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Representation type of surfaces in $\mathbb{P}^3$

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Abstract. The goal of this article is to prove that every surface with a regular point in the three-dimensional projective space of degree at least four, is of wild representation type under the condition that either X is integral or $\mathrm{Pic}(X) \cong \langle \mathcal{O}_X(1) \rangle$; we construct families of arbitrarily large dimension of indecomposable pairwise non-isomorphic ACM vector bundles. On the other hand, we prove that every non-integral ACM scheme of arbitrary dimension at least two, is also very wild in a sense that there exist arbitrarily large dimensional families of pairwise non-isomorphic ACM non-locally free sheaves of rank one.

1. Introduction.

An arithmetically Cohen–Macaulay (for short, ACM) sheaf on a projective scheme X is a coherent sheaf supporting X, which has trivial intermediate cohomology and the stalk at each point whose depth equals the dimension of X. ACM vector bundles correspond to maximal Cohen–Macaulay modules over the associated graded ring and they reflect the properties of the graded ring. It is believed that the category generated by ACM sheaves on X measures the complexity of X. Indeed, a classification of ACM varieties was proposed as *finite, tame or wild* representation type according to the complexity of this category in [**10**] and there are several contributions to this trichotomy such as [**4**], [**8**], [**11**], [**13**]. It is only recent when such a representation type is determined for any ACM reduced scheme; see [**14**].

In this article, we pay our attention to the representation type of surfaces in three-dimensional projective space. Since the ACM vector bundles on smooth surfaces of degree at most two are completely classified due to the work by Horrocks and [**17**], [**18**], we may focus on surfaces of degree at least three. The case of cubic surfaces is dealt in [**5**], [**12**] and the case of quartic surfaces is from [**20**]. Our main result is the following, which implies that the surfaces in Theorem 1.1 are of wild representation type.

Theorem 1.1. *Let $X \subset \mathbb{P}^3$ be a surface, defined as the zero set of a homogeneous polynomial in four variables of degree at least four with $X_{\mathrm{reg}} \neq \emptyset$. Assume further that either* $\mathrm{Pic}(X) = \mathbb{Z}\langle \mathcal{O}_X(1) \rangle$ *or that X is integral. For every even and positive integer r,*

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there exists a family $\{\mathcal{E}_\lambda\}_{\lambda\in\Lambda}$ *of indecomposable ACM vector bundles of rank* r *such that* Λ *is an integral quasi-projective variety with* $\dim\Lambda = r$ *and* $\mathcal{E}_\lambda \not\cong \mathcal{E}_{\lambda'}$ *for all* $\lambda \neq \lambda'$ *in* Λ.

It has to be noticed that although the result in [**14**] is more general than the implication of Theorem 1.1 regarding the wildness of the representation type, Theorem 1.1 provides a concrete way of constructing families of indecomposable ACM 'vector bundles' with prescribed rank, even on singular surfaces.

On the other hand, every non-integral ACM projective scheme of arbitrary dimension at least two, whose associated reduced scheme contains at least one ACM irreducible component, is of 'very wild' representation type, in a sense that there exist arbitrarily large dimensional families of pairwise non-isomorphic ACM non-locally free sheaves of rank one; see Proposition 5.3.

Here we summarize the structure of this article. In Section 2 we collect several definitions and basic results that are used throughout the article. In Section 3 we state the main result in Theorem 3.10, which would automatically imply Theorem 1.1. We also give a proof of Theorem 3.10 in special case and suggest a number of its variation to construct ACM vector bundles. Then we spend the whole Section 4 for the proof of Theorem 3.10; basically we use induction on rank and the main ingredient for the proof is Lemma 4.5 and the use of a monodromy argument. Then we show in Section 5 the wildness of any ACM projective scheme of dimension at least two by investigating non-locally free ideal sheaves.

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2. Preliminary.

