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Asymptotic behavior of eigenfrequencies of a thin elastic rod with non-uniform cross-section

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Abstract. We study the eigenvalue problem of the elliptic operator which arises in the linearized model of the periodic oscillations of a homogeneous and isotropic elastic body. The square of the frequency agrees to the eigenvalue. Particularly, we deal with a thin rod with non-uniform connected cross-section in several cases of boundary conditions. We see that there appear many small eigenvalues which accumulate to 0 as the thinness parameter ε tends to 0. These eigenvalues correspond to the bending mode of vibrations of the thin body. We investigate the asymptotic behavior of these eigenvalues and obtain a characterization formula of the limit equation for $\varepsilon \to 0$.

1. Introduction.

In this paper we analyze the asymptotic behavior of small eigenvalues and eigenfunctions of the linearized elasticity eigenvalue problem of a thin rod with non-uniform cross-section (see Figure 1).

There are many works on such type of spectral problems of singularly deformed domains in these several decades (cf. Courant–Hilbert [**9**], Egorov–Kondratiev [**13**], Maz'ya–Nazarov–Plamenevskij [**20**]). Particularly, eigenvalue problems of vibration of thin elastic bodies like plates and rods are of much importance and interest from PDE theory and engineering point of view (see for example Antman [**1**], Ciarlet [**6**], Cioranescu–Saint Jean Paulin [**8**], Love [**19**], Nazarov [**22**]).

Ciarlet and Kesavan [**7**] pioneered ideas on elastic plates that would further be adapted to the case of thin rods. To name some previous works, Kerdid [**17**] studied the behavior of small eigenvalues of the linearized elasticity eigenvalue problem of a thin rod with constant cross-section. Tambača [**25**] gives a result on the convergence of the eigenvalues and eigenfunctions in the case of a thin curved rod. Both papers consider that the ends of the rod are clamped. Kerdid [**18**] also considered a joint of two rods with one of the ends without clamping.

The purpose of this paper is to give similar results of the behavior of small eigenvalues in more general cases. We obtain the characterization formula, which is derived from a fourth order ordinary differential equation system on the one-dimensional limit set of the thin elastic body. We make full use of the variational characterization of the eigenvalues as well as detailed analysis of the weak formulation of the eigenfunctions. Previous works assumed that the cross-section of the rod was simply connected, constant and the barycenter or "center of mass" to be constant. We will remove these restrictions, so

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the rod has non-uniform connected cross-section. Furthermore, we will consider the case when both ends of the rod are clamped, and also the case when only one end is clamped.

In other similar works on linear elasticity problems that are related to the present paper, Griso ([**14**] among other works) studies the asymptotic behavior of structures made of junctions of curved rods, plates and combinations of both types. Irago–Viaño [**15**] obtained higher order approximations of flexural eigenvalues of a thin straight rod using an asymptotic expansion procedure. Irago–Kerdid–Viaño [**16**] studied the case of high frequency vibrations related to stretching and torsional modes of thin rods. Nazarov [**21**], Nazarov–Slutskii [**23**] and Buttazzo–Cardone–Nazarov [**4**], [**5**] provide an elaborate research on asymptotic expansion methods for anisotropic and non-homogeneous elastic thin rods and plates. The study of eigenvalue problems on thin multi-structures for different equations is common and of much interest in the PDE theory. For example, works like Bunoiu–Cardone–Nazarov [**2**], [**3**] deal with the case of the Poisson equation for junctions of rods and a plate. For an extensive list of references see Ciarlet [**6**].

The present paper is organized as follows. First we explain the setting of the problem in Section 2. In Section 3 we introduce some notations and formulate the three-dimensional eigenvalue problem along with the main result involving the order and the asymptotic behavior of the eigenvalues. In Section 4 we present some preliminaries used during the proof as well as the variational formulation of the main problem. The proof of the order of the eigenvalue is given in Section 5. A lower bound of the limit eigenvalue is shown in Section 6 while an upper bound is given in Section 7. In Section 8 Appendix we give proof to some lemmas and further details on some computations stated in the main body of the paper. Moreover, Sections 4, 5 and 6 are split into two parts, explaining the differences between the two different boundary conditions we consider.

2. Setting.

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain. We want to study the oscillations of an elastic body with the shape of Ω.

We denote by $u = (u_1, u_2, u_3) : \Omega \longrightarrow \mathbb{R}^3$ the displacement vector field associated with the oscillations. Let λ_1, λ_2 be real constants corresponding to the mechanical properties of the elastic body. We assume $\lambda_1 > 0$, $\lambda_2 > 0$ in this paper. We define the tensors

$$\begin{aligned} e(u) &= (e_{ij}(u))_{1\le i,j\le 3} = \left(\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)_{1\le i,j\le 3}, \\ \sigma(u) &= \lambda_1 \operatorname{tr}(e(u)) \operatorname{Id}_3 + 2\lambda_2 e(u), \end{aligned}$$

where tr is the trace of a matrix and Id_3 is the 3×3 identity matrix. $e(u)$ is called the *linearized strain tensor* and $\sigma(u)$ is the *stress tensor* derived from Hooke's law in the case of a homogeneous isotropic elastic body (cf. Ciarlet [**6**]).

With this notation, the operator of the elastic equation is defined as the 2nd order linear elliptic operator

$$L[u] = \operatorname{div}\sigma(u), \qquad \text{i.e.} \qquad (L[u])_i = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\sigma_{ij}(u) \quad (1 \le i \le 3),$$

and the oscillations of an elastic body can be described by the following wave equation

$$\varrho \frac{\partial^2 u}{\partial t^2} = L[u] \tag{1}$$

where $\varrho > 0$ is the density.

Now, we take $\varrho = 1$ and we assume that the oscillations are periodic of period $2\pi/\omega$ ($\omega > 0$). In this case, we can write the displacement field as $u(x,t) = \mathrm{e}^{i\omega t}v(x)$. Thus, $\partial^2 u/\partial t^2 = -\omega^2 u(x,t)$. Putting $\mu = \omega^2$, the wave equation (1) becomes the eigenvalue problem

$$L[v] + \mu v = \mathbf{0}. \tag{2}$$

We now prepare the mathematical setting of our problem. We start presenting the domain $\Omega_\varepsilon = \Omega$, where $\varepsilon > 0$ is a small parameter corresponding to the thickness of the elastic body. Let $l > 0$ and let $B \subseteq \mathbb{R}^2$ be a connected bounded domain such that the boundary is $\mathcal{C}^3$ with $m \in \mathbb{N}$ connected components. We consider the sets

$$\begin{aligned} S &= B \times (0,l), & s_1^{(-)} &= \overline{B} \times \{0\}, \\ s_1^{(+)} &= \overline{B} \times \{l\}, & s_2 &= \partial B \times (0,l). \end{aligned}$$

Note that $\partial S = s_1^{(-)} \cup s_1^{(+)} \cup s_2$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a $\mathcal{C}^3$-diffeomorphism which satisfies the following properties.

i) $F(z) = (F_1(z), F_2(z), z_3) \quad (z = (z_1, z_2, z_3) \in S)$.

ii) $F_i(0,0,z_3) = 0 \quad (i = 1,2, \quad 0 \le z_3 \le l)$.

iii) The determinant of the Jacobian matrix of F is positive for all $z \in S$.

Let $\varepsilon > 0$ be a small positive parameter and define $F^\varepsilon(z) = (\varepsilon F_1(z), \varepsilon F_2(z), z_3)$. With this notation, we consider the following sets in $\mathbb{R}^3$.

$$\Omega_\varepsilon = F^\varepsilon(S), \quad \Gamma_{1,\varepsilon}^{(-)} = F^\varepsilon(s_1^{(-)}), \quad \Gamma_{1,\varepsilon}^{(+)} = F^\varepsilon(s_1^{(+)}), \quad \Gamma_{2,\varepsilon} = F^\varepsilon(s_2).$$

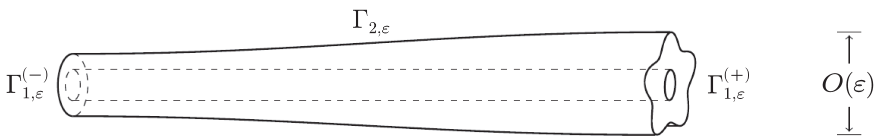

Figure 1. Example of Ω_ε.

We can think of Ω_ε as a slightly smoothly deformed thin cylinder (see Figure 1). It is easy to see $\partial\Omega_\varepsilon = \Gamma_{1,\varepsilon}^{(-)} \cup \Gamma_{1,\varepsilon}^{(+)} \cup \Gamma_{2,\varepsilon}$. Moreover, we obtain Ω_1, $\Gamma_{1,1}^{(-)}$, $\Gamma_{1,1}^{(+)}$, $\Gamma_{2,1}$ just by putting $\varepsilon = 1$ in the previous definition. Note that $\Omega_1 = F(S)$.

Let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$ be the coordinates in the sets Ω_ε, Ω_1 and S, thus obtaining the relation between the coordinates

$$\begin{cases} (x_1, x_2, x_3) = (\varepsilon y_1, \varepsilon y_2, y_3), \\ (y_1, y_2, y_3) = (F_1(z), F_2(z), z_3), \\ (x_1, x_2, x_3) = (\varepsilon F_1(z), \varepsilon F_2(z), z_3). \end{cases} \tag{3}$$

We want to study the small eigenvalues (low-frequency oscillations related to flexural vibrations) associated with the thin elastic body Ω_ε. We denote by $u = (u_1, u_2, u_3) : \Omega_\varepsilon \longrightarrow \mathbb{R}^3$ the displacement vector field associated with the oscillations.

With this notation, the main subject of the present paper is to study the eigenvalues and eigenfunctions when the parameter ε goes to zero of the following eigenvalue problems.

$$\begin{cases} L[u] + \mu u = \mathbf{0} \text{ in } \Omega_\varepsilon \\ u = \mathbf{0} \qquad\qquad \text{on } \Gamma_{1,\varepsilon}^{(-)} \cup \Gamma_{1,\varepsilon}^{(+)} \\ \sigma(u)\,\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma_{2,\varepsilon} \end{cases} \tag{DD}$$

$$\begin{cases} L[u] + \mu u = \mathbf{0} \text{ in } \Omega_\varepsilon \\ u = \mathbf{0} \qquad\qquad \text{on } \Gamma_{1,\varepsilon}^{(-)} \\ \sigma(u)\,\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma_{2,\varepsilon} \cup \Gamma_{1,\varepsilon}^{(+)} \end{cases} \tag{DN}$$

where $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega_\varepsilon$. The case (DD) corresponds to a thin rod with both ends clamped while the case (DN), to a thin rod with only one clamped end.

3. Some notations and main results.

In order to state the main results we first introduce several notations.

Denote $\mathrm{d}y' = \mathrm{d}y_1\mathrm{d}y_2$ and define the set $\widehat{\Omega}(y_3)$ to be the cross-section of $\Omega_1 = F(S)$ at $y_3 \in [0, l\,]$. Furthermore, for $1 \le i, j \le 2$, we define the functions

$$H(y_3) = \int_{\widehat{\Omega}(y_3)} 1\mathrm{d}y', \quad K_i(y_3) = \int_{\widehat{\Omega}(y_3)} y_i\mathrm{d}y', \quad A_{ij}(y_3) = \int_{\widehat{\Omega}(y_3)} y_i y_j \mathrm{d}y' \quad (y_3 \in [0, l\,])$$

and write $Y = \lambda_2(3\lambda_1 + 2\lambda_2)/(\lambda_1 + \lambda_2)$, known as the *Young modulus.*

REMARK 3.1. Set first $z' = (z_1, z_2)$, $\mathrm{d}z' = \mathrm{d}z_1\mathrm{d}z_2$. If we denote by

$$J(z) = \left(\frac{\partial F_i}{\partial z_j}\right)_{1\le i,j\le 3} = \begin{pmatrix} \frac{\partial F_1}{\partial z_1} & \frac{\partial F_1}{\partial z_2} & \frac{\partial F_1}{\partial z_3} \\ \frac{\partial F_2}{\partial z_1} & \frac{\partial F_2}{\partial z_2} & \frac{\partial F_2}{\partial z_3} \\ 0 & 0 & 1 \end{pmatrix}$$

the Jacobian matrix of F and by $J_*(z) = \det(J(z))$ its determinant, then after a change of variables we can also express the previous functions with

$$H(z_3)=\int_B J_*(z',z_3)\mathrm{d}z', \qquad K_i(z_3)=\int_B F_i(z',z_3)J_*(z',z_3)\mathrm{d}z',$$
$$A_{ij}(z_3)=\int_B F_i(z',z_3)F_j(z',z_3)J_*(z',z_3)\mathrm{d}z' \quad (z_3\in[0,l]).$$

REMARK 3.2. Note that the matrix $(A_{ij}(z_3))_{1\le i,j,\le 2}$ is positive definite.

If we denote by $\{\mu_k^{DD}(\varepsilon)\}_{k=1}^{+\infty}$ and $\{\mu_k^{DN}(\varepsilon)\}_{k=1}^{+\infty}$ the eigenvalues of problem (DD) and (DN) respectively, it is known that for any $\varepsilon>0$ there are infinite discrete sequences of positive eigenvalues

$$0<\mu_1^{DD}(\varepsilon)\le\mu_2^{DD}(\varepsilon)\le\cdots\le\mu_k^{DD}(\varepsilon)\le\mu_{k+1}^{DD}(\varepsilon)\le\cdots \text{ with } \lim_{k\to+\infty}\mu_k^{DD}(\varepsilon)=+\infty$$
$$0<\mu_1^{DN}(\varepsilon)\le\mu_2^{DN}(\varepsilon)\le\cdots\le\mu_k^{DN}(\varepsilon)\le\mu_{k+1}^{DN}(\varepsilon)\le\cdots \text{ with } \lim_{k\to+\infty}\mu_k^{DN}(\varepsilon)=+\infty$$

which are arranged in increasing order, counting multiplicities (cf. Courant–Hilbert [**9**], Edmunds–Evans [**12**], Egorov–Kondratiev [**13**]).

Now we present the main results of the paper.

THEOREM 3.3 (Both ends clamped). *Let $\mu_k^{DD}(\varepsilon)$ be the k-th eigenvalue of problem* (DD). *Then the following statements hold for each $k\in\mathbb{N}$.*

a) $\mu_k^{DD}(\varepsilon)=O(\varepsilon^2)$ *as* $\varepsilon\to 0$.

b) *Moreover, we have the limit*

$$\lim_{\varepsilon\to 0}\frac{\mu_k^{DD}(\varepsilon)}{\varepsilon^2}=\Lambda_k^{DD},$$

where Λ_k^{DD} denotes the k-th eigenvalue of the 4th order ordinary differential operator

$$\begin{cases} Y\dfrac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) & -K_1(\tau)\\ A_{21}(\tau) & A_{22}(\tau) & -K_2(\tau)\end{pmatrix}\begin{pmatrix}\dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2}\\ \dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2}\\ \dfrac{\mathrm{d}\eta_3}{\mathrm{d}\tau}\end{pmatrix}\right)=\Lambda\, H(\tau)\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} & (0<\tau<l),\\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(H(\tau)\dfrac{\mathrm{d}\eta_3}{\mathrm{d}\tau}\right)=\dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(K_1(\tau)\dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2}+K_2(\tau)\dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2}\right) & (0<\tau<l),\\ \eta_3(0)=\eta_i(0)=\dfrac{\mathrm{d}\eta_i}{\mathrm{d}\tau}(0)=0 & (i=1,2),\\ \eta_3(l)=\eta_i(l)=\dfrac{\mathrm{d}\eta_i}{\mathrm{d}\tau}(l)=0 & (i=1,2).\end{cases}$$

THEOREM 3.4 (Only one end clamped). *Let $\mu_k^{DN}(\varepsilon)$ be the k-th eigenvalue of problem* (DN). *Then the following statements hold for each $k\in\mathbb{N}$.*

a) $\mu_k^{DN}(\varepsilon)=O(\varepsilon^2)$ *as* $\varepsilon\to 0$.

b) *Moreover, we have the limit*

$$\lim_{\varepsilon \to 0} \frac{\mu_k^{DN}(\varepsilon)}{\varepsilon^2} = \Lambda_k^{DN},$$

where Λ_k^{DN} denotes the k-th eigenvalue of the 4th order ordinary differential operator

$$\begin{cases} Y\dfrac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) & -K_1(\tau) \\ A_{21}(\tau) & A_{22}(\tau) & -K_2(\tau) \end{pmatrix}\begin{pmatrix} \dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} \\ \dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2} \\ \dfrac{\mathrm{d}\eta_3}{\mathrm{d}\tau} \end{pmatrix}\right) = \Lambda\, H(\tau)\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} & (0 < \tau < l), \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(H(\tau)\dfrac{\mathrm{d}\eta_3}{\mathrm{d}\tau}\right) = \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(K_1(\tau)\dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} + K_2(\tau)\dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2}\right) & (0 < \tau < l), \\ \eta_3(0) = \eta_i(0) = \dfrac{\mathrm{d}\eta_i}{\mathrm{d}\tau}(0) = 0 & (i = 1, 2), \\ \dfrac{\mathrm{d}\eta_3}{\mathrm{d}\tau}(l) = \dfrac{\mathrm{d}^2\eta_i}{\mathrm{d}\tau^2}(l) = \dfrac{\mathrm{d}^3\eta_i}{\mathrm{d}\tau^3}(l) = 0 & (i = 1, 2). \end{cases}$$

REMARK 3.5. Note that if the functions $K_i \equiv 0$ for $i = 1, 2$, then the ordinary differential equations in Theorem 3.3 and Theorem 3.4 get simpler. Using the corresponding boundary conditions, the equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(H(\tau)\frac{\mathrm{d}\eta_3}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(K_1(\tau)\frac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} + K_2(\tau)\frac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2}\right) \qquad (0 < \tau < l)$$

yields $\eta_3 \equiv 0$, and hence the ODE in Theorem 3.3 and Theorem 3.4 simplifies to

$$Y\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) \\ A_{12}(\tau) & A_{22}(\tau) \end{pmatrix}\begin{pmatrix} \dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} \\ \dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2} \end{pmatrix}\right) = \Lambda\, H(\tau)\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

with the respective boundary conditions.

