d_{X_n}(x,x') - \theta_{K_n}h_n)_+ - (d_Y(y,y') - \theta_K h)_+| \\
&\le \tau^{(1-t)}_{K_n,N'}((d_{X_n}(x,x') - \theta_{K_n}h_n)_+) + C(\hat{d}_n(x,y) + \hat{d}_n(x',y') + |\theta_{K_n}h_n - \theta_K h|). \quad (5.13)
\end{aligned}
$$

This leads

$$
\begin{aligned}
&- T^{(1-t),0}_{h,K_n,N'}(\bar{\pi}_n|\mu_Y) \\
&\quad\le \int_{X_n\times X_n}\int_{Y\times Y} \frac{\rho_0(y)\rho_1(y')}{\sigma_0^n(x)\sigma_1^n(x')}\tau^{(1-t)}_{K_n,N'}((d_{X_n}(x,x') - \theta_{K_n}h_n)_+)\rho_0(y)^{-1/N'} \\
&\qquad\qquad d\xi^n_{x'}(y')d\xi^n_x(y)d\pi_n(x,x') \\
&\qquad + C\int_{X_n\times X_n}\int_{Y\times Y} \frac{\rho_0(y)\rho_1(y')}{\sigma_0^n(x)\sigma_1^n(x')}\hat{d}_n(x,y)\rho_0(y)^{-1/N'}\, d\xi^n_{x'}(y')d\xi^n_x(y)d\pi_n(x,x') \\
&\qquad + C\int_{X_n\times X_n}\int_{Y\times Y} \frac{\rho_0(y)\rho_1(y')}{\sigma_0^n(x)\sigma_1^n(x')}\hat{d}_n(x',y')\rho_0(y)^{-1/N'}\, d\xi^n_{x'}(y')d\xi^n_x(y)d\pi_n(x,x') \\
&\qquad + C|\theta_{K_n}h_n - \theta_K h|\int_{X_n\times X_n}\int_{Y\times Y} \frac{\rho_0(y)\rho_1(y')}{\sigma_0^n(x)\sigma_1^n(x')}\rho_0(y)^{-1/N'}\, d\xi^n_{x'}(y')d\xi^n_x(y)d\pi_n(x,x') \\
&\quad =: (\mathrm{I}) + C(\mathrm{II}) + C(\mathrm{III}) + C|\theta_{K_n}h_n - \theta_K h|(\mathrm{IV}).
\end{aligned}
$$

We estimate (I)–(IV). By Jensen's inequality,

$$
\begin{aligned}
\text{(I)} &= \int_{X_n\times X_n} \tau^{(1-t)}_{K_n,N'}((d_{X_n}(x,x')-\theta_{K_n}h_n)_+)\sigma^n_0(x)^{-1}\int_Y \rho_0(y)^{1-1/N'}\,d\xi^n_x(y)d\pi_n(x,x')\\
&\le \int_{X_n\times X_n} \tau^{(1-t)}_{K_n,N'}((d_{X_n}(x,x')-\theta_{K_n}h_n)_+)\sigma^n_0(x)^{-1/N'}\,d\pi_n(x,x')\\
&= -T^{(1-t),0}_{h_n,K_n,N'}(\pi_n|\mu_{X_n}).
\end{aligned}
$$

$$
\text{(II)} = \int_{X_n}\int_Y \hat{d}_n(x,y)\rho_0(y)^{1-1/N'}\,d\xi^n_x(y)d\mu_{X_n}(x) \le r^{1-1/N'}\varepsilon. \tag{5.14}
$$