Throughout the article our base field $\mathbf{k}$ is algebraically closed of characteristic 0. We always assume that our projective schemes $X \subset \mathbb{P}^N$ are arithmetically Cohen–Macaulay, namely, $h^1(\mathcal{I}_{X,\mathbb{P}^N}(t)) = 0$ for all $t \in \mathbb{Z}$ and $h^i(\mathcal{O}_X(t)) = 0$ for all $t \in \mathbb{Z}$ and all $i = 1, \dots, \dim X - 1$, of pure dimension at least two. Then by [**25**, Théorème 1 on page 268] all local rings $\mathcal{O}_{X,x}$ are Cohen–Macaulay of dimension $\dim X$. From $h^1(\mathcal{I}_{X,\mathbb{P}^N}) = 0$ we see that X_{red} is connected. Since in all our main result we have $N = \dim X + 1 = 3$, the reader can always assume that X is a surface in $\mathbb{P}^3$, although there are several statements that hold in more general situations. By a surface of degree $m \geq 1$ in $\mathbb{P}^3$, we always mean the zero locus of a homogeneous polynomial of degree m in four variables. For a vector bundle $\mathcal{E}$ of rank $r \in \mathbb{Z}$ on X, we say that $\mathcal{E}$ *splits* if all its indecomposable factors are $\mathcal{O}_X(t)$ for some $t \in \mathbb{Z}$; $\mathcal{E} \cong \bigoplus_{i=1}^r \mathcal{O}_X(t_i)$ for some $t_i \in \mathbb{Z}$ with $i = 1, \dots, r$.

We always fix the embedding $X \subset \mathbb{P}^N$ and the associated polarization $\mathcal{O}_X(1)$. For a coherent sheaf $\mathcal{E}$ on a closed subscheme X of a fixed projective space, we denote $\mathcal{E} \otimes \mathcal{O}_X(t)$ by $\mathcal{E}(t)$ for $t \in \mathbb{Z}$. For another coherent sheaf $\mathcal{G}$, we denote by $\hom_X(\mathcal{F}, \mathcal{G})$ the dimension of $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$, and by $\mathrm{ext}^i_X(\mathcal{F}, \mathcal{G})$ the dimension of $\mathrm{Ext}^i_X(\mathcal{F}, \mathcal{G})$. Finally we denote the canonical sheaf of X by ω_X.

Definition 2.1. A coherent sheaf $\mathcal{E}$ on X is called *arithmetically Cohen–Macaulay* (for short, ACM) if the following conditions hold:

(i) $\mathcal{E}$ is locally Cohen–Macaulay, that is, the stalk $\mathcal{E}_x$ has depth equal to $\dim \mathcal{O}_{X,x}$ for any $x \in X$;

(ii) $H^i(\mathcal{E}(t)) = 0$ for all $t \in \mathbb{Z}$ and $i = 1, \dots, \dim(X) - 1$.

Remark 2.2. In the condition (i) of Definition 2.1, we may only require that the stalk $\mathcal{E}_x$ has positive depth for any point $x \in X$; see [**2**, Remark 2.2] and [**25**, Théorème 1 on page 268].

If $\mathcal{E}$ is a coherent sheaf on a closed subscheme X of a fixed projective space, then we may consider its Hilbert polynomial $\mathrm{P}_{\mathcal{E}}(t) \in \mathbb{Q}[t]$ with the leading coefficient $\mu(\mathcal{E})/d!$, where d is the dimension of $\mathrm{Supp}(\mathcal{E})$ and $\mu = \mu(\mathcal{E})$ is called the *multiplicity* of $\mathcal{E}$. The *normalized* Hilbert polynomial $p_{\mathcal{E}}(t)$ of $\mathcal{E}$ is defined to be the Hilbert polynomial of $\mathcal{E}$ divided by $\mu(\mathcal{E})$.

Definition 2.3. If $\dim \mathrm{Supp}(\mathcal{E}) = \dim(X)$, then the *rank* of $\mathcal{E}$ is defined to be

$$\mathrm{rank}(\mathcal{E}) = \frac{\mu(\mathcal{E})}{\mu(\mathcal{O}_X)}.$$

Otherwise it is defined to be zero.

For an integral scheme X, the rank of $\mathcal{E}$ is the dimension of the stalk $\mathcal{E}_x$ at the generic point $x \in X$. But in general $\mathrm{rank}(\mathcal{E})$ needs not be integer.

Now the following construction of a coherent sheaf with higher rank and almost the same cohomological data as the starting coherent sheaf in Lemma 2.4, is due to [**3**]. In case of some surfaces in $\mathbb{P}^3$ of degree at least two, the construction provides an indecomposable ACM vector bundles of rank three; see Proposition 3.3.

Lemma 2.4. *Let $(X, \mathcal{O}_X(1))$ be an ACM projective scheme of dimension $n \geq 2$. For a fixed coherent sheaf $\mathcal{G}$ with pure depth n on X, assume the existence of $t_0 \in \mathbb{Z}$ such that $s := h^1(\mathcal{G}(t_0)) > 0$. Then the vector space $W := H^1(\mathcal{G}(t_0))$ induces the following unique extension up to isomorphisms*

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(-t_0) \otimes W^\vee \longrightarrow 0 \tag{1}$$

and the sheaf $\mathcal{E}$ in the middle satisfies the following:

(i) *$h^1(\mathcal{E}(t)) = h^1(\mathcal{G}(t))$ for all $t \neq t_0$, and $h^1(\mathcal{E}(t_0)) = 0$;*

(ii) *$h^i(\mathcal{E}(t)) = h^i(\mathcal{G}(t))$ for all $t \in \mathbb{Z}$ and all i with $2 \leq i \leq n-1$.*

If $\mathcal{G}$ is locally free, then $\mathcal{E}$ is locally free.