4. Preliminaries and variational formulation.

In this section we will introduce some notation and some results we will need afterwards during the proof of the main theorems as well as the variational formulation of our main problems.

We start with Korn's inequality (cf. Ciarlet [**6**], Dautray–Lions [**10**]).

PROPOSITION 4.1 (Korn's inequality). *Let Ω be a bounded domain in $\mathbb{R}^3$. If Γ_0 is a measurable subset of the boundary $\partial\Omega$ such that* $\operatorname{area}\Gamma_0 > 0$, *then there exists a constant $C > 0$ such that*

$$\|v\|_{H^1(\Omega,\mathbb{R}^3)} \leq C \left(\sum_{i,j=1}^{3} \|e_{ij}(v)\|^2_{L^2(\Omega)} \right)^{1/2}$$

for any $v = (v_1, v_2, v_3) \in H^1(\Omega, \mathbb{R}^3)$ *with* $v|_{\Gamma_0} = \mathbf{0}$.

Definition 4.2. Let $\phi, \psi \in H^1(\Omega_\varepsilon, \mathbb{R}^3) \backslash \{\mathbf{0}\}$. We define the bilinear form

$$B_\varepsilon[\phi, \psi] = \int_{\Omega_\varepsilon} \left(\lambda_1 \operatorname{div} \phi \operatorname{div} \psi + 2\lambda_2 \sum_{i,j=1}^{3} e_{ij}(\phi) e_{ij}(\psi) \right) \mathrm{d}x$$

and the *Rayleigh quotient* by

$$\mathcal{R}_\varepsilon(\phi) = \frac{B_\varepsilon[\phi, \phi]}{\|\phi\|^2_{L^2(\Omega_\varepsilon, \mathbb{R}^3)}}.$$

It is easy to see that the Rayleigh quotient satisfies $\mathcal{R}_\varepsilon(c\phi) = \mathcal{R}_\varepsilon(\phi)$ for all $c > 0$ (homogeneity condition).

From now on let $k \in \mathbb{N}$. We set $\mathcal{H}_{k-1}(\,\cdot\,, \mathbb{R}^3)$ the set of all linear subspaces of dimension $k-1$ of $L^2(\,\cdot\,, \mathbb{R}^3)$. We now introduce the so-called *Max-Min principle*, which we use to characterize the eigenvalues of (DD) and (DN).

Proposition 4.3 (Max-Min principle). *Let* $\mathcal{W}_\varepsilon, \mathcal{W}'_\varepsilon$ *be the function spaces*

$$\mathcal{W}_\varepsilon = \{\phi \in H^1(\Omega_\varepsilon, \mathbb{R}^3) \mid \phi = \mathbf{0} \text{ on } \Gamma^{(-)}_{1,\varepsilon} \cup \Gamma^{(+)}_{1,\varepsilon}\},$$
$$\mathcal{W}'_\varepsilon = \{\phi \in H^1(\Omega_\varepsilon, \mathbb{R}^3) \mid \phi = \mathbf{0} \text{ on } \Gamma^{(-)}_{1,\varepsilon}\}.$$

Then the k-th eigenvalues are characterized as follows:

$$\mu_k^{DD}(\varepsilon) = \sup_{X \in \mathcal{H}_{k-1}(\Omega_\varepsilon, \mathbb{R}^3)} \inf\{\mathcal{R}_\varepsilon(\phi) \mid \phi \in \mathcal{W}_\varepsilon \backslash \{\mathbf{0}\}, \phi \perp X \text{ in } L^2(\Omega_\varepsilon, \mathbb{R}^3)\}, \tag{4}$$
$$\mu_k^{DN}(\varepsilon) = \sup_{X \in \mathcal{H}_{k-1}(\Omega_\varepsilon, \mathbb{R}^3)} \inf\{\mathcal{R}_\varepsilon(\phi) \mid \phi \in \mathcal{W}'_\varepsilon \backslash \{\mathbf{0}\}, \phi \perp X \text{ in } L^2(\Omega_\varepsilon, \mathbb{R}^3)\}. \tag{5}$$

Recall that $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are used as the coordinates in Ω_ε and $\Omega_1 = F(S)$, respectively with the relation given in (3). We change the variables to transform Ω_ε into $F(S)$. We now compute the new stress and strain tensors in terms of the new variables in $F(S)$.

We begin to study the problem by variational methods. In order to consider the stress and strain tensors in terms of y, we introduce the scaling and change of variable

$$u_1 = \varepsilon U_1,\ u_2 = \varepsilon U_2,\ u_3 = \varepsilon^2 U_3.$$

We obtain the following expressions of $e_{ij}(u)$.

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{1}{\varepsilon}\frac{\partial u_i}{\partial y_j} + \frac{1}{\varepsilon}\frac{\partial u_j}{\partial y_i}\right) = \frac{1}{2}\left(\frac{\partial U_i}{\partial y_j} + \frac{\partial U_j}{\partial y_i}\right)$$

$$e_{i3}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial y_3} + \frac{1}{\varepsilon}\frac{\partial u_3}{\partial y_i}\right) = \frac{1}{2}\left(\varepsilon\frac{\partial U_i}{\partial y_3} + \varepsilon\frac{\partial U_3}{\partial y_i}\right) \quad (1 \le i,j \le 2)$$
$$e_{33}(u) = \frac{\partial u_3}{\partial x_3} = \frac{\partial u_3}{\partial y_3} = \varepsilon^2\frac{\partial U_3}{\partial y_3}.$$

We observe that after the change of variables we just introduced, we rewrote the strain tensor $e_{ij}(u)$ in terms of $U = (U_1, U_2, U_3)$. Therefore, for $1 \le i,j \le 2$ we can define

$$E_{ij}(U) = \frac{1}{2}\left(\frac{\partial U_i}{\partial y_j} + \frac{\partial U_j}{\partial y_i}\right),\ E_{i3}(U) = \frac{1}{2}\left(\frac{\partial U_i}{\partial y_3} + \frac{\partial U_3}{\partial y_i}\right), \quad E_{33}(U) = \frac{\partial U_3}{\partial y_3}.$$

Note also that since we have symmetry, i.e. $e_{ij}(u) = e_{ji}(u)$ $(1 \le i,j \le 3)$, we also define $E_{3i}(U) = E_{i3}(U)$ $(i = 1,2)$. With this notation, we have the relation

$$e_{ij}(u) = E_{ij}(U), \quad e_{i3}(u) = \varepsilon E_{i3}(U) \quad (1 \le i,j \le 2), \quad e_{33}(u) = \varepsilon^2 E_{33}(U). \tag{6}$$

Furthermore, using (6), we proceed to write the divergence in terms of U.

$$\begin{aligned}\mathrm{div}(u) &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = e_{11}(u) + e_{22}(u) + e_{33}(u)\\ &= E_{11}(U) + E_{22}(U) + \varepsilon^2 E_{33}(U).\end{aligned} \tag{7}$$

Our next step is to rewrite the Rayleigh quotient and to describe the eigenvalues in terms of y. We distinguish between the (DD) case and the (DN) case.

4.1. (DD) case.

Recall the set

$$\mathcal{W}_\varepsilon = \left\{\phi \in H^1(\Omega_\varepsilon, \mathbb{R}^3) \mid \phi = \mathbf{0} \text{ on } \Gamma_{1,\varepsilon}^{(-)} \cup \Gamma_{1,\varepsilon}^{(+)}\right\}$$

introduced in Proposition 4.3. For every $\phi \in \mathcal{W}_\varepsilon$ we set $B_\varepsilon[\phi,\phi]$ and $\mathcal{R}_\varepsilon$ as in Definition 4.2. We change the unknown variables $\phi = \phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x))$ into $\Phi = \Phi(y) = (\Phi_1(y), \Phi_2(y), \Phi_3(y))$ by $\phi_i(x) = \varepsilon\Phi_i(y)$ $(i = 1,2)$, $\phi_3(x) = \varepsilon^2\Phi_3(y)$ according to the coordinate change $x = (\varepsilon y_1, \varepsilon y_2, y_3)$ described in (3). Define now the set

$$\mathcal{W}_1 = \left\{\Phi \in H^1(F(S), \mathbb{R}^3) \mid \Phi = \mathbf{0} \text{ on } \Gamma_{1,1}^{(-)} \cup \Gamma_{1,1}^{(+)}\right\}. \tag{8}$$

We want to describe the k-th eigenvalue $\mu_k^{DD}(\varepsilon)$ in terms of the new spaces and functions after the change of variables. Note that $\phi \in \mathcal{W}_\varepsilon$ if and only if $\Phi \in \mathcal{W}_1$. Thus, using this fact together with the relations (6) and (7), and substituting them into $B_\varepsilon[\phi,\phi]$ and $\mathcal{R}_\varepsilon(\phi)$, for every $\Phi \in \mathcal{W}_1$ we define

$$\begin{aligned}\widetilde{B}_\varepsilon[\Phi,\Phi] = \int_{F(S)} \Bigg\{&\lambda_1\left(E_{11}(\Phi) + E_{22}(\Phi) + \varepsilon^2 E_{33}(\Phi)\right)^2\\ &+ 2\lambda_2\left(\sum_{i,j=1}^{2} E_{ij}(\Phi)^2 + 2\varepsilon^2\sum_{i=1}^{2} E_{i3}(\Phi)^2 + \varepsilon^4 E_{33}(\Phi)^2\right)\Bigg\}\varepsilon^2\mathrm{d}y,\end{aligned} \tag{9}$$

$$\widetilde{\mathcal{R}}_\varepsilon(\Phi) = \frac{\widetilde{B}_\varepsilon[\Phi,\Phi]}{\displaystyle\int_{F(S)} \left(\varepsilon^2\Phi_1^2 + \varepsilon^2\Phi_2^2 + \varepsilon^4\Phi_3^2\right)\varepsilon^2 \mathrm{d}y}. \tag{10}$$

Furthermore, for all $\Phi, \Psi \in \mathcal{W}_1$ we say that $\Phi \perp_\varepsilon \Psi$ if and only if

$$\int_{F(S)} (\Phi_1\Psi_1 + \Phi_2\Psi_2 + \varepsilon^2\Phi_3\Psi_3)\mathrm{d}y = 0.$$

Due to this definition, $\phi \perp \psi$ if and only if $\Phi \perp_\varepsilon \Psi$. For every $Z \in \mathcal{H}_{k-1}(F(S), \mathbb{R}^3)$ we define the set

$$Z^{\perp_\varepsilon} = \{\Phi \in \mathcal{W}_1 \mid \Phi \perp_\varepsilon \Psi \text{ for all } \Psi \in Z\},$$

which is a closed subspace of $\mathcal{W}_1$.

Using the Max-Min principle (Proposition 4.3), after the change of variables, the characterization (4) of $\mu_k^{DD}(\varepsilon)$ can be rewritten as

$$\mu_k^{DD}(\varepsilon) = \sup_{Z\in\mathcal{H}_{k-1}(F(S),\mathbb{R}^3)} \inf\{\widetilde{\mathcal{R}}_\varepsilon(\Phi) \mid \Phi \in \mathcal{W}_1\backslash\{\mathbf{0}\}, \Phi \in Z^{\perp_\varepsilon}\}. \tag{11}$$

4.2. (DN) case.

For the case of the eigenvalues $\mu_k^{DN}(\varepsilon)$, note that we can similarly characterize $\mu_k^{DN}(\varepsilon)$ with

$$\mu_k^{DN}(\varepsilon) = \sup_{Z\in\mathcal{H}_{k-1}(F(S),\mathbb{R}^3)} \inf\{\widetilde{\mathcal{R}}_\varepsilon(\Phi) \mid \Phi \in \mathcal{W}_1'\backslash\{\mathbf{0}\}, \Phi \in Z^{\perp_\varepsilon}\} \tag{12}$$

where

$$\mathcal{W}_1' = \left\{\Phi \in H^1(F(S), \mathbb{R}^3) \mid \Phi = \mathbf{0} \text{ on } \Gamma_{1,1}^{(-)}\right\}. \tag{13}$$

5. Proof of the order of the eigenvalues.

5.1. (DD) case.

We begin showing that $\mu_k^{DD}(\varepsilon) = O(\varepsilon^2)$ as $\varepsilon \to 0$. In order to do so, we will find an upper bound of the eigenvalue $\mu_k^{DD}(\varepsilon)$ using the Max-Min principle and (11).

Let us take test functions $\Upsilon^{(s)} = \Upsilon^{(s)}(y) = (\Upsilon_1^{(s)}(y), \Upsilon_2^{(s)}(y), \Upsilon_3^{(s)}(y))$ $(s \in \mathbb{N})$ as follows:

$$\begin{aligned}
\Upsilon_1^{(s)}(y) &= \eta_1^{(s)}(y_3), \\
\Upsilon_2^{(s)}(y) &= \eta_2^{(s)}(y_3), \\
\Upsilon_3^{(s)}(y) &= \eta_3^{(s)}(y_3) - y_1\frac{\mathrm{d}\eta_1^{(s)}}{\mathrm{d}y_3} - y_2\frac{\mathrm{d}\eta_2^{(s)}}{\mathrm{d}y_3},
\end{aligned}$$

where $\{\eta_1^{(s)}, \eta_2^{(s)}, \eta_3^{(s)}\}_{s\in\mathbb{N}}$ is a linearly independent system satisfying

$$\begin{aligned}
&\eta_1^{(s)}, \eta_2^{(s)} \in H^2((0,l)), \eta_3^{(s)} \in H^1((0,l)),\\
&\eta_i^{(s)}(0) = \eta_i^{(s)}(l) = 0 \quad (i = 1,2,3),\\
&\frac{\mathrm{d}\eta_i^{(s)}}{\mathrm{d}z_3}(0) = \frac{\mathrm{d}\eta_i^{(s)}}{\mathrm{d}z_3}(l) = 0 \quad (i = 1,2).
\end{aligned}$$

Choose an arbitrary $Z \in \mathcal{H}_{k-1}(F(S), \mathbb{R}^3)$ and let $\widetilde{Z} = L.H.[\Upsilon^{(1)}, \Upsilon^{(2)}, \ldots, \Upsilon^{(k)}]$ denote the minimal linear space that contains the set $\{\Upsilon^{(1)}, \Upsilon^{(2)}, \ldots, \Upsilon^{(k)}\}$. Since each $\Upsilon^{(s)} \in \mathcal{W}_1$ (for all $s \in \mathbb{N}$), we have that $\widetilde{Z} \subseteq \mathcal{W}_1$. Since $\dim Z < \dim \widetilde{Z}$, there exist a function $\Psi \in \widetilde{Z} \cap Z^{\perp_\varepsilon}$ and a vector $(c_1, \ldots, c_k) = (c_1(\varepsilon), \ldots, c_k(\varepsilon)) \in \mathbb{R}^k \backslash \{\mathbf{0}\}$ such that

$$\Psi = \sum_{s=1}^{k} c_s(\varepsilon) \Upsilon^{(s)}. \tag{14}$$

Note that since both $\widetilde{Z}$ and $Z^{\perp_\varepsilon}$ are subsets of $\mathcal{W}_1$, we have also that $\Psi \in \mathcal{W}_1$ and due to the fact that $(c_1, \ldots, c_k) \in \mathbb{R}^k \backslash \{\mathbf{0}\}$ we deduce that $\Psi \in \mathcal{W}_1 \backslash \{\mathbf{0}\}$, so we can apply $\widetilde{\mathcal{R}}_\varepsilon$ to Ψ (cf. (10)).