By Jensen's inequality and (5.3),

$$
\begin{aligned}
\text{(III)} &\le \int_{X_n\times X_n}\int_Y \rho_1(y')\sigma^n_0(x)^{-1/N'}\hat{d}_n(x',y')\sigma^n_1(x')^{-1}\,d\xi^n_{x'}(y')d\pi_n(x,x')\\
&\le \left(\int_{X_n\times X_n}\int_Y \rho_1(y')^{N'}\sigma^n_0(x)^{-1}\sigma^n_1(x')^{-1}\,d\xi^n_{x'}(y')d\pi_n(x,x')\right)^{1/N'}\\
&\quad\times\left(\int_{X_n\times X_n}\int_Y \hat{d}_n(x',y')^{N'/(N'-1)}\sigma^n_1(x')^{-1}\,d\xi^n_{x'}(y')d\pi_n(x,x')\right)^{1-1/N'}\\
&\le \left(r^{N'-1}\int_{X_n\times X_n}\int_Y \rho_1(y')\sigma^n_0(x)^{-1}\sigma^n_1(x')^{-1}\,d\xi^n_{x'}(y')d\pi_n(x,x')\right)^{1/N'}\\
&\quad\times\left(\int_{X_n}\int_Y \hat{d}_n(x',y')^{N'/(N'-1)}\sigma^n_1(x')^{-1}\,d\xi^n_{x'}(y')d\mu^n_1(x')\right)^{1-1/N'}\\
&= r^{1-1/N'}\left(\int_{X_n\times Y}\hat{d}_n(x',y')^{N'/(N'-1)}\,d\hat{\pi}_n(x',y')\right)^{1-1/N'}\\
&\le r^{1-1/N'}\max\left\{\varepsilon,(2\tilde{L}+\sqrt{\varepsilon})^{2/N'-1}\varepsilon^{2-2/N'}\right\}.
\end{aligned}\tag{5.15}
$$

In the second inequality, we consider $\sigma^n_1(x)^{-1}\,d\xi^n_{x'}(y')d\pi_n(x,x')$ as a new measure and apply Hölder's inequality to $\rho_1(y')\sigma^n_0(x)^{-1/N'}$ and $\hat{d}_n(x',y')$. In the last inequality, Hölder's inequality implies

$$
\begin{aligned}
&\left(\int_{X_n\times Y}\hat{d}_n(x',y')^{N'/(N'-1)}\,d\hat{\pi}_n(x',y')\right)^{1-1/N'}\\
&\qquad\le \begin{cases}\varepsilon & \text{if } N'\ge 2,\\ (2\tilde{L}+\sqrt{\varepsilon})^{2/N'-1}\varepsilon^{2-2/N'} & \text{if } 1<N'<2.\end{cases}
\end{aligned}
$$

By Jensen's inequality,

$$
\begin{aligned}
\text{(IV)} &= \int_{X_n\times X_n}\int_Y \sigma^n_0(x)^{-1}\rho_0(y)^{1-1/N'}\,d\xi^n_x(y)d\pi_n(x,x')\\
&\le \int_{X_n\times X_n}\sigma^n_0(x)^{-1/N'}\,d\pi_n(x,x')\\
&\le 1.
\end{aligned}
$$

Thus we obtain

$$\begin{aligned} -T^{(1-t),0}_{h,K_n,N'}(\bar\pi_n|\mu_Y) &\le -T^{(1-t),0}_{h_n,K_n,N'}(\pi_n|\mu_{X_n}) + 2Cr^{1-1/N'}\max\left\{\varepsilon,(2\tilde L+\sqrt{\varepsilon})^{2/N'-1}\varepsilon^{2-2/N'}\right\} \\ &\quad + C|\theta_{K_n}h_n-\theta_K h|. \end{aligned} \tag{5.16}$$

Put

$$\begin{aligned} \tilde C_{+,N'} &:= (\operatorname{diam}Y)\sqrt{\frac{K+\varepsilon}{N'-1}} < \pi, \quad \tilde C_{-,N'} := \left(\operatorname{diam}Y + \sup_{n\in\mathbb N} h_n\right)\sqrt{\frac{1}{N'-1}\sup_{n\in\mathbb N}|K_n|}, \\ C_{*,N'} &:= \sup_{t\in[0,1]}\sup_{\alpha\in[0,\tilde C_{*,N'}]}\left|\frac{d}{d\alpha}t^{1/N'}\left(\frac{\sin t\alpha}{\sin\alpha}\right)^{1-1/N'}\right|, \quad *\in\{+,-\}, \\ \hat C_{N'} &:= \max\{C_{+,N'},C_{-,N'}\}. \end{aligned}$$