The construction of ACM vector bundles in Proposition 2.5 and the one in Proposition 3.3 is an extension of the method in [**7**, Remark 4.3].

Proposition 2.5. *Let $X \subset \mathbb{P}^N$ be a projective Gorenstein scheme with pure dimension two and pure depth two such that $h^1(\mathcal{O}_X(t)) = 0$ for all $t \in \mathbb{Z}$ and $h^1(\mathcal{I}_{X,\mathbb{P}^N}) = 0$.*

Assume $X_{\mathrm{reg}} \neq \emptyset$ *and fix* $p \in X_{\mathrm{reg}}$. *Then there exists an ACM vector bundle* $\mathcal{E}_p$ *of rank two on* X *fitting into the exact sequence*

$$0 \longrightarrow \omega_X(1) \longrightarrow \mathcal{E}_p \longrightarrow \mathcal{I}_{p,X} \longrightarrow 0. \tag{2}$$

Moreover, if $\deg(\omega_X) + \deg(X) \geq 0$ *and* $p, q \in X_{\mathrm{reg}}$ *with* $p \neq q$, *then we have* $\mathcal{E}_p \not\cong \mathcal{E}_q$.

Proof. Since X is Gorenstein, $\omega_X(1)$ is a line bundle and we get

$$\mathrm{Ext}^1_X(\mathcal{I}_{p,X}, \omega_X(1)) \cong H^1(\mathcal{I}_{p,X}(-1))^\vee \cong \mathbf{k}.$$

So up to isomorphism there exists a unique sheaf $\mathcal{E}_p$ fitting into an extension (2) with a nonzero extension class. Since $h^0(\mathcal{O}_X(-1)) = 0$ and $p \in X_{\mathrm{reg}}$, the Cayley–Bacharach condition is satisfied for (2) and so $\mathcal{E}_p$ is locally free; see [**6**]. Note that the restriction map

$$H^0(\mathcal{O}_X(t)) \longrightarrow H^0(\mathcal{O}_X(t)_{|\{p\}})$$

is surjective for any $t \geq 0$. This implies that $h^1(\mathcal{I}_{p,X}(t)) = 0$ for any $t \geq 0$, because we have $h^1(\mathcal{O}_X(t)) = 0$. Then we see from (2) that $h^1(\mathcal{E}_p(t)) = 0$ for any $t \geq 0$. On the other hand, from $\det(\mathcal{E}_p) \cong \omega_X(1)$, we get that $h^1(\mathcal{E}_p(t)) = h^1(\mathcal{E}_p^\vee \otimes \omega_X(-t)) = h^1(\mathcal{E}_p(-t-1)) = 0$ for $t < 0$ by Serre's duality. Thus $\mathcal{E}_p$ is ACM.

For the second assertion, assume $\mathcal{E}_p \cong \mathcal{E}_q$. From the assumption $\deg(\omega_X(1)) \geq 0$, we get $h^0(\omega_X^\vee(-1)) \leq 1$ with equality if and only if $\omega_X \cong \mathcal{O}_X(-1)$. In particular, we have $h^0(\mathcal{I}_{p,X} \otimes \omega_X^\vee(-1)) = 0$. Then from the assumption $h^1(\mathcal{O}_X) = 0$ and (2), we get $h^0(\mathcal{E}_p \otimes \omega_X^\vee(-1)) = 1$ and that p is the only zero of a nonzero section of $H^0(\mathcal{E}_p \otimes \omega_X^\vee(-1))$. Thus we get $p = q$. □

Theorem 2.6. *Let* $X \subset \mathbb{P}^N$ *be a projective Gorenstein scheme with pure dimension two and pure depth two, satisfying that*

- $h^1(\mathcal{O}_X(t)) = 0$ *for all* $t \in \mathbb{Z}$ *and* $h^1(\mathcal{I}_{X,\mathbb{P}^N}) = 0$;
- $X_{\mathrm{reg}} \neq \emptyset$ *and* $\deg(\omega_X) + \deg(X) \geq 0$.