Using the definition of $\Upsilon^{(s)}$ we compute

$$E_{ij}(\Upsilon^{(s)}) = 0, \tag{15}$$

$$E_{i3}(\Upsilon^{(s)}) = \frac{1}{2}\left(\frac{\partial \Upsilon_i^{(s)}}{\partial y_3} + \frac{\partial \Upsilon_3^{(s)}}{\partial y_i}\right) = \frac{1}{2}\left(\frac{\mathrm{d}\eta_i^{(k)}}{\mathrm{d}z_3} - \frac{\mathrm{d}\eta_i^{(k)}}{\mathrm{d}z_3}\right) = 0 \quad (1 \le i, j \le 2). \tag{16}$$

Now we want to calculate $\widetilde{\mathcal{R}}_\varepsilon(\Psi)$. Using the linearity of the operator E_{ij}, (15) and (16), we see that

$$E_{ij}(\Psi) = \sum_{s=1}^{k} c_s(\varepsilon) E_{ij}(\Upsilon^{(s)}) = 0, \quad E_{i3}(\Psi) = \sum_{s=1}^{k} c_s(\varepsilon) E_{i3}(\Upsilon^{(s)}) = 0 \quad (1 \le i, j \le 2). \tag{17}$$

Hence, using (17) and the definition in (9), we get

$$\begin{aligned}
\widetilde{B}_\varepsilon[\Psi, \Psi] &= \int_{F(S)} \Bigg\{ \lambda_1 \left(E_{11}(\Psi) + E_{22}(\Psi) + \varepsilon^2 E_{33}(\Psi)\right)^2 \\
&\qquad + 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(\Psi)^2 + 2\varepsilon^2 \sum_{i=1}^{2} E_{i3}(\Psi)^2 + \varepsilon^4 E_{33}(\Psi)^2 \right) \Bigg\} \varepsilon^2 \mathrm{d}y \\
&= \int_{F(S)} \left(\lambda_1 \left(\varepsilon^2 E_{33}(\Psi)\right)^2 + 2\lambda_2 \left(\varepsilon^4 E_{33}(\Psi)^2\right) \right) \varepsilon^2 \mathrm{d}y \\
&= \varepsilon^6 \int_{F(S)} (\lambda_1 + 2\lambda_2) E_{33}(\Psi)^2 \mathrm{d}y.
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
\widetilde{\mathcal{R}}_\varepsilon(\Psi) &= \frac{\varepsilon^6 \displaystyle\int_{F(S)} (\lambda_1+2\lambda_2)E_{33}(\Psi)^2 \mathrm{d}y}{\displaystyle\int_{F(S)} \left(\varepsilon^2\Psi_1^2+\varepsilon^2\Psi_2^2+\varepsilon^4\Psi_3^2\right)\varepsilon^2 \mathrm{d}y} = \frac{\varepsilon^6}{\varepsilon^4} \frac{\displaystyle\int_{F(S)} (\lambda_1+2\lambda_2)E_{33}(\Psi)^2 \mathrm{d}y}{\displaystyle\int_{F(S)} \left(\Psi_1^2+\Psi_2^2+\varepsilon^2\Psi_3^2\right) \mathrm{d}y} \\
&\leq \varepsilon^2 \frac{\displaystyle\int_{F(S)} (\lambda_1+2\lambda_2)E_{33}(\Psi)^2 \mathrm{d}y}{\displaystyle\int_{F(S)} \left(\Psi_1^2+\Psi_2^2\right) \mathrm{d}y}.
\end{aligned}
$$

Now substitute the definition (14) into the previous equation to obtain

$$
\widetilde{\mathcal{R}}_\varepsilon(\Psi) \leq \varepsilon^2 \frac{\displaystyle\int_{F(S)} (\lambda_1+2\lambda_2) \sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)E_{33}(\Upsilon^{(p)})E_{33}(\Upsilon^{(q)})\mathrm{d}y}{\displaystyle\int_{F(S)} \sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\left(\Upsilon_1^{(p)}\Upsilon_1^{(q)}+\Upsilon_2^{(p)}\Upsilon_2^{(q)}\right)\mathrm{d}y}. \tag{18}
$$

Let us put

$$
\gamma_{pq} = \int_{F(S)} E_{33}(\Upsilon^{(p)})E_{33}(\Upsilon^{(q)})\mathrm{d}y, \quad \widehat{\gamma}_{pq} = \int_{F(S)} \left(\Upsilon_1^{(p)}\Upsilon_1^{(q)}+\Upsilon_2^{(p)}\Upsilon_2^{(q)}\right)\mathrm{d}y.
$$

Note that since we chose the system $\{\eta_1^{(s)},\eta_2^{(s)},\eta_3^{(s)}\}_{s\in\mathbb{N}}$ to be linearly independent and by the symmetry $\gamma_{pq}=\gamma_{qp}$, $\widehat{\gamma}_{pq}=\widehat{\gamma}_{qp}$, we have that $(\gamma_{pq})_{1\leq p,q\leq k}$ and $(\widehat{\gamma}_{pq})_{1\leq p,q\leq k}$ are positive definite matrices. Therefore, all of its eigenvalues are positive. Let γ_* be the biggest eigenvalue of $(\gamma_{pq})_{1\leq p,q\leq k}$ and $\widehat{\gamma}_*$, the smallest eigenvalue of $(\widehat{\gamma}_{pq})_{1\leq p,q\leq k}$. With this notation, we have the bounds

$$
\begin{aligned}
\sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\gamma_{pq} &\leq \gamma_*(c_1(\varepsilon)^2+\cdots+c_k(\varepsilon)^2), \\
\sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\widehat{\gamma}_{pq} &\geq \widehat{\gamma}_*(c_1(\varepsilon)^2+\cdots+c_k(\varepsilon)^2).
\end{aligned}
$$

Therefore, (18) becomes

$$
\begin{aligned}
\widetilde{\mathcal{R}}_\varepsilon(\Psi) &\leq \varepsilon^2 \frac{(\lambda_1+2\lambda_2)\displaystyle\sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\gamma_{pq}}{\displaystyle\sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\widehat{\gamma}_{pq}} \leq \varepsilon^2 \frac{(\lambda_1+2\lambda_2)\gamma_*(c_1(\varepsilon)^2+\cdots+c_k(\varepsilon)^2)}{\widehat{\gamma}_*(c_1(\varepsilon)^2+\cdots+c_k(\varepsilon)^2)} \\
&= \varepsilon^2 \frac{(\lambda_1+2\lambda_2)\gamma_*}{\widehat{\gamma}_*}.
\end{aligned}
$$

Put $C=(\lambda_1+2\lambda_2)\gamma_*/\widehat{\gamma}_*$. We obtained that for a certain $\Psi\in\mathcal{W}_1$ there exists a positive constant C independent of ε and independent of the choice of Z such that

$\widetilde{\mathcal{R}}_\varepsilon(\Psi) \le \varepsilon^2 C$. Thus, taking the infimum, we have

$$\inf\{\widetilde{\mathcal{R}}_\varepsilon(\Phi) \mid \Phi \in \mathcal{W}_1\backslash\{\mathbf{0}\}, \Phi \in Z^{\perp_\varepsilon}\} \le \widetilde{\mathcal{R}}_\varepsilon(\Psi) \le \varepsilon^2 C.$$

Since Z was arbitrary and C does not depend on the choice of Z, we can take the supremum on both sides over $\mathcal{H}_{k-1}(F(S),\mathbb{R}^3)$ to obtain

$$0 \le \mu_k^{DD}(\varepsilon) = \sup_{Z\in\mathcal{H}_{k-1}(F(S),\mathbb{R}^3)} \left\{\inf\{\widetilde{\mathcal{R}}_\varepsilon(\Phi) \mid \Phi \in \mathcal{W}_1\backslash\{\mathbf{0}\}, \Phi \in Z^{\perp_\varepsilon}\}\right\} \le \varepsilon^2 C.$$

Here we used the characterization (11) deduced in the previous section. Therefore we obtain

$$\mu_k^{DD}(\varepsilon) = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0$$

which proves Theorem 3.3-a).

5.2. (DN) case.

For the case of the eigenvalues $\mu_k^{DN}(\varepsilon)$, note that due to the definition of the sets $\mathcal{W}_1$ and $\mathcal{W}_1'$ (see (8) and (13)), we see that $\mathcal{W}_1 \subseteq \mathcal{W}_1'$, therefore, the infimum over $\mathcal{W}_1'$ is not greater than over $\mathcal{W}_1$. Thus $0 \le \mu_k^{DN}(\varepsilon) \le \mu_k^{DD}(\varepsilon)$ and Theorem 3.4-a) also holds.

6. Weak formulation and deduction of the limit ODE.

The weak formulation of the equation of (DD) and (DN) is

$$\int_{\Omega_\varepsilon} \left(\lambda_1 \operatorname{div} u \operatorname{div} v + 2\lambda_2 \sum_{i,j=1}^{3} e_{ij}(u)e_{ij}(v)\right) \mathrm{d}x = \mu \int_{\Omega_\varepsilon} \sum_{i=1}^{3} u_i v_i \mathrm{d}x.$$

Here μ is an eigenvalue, u is the corresponding eigenfunction and $v = (v_1, v_2, v_3) \in \mathcal{W}_\varepsilon$ (or $\mathcal{W}_\varepsilon'$) is a test function. By the change of the variable given in (3) together with $u_i = \varepsilon U_i$, $v_i = \varepsilon V_i$ $(i = 1, 2)$ and $u_3 = \varepsilon^2 U_3$, $v_3 = \varepsilon^2 V_3$, the previous weak formulation is rewritten in terms of y as follows.

$$\begin{aligned}
&\int_{F(S)} \Bigg\{\lambda_1 \left(E_{11}(U) + E_{22}(U) + \varepsilon^2 E_{33}(U)\right)\left(E_{11}(V) + E_{22}(V) + \varepsilon^2 E_{33}(V)\right) \\
&\qquad + 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(U)E_{ij}(V) + 2\varepsilon^2 \sum_{i=1}^{2} E_{i3}(U)E_{i3}(V) + \varepsilon^4 E_{33}(U)E_{33}(V)\right)\Bigg\} \mathrm{d}y \\
&= \mu \int_{F(S)} \left(\varepsilon^2 U_1 V_1 + \varepsilon^2 U_2 V_2 + \varepsilon^4 U_3 V_3\right) \mathrm{d}y. \qquad (19)
\end{aligned}$$

6.1. (DD) case.

The proofs for the (DD) case and the (DN) case are very similar. Therefore, for simplicity, we will analyze the (DD) case and explain the main differences afterwards. In this section, to simplify the notation, let us write $\mu_k(\varepsilon)$ instead of $\mu_k^{DD}(\varepsilon)$.

Let $\{\Phi_\varepsilon^{(k)}\}_{k=1}^{+\infty} = \{(\Phi_{1,\varepsilon}^{(k)}, \Phi_{2,\varepsilon}^{(k)}, \Phi_{3,\varepsilon}^{(k)})\}_{k=1}^{+\infty}$ be the corresponding eigenfunctions of the eigenvalues $\{\mu_k(\varepsilon)\}_{k=1}^{+\infty}$ and such that

$$\int_{F(S)} \left((\Phi_{1,\varepsilon}^{(k)})^2 + (\Phi_{2,\varepsilon}^{(k)})^2 + (\Phi_{3,\varepsilon}^{(k)})^2\right) \mathrm{d}y = 1.$$

Now we put $U = V = \Phi_\varepsilon^{(k)}$ in (19) so that we get

$$\begin{aligned} \int_{F(S)} \Bigg\{ & \lambda_1 \left(E_{11}(\Phi_\varepsilon^{(k)}) + E_{22}(\Phi_\varepsilon^{(k)}) + \varepsilon^2 E_{33}(\Phi_\varepsilon^{(k)})\right)^2 \\ & + 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(\Phi_\varepsilon^{(k)})^2 + 2\varepsilon^2 \sum_{i=1}^{2} E_{i3}(\Phi_\varepsilon^{(k)})^2 + \varepsilon^4 E_{33}(\Phi_\varepsilon^{(k)})^2 \right) \Bigg\} \mathrm{d}y \\ = \mu_k(\varepsilon) & \int_{F(S)} \left(\varepsilon^2 (\Phi_{1,\varepsilon}^{(k)})^2 + \varepsilon^2 (\Phi_{2,\varepsilon}^{(k)})^2 + \varepsilon^4 (\Phi_{3,\varepsilon}^{(k)})^2\right) \mathrm{d}y. \end{aligned} \tag{20}$$

Note that by the choice of the $\{\Phi_\varepsilon^{(k)}\}_{k=1}^{+\infty}$ and by Theorem 3.3-a), i.e. $\mu_k(\varepsilon) = O(\varepsilon^2)$ as $\varepsilon \to 0$, we can see that the right-hand side of (20) is $O(\varepsilon^4)$ as $\varepsilon \to 0$. Therefore, the left-hand side must also satisfy the same condition and we conclude that

$$E_{ij}(\Phi_\varepsilon^{(k)}) = O(\varepsilon^2), \quad E_{i3}(\Phi_\varepsilon^{(k)}) = O(\varepsilon), \quad E_{33}(\Phi_\varepsilon^{(k)}) = O(1) \tag{21}$$

in the $L^2(F(S), \mathbb{R}^3)$ sense for $1 \le i, j \le 2$. Combining this fact with Korn's inequality (Proposition 4.1), we can see that $\Phi_\varepsilon^{(k)}$ is bounded in $H^1(F(S), \mathbb{R}^3)$. Let $\{\varepsilon_p\}_{p=1}^{+\infty}$ be any positive sequence such that $\varepsilon_p \to 0$ as $p \to +\infty$. Then, using the previous facts, there exists a subsequence $\{\varepsilon_{p(q)}\}_{q=1}^{+\infty}$ such that

$$\lim_{q \to +\infty} \Phi_{\varepsilon_{p(q)}}^{(k)} = \Phi^{(k)} \text{ weakly in } H^1(F(S), \mathbb{R}^3).$$

Moreover, from Rellich's theorem, we have

$$\lim_{q \to +\infty} \Phi_{\varepsilon_{p(q)}}^{(k)} = \Phi^{(k)} \text{ in } L^2(F(S), \mathbb{R}^3) \text{ with } \|\Phi^{(k)}\|_{L^2(F(S),\mathbb{R}^3)} = 1,$$

so we have non-trivial limit functions $\{\Phi^{(k)}\}_{k=1}^{+\infty} = \{(\Phi_1^{(k)}, \Phi_2^{(k)}, \Phi_3^{(k)})\}_{k=1}^{+\infty}$, which form an orthonormal basis of $L^2(F(S), \mathbb{R}^3)$. For $1 \le i, j \le 2$, we now set

$$\kappa_{ij}^\varepsilon = \frac{1}{\varepsilon^2} E_{ij}(\Phi_\varepsilon^{(k)}), \quad \kappa_{i3}^\varepsilon = \frac{1}{\varepsilon} E_{i3}(\Phi_\varepsilon^{(k)}), \quad \kappa_{33}^\varepsilon = E_{33}(\Phi_\varepsilon^{(k)}).$$

Furthermore, we define $\kappa_{3i}^\varepsilon = \kappa_{i3}^\varepsilon$. We remark that for $1 \le i, j \le 3$, each κ_{ij}^ε depends also on k. Due to (21) we have that $\kappa_{ij}^\varepsilon = O(1)$ $(1 \le i, j, \le 3)$ as $\varepsilon \to 0$ in the $L^2(F(S), \mathbb{R}^3)$ sense, that is, κ_{ij}^ε are bounded in $L^2(F(S), \mathbb{R}^3)$. Therefore, there exists a further subsequence $\{\varepsilon_{p(q(n))}\}_{n=1}^{+\infty}$ such that

$$\lim_{n \to +\infty} \kappa_{ij}^{\varepsilon_{p(q(n))}} = \kappa_{ij} \text{ weakly in } L^2(F(S), \mathbb{R}^3) \quad (1 \le i, j \le 3).$$

Note again, that each κ_{ij} still depends on k. Furthermore, in virtue of Theorem 3.3-a) there exists a constant c such that $\mu_k(\varepsilon)/\varepsilon^2 \le c$ and we conclude that there exist an even further subsequence $\{\zeta_r\}_{r=1}^{+\infty} \subseteq \{\varepsilon_{p(q(n))}\}_{n=1}^{+\infty}$ and a constant $\widetilde{\Lambda}_k$ that satisfy

$$\lim_{r\to+\infty} \frac{\mu_k(\zeta_r)}{\zeta_r^2} = \widetilde{\Lambda}_k. \tag{22}$$

This proves the existence of the limit for a subsequence of $\{\varepsilon_p\}_{p=1}^{+\infty}$.