Note that by the fundamental theorem of calculus,

$$\begin{aligned} &\tau^{(1-t)}_{K_n,N'}((d_Y(y,y')-\theta_K h)_+) \\ &\quad \ge \tau^{(1-t)}_{K,N'}((d_Y(y,y')-\theta_K h)_+) - \frac{\hat C_{N'}(\operatorname{diam}Y+\theta_K h)}{\sqrt{N'-1}}\left|\sqrt{|K_n|}-\sqrt{|K|}\right|, \end{aligned}$$

and then

$$\begin{aligned} &-T^{(1-t),0}_{h,K_n,N'}(\bar\pi_n|\mu_Y) \\ &\quad \ge -T^{(1-t),0}_{h,K,N'}(\bar\pi_n|\mu_Y) + \frac{\hat C_{N'}(\operatorname{diam}Y+\theta_K h)}{\sqrt{N'-1}}\left|\sqrt{|K_n|}-\sqrt{|K|}\right|S_{N'}(\nu_0|\mu_Y). \end{aligned} \tag{5.17}$$

Then (5.16), (5.17), and Lemma 3.6 together imply

$$\begin{aligned} T^{(1-t),0}_{h_n,K_n,N'}(\pi_n|\mu_{X_n}) &\le T^{(1-t),0}_{h,K,N'}(\bar\pi_n|\mu_Y) + 2Cr^{1-1/N'}\max\left\{\varepsilon,(2\tilde L+\sqrt{\varepsilon})^{2/N'-1}\varepsilon^{2-2/N'}\right\} \\ &\quad + |\theta_{K_n}h_n-\theta_K h| + \frac{\hat C_{N'}(\operatorname{diam}Y+\theta_K h)}{\sqrt{N'-1}}\left|\sqrt{|K_n|}-\sqrt{|K|}\right|. \end{aligned}$$

Similarly,

$$\begin{aligned} T^{(t),1}_{h_n,K_n,N'}(\pi_n|\mu_{X_n}) &\le T^{(t),1}_{h,K,N'}(\bar\pi_n|\mu_Y) + 2Cr^{1-1/N'}\max\left\{\varepsilon,(2\tilde L+\sqrt{\varepsilon})^{2/N'-1}\varepsilon^{2-2/N'}\right\} \\ &\quad + |\theta_{K_n}h_n-\theta_K h| + \frac{\hat C_{N'}(\operatorname{diam}Y+\theta_K h)}{\sqrt{N'-1}}\left|\sqrt{|K_n|}-\sqrt{|K|}\right|. \end{aligned}$$

Combining (5.8), (5.12), and these inequalities, we obtain

$$\begin{aligned} S_{N'}(\nu^n_t|\mu_Y) &\le T^{(t)}_{h,K,N'}(\bar\pi_n|\mu_Y) + 4Cr^{1-1/N'}\max\left\{\varepsilon,(2\tilde L+\sqrt{\varepsilon})^{2/N'-1}\varepsilon^{2-2/N'}\right\} \\ &\quad + 2|\theta_{K_n}h_n-\theta_K h| + \frac{2\hat C_{N'}(\operatorname{diam}Y+\theta_K h)}{\sqrt{N'-1}}\left|\sqrt{|K_n|}-\sqrt{|K|}\right|. \end{aligned} \tag{5.18}$$

On the other hand, by the triangle inequality, Minkowski's inequality, Lemma 3.9, and