Then there exists a two-dimensional family of pairwise non-isomorphic ACM vector bundles of rank two on X *whose very general member is indecomposable*; *here "very general" means outside countably many proper subvarieties.*

Proof. By assumption X_{reg} is a two-dimensional quasi-projective smooth variety. By Proposition 2.5 there is a flat family of ACM vector bundles $\{\mathcal{E}_p\}_{p \in X_{\mathrm{reg}}}$ of rank two such that if $p, q \in X_{\mathrm{reg}}$ and $p \neq q$, then $\mathcal{E}_p \not\cong \mathcal{E}_q$. Thus it is sufficient to prove that each $\mathcal{E}_p$ is indecomposable. Assume that $\mathcal{E}_p$ is decomposable. Since $\mathcal{E}_p$ is a vector bundle of rank two, we get $\mathcal{E}_p \cong \mathcal{A}_1 \oplus \mathcal{A}_2$ with each $\mathcal{A}_i$ a line bundle. Without loss of generality we assume $h^0(\mathcal{A}_1 \otimes \omega_X^\vee(1)) \geq h^0(\mathcal{A}_2 \otimes \omega_X^\vee(1))$. Since $h^0(\mathcal{E}_p \otimes \omega_X^\vee(1)) = 1$ from the proof of Proposition 2.5, we get $h^0(\mathcal{A}_1 \otimes \omega_X^\vee(1)) = 1$ and $h^0(\mathcal{A}_2 \otimes \omega_X^\vee(1)) = 0$. Thus a nonzero section σ of $\mathcal{E}_p \otimes \omega_X^\vee(1)$ has either no zero or an effective Cartier divisor of X as its zero locus, contradicting the fact that σ vanishes only at p, as shown in the proof of Proposition 2.5. □

Throughout the article, as in Proposition 2.5, our construction of ACM sheaf of rank two on X is in terms of the following extension

$$0 \longrightarrow \omega_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z,X}(a) \longrightarrow 0 \tag{3}$$

with Z a locally complete intersection of codimension two in X and $a \in \mathbb{Z}$. Such extensions are parametrized by $\mathrm{Ext}^1_X(\mathcal{I}_{Z,X}(a), \omega_X)$. In case when X is a surface, the coboundary map associated to (3) is

$$\delta_1 : H^1(\mathcal{I}_{Z,X}(a)) \longrightarrow H^2(\omega_X) \cong \mathbf{k}$$

and by Serre's duality in [**16**, Theorem 3.12] its dual is

$$\mathbf{k} \cong \mathrm{Hom}_X(\omega_X, \omega_X) \longrightarrow \mathrm{Ext}^1_X(\mathcal{I}_{Z,X}(a), \omega_X),$$

which is obtained by applying the functor $\mathrm{Hom}_X(-, \omega_X)$ to (3). Thus the coboundary map δ_1 is surjective if and only if (3) is a non-trivial extension. Since we assume $h^1(\mathcal{O}_X) = h^1(\omega_X) = 0$, this implies that $h^1(\mathcal{E}) = h^1(\mathcal{I}_{Z,X}(a)) - 1$.

3. ACM vector bundle on surfaces in $\mathbb{P}^3$.

We always assume that $X \subset \mathbb{P}^3$ is a surface of degree m, not necessarily smooth. In particular, its dualizing sheaf is $\omega_X \cong \mathcal{O}_X(m-4)$ and we get $h^2(\mathcal{O}_X) = \binom{m-1}{3}$. We also have $h^0(\mathcal{O}_X) = 1$ and $h^1(\mathcal{O}_X) = 0$.

Lemma 3.1. *Each line bundle $\mathcal{O}_X(t)$ with $t \in \mathbb{Z}$, is stable as an $\mathcal{O}_{\mathbb{P}^3}$-sheaf with pure depth* 2.

Proof. It is enough to deal with the case $t = 0$. Assume the contrary and take a subsheaf $\mathcal{A} \subsetneq \mathcal{O}_X$ such that $\mathcal{B} := \mathcal{O}_X/\mathcal{A}$ has depth 2 and normalized Hilbert polynomial at least the one of $\mathcal{O}_X$. Since $\mathcal{B}$ is a quotient of $\mathcal{O}_X$ with depth 2 and X has no embedded component, we get $\mathcal{B} \cong \mathcal{O}_T$ for T a union of some of the irreducible components of X_{red} with at most the multiplicities appearing in X. This implies that $T \in |\mathcal{O}_{\mathbb{P}^3}(d)|$ for some integer d with $1 \le d < m$. Now the Hilbert polynomial of $\mathcal{O}_X$ is

$$\begin{aligned}\mathrm{P}_{\mathcal{O}_X}(t) &= \binom{t+3}{3} - \binom{t-m+3}{3} \\ &= \left(\frac{m}{2}\right)t^2 + \left(2m - \frac{m^2}{2}\right)t + \left(\frac{m^3}{6} - m^2 + \frac{11m}{6}\right).\end{aligned}$$