We will characterize $\{\widetilde{\Lambda}_k\}_{k=1}^{+\infty}$. We take particular test functions and deduce several conditions for the limit functions $\Phi^{(k)}$ and κ_{ij}. Now put $U = \Phi^{(k)}_{\zeta_r}$, $\varepsilon = \zeta_r$, substitute them into (19) and after dividing both sides by ζ_r^2 we obtain

$$\begin{aligned}\int_{F(S)} \Bigg\{ &\lambda_1(\kappa_{11}^{\zeta_r} + \kappa_{22}^{\zeta_r} + \kappa_{33}^{\zeta_r}) \left(E_{11}(V) + E_{22}(V) + \zeta_r^2 E_{33}(V)\right) \\ &+ 2\lambda_2 \left(\sum_{i,j=1}^{2} \kappa_{ij}^{\zeta_r} E_{ij}(V) + 2\zeta_r \sum_{i=1}^{2} \kappa_{i3}^{\zeta_r} E_{i3}(V) + \zeta_r^2 \kappa_{33}^{\zeta_r} E_{33}(V) \right) \Bigg\} \mathrm{d}y \\ = \mu_k(\zeta_r) \int_{F(S)} &\left(\Phi^{(k)}_{1,\zeta_r} V_1 + \Phi^{(k)}_{2,\zeta_r} V_2 + \zeta_r^2 \Phi^{(k)}_{3,\zeta_r} V_3 \right) \mathrm{d}y \end{aligned} \tag{23}$$

for any test function $V = (V_1, V_2, V_3) \in \mathcal{W}_1$. By letting $r \to +\infty$ in (23), we get

$$\int_{F(S)} \left(\lambda_1(\kappa_{11} + \kappa_{22} + \kappa_{33}) \left(E_{11}(V) + E_{22}(V)\right) + 2\lambda_2 \sum_{i,j=1}^{2} \kappa_{ij} E_{ij}(V) \right) \mathrm{d}y = 0. \tag{24}$$

Next we choose $V_2 = 0$. We see that $E_{22}(V) = 0$, and since $\kappa_{12} = \kappa_{21}$, (24) becomes

$$\int_{F(S)} \left\{ \lambda_1 \sum_{p=1}^{3} \kappa_{pp} \frac{\partial V_1}{\partial y_1} + 2\lambda_2 \left(\kappa_{11} \frac{\partial V_1}{\partial y_1} + \kappa_{12} \frac{\partial V_1}{\partial y_2} \right) \right\} \mathrm{d}y = 0$$

$$\int_{F(S)} \left\{ \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{11} \right) \frac{\partial V_1}{\partial y_1} + 2\lambda_2 \kappa_{12} \frac{\partial V_1}{\partial y_2} \right\} \mathrm{d}y = 0. \tag{25}$$

By integration by parts in (25) we obtain

$$-\int_{F(S)} \left\{ \frac{\partial}{\partial y_1} \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{11} \right) V_1 + \frac{\partial}{\partial y_2} \left(2\lambda_2 \kappa_{12}\right) V_1 \right\} \mathrm{d}y = 0$$

$$-\int_{F(S)} \left\{ \frac{\partial}{\partial y_1} \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{11} \right) + \frac{\partial}{\partial y_2} \left(2\lambda_2 \kappa_{12}\right) \right\} V_1 \mathrm{d}y = 0.$$

In fact, due to the arbitrariness of V_1 we have

$$\frac{\partial}{\partial y_1} \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{11} \right) + \frac{\partial}{\partial y_2} \left(2\lambda_2 \kappa_{12}\right) = 0 \tag{26}$$

in the distribution sense. Similarly, letting $V_1 = 0$ we also deduce that

$$\int_{F(S)} \left\{ (2\lambda_2 \kappa_{12}) \frac{\partial V_2}{\partial y_1} + \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{22} \right) \frac{\partial V_2}{\partial y_2} \right\} \mathrm{d}y = 0, \tag{27}$$

$$\frac{\partial}{\partial y_1} (2\lambda_2 \kappa_{12}) + \frac{\partial}{\partial y_2} \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{22} \right) = 0. \tag{28}$$

We write

$$\begin{aligned} \alpha_1 &= \lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{11}, \qquad & \alpha_2 &= 2\lambda_2 \kappa_{12}, \\ \beta_1 &= 2\lambda_2 \kappa_{12}, & \beta_2 &= \lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{22}, \end{aligned} \tag{29}$$

so that (25), (26), (27) and (28) become

$$\int_{F(S)} \left(\alpha_1 \frac{\partial V_1}{\partial y_1} + \alpha_2 \frac{\partial V_1}{\partial y_2} \right) \mathrm{d}y = 0, \quad \int_{F(S)} \left(\beta_1 \frac{\partial V_2}{\partial y_1} + \beta_2 \frac{\partial V_2}{\partial y_2} \right) \mathrm{d}y = 0, \tag{30}$$

$$\frac{\partial \alpha_1}{\partial y_1} = -\frac{\partial \alpha_2}{\partial y_2}, \quad \frac{\partial \beta_1}{\partial y_1} = -\frac{\partial \beta_2}{\partial y_2}. \tag{31}$$

Note however that the functions V_1 and V_2 in (30) are arbitrary test functions. Therefore, for every $\phi \in H^1(F(S))$ with $\phi = 0$ on $\Gamma_{1,1}^{(+)} \cup \Gamma_{1,1}^{(-)}$, we have

$$\int_{F(S)} \left(\alpha_1 \frac{\partial \phi}{\partial y_1} + \alpha_2 \frac{\partial \phi}{\partial y_2} \right) \mathrm{d}y = 0, \quad \int_{F(S)} \left(\beta_1 \frac{\partial \phi}{\partial y_1} + \beta_2 \frac{\partial \phi}{\partial y_2} \right) \mathrm{d}y = 0. \tag{32}$$

We will now use the following lemma.

Lemma 6.1. *Assume that properties* (31) *and* (32) *are satisfied. Then the following statements hold.*

a) *There exist functions $h_1, h_2 \in L^2(F(S))$ such that $\partial h_p / \partial y_j \in L^2(F(S))$ for $1 \leq j$, $p \leq 2$ and*

$$\frac{\partial h_1}{\partial y_1} = -\alpha_2, \ \frac{\partial h_1}{\partial y_2} = \alpha_1, \ \frac{\partial h_2}{\partial y_1} = -\beta_2, \ \frac{\partial h_2}{\partial y_2} = \beta_1. \tag{33}$$

Moreover, h_1, h_2 take values on the boundary and $h_p|_{\Gamma_{2,1}} \in L^2(\Gamma_{2,1})$ for $p = 1, 2$.

b) *Write $\Gamma_{2,1} = g_1 \cup \cdots \cup g_m$ where each g_i is the i-th connected component of $\Gamma_{2,1}$ ($m \in \mathbb{N}$, $i = 1, \ldots, m$). Then, for $i = 1, \ldots, m$ the functions $h_1|_{g_i}, h_2|_{g_i}$ do not depend on (y_1, y_2) along g_i.*

For the proof of this lemma see Section 8 Appendix. Let us use the functions h_1 and h_2 given by this lemma. From (29) and (33), we note

$$\frac{\partial h_1}{\partial y_1}+\frac{\partial h_2}{\partial y_2}=\beta_1-\alpha_2=0, \tag{34}$$

$$\frac{\partial h_1}{\partial y_2}-\frac{\partial h_2}{\partial y_1}=\alpha_1+\beta_2=2\lambda_1\sum_{p=1}^{3}\kappa_{pp}+2\lambda_2(\kappa_{11}+\kappa_{22}). \tag{35}$$

For brevity, let us write

$$Q=\frac{\partial h_1}{\partial y_2}-\frac{\partial h_2}{\partial y_1}.$$

We rewrite the equality (35) with Q and we calculate

$$Q=2\left(\lambda_1\sum_{p=1}^{3}\kappa_{pp}+\lambda_2(\kappa_{11}+\kappa_{22})\right)=2\left((\lambda_1+\lambda_2)\sum_{p=1}^{3}\kappa_{pp}-\lambda_2\kappa_{33}\right)$$

$$\lambda_1 Q=2\left(\lambda_1(\lambda_1+\lambda_2)\sum_{p=1}^{3}\kappa_{pp}-\lambda_1\lambda_2\kappa_{33}\right)$$

$$\lambda_1 Q+2\lambda_2(3\lambda_1+2\lambda_2)\kappa_{33}=2(\lambda_1+\lambda_2)\left(\lambda_1\sum_{p=1}^{3}\kappa_{pp}+2\lambda_2\kappa_{33}\right).$$

Eventually, we obtain

$$\frac{\lambda_1}{2(\lambda_1+\lambda_2)}Q+\frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2}\kappa_{33}=\lambda_1\sum_{p=1}^{3}\kappa_{pp}+2\lambda_2\kappa_{33}. \tag{36}$$

This computation will be useful afterwards.

We go back to (23) with some particular test functions. Take functions $\rho_1=\rho_1(y_3)$, $\rho_2=\rho_2(y_3),\rho_3=\rho_3(y_3)$ such that

$$\begin{aligned}
&\rho_1,\rho_2\in H^2((0,l)),\quad \rho_3\in H^1((0,l)),\\
&\rho_i(0)=\rho_i(l)=0 \qquad (i=1,2,3),\\
&\frac{\mathrm{d}\rho_i}{\mathrm{d}y_3}(0)=\frac{\mathrm{d}\rho_i}{\mathrm{d}y_3}(l)=0 \qquad (i=1,2),
\end{aligned}$$

and put a test function $V=(V_1,V_2,V_3)\in\mathcal{W}_1$ by

$$\begin{aligned}
&V_1(y)=\rho_1(y_3),\\
&V_2(y)=\rho_2(y_3),\\
&V_3(y)=\rho_3(y_3)-y_1\frac{\mathrm{d}\rho_1}{\mathrm{d}y_3}-y_2\frac{\mathrm{d}\rho_2}{\mathrm{d}y_3}.
\end{aligned}$$

For this test function we note that $E_{ij}(V)=0$, $E_{i3}(V)=0$ for $1\le i,j\le 2$ (see the computations in (15) and (16)). Substituting the new test function into (23), dividing both sides by ζ_r^2, letting $r\to+\infty$ and using (22) we deduce

$$\int_{F(S)} \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{33} \right) E_{33}(V)\mathrm{d}y = \widetilde{\Lambda}_k \int_{F(S)} \left(\Phi_1^{(k)} \rho_1 + \Phi_2^{(k)} \rho_2 \right) \mathrm{d}y. \tag{37}$$

Now we begin the next step to characterize the behavior of the eigenvalue limit. We substitute (36) into (37) to get

$$\int_{F(S)} \left(\frac{\lambda_1}{2(\lambda_1+\lambda_2)} Q + \frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2} \kappa_{33} \right) E_{33}(V)\mathrm{d}y = \widetilde{\Lambda}_k \int_{F(S)} \left(\Phi_1^{(k)} \rho_1 + \Phi_2^{(k)} \rho_2 \right) \mathrm{d}y. \tag{38}$$

Using the above test function V, we have

$$E_{33}(V) = \frac{\partial V_3}{\partial y_3} = \frac{\mathrm{d}\rho_3}{\mathrm{d}y_3} - y_1 \frac{\mathrm{d}^2\rho_1}{\mathrm{d}y_3^2} - y_2 \frac{\mathrm{d}^2\rho_2}{\mathrm{d}y_3^2}. \tag{39}$$

Define $\mathrm{d}y' = \mathrm{d}y_1\mathrm{d}y_2$ and let $\widehat{\Omega}(y_3)$ be the the cross-section of $F(S)$ at $y_3 \in [0,l]$. We now look into equation (38) and we rewrite

$$\begin{aligned}
\int_{F(S)} QE_{33}(V)\mathrm{d}y &= \int_0^l \int_{\widehat{\Omega}(y_3)} Q \left(\frac{\mathrm{d}\rho_3}{\mathrm{d}y_3} - y_1 \frac{\mathrm{d}^2\rho_1}{\mathrm{d}y_3^2} - y_2 \frac{\mathrm{d}^2\rho_2}{\mathrm{d}y_3^2} \right) \mathrm{d}y'\mathrm{d}y_3 \\
&= \int_0^l \left(\int_{\widehat{\Omega}(y_3)} Q \frac{\mathrm{d}\rho_3}{\mathrm{d}y_3} \mathrm{d}y' + \int_{\widehat{\Omega}(y_3)} Q y_1 \frac{\mathrm{d}^2\rho_1}{\mathrm{d}y_3^2} \mathrm{d}y' + \int_{\widehat{\Omega}(y_3)} Q y_2 \frac{\mathrm{d}^2\rho_2}{\mathrm{d}y_3^2} \mathrm{d}y' \right) \mathrm{d}y_3 \\
&= \int_0^l \left(\frac{\mathrm{d}\rho_3}{\mathrm{d}y_3} \int_{\widehat{\Omega}(y_3)} Q\mathrm{d}y' + \frac{\mathrm{d}^2\rho_1}{\mathrm{d}y_3^2} \int_{\widehat{\Omega}(y_3)} Q y_1 \mathrm{d}y' + \frac{\mathrm{d}^2\rho_2}{\mathrm{d}y_3^2} \int_{\widehat{\Omega}(y_3)} Q y_2 \mathrm{d}y' \right) \mathrm{d}y_3. \quad (40)
\end{aligned}$$

We will now use the following lemma (see the proof in Section 8 Appendix).

LEMMA 6.2. *With the same notation as above, for every $y_3 \in [0,l]$ it holds that*

$$\int_{\widehat{\Omega}(y_3)} Q\mathrm{d}y' = 0, \qquad \int_{\widehat{\Omega}(y_3)} Q y_i \mathrm{d}y' = 0 \quad (i=1,2).$$

Using this lemma, we see that (40) becomes

$$\int_{F(S)} QE_{33}(V)\mathrm{d}y = 0.$$

As a consequence, (38) simplifies to

$$\int_{F(S)} \frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2} \kappa_{33} E_{33}(V)\mathrm{d}y = \widetilde{\Lambda}_k \int_{F(S)} \left(\Phi_1^{(k)} \rho_1 + \Phi_2^{(k)} \rho_2 \right) \mathrm{d}y. \tag{41}$$

We will now proceed to compute κ_{33}. Recall that $\kappa_{33} = E_{33}(\Phi^{(k)})$. We know by (21) that $E_{ij}(\Phi^{(k)}) = E_{i3}(\Phi^{(k)}) = 0$ for $1 \le i,j \le 2$. This will help us find a more explicit form of the functions $\Phi^{(k)}$. In order to solve the partial differential equation in the weak sense for $\Phi^{(k)}$, we first write

$$E_{ij}(\Phi^{(k)}) = \frac{1}{2}\left(\frac{\partial \Phi_i^{(k)}}{\partial y_j} + \frac{\partial \Phi_j^{(k)}}{\partial y_i}\right), \qquad E_{i3}(\Phi^{(k)}) = \frac{1}{2}\left(\frac{\partial \Phi_i^{(k)}}{\partial y_3} + \frac{\partial \Phi_3^{(k)}}{\partial y_i}\right).$$

For $i = 1, 2$, from $E_{ii}(\Phi^{(k)}) = 0$ we have $\partial \Phi_i^{(k)}/\partial y_i = 0$ and therefore we deduce that $\Phi_i^{(k)}$ does not depend on y_i. By $E_{12}(\Phi^{(k)}) = 0$, we see

$$\frac{\partial \Phi_1^{(k)}}{\partial y_2} + \frac{\partial \Phi_2^{(k)}}{\partial y_1} = 0 \quad \text{and thus} \quad \frac{\partial \Phi_1^{(k)}}{\partial y_2} = -\frac{\partial \Phi_2^{(k)}}{\partial y_1} \quad \text{in } F(S).$$

Note that since $\Phi_i^{(k)}$ does not depend on y_i, $\partial \Phi_1^{(k)}/\partial y_2$ does not depend on y_1 and $\partial \Phi_2^{(k)}/\partial y_1$ does not depend on y_2. Due to the relation we found in the previous equation, we conclude that there exists a function $\xi^{(k)}(y_3) \in L^2((0,l))$ depending only on y_3 such that

$$\frac{\partial \Phi_1^{(k)}}{\partial y_2} = -\frac{\partial \Phi_2^{(k)}}{\partial y_1} = -\xi^{(k)}(y_3).$$

For further details see Section 8 Appendix Proposition 8.1. Hence, there exist functions $\eta_1^{(k)}(y_3)$, $\eta_2^{(k)}(z_3) \in H^1((0,l))$ that depend only on y_3 such that

$$\Phi_1^{(k)}(y) = -\xi^{(k)}(y_3)y_2 + \eta_1^{(k)}(y_3), \quad \Phi_2^{(k)}(y) = \xi^{(k)}(y_3)y_1 + \eta_2^{(k)}(y_3) \quad (i = 1, 2).$$

Applying the boundary conditions, we see $\xi^{(k)}(0) = 0$. Moreover, due to $E_{i3}(\Phi^{(k)}) = 0$,

$$\frac{\partial \Phi_3^{(k)}}{\partial y_1} = -\frac{\partial \Phi_1^{(k)}}{\partial y_3} = y_2\frac{\mathrm{d}\xi^{(k)}}{\mathrm{d}y_3} - \frac{\mathrm{d}\eta_1^{(k)}}{\mathrm{d}y_3}, \qquad \frac{\partial \Phi_3^{(k)}}{\partial y_2} = -\frac{\partial \Phi_2^{(k)}}{\partial y_3} = -y_1\frac{\mathrm{d}\xi^{(k)}}{\mathrm{d}y_3} - \frac{\mathrm{d}\eta_2^{(k)}}{\mathrm{d}y_3}.$$

Differentiating the first equation with respect to y_2 and the second equation with respect to y_1 and comparing the two results, we see that $\mathrm{d}\xi^{(k)}/\mathrm{d}y_3 = 0$, and therefore, $\xi^{(k)}$ is a constant. However, by the boundary condition we know that $\xi^{(k)}(0) = 0$, thus we see that in fact $\xi^{(k)} \equiv 0$. Hence,

$$\frac{\partial \Phi_3^{(k)}}{\partial y_1} = -\frac{\mathrm{d}\eta_1^{(k)}}{\mathrm{d}y_3}, \qquad \frac{\partial \Phi_3^{(k)}}{\partial y_2} = -\frac{\mathrm{d}\eta_2^{(k)}}{\mathrm{d}y_3}.$$