(5.6),

$$
\begin{aligned}
&\left(\int_{Y\times Y}(d_Y(y,y')-\theta_K h)_+^2\,d\bar{\pi}_n(y,y')\right)^{1/2}\\
&\quad\le\left(\int_{X_n\times X_n}(d_{X_n}(x,x')-\theta_{K_n}h_n)_+^2\,d\pi_n(x,x')\right)^{1/2}\\
&\qquad+\left(\int_{X_n\times X_n}\int_{Y\times Y}\frac{\rho_0(y)\rho_1(y')}{\sigma_0^n(x)\sigma_1^n(x')}\hat{d}_n(x,y)^2\,d\xi_{x'}^n(y')d\xi_x^n(y)d\pi_n(x,x')\right)^{1/2}\\
&\qquad+\left(\int_{X_n\times X_n}\int_{Y\times Y}\frac{\rho_0(y)\rho_1(y')}{\sigma_0^n(x)\sigma_1^n(x')}\hat{d}_n(x',y')^2\,d\xi_{x'}^n(y')d\xi_x^n(y)d\pi_n(x,x')\right)^{1/2}\\
&\qquad+|\theta_{K_n}h_n-\theta_K h|\\
&\quad=W_2^{\theta_{K_n}h_n}(\mu_0^n,\mu_1^n)+\left(\int_{X_n\times Y}\rho_0(y)\hat{d}_n(x,y)^2\,d\hat{\pi}_n(x,y)\right)^{1/2}\\
&\qquad+\left(\int_{X_n\times Y}\rho_1(y')\hat{d}_n(x',y')^2\,d\hat{\pi}_n(x',y')\right)^{1/2}+|\theta_{K_n}h_n-\theta_K h|\\
&\quad\le W_2^{\theta_{K_n}h_n}(\mu_0^n,\mu_1^n)+2\sqrt{r}\varepsilon+|\theta_{K_n}h_n-\theta_K h|\\
&\quad\le W_2^{\theta_K h}(\mu_0,\mu_1)+2(1+\sqrt{r})\varepsilon+2|\theta_{K_n}h_n-\theta_K h|. \qquad (5.19)
\end{aligned}
$$

By the compactness of $\Pi(\nu_0,\nu_1)$, (5.9), and Lemma 3.4, two sequences $\{\bar{\pi}_n\}_{n=1}^\infty$ and $\{\nu_t^n\}_{n=1}^\infty$ are both tight. We denote their weak limits by $\pi^{r,\varepsilon}$ and $\nu_t^{r,\varepsilon}$, respectively. Therefore, Lemma 3.8 (4), Lemma 3.6, Lemma 5.2, (5.11), (5.18), and (5.19) together imply the statement. This completes the proof of Claim 5.3. □

Two measures $\pi^{r,\varepsilon}\in\Pi(\nu_0,\nu_1)$ and $\nu_t^{r,\varepsilon}\in\mathcal{P}_2^{ac}(Y)$ are as in Claim 5.3. By Lemma 3.8 (1), (5.9), and Lemma 3.4, two sets $\{\pi^{r,\varepsilon}\}_{\varepsilon>0}$ and $\{\nu_t^{r,\varepsilon}\}_{\varepsilon>0}$ are tight. Taking limits as $\varepsilon\to 0$, we denote their weak convergent limits by $\pi^r\in\Pi(\nu_0,\nu_1)$ and $\nu_t^r\in\mathcal{P}_2^{ac}(Y)$, respectively. Therefore, combining Claim 5.3 (1)–(3), Lemma 3.8 (4), Lemma 3.6, and Lemma 5.2, the optimal coupling $\pi^r\in(\theta_K h)$-$\mathrm{Opt}(\nu_0,\nu_1)$ and the family of measures $(\nu_t^r)_{t\in(0,1)}$ satisfy the definition of h-$\mathrm{CD}(K,N)$. Note that $(\nu_t^r)_{t\in(0,1)}$ is an h-rough geodesic between ν_0 and ν_1.