Since $(\partial/\partial y_2)(-\mathrm{d}\eta_1^{(k)}/\mathrm{d}y_3) = (\partial/\partial y_1)(-\mathrm{d}\eta_2^{(k)}/\mathrm{d}y_3) = 0$ we can solve for $\Phi_3^{(k)}$, and we get the solution

$$\begin{aligned} \Phi_1^{(k)}(y) &= \eta_1^{(k)}(y_3), \\ \Phi_2^{(k)}(y) &= \eta_2^{(k)}(y_3), \\ \Phi_3^{(k)}(y) &= \eta_3^{(k)}(y_3) - y_1\frac{\mathrm{d}\eta_1^{(k)}}{\mathrm{d}y_3} - y_2\frac{\mathrm{d}\eta_2^{(k)}}{\mathrm{d}y_3}. \end{aligned} \tag{42}$$

Now we are able to compute

$$\kappa_{33} = E_{33}(\Phi^{(k)}) = \frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}y_3} - y_1\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}y_3^2} - y_2\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}y_3^2}. \tag{43}$$

For commodity, let us put $Y = \lambda_2(3\lambda_1 + 2\lambda_2)/(\lambda_1 + \lambda_2)$. We substitute (39) and (43) into (41), so it becomes

$$\begin{aligned}\int_{F(S)} Y \left(\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}y_3} - y_1\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}y_3^2} - y_2\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}y_3^2}\right)\left(\frac{\mathrm{d}\rho_3}{\mathrm{d}y_3} - y_1\frac{\mathrm{d}^2\rho_1}{\mathrm{d}y_3^2} - y_2\frac{\mathrm{d}^2\rho_2}{\mathrm{d}y_3^2}\right)\mathrm{d}y \\ = \widetilde{\Lambda}_k \int_{F(S)} \left(\eta_1^{(k)}\rho_1 + \eta_2^{(k)}\rho_2\right)\mathrm{d}y. \qquad (44)\end{aligned}$$

Let us now analyze the integrals of (44). For $1 \le i, j \le 2$ let us define the following functions.

$$\begin{aligned} H = H(y_3) = \int_{\widehat{\Omega}(y_3)} 1\mathrm{d}y', \qquad K_i = K_i(y_3) = \int_{\widehat{\Omega}(y_3)} y_i\mathrm{d}y', \\ A_{ij} = A_{ij}(y_3) = \int_{\widehat{\Omega}(y_3)} y_iy_j\mathrm{d}y' \quad (y_3 \in [0,l])\,. \end{aligned} \qquad (45)$$

With this notation and using integration by parts accordingly, we have

$$\begin{aligned}
&\int_{F(S)} \frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}y_3}\frac{\mathrm{d}\rho_3}{\mathrm{d}y_3}\mathrm{d}y = \int_0^l H\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\frac{\mathrm{d}\rho_3}{\mathrm{d}z_3}\mathrm{d}z_3 = -\int_0^l \frac{\mathrm{d}}{\mathrm{d}z_3}\left(H\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)\rho_3\mathrm{d}z_3,\\
&\int_{F(S)} y_i\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}y_3}\frac{\mathrm{d}^2\rho_i}{\mathrm{d}y_3^2}\mathrm{d}y = \int_0^l K_i\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\frac{\mathrm{d}^2\rho_i}{\mathrm{d}z_3^2}\mathrm{d}z_3 = \int_0^l \frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(K_i\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)\rho_i\mathrm{d}z_3,\\
&\int_{F(S)} y_i\frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}y_3^2}\frac{\mathrm{d}\rho_3}{\mathrm{d}y_3}\mathrm{d}y = \int_0^l K_i\frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}\frac{\mathrm{d}\rho_3}{\mathrm{d}z_3}\mathrm{d}z_3 = -\int_0^l \frac{\mathrm{d}}{\mathrm{d}z_3}\left(K_i\frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}\right)\rho_3\mathrm{d}z_3,\\
&\int_{F(S)} y_iy_j\frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}y_3^2}\frac{\mathrm{d}^2\rho_j}{\mathrm{d}y_3^2}\mathrm{d}y = \int_0^l A_{ij}\frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}\frac{\mathrm{d}^2\rho_j}{\mathrm{d}z_3^2}\mathrm{d}z_3 = \int_0^l \frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{ij}\frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}\right)\rho_j\mathrm{d}z_3,\\
&\widetilde{\Lambda}_k\int_{F(S)}\left(\eta_1^{(k)}\rho_1 + \eta_2^{(k)}\rho_2\right)\mathrm{d}y = \widetilde{\Lambda}_k\int_0^l H\left(\eta_1^{(k)}\rho_1 + \eta_2^{(k)}\rho_2\right)\mathrm{d}z_3.
\end{aligned}$$

Plugging this into (44) and rearranging it we obtain

$$\begin{aligned}
Y\int_0^l \Bigg\{&\frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{11}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + A_{12}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} - K_1\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)\rho_1\\
&+\frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{12}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + A_{22}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} - K_2\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)\rho_2\\
&+\frac{\mathrm{d}}{\mathrm{d}z_3}\left(K_1\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2} + K_2\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2} - H\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)\rho_3\Bigg\}\mathrm{d}z_3 = \widetilde{\Lambda}_k\int_0^l H\left(\eta_1^{(k)}\rho_1 + \eta_2^{(k)}\rho_2\right)\mathrm{d}z_3.
\end{aligned} \qquad (46)$$

Choosing $\rho_1, \rho_2 = 0$, we see that

$$Y\int_0^l \frac{\mathrm{d}}{\mathrm{d}z_3}\left(K_1\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2}+K_2\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2}-H\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)\rho_3\mathrm{d}z_3=0. \tag{47}$$

Note now that (47) holds for all $\rho_3\in H_0^1((0,l))$, so we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}z_3}\left(K_1\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2}+K_2\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2}-H\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)=0,$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}z_3}\left(H\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)=\frac{\mathrm{d}}{\mathrm{d}z_3}\left(K_1\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2}+K_2\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2}\right). \tag{48}$$

Plugging (47) into (46), we get

$$\begin{aligned}Y\int_0^l\Bigg\{&\frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{11}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2}+A_{12}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2}-K_1\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)\rho_1\\&+\frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{12}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2}+A_{22}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2}-K_2\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)\rho_2\Bigg\}\mathrm{d}z_3=\widetilde{\Lambda}_k\int_0^l H\left(\eta_1^{(k)}\rho_1+\eta_2^{(k)}\rho_2\right)\mathrm{d}z_3.\end{aligned} \tag{49}$$

Now taking $\rho_2=0$ in (49), we see

$$Y\int_0^l \frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{11}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2}+A_{12}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2}-K_1\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)\rho_1\mathrm{d}z_3=\widetilde{\Lambda}_k\int_0^l H\eta_1^{(k)}\rho_1\mathrm{d}z_3.$$

Since ρ_1 is arbitrary, we conclude that

$$Y\frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{11}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2}+A_{12}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2}-K_1\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)=\widetilde{\Lambda}_k H\eta_1^{(k)}. \tag{50}$$

Similarly, with the same argument but taking $\rho_1=0$, we get

$$Y\frac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(A_{21}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2}+A_{22}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2}-K_2\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)=\widetilde{\Lambda}_k H\eta_2^{(k)}. \tag{51}$$

Combining the equations (48), (50) and (51) we obtain the system of differential equations

$$\begin{cases} Y\dfrac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(\begin{pmatrix} A_{11} & A_{12} & -K_1 \\ A_{21} & A_{22} & -K_2 \end{pmatrix}\begin{pmatrix} \dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}z_3^2} \\ \dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}z_3^2} \\ \dfrac{\mathrm{d}\eta_3}{\mathrm{d}z_3} \end{pmatrix}\right) = \widetilde{\Lambda}\, H \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} & (0 < z_3 < l), \\ \dfrac{\mathrm{d}}{\mathrm{d}z_3}\left(H\dfrac{\mathrm{d}\eta_3}{\mathrm{d}z_3}\right) = \dfrac{\mathrm{d}}{\mathrm{d}z_3}\left(K_1\dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}z_3^2} + K_2\dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}z_3^2}\right) & (0 < z_3 < l). \end{cases} \tag{52}$$

We now discuss the boundary conditions of the functions $\eta_i^{(k)}$ for $i = 1, 2, 3$ for the (DD) case, that is, the case with both ends clamped. Then, we know that $\Phi^{(k)}(y_1, y_2, 0) = 0$ and $\Phi^{(k)}(y_1, y_2, l) = 0$. From (42) we can deduce that

$$\begin{cases} \eta_3^{(k)}(0) = \eta_i^{(k)}(0) = \dfrac{\mathrm{d}\eta_i^{(k)}}{\mathrm{d}z_3}(0) = 0 \\ \eta_3^{(k)}(l) = \eta_i^{(k)}(l) = \dfrac{\mathrm{d}\eta_i^{(k)}}{\mathrm{d}z_3}(l) = 0 \end{cases} \quad (i = 1, 2). \tag{dd}$$

Let $\{\Lambda_{k^*}\}_{k^*=1}^{+\infty}$ be the set of eigenvalues of problem (52) with (dd) boundary conditions. Then, we have proved that $\widetilde{\Lambda}_k \in \{\Lambda_{k^*}\}_{k^*=1}^{+\infty}$, and more generally $\{\widetilde{\Lambda}_k\}_{k=1}^{+\infty} \subseteq \{\Lambda_{k^*}\}_{k^*=1}^{+\infty}$. Thus, we can assure that

$$\widetilde{\Lambda}_k \geq \Lambda_k \qquad (k \geq 1). \tag{53}$$

It still remains to prove that $\widetilde{\Lambda}_k \leq \Lambda_k$ for $k \geq 1$ (cf. Section 7).

6.2. (DN) case.

We will cover now the case of $\mu_k^{DN}(\varepsilon)$. The proof is pretty similar to the case of $\mu_k^{DD}(\varepsilon)$ with some minor changes, specially on the boundary.

The function space $\mathcal{W}_1$ changes to

$$\mathcal{W}_1' = \{\phi \in H^1(F(S), \mathbb{R}^3) \mid \phi = \mathbf{0} \text{ on } \Gamma_{1,1}^{(-)}\},$$

and the test functions chosen during the proof, now only vanish on $\Gamma_{1,1}^{(-)}$. In particular, $\rho_i(0) = 0$ for $i = 1, 2, 3$ and $\mathrm{d}\rho_i/\mathrm{d}z_3(0) = 0$ for $i = 1, 2$. Let us now discuss the boundary conditions of the functions $\eta_i^{(k)}$ for $i = 1, 2, 3$. With the same argument as before, on the clamped end, we easily see that $\eta_i^{(k)}(0) = 0$ for $i = 1, 2, 3$ and $\mathrm{d}\eta_i^{(k)}/\mathrm{d}z_3(0) = 0$ for $i = 1, 2$. We go back to (44) and put $\rho_2 = 0$ and $\rho_3 = 0$, to obtain

$$Y\int_{F(S)}\left(-\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}y_3} + y_1\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}y_3^2} + y_2\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}y_3^2}\right) y_1\frac{\mathrm{d}^2\rho_1}{\mathrm{d}y_3^2}\mathrm{d}y = \widetilde{\Lambda}_k\int_{F(S)}\eta_1^{(k)}\rho_1\mathrm{d}y.$$

Using the definition (45) of the functions H, K_i and A_{ij} for $1 \leq i, j \leq 2$, we transform the previous equation into

$$Y\int_0^l\left(-K_1\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}+A_{11}\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2}+A_{12}\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2}\right)\frac{\mathrm{d}^2\rho_1}{\mathrm{d}z_3^2}\mathrm{d}z_3=\widetilde{\Lambda}_k\int_0^l H\eta_1^{(k)}\rho_1\mathrm{d}z_3. \tag{54}$$

To simplify notation we write

$$\begin{aligned}
P_i(z_3)&=-K_i(z_3)\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}+A_{i1}(z_3)\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2}+A_{i2}(z_3)\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2}\qquad (i=1,2),\\
P_3(z_3)&=H(z_3)\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}-K_1(z_3)\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}z_3^2}-K_2(z_3)\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}z_3^2}.
\end{aligned}$$

We use integration by parts two times in (54) to obtain

$$Y\left(\left[P_1(z_3)\frac{\mathrm{d}\rho_1}{\mathrm{d}z_3}\right]_0^l-\left[\frac{\mathrm{d}P_1}{\mathrm{d}z_3}\rho_1(z_3)\right]_0^l+\int_0^l\frac{\mathrm{d}^2P_1}{\mathrm{d}z_3^2}\rho_1\mathrm{d}z_3\right)=\widetilde{\Lambda}_k\int_0^l H\eta_1^{(k)}\rho_1\mathrm{d}z_3.$$

Using (50), we see that the previous equation becomes

$$Y\left(\left[P_1(z_3)\frac{\mathrm{d}\rho_1}{\mathrm{d}z_3}\right]_0^l-\left[\frac{\mathrm{d}P_1}{\mathrm{d}z_3}\rho_1(z_3)\right]_0^l\right)=0.$$

Note that in the (DD) case, we can see that all terms above vanish. However, in the (DN) case we have that $\rho_1(0)=0$ and $\mathrm{d}\rho_1/\mathrm{d}z_3(0)=0$. Therefore

$$P_1(l)\frac{\mathrm{d}\rho_1}{\mathrm{d}z_3}(l)-\frac{\mathrm{d}P_1}{\mathrm{d}z_3}(l)\rho_1(l)=0.$$

Using proper test functions ρ_1, we conclude $P_1(l)=0$ and $\mathrm{d}P_1/\mathrm{d}z_3(l)=0$. In a similar fashion, choosing $\rho_1=0$ and $\rho_3=0$, we deduce $P_2(l)=0$ and $\mathrm{d}P_2/\mathrm{d}z_3(l)=0$. Finally, taking $\rho_1=0$ and $\rho_2=0$, we get $P_3(l)=0$. Moreover, from (48), we also get $\mathrm{d}P_3/\mathrm{d}z_3(l)=0$. Thus, we have seen that $P_i(l)=0$ and $\mathrm{d}P_i/\mathrm{d}z_3(l)=0$ for $i=1,2,3$ and therefore solving the systems we obtain

$$\frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}(l)=\frac{\mathrm{d}^3\eta_i^{(k)}}{\mathrm{d}z_3^3}(l)=0\quad (i=1,2),\qquad \frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}(l)=\frac{\mathrm{d}^2\eta_3^{(k)}}{\mathrm{d}z_3^2}(l)=0. \tag{55}$$

To sum up, we have the boundary conditions

$$\begin{cases}\eta_3^{(k)}(0)=\eta_i^{(k)}(0)=\dfrac{\mathrm{d}\eta_i^{(k)}}{\mathrm{d}z_3}(0)=0\\[2mm] \dfrac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}(l)=\dfrac{\mathrm{d}^2\eta_3^{(k)}}{\mathrm{d}z_3^2}(l)=\dfrac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}(l)=\dfrac{\mathrm{d}^3\eta_i^{(k)}}{\mathrm{d}z_3^3}(l)=0\end{cases}\qquad (i=1,2). \tag{dn}$$

REMARK 6.3. It can be shown that the condition $\mathrm{d}\eta_3^{(k)}/\mathrm{d}z_3^2=0$ is not independent and can be deduced from the other conditions and equations. Thus we can drop it when stating the result.

7. Upper bound for the limit eigenvalues.

We now start to prove that $\widetilde{\Lambda}_k \le \Lambda_k$. Consider the system of ordinary differential equations

$$\begin{cases} Y \dfrac{\mathrm{d}^2}{\mathrm{d}z_3^2}\left(\begin{pmatrix} A_{11} & A_{12} & -K_1 \\ A_{21} & A_{22} & -K_2 \end{pmatrix}\begin{pmatrix} \dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}z_3^2} \\ \dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}z_3^2} \\ \dfrac{\mathrm{d}\eta_3}{\mathrm{d}z_3} \end{pmatrix}\right) = \Lambda\, H \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} & (0 < z_3 < l), \\ \dfrac{\mathrm{d}}{\mathrm{d}z_3}\left(H \dfrac{\mathrm{d}\eta_3}{\mathrm{d}z_3}\right) = \dfrac{\mathrm{d}}{\mathrm{d}z_3}\left(K_1 \dfrac{\mathrm{d}^2\eta_1}{\mathrm{d}z_3^2} + K_2 \dfrac{\mathrm{d}^2\eta_2}{\mathrm{d}z_3^2}\right) & (0 < z_3 < l). \end{cases} \tag{56}$$

where $Y = \lambda_2(3\lambda_1 + 2\lambda_2)/(\lambda_1 + \lambda_2)$. In a very similar fashion as before, we first consider the (DD) case, so we assume the functions satisfy the (dd) boundary condition.

Let Λ_k be the k-th eigenvalue of the problem (56) with (dd) boundary condition and $\eta^{(k)} = (\eta_1^{(k)}, \eta_2^{(k)}, \eta_3^{(k)})$ its associated eigenfunction. By the relation we have in (56), $\eta_3^{(k)}$ satisfies $(\mathrm{d}/\mathrm{d}z_3)(H(\mathrm{d}\eta_3^{(k)}/\mathrm{d}z_3)) = (\mathrm{d}/\mathrm{d}z_3)(K_1(\mathrm{d}^2\eta_1^{(k)}/\mathrm{d}z_3^2) + K_2(\mathrm{d}^2\eta_2^{(k)}/\mathrm{d}^2 z_3^2))$.

We recall that $\widetilde{\Lambda}_k = \lim_{r\to+\infty}(1/\zeta_r^2)\mu_k(\zeta_r)$ (see (22)) and the eigenvalue $\mu_k(\varepsilon)$ can be characterized by the Rayleigh's quotient via

$$\mu_k(\varepsilon) = \sup_{Z\in\mathcal{H}_{k-1}(F(S),\mathbb{R}^3)} \inf\{\widetilde{\mathcal{R}}_\varepsilon(\Phi) \mid \Phi \in \mathcal{W}_1\backslash\{\mathbf{0}\}, \Phi \in Z^{\perp_\varepsilon}\}$$

(see (11)). We want to show that $\widetilde{\Lambda}_k \le \Lambda_k$.

We multiply the system (56) by (η_1, η_2) and integrate over the interval $(0, l)$. Applying the integration by parts we obtain

$$Y \int_0^l \left(\sum_{i,j=1}^2 A_{ij} \frac{\mathrm{d}^2\eta_i}{\mathrm{d}z_3^2}\frac{\mathrm{d}^2\eta_j}{\mathrm{d}z_3^2} - \sum_{i=1}^2 K_i \frac{\mathrm{d}^2\eta_i}{\mathrm{d}z_3^2}\frac{\mathrm{d}\eta_3}{\mathrm{d}z_3}\right)\mathrm{d}z_3 = \Lambda \int_0^l H\left(\eta_1^2 + \eta_2^2\right)\mathrm{d}z_3.$$

Using the relationship between η_3 and (η_1, η_2) we have in (52), we deduce that

$$Y\int_0^l \left(\sum_{i,j=1}^2 A_{ij} \frac{\mathrm{d}^2\eta_i}{\mathrm{d}z_3^2}\frac{\mathrm{d}^2\eta_j}{\mathrm{d}z_3^2} - 2\sum_{i=1}^2 K_i \frac{\mathrm{d}^2\eta_i}{\mathrm{d}z_3^2}\frac{\mathrm{d}\eta_3}{\mathrm{d}z_3} + H\left(\frac{\mathrm{d}\eta_3}{\mathrm{d}z_3}\right)^2\right)\mathrm{d}z_3 = \Lambda\int_0^l H\left(\eta_1^2 + \eta_2^2\right)\mathrm{d}z_3.$$

Therefore, if $\eta^{(k)} = (\eta_1^{(k)}, \eta_2^{(k)})$ is the k-th eigenfunction of the ordinary differential equation (56), we have that

$$\Lambda_k = \frac{Y\displaystyle\int_0^l \left(\sum_{i,j=1}^2 A_{ij} \frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}\frac{\mathrm{d}^2\eta_j^{(k)}}{\mathrm{d}z_3^2} - 2\sum_{i=1}^2 K_i \frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3} + H\left(\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)^2\right)\mathrm{d}z_3}{\displaystyle\int_0^l H\left(\left(\eta_1^{(k)}\right)^2 + \left(\eta_2^{(k)}\right)^2\right)\mathrm{d}z_3}. \tag{57}$$

Recall now the Rayleigh's quotient $\widetilde{\mathcal{R}}_\varepsilon$ introduced in (10). We now try new test functions $\Theta(y) = \Theta = (\Theta_1, \Theta_2, \Theta_3)$, $\phi(y) = \phi = (\phi_1, \phi_2, \phi_3)$ given by

$$\begin{aligned} \Theta_i &= \eta_i + \varepsilon^2 \phi_i \qquad (i = 1, 2), \\ \Theta_3 &= \eta_3 - y_1 \frac{\mathrm{d}\eta_1}{\mathrm{d}y_3} - y_2 \frac{\mathrm{d}\eta_2}{\mathrm{d}y_3} + \varepsilon \phi_3, \end{aligned}$$

where the functions η_i for $i = 1, 2, 3$ depend only on y_3. The choice of these test functions comes from the fact that we want $E_{ij}(\Theta)$ to satisfy (21). Indeed, since for $1 \le i, j \le 2$ we have $E_{ij}(\eta) = 0$ and $E_{i3}(\eta) = 0$, we calculate

$$\begin{aligned} E_{ij}(\Theta) &= \varepsilon^2 E_{ij}(\phi), \\ E_{i3}(\Theta) &= \frac{1}{2}\left(\varepsilon^2 \frac{\partial \phi_i}{\partial y_3} + \varepsilon \frac{\partial \phi_3}{\partial y_i}\right) \qquad (1 \le i, j \le 2), \\ E_{33}(\Theta) &= \frac{\mathrm{d}\eta_3}{\mathrm{d}y_3} - y_1 \frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2} - y_2 \frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2} + \varepsilon \frac{\partial \phi_3}{\partial y_3}. \end{aligned}$$

For brevity we write $N = \mathrm{d}\eta_3/\mathrm{d}y_3 - y_1(\mathrm{d}^2\eta_1/\mathrm{d}y_3^2) - y_2(\mathrm{d}^2\eta_2/\mathrm{d}y_3^2)$. Knowing this, we compute $\widetilde{\mathcal{R}}_\varepsilon(\Theta)$.

$$\widetilde{\mathcal{R}}_\varepsilon(\Theta) = \frac{\displaystyle\int_{F(S)} \left(\lambda_1 \left(\varepsilon^2 \frac{\partial \phi_1}{\partial y_1} + \varepsilon^2 \frac{\partial \phi_2}{\partial y_2} + \varepsilon^2 N + \varepsilon^3 \frac{\partial \phi_3}{\partial y_3}\right)^2 + 2\lambda_2 \sum_{i,j=1}^{2} \varepsilon^4 E_{ij}(\phi)^2\right) \mathrm{d}y}{\displaystyle\int_{F(S)} \left(\varepsilon^2(\eta_1 + \varepsilon^2\phi_1)^2 + \varepsilon^2(\eta_2 + \varepsilon^2\phi_2)^2 + \varepsilon^4 \left(\eta_3 - y_1 \frac{\mathrm{d}\eta_1}{\mathrm{d}y_3} - y_2 \frac{\mathrm{d}\eta_2}{\mathrm{d}y_3} + \varepsilon\phi_3\right)^2\right) \mathrm{d}y}$$
$$+ \frac{\displaystyle\int_{F(S)} 2\lambda_2 \left(2\varepsilon^2 \sum_{i=1}^{2} \frac{1}{4}\left(\varepsilon^2 \frac{\partial \phi_i}{\partial y_3} + \varepsilon \frac{\partial \phi_3}{\partial y_i}\right)^2 + \varepsilon^4 \left(N + \varepsilon \frac{\partial \phi_3}{\partial y_3}\right)^2\right) \mathrm{d}y}{\displaystyle\int_{F(S)} \left(\varepsilon^2(\eta_1 + \varepsilon^2\phi_1)^2 + \varepsilon^2(\eta_2 + \varepsilon^2\phi_2)^2 + \varepsilon^4 \left(\eta_3 - y_1 \frac{\mathrm{d}\eta_1}{\mathrm{d}y_3} - y_2 \frac{\mathrm{d}\eta_2}{\mathrm{d}y_3} + \varepsilon\phi_3\right)^2\right) \mathrm{d}y}.$$

Multiplying by $1/\varepsilon^2$ and taking the limit $\varepsilon \to 0$, we see

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \widetilde{\mathcal{R}}_\varepsilon(\Theta) = \frac{\displaystyle\int_{F(S)} \lambda_1 \left(\frac{\partial \phi_1}{\partial y_1} + \frac{\partial \phi_2}{\partial y_2} + N\right)^2 \mathrm{d}y}{\displaystyle\int_{F(S)} (\eta_1^2 + \eta_2^2) \mathrm{d}y} + \frac{\displaystyle\int_{F(S)} 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(\phi)^2 + \frac{1}{2}\sum_{i=1}^{2} \left(\frac{\partial \phi_3}{\partial y_i}\right)^2 + N^2\right) \mathrm{d}y}{\displaystyle\int_{F(S)} (\eta_1^2 + \eta_2^2) \mathrm{d}y}. \tag{58}$$

We want to find the $\phi = (\phi_1, \phi_2, \phi_3)$ that minimizes the numerator in (58)

$$\mathcal{M}(\phi)=\int_{F(S)}\left(\lambda_1\left(\frac{\partial\phi_1}{\partial y_1}+\frac{\partial\phi_2}{\partial y_2}+N\right)^2+2\lambda_2\left(\sum_{i,j=1}^{2}E_{ij}(\phi)^2+\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\phi_3}{\partial y_i}\right)^2+N^2\right)\right)\mathrm{d}y.$$

In order to minimize $\mathcal{M}$, we put the test function ϕ as follows.

$$\phi_i(y)=\sum_{p,q=1}^{2}\alpha_{pq}^{(i)}y_py_q+\sum_{p=1}^{2}\beta_p^{(i)}y_p \qquad (i=1,2), \tag{59}$$

$$\phi_3(y)=0 \tag{60}$$

where $\alpha_{pq}^{(i)}$ and $\beta_p^{(i)}$ depend only on y_3 for $1\le p,q,i\le 2$ and satisfy $\alpha_{12}^{(i)}=\alpha_{21}^{(i)}$ for $i=1,2$.

If we substitute this test function into $\mathcal{M}$, we obtain an expression that can be written as a polynomial of degree 2 on the variables $\alpha_{pq}^{(i)}$ and $\beta_p^{(i)}$ for $1\le i,p,q\le 2$ (in total there are 10 variables). Thus, it can be further rewritten as $\int_0^l(\alpha^T\mathcal{X}\alpha+\mathcal{Y}\alpha)\mathrm{d}y_3$ for a certain matrix valued function $\mathcal{X}$ and a certain vector valued function $\mathcal{Y}$ (for the explicit forms of $\mathcal{X}$ and $\mathcal{Y}$ see Appendix Remark 8.2) with

$$\alpha=\left(\alpha_{11}^{(1)},\alpha_{12}^{(1)},\alpha_{22}^{(1)},\alpha_{11}^{(2)},\alpha_{12}^{(2)},\alpha_{22}^{(2)},\beta_1^{(1)},\beta_2^{(1)},\beta_1^{(2)},\beta_2^{(2)}\right)^T.$$

Since we want the minimum, we differentiate the expression $\int_0^l(\alpha^T\mathcal{X}\alpha+\mathcal{Y}\alpha)\mathrm{d}y_3$ with respect to α and solve the linear system $2\mathcal{X}\alpha+\mathcal{Y}=0$ for α. After long but simple calculations we obtain

$$\alpha_{11}^{(1)}=\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2},\quad \alpha_{12}^{(1)}=\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2},\ \alpha_{22}^{(1)}=-\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2},$$

$$\alpha_{11}^{(2)}=-\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2},\ \alpha_{12}^{(2)}=\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2},\ \alpha_{22}^{(2)}=\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2},$$

$$\beta_1^{(1)}=-\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}\eta_3}{\mathrm{d}y_3},\quad \beta_2^{(2)}=-\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mathrm{d}\eta_3}{\mathrm{d}y_3}.$$

In fact, the matrix $\mathcal{X}$ in the system is degenerate and we additionally obtain the condition $\beta_1^{(2)}+\beta_2^{(1)}=0$. It can also be checked that the minimum obtained is always the same, so to simplify, we put $\beta_1^{(2)}=0$ and $\beta_2^{(1)}=0$. Therefore, recalling (59), we obtain

$$\begin{aligned}
\phi_1(y)&=\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\left(\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2}y_1^2+2\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}y_1y_2-\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2}y_2^2-2\frac{\mathrm{d}\eta_3}{\mathrm{d}y_3}y_1\right),\\
\phi_2(y)&=\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\left(-\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}y_1^2+2\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2}y_1y_2+\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}y_2^2-2\frac{\mathrm{d}\eta_3}{\mathrm{d}y_3}y_2\right),\\
\phi_3(y)&=0.
\end{aligned} \tag{61}$$

Substituting (61) into (58) and after long but elementary computations, we obtain the minimum

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\widetilde{\mathcal{R}}_\varepsilon(\Theta)=\frac{\displaystyle\int_{F(S)}\frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2}N^2\mathrm{d}y}{\displaystyle\int_{F(S)}(\eta_1^2+\eta_2^2)\mathrm{d}y}. \tag{62}$$

Substituting $(\eta_1,\eta_2,\eta_3)=(\eta_1^{(k)},\eta_2^{(k)},\eta_3^{(k)})$ and the definition of N into (62) and integrating by parts, we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\widetilde{\mathcal{R}}_\varepsilon(\Theta)=\frac{Y\displaystyle\int_0^l\left(\sum_{i,j=1}^2 A_{ij}\frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}\frac{\mathrm{d}^2\eta_j^{(k)}}{\mathrm{d}z_3^2}-2\sum_{i=1}^2 K_i\frac{\mathrm{d}^2\eta_i^{(k)}}{\mathrm{d}z_3^2}\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}+H\left(\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}z_3}\right)^2\right)\mathrm{d}z_3}{\displaystyle\int_0^l H\left(\left(\eta_1^{(k)}\right)^2+\left(\eta_2^{(k)}\right)^2\right)\mathrm{d}z_3},$$

which, from (57), turns out to be

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\widetilde{\mathcal{R}}_\varepsilon(\Theta)=\Lambda_k. \tag{63}$$

Our next goal is to use the Max-Min method to prove the desired inequality $\widetilde{\Lambda}_k\le\Lambda_k$. First, we consider the eigenfunction $\eta^{(k)}=(\eta_1^{(k)},\eta_2^{(k)},\eta_3^{(k)})$ corresponding to the eigenvalue Λ_k of problem (56) with (dd) boundary condition. We also choose the functions $\eta^{(k)}$ so that

$$\int_{F(S)}\left(\eta_1^{(k)}\eta_1^{(k')}+\eta_2^{(k)}\eta_2^{(k')}\right)\mathrm{d}y=\delta(k,k'), \tag{64}$$

where δ is the Kronecker delta. We define

$$N_k=\frac{\mathrm{d}\eta_3^{(k)}}{\mathrm{d}y_3}-y_1\frac{\mathrm{d}^2\eta_1^{(k)}}{\mathrm{d}y_3^2}-y_2\frac{\mathrm{d}^2\eta_2^{(k)}}{\mathrm{d}y_3^2}. \tag{65}$$

Using the weak formulation of (56), we know that

$$Y\int_{F(S)}N_kN_{k'}\mathrm{d}y=\Lambda_k\delta(k,k'). \tag{66}$$

Let us consider the test functions

$$\Phi_i^{(s)}=\eta_i^{(s)}+\varepsilon^2\phi_i^{(s)}\qquad(i=1,2),$$
$$\Phi_3^{(s)}=\eta_3^{(s)}-y_1\frac{\mathrm{d}\eta_1^{(s)}}{\mathrm{d}y_3}-y_2\frac{\mathrm{d}\eta_2^{(s)}}{\mathrm{d}y_3},$$

with $s\in\mathbb{N}$ and

$$\phi_1^{(s)}=\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\left(\frac{\mathrm{d}^2\eta_1^{(s)}}{\mathrm{d}y_3^2}y_1^2+2\frac{\mathrm{d}^2\eta_2^{(s)}}{\mathrm{d}y_3^2}y_1y_2-\frac{\mathrm{d}^2\eta_1^{(s)}}{\mathrm{d}y_3^2}y_2^2-2\frac{\mathrm{d}\eta_3^{(s)}}{\mathrm{d}y_3}y_1\right), \tag{67}$$

$$\phi_2^{(s)} = \frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\left(-\frac{\mathrm{d}^2\eta_2^{(s)}}{\mathrm{d}y_3^2}y_1^2 + 2\frac{\mathrm{d}^2\eta_1^{(s)}}{\mathrm{d}y_3^2}y_1y_2 + \frac{\mathrm{d}^2\eta_2^{(s)}}{\mathrm{d}y_3^2}y_2^2 - 2\frac{\mathrm{d}\eta_3^{(s)}}{\mathrm{d}y_3}y_2\right). \tag{68}$$

Choose an arbitrary $Z \in \mathcal{H}_{k-1}(F(S),\mathbb{R}^3)$ and let $\widetilde{Z} = L.H.[\Phi^{(1)},\Phi^{(2)},\dots,\Phi^{(k)}]$ be the minimal linear space that contains the set $\{\Phi^{(1)},\Phi^{(2)},\dots,\Phi^{(k)}\}$. Note that $\dim\widetilde{Z} = k$ and that each $\Phi^{(s)} \in \mathcal{W}_1$ (for all $s \in \mathbb{N}$), so we have that $\widetilde{Z} \subseteq \mathcal{W}_1$. Since $\dim Z < \dim\widetilde{Z}$, we know that there exist a function $\Psi = (\Psi_1,\Psi_2,\Psi_3) \in \widetilde{Z} \cap Z^{\perp_\varepsilon}$ and a vector $(c_1,\dots,c_k) = (c_1(\varepsilon),\dots,c_k(\varepsilon)) \in \mathbb{R}^k\backslash\{\mathbf{0}\}$ such that

$$\Psi = \sum_{s=1}^{k} c_s(\varepsilon)\Phi^{(s)}.$$

Note that since both $\widetilde{Z}$ and $Z^{\perp_\varepsilon}$ are subsets of $\mathcal{W}_1$, we have also that $\Psi \in \mathcal{W}_1$ and due to the fact that $(c_1(\varepsilon),\dots,c_k(\varepsilon)) \in \mathbb{R}^k\backslash\{\mathbf{0}\}$ we deduce that $\Psi \in \mathcal{W}_1\backslash\{\mathbf{0}\}$, so we can apply $\widetilde{\mathcal{R}}_\varepsilon$ to Ψ. We compute

$$E_{ii}(\Psi) = -\varepsilon^2\sum_{s=1}^{k} c_s(\varepsilon)\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}N_s, \qquad E_{i3}(\Psi) = \varepsilon^2\sum_{s=1}^{k} c_s(\varepsilon)E_{i3}(\phi) \qquad (1 \le i \le 2),$$
$$E_{12}(\Psi) = E_{21}(\Psi) = 0, \qquad E_{33}(\Psi) = \sum_{s=1}^{k} c_s(\varepsilon)N_s.$$