We consider the general case where $|\theta_{K_n}h_n-\theta_K h|\to 0$ as $n\to\infty$. Take $\nu_0=\rho_0\mu_Y,\nu_1=\rho_1\mu_Y\in\mathcal{P}_2^{ac}(Y)$. For $r>0$, we set

$$
\begin{aligned}
&r'=r'(r):=\max_{i=0,1}\{\nu_i(\{\rho_i\le r\})^{-1}\}r,\\
&\nu_i^{r'}:=\nu_i(\{\rho_i\le r\})^{-1}\nu_i|_{\{\rho_i\le r\}}=\rho_i^{r'}\mu_Y\in\mathcal{P}_2^{ac}(Y),\quad i=0,1,\\
&\pi_i^{r'}:=(\mathrm{id}_Y,\mathrm{id}_Y)_*\nu_i|_{\{\rho_i\le r\}}+\nu_i|_{\{\rho_i>r\}}\otimes\nu_i^{r'}\in\Pi(\nu_i,\nu_i^{r'}),
\end{aligned}
$$

where $(\mathrm{id}_Y,\mathrm{id}_Y):Y\ni y\mapsto(y,y)\in Y\times Y$. Since

$$
W_2(\nu_i,\nu_i^{r'})^2\le\int_{Y\times Y}d_Y(y,y')^2\,d\pi_i^{r'}(y,y')\le(\operatorname{diam}Y)^2\nu_i(\{\rho_i>r\}),
$$

we have $W_2(\nu_i, \nu_i^{r'}) \to 0$ as $r \to \infty$. We also have $\|\rho_i^{r'}\|_\infty \le r'$. Apply the above discussion to $\nu_0^{r'}$ and $\nu_1^{r'}$, we obtain a $(\theta_K h)$-optimal coupling $\pi^{r'} \in \Pi(\nu_0^{r'}, \nu_1^{r'})$ and a family of measures $(\nu_t^{r'})_{t\in(0,1)}$ such that for any $t \in [0,1]$ and any $N' > N$, we have

$$W_2\big(\nu_t^{r'}, \nu_i^{r'}\big) \le t^{1-i}(1-t)^i W_2\big(\nu_0^{r'}, \nu_1^{r'}\big) + h, \tag{5.20}$$

$$S_{N'}\big(\nu_t^{r'}|\mu_Y\big) \le T_{h,K,N'}^{(1-t),0}\big(\pi^{r'}|\mu_Y\big) + T_{h,K,N'}^{(t),1}\big(\pi^{r'}|\mu_Y\big). \tag{5.21}$$

By the compactness of Y, the set $\{\nu_t^{r'}\}_{r>0}$ is tight. Denote its weak limit by ν_t, i.e., $\nu_t^{r'}$ converges weakly to ν_t as $r \to \infty$. By Lemma 3.8 (4) and $W_2(\nu_i, \nu_i^{r'}) \to 0$, we obtain that $(\nu_t)_{t\in(0,1)}$ is an h-rough geodesic between ν_0 and ν_1. Since d_Y is bounded and $\nu_i^{r'}$ converges weakly to ν_i $(i = 0, 1)$, the measure $\pi^{r'}$ converges weakly to a $(\theta_K h)$-optimal coupling π of ν_0 and ν_1 as $r \to \infty$. For any $\varepsilon' > 0$, there is a bounced continuous function $\varphi : Y \to \mathbb{R}$ such that

$$\int_Y |\rho_0^{-1/N'} - \varphi|\, d\nu_0 < \varepsilon',$$

and then

$$\int_Y |\rho_0^{-1/N'}\mathbf{1}_{\{\rho_0\le r\}} - \varphi|\, d\nu_0^{r'} < \frac{\varepsilon'}{\nu_0(\{\rho_0 \le r\})},$$

where $\mathbf{1}_{\{\rho_0\le r\}}$ is the characteristic function of the set $\{\rho_0 \le r\} \subset Y$. Put

$$T_0 := \sup_{(y,y')\in Y^2} \tau_{K,N'}^{(1-t)}((d_Y(y,y') - \theta_K h)_+) \in [0,\infty).$$