Using these computations, the numerator of the Rayleigh quotient $\widetilde{\mathcal{R}}_\varepsilon(\Psi)$ is

$$\begin{aligned}
&\int_{F(S)}\left(\lambda_1\left(\varepsilon^2\left(\sum_{s=1}^{k} c_s(\varepsilon)\frac{\lambda_2}{\lambda_1+\lambda_2}N_s\right)\right)^2 + 2\lambda_2\left(\sum_{i=1}^{2}\varepsilon^4\left(\sum_{s=1}^{k} c_s(\varepsilon)\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}N_s\right)^2\right)\right)\mathrm{d}y\\
&\qquad + \int_{F(S)} 2\lambda_2\left(2\varepsilon^4\sum_{i=1}^{2}\frac{1}{4}\left(\varepsilon^2\sum_{s=1}^{k} c_s(\varepsilon)E_{i3}(\phi)\right)^2 + \varepsilon^4\left(\sum_{s=1}^{k} c_s(\varepsilon)N_s\right)^2\right)\mathrm{d}y\\
&= \varepsilon^4\sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\int_{F(S)} YN_pN_q\mathrm{d}y + \varepsilon^6\sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\widetilde{\kappa}(p,q,\varepsilon)
\end{aligned} \tag{69}$$

for some functions $\widetilde{\kappa}(p,q,\varepsilon) = O(1)$ as $\varepsilon \to 0$. Note that these functions $\widetilde{\kappa}(p,q,\varepsilon)$ do not depend on the choice of Z. Due to (66), it follows that (69) becomes

$$\begin{aligned}
&\varepsilon^4\sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\int_{F(S)} YN_pN_q\mathrm{d}y + \varepsilon^6\sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\widetilde{\kappa}(p,q,\varepsilon)\\
&\qquad\qquad = \varepsilon^4\sum_{p=1}^{k} c_p(\varepsilon)^2\Lambda_p + \varepsilon^6\sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\widetilde{\kappa}(p,q,\varepsilon).
\end{aligned} \tag{70}$$

Note also that the denominator of $\mathcal{R}_\varepsilon(\Psi)$ satisfies

$$
\begin{aligned}
&\varepsilon^2 \int_{F(S)} \left(\Psi_1^2+\Psi_2^2+\varepsilon^2\Psi_3^2\right) \mathrm{d}y \geq \varepsilon^2 \int_{F(S)} \left(\Psi_1^2+\Psi_2^2\right) \mathrm{d}y \\
&\quad = \varepsilon^2 \int_{F(S)} \left(\left(\sum_{s=1}^{k} c_k(\varepsilon)(\eta_1^{(s)}+\varepsilon^2\phi_1^{(s)}) \right)^2 + \left(\sum_{s=1}^{k} c_k(\varepsilon)(\eta_2^{(s)}+\varepsilon^2\phi_2^{(s)}) \right)^2 \right) \mathrm{d}y \\
&\quad = \varepsilon^2 \int_{F(S)} \sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon) \left(\sum_{n=1}^{2} (\eta_n^{(p)}+\varepsilon^2\phi_n^{(p)})(\eta_n^{(q)}+\varepsilon^2\phi_n^{(q)}) \right) \mathrm{d}y \\
&\quad = \varepsilon^2 \sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon) \int_{F(S)} (\eta_1^{(p)}\eta_1^{(q)}+\eta_2^{(p)}\eta_2^{(q)})\mathrm{d}y + \varepsilon^4 \sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\widehat{\kappa}(p,q,\varepsilon) \qquad (71)
\end{aligned}
$$

for certain functions $\widehat{\kappa}(p,q,\varepsilon)=O(1)$ as $\varepsilon\to 0$. Note again that the functions $\widehat{\kappa}(p,q,\varepsilon)$ do not depend on the choice of Z. By the homogeneity property of the Rayleigh's quotient we may assume without loss of generality that $\sum_{p=1}^{k} c_p(\varepsilon)^2=1$. Thus we have $|c_p(\varepsilon)|\leq 1$ for $1\leq p\leq k$. Combining this fact with the orthogonality in (64), we get

$$
\begin{aligned}
&\varepsilon^2 \sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon) \int_{F(S)} (\eta_1^{(p)}\eta_1^{(q)}+\eta_2^{(p)}\eta_2^{(q)})\mathrm{d}y + \varepsilon^4 \sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\widehat{\kappa}(p,q,\varepsilon) \\
&\quad = \varepsilon^2 \sum_{p=1}^{k} c_p(\varepsilon)^2 + \varepsilon^4 \sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\widehat{\kappa}(p,q,\varepsilon) = \varepsilon^2 + \varepsilon^4 \sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\widehat{\kappa}(p,q,\varepsilon) \\
&\quad \geq \varepsilon^2 - \varepsilon^4 \sum_{p,q=1}^{k} |c_p(\varepsilon)||c_q(\varepsilon)||\widehat{\kappa}(p,q,\varepsilon)| \geq \varepsilon^2 - \varepsilon^4 \sum_{p,q=1}^{k} |\widehat{\kappa}(p,q,\varepsilon)|. \qquad (72)
\end{aligned}
$$

Therefore, with (71) and (72), we deduce that

$$
\varepsilon^2 \int_{F(S)} \left(\Psi_1^2+\Psi_2^2+\varepsilon^2\Psi_3^2\right) \mathrm{d}y \geq \varepsilon^2 - \varepsilon^4 \sum_{p,q=1}^{k} |\widehat{\kappa}(p,q,\varepsilon)|. \qquad (73)
$$

Using (70) and the bound (73), we obtain

$$
\begin{aligned}
\frac{1}{\varepsilon^2}\widetilde{\mathcal{R}}_\varepsilon(\Psi) &\leq \frac{1}{\varepsilon^2} \frac{\varepsilon^4 \sum_{p=1}^{k} c_p(\varepsilon)^2\Lambda_p + \varepsilon^6 \sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\widetilde{\kappa}(p,q,\varepsilon)}{\varepsilon^2 - \varepsilon^4 \sum_{p,q=1}^{k} |\widehat{\kappa}(p,q,\varepsilon)|} \\
&\leq \frac{\Lambda_k \sum_{p=1}^{k} c_p(\varepsilon)^2 + \varepsilon^2 \sum_{p,q=1}^{k} c_p(\varepsilon)c_q(\varepsilon)\widetilde{\kappa}(p,q,\varepsilon)}{1 - \varepsilon^2 \sum_{p,q=1}^{k} |\widehat{\kappa}(p,q,\varepsilon)|} \leq \frac{\Lambda_k + \varepsilon^2 \sum_{p,q=1}^{k} |\widetilde{\kappa}(p,q,\varepsilon)|}{1 - \varepsilon^2 \sum_{p,q=1}^{k} |\widehat{\kappa}(p,q,\varepsilon)|} \qquad (74)
\end{aligned}
$$

provided that the denominator is positive (this is possible because ε is a small real parameter). Let us denote the right hand side of the previous inequality

$$\mathfrak{L}_k(\varepsilon) = \frac{\Lambda_k + \varepsilon^2 \sum_{p,q=1}^{k} |\widetilde{\kappa}(p,q,\varepsilon)|}{1 - \varepsilon^2 \sum_{p,q=1}^{k} |\widehat{\kappa}(p,q,\varepsilon)|}.$$

Note once again that $\mathfrak{L}_k(\varepsilon)$ does not depend on the choice of Z. We know from (74) that

$$\frac{1}{\varepsilon^2} \inf\{\widetilde{\mathcal{R}}_\varepsilon(\Phi) \mid \Phi \in \mathcal{W}_1 \backslash \{\mathbf{0}\}, \Phi \in Z^{\perp_\varepsilon}\} \leq \frac{1}{\varepsilon^2} \widetilde{\mathcal{R}}_\varepsilon(\Psi) \leq \mathfrak{L}_k(\varepsilon).$$

Since $Z \in \mathcal{H}_{k-1}(F(S), \mathbb{R}^3)$ was arbitrary, we take the supremum over $\mathcal{H}_{k-1}(F(S), \mathbb{R}^3)$, so we obtain the upper estimate

$$\frac{1}{\varepsilon^2} \mu_k(\varepsilon) \leq \mathfrak{L}_k(\varepsilon).$$

Taking the limit $\varepsilon \to 0$ and using (22), we have

$$\widetilde{\Lambda}_k \leq \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mu_k(\varepsilon) \leq \limsup_{\varepsilon \to 0} \mathfrak{L}_k(\varepsilon) = \Lambda_k,$$

which agrees to the desired inequality $\widetilde{\Lambda}_k \leq \Lambda_k$ $(k \geq 1)$. We combine this fact together with (53) to conclude that

$$\widetilde{\Lambda}_k = \Lambda_k \qquad (k \geq 1).$$

We only proved $\lim_{r \to +\infty} \mu_k(\zeta_r)/\zeta_r^2 = \widetilde{\Lambda}_k$ for a certain subsequence $\{\zeta_r\}_{r=1}^{+\infty} \subseteq \{\varepsilon_p\}_{p=1}^{+\infty}$, but note that we have shown that $\widetilde{\Lambda}_k = \Lambda_k$ independently of the first chosen sequence $\{\varepsilon_p\}_{p=1}^{+\infty}$. Since this sequence was arbitrary, we can see that in fact for every $k \geq 1$ we have

$$\lim_{\varepsilon \to 0} \frac{\mu_k(\varepsilon)}{\varepsilon^2} = \widetilde{\Lambda}_k.$$

Similarly, we prove the same result in the case (DN).

8. Appendix.

In this appendix we give the proofs of Lemma 6.1 and Lemma 6.2 and some additional facts which we used before in the proof of the main results.

Proof of Lemma 6.1. a) Let $\phi, \psi \in \mathcal{C}_0^{+\infty}(\mathbb{R})$ such that $\int_{\mathbb{R}} \psi(t) \mathrm{d}t = 1$ and $\int_{\mathbb{R}} \phi(t) \mathrm{d}t = 1$. For any $\Phi \in \mathcal{C}_0^{+\infty}(\mathbb{R}^3)$ with $\mathrm{supp}(\Phi) \subseteq F(S)$, we construct h_1 such that

$$\langle h_1, \Phi \rangle = \left(\alpha_2, \widehat{\Phi}\right)_{L^2(F(S))} - \left(\alpha_1, \int_{\mathbb{R}} \widehat{\widehat{\Phi}}(s, y_2, y_3) \mathrm{d}s \, \phi(y_1)\right)_{L^2(F(S))}$$

where

$$\widehat{\Phi}(y) = \int_{-\infty}^{y_1} \left(\Phi(t, y_2, y_3) - \left(\int_{\mathbb{R}} \Phi(s, y_2, y_3) \mathrm{d}s \right) \phi(t) \right) \mathrm{d}t,$$
$$\widehat{\widehat{\Phi}}(y) = \int_{-\infty}^{y_2} \left(\Phi(y_1, \tau, y_3) - \left(\int_{\mathbb{R}} \Phi(y_1, t, y_3) \mathrm{d}t \right) \psi(\tau) \right) \mathrm{d}\tau.$$

Note $\langle h_1, \cdot \rangle$ denotes the linear functional on $\mathcal{C}_0^{+\infty}(F(S))$. With these definitions, the following holds.

$$\widehat{\frac{\partial \Phi}{\partial y_1}} = \Phi(y), \quad \widehat{\widehat{\frac{\partial \Phi}{\partial y_1}}} = 0, \quad \widehat{\widehat{\frac{\partial \Phi}{\partial y_2}}} = \Phi(y).$$

Using these facts and combining it with property (31), we can see after some computations that

$$\left\langle h_1, \frac{\partial \Phi}{\partial y_1} \right\rangle = (\alpha_2, \Phi)_{L^2(F(S))} \quad \text{and} \quad \left\langle h_1, \frac{\partial \Phi}{\partial y_2} \right\rangle = -(\alpha_1, \Phi)_{L^2(F(S))}$$

which proves $\partial h_1 / \partial y_2 = \alpha_1$ and $\partial h_1 / \partial y_1 = -\alpha_2$ in the distribution sense. Moreover, it can also be shown that $|\langle h_1, \Phi \rangle| \leq C \|\Phi\|_{L^2(F(S))}$ for some constant $C > 0$. Using that $\mathcal{C}_0^{+\infty}(F(S))$ is dense in $L^2(F(S))$ and Riesz's Theorem we deduce that $h_1 \in L^2(F(S))$. Furthermore, since $\partial h_1 / \partial y_1$, $\partial h_1 / \partial y_2$ belong to $L^2(F(S))$, we can take values on the boundary and $h_1|_{\partial F(S)} \in L^2(\partial F(S))$. Similar arguments can be done for h_2. This proves item a) of the lemma.

b) We change variables according to (3) and work with z in S. Before beginning with the proof of this item we introduce some notation. Recall that B was an arbitrary connected bounded domain in $\mathbb{R}^2$ and that $s_2 = \partial B \times (0, l)$. Write $\partial B = b_1 \cup \cdots \cup b_m$ where b_i are its connected components. With this notation, for $i = 1, \ldots, m$ we define $\varsigma_i = b_i \times (0, l)$ so that $s_2 = \varsigma_1 \cup \cdots \cup \varsigma_m$. We parametrize the boundary ∂B by the arclength θ and, accordingly, each b_i by θ_i. Through this notes, $n = (n_1, n_2, n_3)$ will denote the unit outward normal vector on s_2.

Let $\widetilde{h}_1(z) = h_1(F(z))$ and let $\widetilde{\phi} = \widetilde{\phi}(z) \in \mathcal{C}^{+\infty}(\overline{S})$ be a smooth test function such that $\widetilde{\phi}(z_1, z_2, 0) = \widetilde{\phi}(z_1, z_2, l) = 0$, namely, $\widetilde{\phi}|_{s_1^{(+)} \cup s_1^{(-)}} = 0$. We compute

$$\begin{aligned}
\int_{s_2} \widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial \theta} \mathrm{d}A &= \int_{s_2} \widetilde{h}_1 \left(\frac{\partial \widetilde{\phi}}{\partial z_1} \frac{\partial z_1}{\partial \theta} + \frac{\partial \widetilde{\phi}}{\partial z_2} \frac{\partial z_2}{\partial \theta} \right) \mathrm{d}A = \int_{s_2} \widetilde{h}_1 \left(-n_2 \frac{\partial \widetilde{\phi}}{\partial z_1} + n_1 \frac{\partial \widetilde{\phi}}{\partial z_2} \right) \mathrm{d}A \\
&= \int_{s_2} \left(n_1 \widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial z_2} - n_2 \widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial z_1} \right) \mathrm{d}A \\
&= \int_S \left(\frac{\partial}{\partial z_1} \left(\widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial z_2} \right) - \frac{\partial}{\partial z_2} \left(\widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial z_1} \right) \right) \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 = \int_S \left(\frac{\partial \widetilde{h}_1}{\partial z_1} \frac{\partial \widetilde{\phi}}{\partial z_2} - \frac{\partial \widetilde{h}_1}{\partial z_2} \frac{\partial \widetilde{\phi}}{\partial z_1} \right) \mathrm{d}z.
\end{aligned}$$

With the change of variables $(y_1, y_2, y_3) = (F_1(z), F_2(z), z_3)$ and (31), with some computations it can be seen that

$$\int_S \left(\frac{\partial \widetilde{h}_1}{\partial z_1} \frac{\partial \widetilde{\phi}}{\partial z_2} - \frac{\partial \widetilde{h}_1}{\partial z_2} \frac{\partial \widetilde{\phi}}{\partial z_1} \right) \mathrm{d}z = - \int_{F(S)} \left(\alpha_2 \frac{\partial \phi}{\partial y_2} + \alpha_1 \frac{\partial \phi}{\partial y_1} \right) \mathrm{d}y$$

where $\phi \in \mathcal{C}^{+\infty}(\overline{F(S)})$. Due to (32), we conclude

$$\int_{s_2} \widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial \theta} \mathrm{d}A = \sum_{j=1}^{m} \int_{\varsigma_j} \widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial \theta_j} \mathrm{d}A = 0. \tag{75}$$