Thus

$$\begin{aligned}
&- T_{h,K,N'}^{(1-t),0}\big(\pi^{r'}|\mu_Y\big)\\
&\quad= (\nu_0(\{\rho_0 \le r\}))^{1/N'} \int_{Y\times Y} \tau_{K,N'}^{(1-t)}((d_Y(y,y') - \theta_K h)_+)\rho_0^{-1/N'}(y)\mathbf{1}_{\{\rho_0\le r\}}\, d\pi^{r'}(y,y')\\
&\quad\ge (\nu_0(\{\rho_0 \le r\}))^{1/N'} \int_{Y\times Y} \tau_{K,N'}^{(1-t)}((d_Y(y,y') - \theta_K h)_+)\varphi(y)\, d\pi^{r'}(y,y')\\
&\qquad - T_0\varepsilon'(\nu_0(\{\rho_0 \le r\}))^{1/N'-1},
\end{aligned}$$

and then

$$\begin{aligned}
\limsup_{r\to\infty} T_{h,K,N'}^{(1-t),0}\big(\pi^{r'}|\mu_Y\big) &\le -\int_{Y\times Y} \tau_{K,N'}^{(1-t)}((d_Y(y,y') - \theta_K h)_+)\varphi(y)\, d\pi(y,y') + T_0\varepsilon'\\
&\le T_{h,K,N'}^{(1-t),0}(\pi|\mu_Y) + 2T_0\varepsilon'.
\end{aligned}$$

Since $\varepsilon' > 0$ is arbitrary,

$$\limsup_{r\to\infty} T_{h,K,N'}^{(1-t),0}\big(\pi^{r'}|\mu_Y\big) \le T_{h,K,N'}^{(1-t),0}(\pi|\mu_Y).$$

Similarly, we obtain

$$\limsup_{r\to\infty} T^{(t),1}_{h,K,N'}\big(\pi^{r'}|\mu_Y\big) \le T^{(t),1}_{h,K,N'}(\pi|\mu_Y).$$

Therefore, the above inequalities and Lemma 3.6 together imply (3.5) for ν_0 and ν_1. We conclude that Y satisfies h-CD(K,N) when $|\theta_{K_n}h_n - \theta_K h| \to 0$ as $n \to \infty$.

In the same way, the proof of general case where $|\theta_{K_n}h_n - \theta_K h|$ does not converge to 0 follows from the next claim.

Claim 5.4. *We assume that $|\theta_{K_n}h_n - \theta_K h|$ does not converge to 0, particularly $K = 0$. There exists $(\nu_t^{r,\varepsilon})_{t\in(0,1)} \subset \mathcal{P}_2^{ac}(Y)$ such that,*

(1) $W_2(\nu_t^{r,\varepsilon},\nu_i) \le t^{1-i}(1-t)^i W_2(\nu_0,\nu_1) + h + 4\varepsilon, \quad i = 0,1,$

(2) *for any $N' > N + \varepsilon$,*

$$\begin{aligned}&S_{N'}(\nu_t^{r,\varepsilon}|\mu_Y)\\ &\quad\le (1-t)S_{N'}(\nu_0|\mu_Y) + tS_{N'}(\nu_1|\mu_Y) + 4Cr^{1-1/N'}\max\big\{\varepsilon,(2\tilde{L}+\sqrt{\varepsilon})^{2/N'-1}\varepsilon^{2-2/N'}\big\}.\end{aligned}$$

Proof. Take $N' > N+\varepsilon$. We may assume $N' > N_n$ and $|K_n| < \varepsilon$ for sufficiently large n. By the fundamental theorem of calculus,

$$\begin{aligned}&\tau^{(1-t)}_{K_n,N'}((d_{X_n}(x,x') - \theta_{K_n}h_n)_+)\\ &\qquad\ge \tau^{(1-t)}_{K_n,N'}((d_Y(y,y') - \theta_{K_n}h_n)_+) - C(\hat{d}_n(x,y) + \hat{d}_n(x',y')),\end{aligned}$$

and

$$\Big|(1-t) - \tau^{(1-t)}_{K_n,N'}((d_Y(y,y') - \theta_{K_n}h_n)_+)\Big| \le \hat{C}_{N'}(\operatorname{diam} Y + h_n)\sqrt{\frac{|K_n|}{N'-1}}.$$