For any $i = 1, \ldots, m$, choose a test function $\widetilde{\phi}$ such that $\widetilde{\phi}|_{\varsigma_j} \equiv 0$ for $j \neq i$. Then (75) becomes

$$\sum_{j=1}^{m} \int_{\varsigma_j} \widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial \theta_j} \mathrm{d}A = \int_{\varsigma_i} \widetilde{h}_1 \frac{\partial \widetilde{\phi}}{\partial \theta_i} \mathrm{d}A = 0. \tag{76}$$

We will now show that $\widetilde{h}_1|_{\varsigma_i}$ does not depend on (z_1, z_2) over ς_i for $i = 1, \ldots, m$. Let $\phi = \phi(\theta, z_3) \in \mathcal{C}^{+\infty}(s_2)$ be a test function such that $\phi(\theta, 0) = \phi(\theta, l) = 0$. We define $\widehat{\phi}$ and χ such that for $i = 1, \ldots, m$

$$\widehat{\phi}|_{\varsigma_i} = \phi|_{\varsigma_i} - \int_{b_i} \phi(\widetilde{\theta}, z_3) \mathrm{d}\widetilde{\theta}, \qquad \chi|_{\varsigma_i} = \int_0^{\theta_i} \widehat{\phi}(\widehat{\theta}, z_3) \mathrm{d}\widehat{\theta}.$$

We compute

$$\begin{aligned} \int_{s_2} \widetilde{h}_1 \phi(\theta, z_3) \mathrm{d}A &= \sum_{j=1}^{m} \int_{\varsigma_j} \widetilde{h}_1 \phi(\theta_j, z_3) \mathrm{d}A \\ &= \sum_{j=1}^{m} \int_{\varsigma_j} \widetilde{h}_1 \left(\phi(\theta, z_3) - \int_{b_j} \phi(\widetilde{\theta}, z_3) \mathrm{d}\widetilde{\theta} + \int_{b_j} \phi(\widetilde{\theta}, z_3) \mathrm{d}\widetilde{\theta} \right) \mathrm{d}A \\ &= \sum_{j=1}^{m} \int_{\varsigma_j} \widetilde{h}_1 \left(\frac{\partial \chi}{\partial \theta_j}(\theta_j, z_3) + \int_{b_j} \phi(\widetilde{\theta}_j, z_3) \mathrm{d}\widetilde{\theta} \right) \mathrm{d}A. \end{aligned} \tag{77}$$

From (75), we can easily see that for any $j = 1, \ldots, m$

$$\int_{\varsigma_j} \widetilde{h}_1 \frac{\partial \chi}{\partial \theta_j}(\theta_j, z_3) \mathrm{d}A = 0.$$

Therefore, we continue the computations in (77) and we obtain

$$\begin{aligned} \sum_{j=1}^{m} \int_{\varsigma_j} \widetilde{h}_1 \phi(\theta_j, z_3) \mathrm{d}A &= \sum_{j=1}^{m} \int_{\varsigma_j} \widetilde{h}_1(\theta_j, z_3) \left(\int_{b_j} \phi(\widetilde{\theta}_j, z_3) \mathrm{d}\widetilde{\theta} \right) \mathrm{d}\theta \mathrm{d}z_3 \\ &= \sum_{j=1}^{m} \int_{\varsigma_j} \phi(\widetilde{\theta}, z_3) \left(\int_{b_j} \widetilde{h}_1(\theta_j, z_3) \mathrm{d}\theta_j \right) \mathrm{d}\widetilde{\theta} \mathrm{d}z_3 \end{aligned}$$

$$= \sum_{j=1}^{m} \int_{\varsigma_j} \phi(\theta_j, z_3) \left(\int_{b_j} \widetilde{h}_1(\widetilde{\theta}, z_3) \mathrm{d}\widetilde{\theta} \right) \mathrm{d}\theta \mathrm{d}z_3,$$

where we used Fubini's Theorem and we renamed the variables θ_j and $\widetilde{\theta}$. Sending it all to the left-hand side, we see

$$\sum_{j=1}^{m} \int_{\varsigma_j} \left(\widetilde{h}_1(\theta_j, z_3) - \int_{b_j} \widetilde{h}_1(\widetilde{\theta}, z_3) \mathrm{d}\widetilde{\theta} \right) \phi(\theta_j, z_3) \mathrm{d}\theta_j \mathrm{d}z_3 = 0.$$

For any $i = 1, \dots, m$, we choose a test function ϕ such that $\phi|_{\varsigma_j} \equiv 0$ for $j \neq i$ so that the previous equation becomes

$$\int_{\varsigma_i} \left(\widetilde{h}_1(\theta_i, z_3) - \int_{b_i} \widetilde{h}_1(\widetilde{\theta}, z_3) \mathrm{d}\widetilde{\theta} \right) \phi(\theta_i, z_3) \mathrm{d}\theta_i \mathrm{d}z_3 = 0.$$

Since $\phi|_{\varsigma_i}$ is arbitrary, we conclude that

$$\widetilde{h}_1|_{\varsigma_i} = \int_{b_i} \widetilde{h}_1(\widetilde{\theta}, z_3) \mathrm{d}\widetilde{\theta},$$

hence $\widetilde{h}_1|_{\varsigma_i}$ does not depend on θ_i, that is, it does not depend on (z_1, z_2) along ς_i. Therefore, using the regularity of F, we conclude that $h_1|_{g_i}$ does not depend on (y_1, y_2) along g_i. All of the above calculations can be made similarly to prove that $h_2|_{g_i}$ does not depend on (y_1, y_2) along g_i. $\square$

Proof of Lemma 6.2. Let $n = (n_1, n_2)$ be the unit outward normal vector on $\partial\widehat{\Omega}(y_3)$ and write $\partial\widehat{\Omega}(y_3) = \widehat{g}_1(y_3) \cup \cdots \cup \widehat{g}_m(y_3)$, where $\widehat{g}_j(y_3)$ are the connected components of $\partial\widehat{\Omega}(y_3)$ ($j = 1, \dots, m$). We use the divergence theorem for the 2-dimensional bounded domain enclosed by $\widehat{g}_j(y_3)$ to see that for every $y_3 \in [0, l]$ and $j = 1, \dots, m$ we have

$$\int_{\widehat{g}_j(y_3)} n_i \mathrm{d}L = 0, \qquad (i = 1, 2) \tag{78}$$

$$\int_{\widehat{g}_j(y_3)} y_2 n_1 \mathrm{d}L = 0, \qquad \int_{\widehat{g}_j(y_3)} y_1 n_2 \mathrm{d}L = 0, \tag{79}$$

$$\int_{\widehat{g}_j(y_3)} (y_2 n_2 - y_1 n_1) \mathrm{d}L = 0. \tag{80}$$

Throughout the next computations, we will use the fact that for $j = 1, \dots, m$ we have that $h_1|_{\widehat{g}_i(y_3)}$, $h_2|_{\widehat{g}_i(y_3)}$ do not depend on $y' = (y_1, y_2)$ along $\widehat{g}_i(y_3)$ (see Lemma 6.1-b)), so we can write $h_p|_{\widehat{g}_j(y_3)} = h_p|_{\widehat{g}_j(y_3)}(y_3)$ for $p = 1, 2$. Using the divergence theorem, we first calculate

$$\int_{\widehat{\Omega}(y_3)} Q \mathrm{d}y' = \int_{\widehat{\Omega}(y_3)} \left(\frac{\partial h_1}{\partial y_2} - \frac{\partial h_2}{\partial y_1} \right) \mathrm{d}y' = \int_{\partial\widehat{\Omega}(y_3)} (h_1 n_2 - h_2 n_1) \mathrm{d}L$$

$$= \sum_{j=1}^{m} \int_{\widehat{g}_j(y_3)} (h_1 n_2 - h_2 n_1)\,\mathrm{d}L$$
$$= \sum_{j=1}^{m} \left(h_1|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} n_2 \mathrm{d}L - h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} n_1 \mathrm{d}L \right) = 0.$$

The last equality is due to (78). We have seen that

$$\int_{\widehat{\Omega}(y_3)} Q \mathrm{d}y' = 0. \tag{81}$$

We now proceed to prove that $\int_{\widehat{\Omega}(y_3)} Q y_i \mathrm{d}y' = 0$ for $i = 1, 2$. For that purpose, from (33) and (34), we see that

$$\int_{\widehat{\Omega}(y_3)} \left(\frac{\partial h_1}{\partial y_1} + \frac{\partial h_2}{\partial y_2} \right) y_1 \mathrm{d}y' = 0$$
$$\int_{\partial\widehat{\Omega}(y_3)} (y_1 h_2 n_2 + y_1 h_1 n_1)\,\mathrm{d}L - \int_{\widehat{\Omega}(y_3)} h_1 \mathrm{d}y' = 0$$
$$\sum_{j=1}^{m} \left(h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_2 \mathrm{d}L + h_1|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_1 \mathrm{d}L \right) - \int_{\widehat{\Omega}(y_3)} h_1 \mathrm{d}y' = 0$$
$$\sum_{j=1}^{m} \left(h_1|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_1 \mathrm{d}L \right) - \int_{\widehat{\Omega}(y_3)} h_1 \mathrm{d}y' = 0$$

where we used (79) in the last step. Therefore

$$\sum_{j=1}^{m} \left(h_1|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_1 \mathrm{d}L \right) = \int_{\widehat{\Omega}(y_3)} h_1 \mathrm{d}y'. \tag{82}$$

Similarly, again from (34), we see

$$\int_{\widehat{\Omega}(y_3)} \left(\frac{\partial h_1}{\partial y_1} + \frac{\partial h_2}{\partial y_2} \right) y_2 \mathrm{d}y' = 0$$

and we get

$$\sum_{j=1}^{m} \left(h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_2 n_2 \mathrm{d}L \right) = \int_{\widehat{\Omega}(y_3)} h_2 \mathrm{d}y'. \tag{83}$$

Using integration by parts and (79) again, we compute

$$\int_{\widehat{\Omega}(y_3)} Q y_1 \mathrm{d}y' = \int_{\widehat{\Omega}(y_3)} \left(\frac{\partial h_1}{\partial z_2} - \frac{\partial h_2}{\partial z_1} \right) y_1 \mathrm{d}y' \tag{84}$$
$$= \int_{\partial\widehat{\Omega}(y_3)} (y_1 h_1 n_2 - y_1 h_2 n_1)\,\mathrm{d}L - \int_{\widehat{\Omega}(y_3)} -h_2 \mathrm{d}y'$$

$$= \sum_{j=1}^{m} \left(h_1|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_2 \mathrm{d}L - h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_1 \mathrm{d}L \right) + \int_{\widehat{\Omega}(y_3)} h_2 \mathrm{d}y'$$

$$= \sum_{j=1}^{m} \left(-h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_1 \mathrm{d}L \right) + \int_{\widehat{\Omega}(y_3)} h_2 \mathrm{d}y'. \tag{85}$$

Using the relation found in (83) and property (80), the equation (84) becomes

$$\sum_{j=1}^{m} \left(-h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_1 \mathrm{d}L \right) + \int_{\widehat{\Omega}(y_3)} h_2 \mathrm{d}y'$$
$$= \sum_{j=1}^{m} \left(-h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_1 n_1 \mathrm{d}L \right) + \sum_{j=1}^{m} \left(h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} y_2 n_2 \mathrm{d}L \right)$$
$$= \sum_{j=1}^{m} \left(h_2|_{\widehat{g}_j(y_3)} \int_{\widehat{g}_j(y_3)} (y_2 n_2 - y_1 n_1) \mathrm{d}L \right) = 0$$

and we see that $\int_{\widehat{\Omega}(y_3)} Q y_1 \mathrm{d}y' = 0$. In a similar way, using (79), (80) and (82), we can prove that $\int_{\widehat{\Omega}(y_3)} Q y_2 \mathrm{d}y' = 0$. □

Proposition 8.1. *Let $\widetilde{\Omega}$ be a domain in $\mathbb{R}^2$ and let $V_1(y_1, y_2), V_2(y_1, y_2) \in \mathcal{D}'(\widetilde{\Omega})$. If*

$$\frac{\partial V_i}{\partial y_j} + \frac{\partial V_j}{\partial y_i} = 0 \quad \text{for } 1 \le i, j \le 2$$

in the distribution sense, then there exist constants $C_1, C_2, C_3 \in \mathbb{R}$ such that

$$V_1(y_1, y_2) = -C_3 y_2 + C_1, \quad V_2(y_1, y_2) = C_3 y_1 + C_2.$$

Proof. The idea of the proof is to use a 2-dimensional version of the fact that if for $V = (V_1, V_2, V_3)$ and $1 \le i, j \le 3$ we have $E_{ij}(V) = (\partial V_i/\partial y_j + \partial V_j/\partial y_i)/2 = 0$, then $V = \mathcal{O}y + C$, where $\mathcal{O} \in M_{3\times 3}(\mathbb{R})$ is an anti-symmetric matrix and $C \in \mathbb{R}^3$ is a constant vector. In addition, this can be shown using that

$$\frac{\partial^2 V_i}{\partial y_j \partial y_k} = \frac{\partial E_{ik}(V)}{\partial y_j} + \frac{\partial E_{ij}(V)}{\partial y_k} - \frac{\partial E_{jk}(V)}{\partial y_i} \quad (1 \le i, j, k \le 3).$$

Further details can be seen in Duvaut–Lion [**11**] and Schwartz [**24**]. □

Remark 8.2. We present here the explicit forms of the matrix $\mathcal{X}$ and the vector $\mathcal{Y}$ used in Section 7 in order to find a minimum.

$$\mathcal{X} = \begin{pmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ \mathcal{X}_2^T & \mathcal{X}_3 \end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{pmatrix},$$

where

$$\mathcal{X}_1=\begin{pmatrix} (4\lambda_1+8\lambda_2)A_{11} & (4\lambda_1+8\lambda_2)A_{12} & 0 & 0 & 4\lambda_1 A_{11} & 4\lambda_1 A_{12} \\ (4\lambda_1+8\lambda_2)A_{12} & 4\lambda_2 A_{11}+(4\lambda_1+8\lambda_2)A_{22} & 4\lambda_2 A_{12} & 4\lambda_2 A_{11} & (4\lambda_1+4\lambda_2)A_{12} & 4\lambda_1 A_{22} \\ 0 & 4\lambda_2 A_{12} & 4\lambda_2 A_{22} & 4\lambda_2 A_{12} & 4\lambda_2 A_{22} & 0 \\ 0 & 4\lambda_2 A_{11} & 4\lambda_2 A_{12} & 4\lambda_2 A_{11} & 4\lambda_2 A_{12} & 0 \\ 4\lambda_1 A_{11} & (4\lambda_1+4\lambda_2)A_{12} & 4\lambda_2 A_{22} & 4\lambda_2 A_{12} & (4\lambda_1+8\lambda_2)A_{11}+4\lambda_2 A_{22} & (4\lambda_1+8\lambda_2)A_{12} \\ 4\lambda_1 A_{12} & 4\lambda_1 A_{22} & 0 & 0 & (4\lambda_1+8\lambda_2)A_{12} & (4\lambda_1+8\lambda_2)A_{22} \end{pmatrix},$$

$$\mathcal{X}_2=\begin{pmatrix} (2\lambda_1+4\lambda_2)K_1 & 0 & 0 & 2\lambda_1 K_1 \\ (2\lambda_1+4\lambda_2)K_2 & 2\lambda_2 K_1 & 2\lambda_2 K_1 & 2\lambda_1 K_2 \\ 0 & 2\lambda_2 K_2 & 2\lambda_2 K_2 & 0 \\ 0 & 2\lambda_2 K_1 & 2\lambda_2 K_1 & 0 \\ 2\lambda_1 K_1 & 2\lambda_2 K_2 & 2\lambda_2 K_2 & (2\lambda_1+4\lambda_2)K_1 \\ 2\lambda_1 K_2 & 0 & 0 & (2\lambda_1+4\lambda_2)K_2 \end{pmatrix}, \quad \mathcal{X}_3=\begin{pmatrix} (\lambda_1+2\lambda_2)H & 0 & 0 & \lambda_1 H \\ 0 & \lambda_2 H & \lambda_2 H & 0 \\ 0 & \lambda_2 H & \lambda_2 H & 0 \\ \lambda_1 H & 0 & 0 & (\lambda_1+2\lambda_2)H \end{pmatrix},$$

$$\mathcal{Y}_1=\begin{pmatrix} 4\lambda_1\gamma_1 \\ 4\lambda_1\gamma_2 \\ 0 \\ 0 \\ 4\lambda_1\gamma_1 \\ 4\lambda_1\gamma_2 \end{pmatrix}, \quad \mathcal{Y}_2=\begin{pmatrix} 2\lambda_1\gamma_0 \\ 0 \\ 0 \\ 2\lambda_1\gamma_0 \end{pmatrix} \quad \text{with} \quad \begin{cases} \gamma_0 = H\frac{\mathrm{d}\eta_3}{\mathrm{d}y_3} - K_1\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2} - K_2\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}, \\ \gamma_1 = K_1\frac{\mathrm{d}\eta_3}{\mathrm{d}y_3} - A_{11}\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2} - A_{12}\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}, \\ \gamma_2 = K_2\frac{\mathrm{d}\eta_3}{\mathrm{d}y_3} - A_{12}\frac{\mathrm{d}^2\eta_1}{\mathrm{d}y_3^2} - A_{22}\frac{\mathrm{d}^2\eta_2}{\mathrm{d}y_3^2}. \end{cases}$$

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