These inequalities, Jensen's inequality, (5.14), (5.15), and Lemma 3.6 together imply

$$\begin{aligned}
&-(1-t)S_{N'}(\nu_0|\mu_Y)\\
&\quad= \int_{Y\times Y}(1-t)\rho_0(y)^{-1/N'}\,d\bar{\pi}(y,y')\\
&\quad\le \int_{X_n\times X_n}\int_{Y\times Y}\tau^{(1-t)}_{K_n,N'}((d_{X_n}(x,x') - \theta_{K_n}h_n)_+)\rho_0(y)^{-1/N'}\frac{\rho_0(y)\rho_1(y')}{\sigma_0^n(x)\sigma_1^n(x')}\\
&\qquad\qquad\qquad\qquad\qquad\qquad d\xi^n_{x'}(y')d\xi^n_x(y)d\pi_n(x,x')\\
&\qquad + C\int_{X_n\times X_n}\int_{Y\times Y}\frac{\rho_0(y)\rho_1(y')}{\sigma_0^n(x)\sigma_1^n(x')}\hat{d}_n(x,y)\rho_0(y)^{-1/N'}\,d\xi^n_{x'}(y')d\xi^n_x(y)d\pi_n(x,x')\\
&\qquad + C\int_{X_n\times X_n}\int_{Y\times Y}\frac{\rho_0(y)\rho_1(y')}{\sigma_0^n(x)\sigma_1^n(x')}\hat{d}_n(x',y')\rho_0(y)^{-1/N'}\,d\xi^n_{x'}(y')d\xi^n_x(y)d\pi_n(x,x')\\
&\qquad - \hat{C}_{N'}(\operatorname{diam} Y + h_n)\sqrt{\frac{|K_n|}{N'-1}}S_{N'}(\nu_0|\mu_Y)\\
&\quad\le -T^{(1-t),0}_{h_n,K_n,N'}(\pi_n|\mu_{X_n}) + 2Cr^{1-1/N'}\max\big\{\varepsilon,(2\tilde{L}+\sqrt{\varepsilon})^{2/N'-1}\varepsilon^{2-2/N'}\big\}\\
&\qquad + \hat{C}_{N'}(\operatorname{diam} Y + h_n)\sqrt{\frac{|K_n|}{N'-1}}.
\end{aligned}$$

Similarly,

$$-tS_{N'}(\nu_1|\mu_Y) \le -T^{(t),1}_{h_n,K_n,N'}(\pi_n|\mu_{X_n}) + 2Cr^{1-1/N'}\max\left\{\varepsilon, (2\tilde{L}+\sqrt{\varepsilon})^{2/N'-1}\varepsilon^{2-2/N'}\right\} + \hat{C}_{N'}(\operatorname{diam} Y + h_n)\sqrt{\frac{|K_n|}{N'-1}}.$$

Combining (5.8), (5.12), and these inequalities, we obtain

$$\begin{aligned} S_{N'}(\nu_t^n|\mu_Y) \le{}& (1-t)S_{N'}(\nu_0|\mu_Y) + tS_{N'}(\nu_1|\mu_Y) \\ &+ 4Cr^{1-1/N'}\max\left\{\varepsilon, (2\tilde{L}+\sqrt{\varepsilon})^{2/N'-1}\varepsilon^{2-2/N'}\right\} \\ &+ 2\hat{C}_{N'}(\operatorname{diam} Y + h_n)\sqrt{\frac{|K_n|}{N'-1}}. \end{aligned} \tag{5.22}$$

By (5.9) and Lemma 3.4, the sequence $\{\nu_t^n\}_{n=1}^\infty$ is tight. We denote its weak limit by $\nu_t^{r,\varepsilon}$. Therefore, Lemma 3.8 (4), Lemma 3.6, (5.11), and (5.22) together imply the statement. This completes the proof of Claim 5.4. □

The proof of the theorem is now complete. □

Remark 5.5. Note that we only use the compactness of Y for tightness of $\{\nu_t^{r'}\}_{r>0}$.

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