Similarly, we get the Hilbert polynomial $\mathrm{P}_{\mathcal{O}_T}(t)$ of $\mathcal{O}_T$ by replacing m in $\mathrm{P}_{\mathcal{O}_X}(t)$ by d. Then we see that $p_{\mathcal{O}_X}(t) < p_{\mathcal{O}_T}(t)$ for $t \gg 0$, a contradiction. □

Remark 3.2. If either $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$ or X is integral, then every line bundle is stable. Note also that the proof of Lemma 3.1 shows that the ideal sheaf $\mathcal{I}_{Z,X}$ for any zero-dimensional subscheme $Z \subset X$, is also stable. If X is integral, then any sheaf of rank 1 with positive depth is stable. Thus these sheaves are indecomposable.

Proposition 3.3. *Let $X \subset \mathbb{P}^3$ be a surface of degree $m \geq 2$ with $X_{\mathrm{reg}} \neq \emptyset$. Fix $p \in X_{\mathrm{reg}}$, and let $\mathcal{E}_p$ be the unique non-trivial extension*

$$0 \longrightarrow \mathcal{O}_X(m-3) \longrightarrow \mathcal{E}_p \longrightarrow \mathcal{I}_{p,X} \longrightarrow 0. \tag{4}$$

Then $\mathcal{E}_p$ is an ACM vector bundle of rank two on X and $\mathcal{E} \not\cong \mathcal{O}_X(a) \oplus \mathcal{O}_X(b)$ for any $a, b \in \mathbb{Z}$. If one of the following holds, then $\mathcal{E}$ is indecomposable.

(i) $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$,

(ii) *$\mathcal{O}_X(t)$ for $t \in \mathbb{Z}$ are the only ACM line bundles on X, or*

(iii) *$m \geq 4$ and X is integral.*

Proof. By Proposition 2.5 it remains to deal with indecomposability of $\mathcal{E}_p$. First show that there are no integers a, b such that $\mathcal{E}_p \cong \mathcal{O}_X(a) \oplus \mathcal{O}_X(b)$. Assume that such a, b exist, say $a \geq b$. Since $h^0(\mathcal{E}_p(3-m)) = 1$ and $h^0(\mathcal{E}_p(2-m)) = 0$, we get $(a, b) = (m-3, 0)$ and $m \geq 3$. Then we get $h^0(\mathcal{E}_p) = \binom{m}{3} + 1$, while (4) gives $h^0(\mathcal{E}_p) = \binom{m}{3}$.

Now assume that $\mathcal{E}_p$ is decomposable. Since $\mathcal{E}_p$ is locally free and it has rank 2, we have $\mathcal{E}_p \cong \mathcal{A}_1 \oplus \mathcal{A}_2$ with each $\mathcal{A}_i \in \mathrm{Pic}(X)$. Since $\mathcal{E}_p$ is ACM, each $\mathcal{A}_i$ is ACM. In cases (i) and (ii) the assertion holds by above. Thus we assume the case (iii). By Lemma 3.1 and Remark 3.2, (4) is the Harder–Narasimhan filtration of $\mathcal{E}_p$. Applying the functor $\mathrm{Hom}_X(\mathcal{E}_p, -)$ to (4), we get

$$0 \longrightarrow \mathrm{Hom}_X(\mathcal{E}_p, \mathcal{O}_X(m-3)) \longrightarrow \mathrm{Hom}_X(\mathcal{E}_p, \mathcal{E}_p) \longrightarrow \mathrm{Hom}_X(\mathcal{E}_p, \mathcal{I}_{p,X}) \longrightarrow \mathrm{Ext}^1_X(\mathcal{E}_p, \mathcal{O}_X(m-3)).$$

Note that $\hom_X(\mathcal{E}_p, \mathcal{O}_X(m-3)) = h^2(\mathcal{E}_p(-1)) = h^0(\mathcal{E}_p) = \binom{m}{3}$ by Serre's duality. By applying the functor $\mathrm{Hom}_X(-, \mathcal{I}_{p,X})$ to (4), we get

$$\hom_X(\mathcal{E}_p, \mathcal{I}_{p,X}) = \hom_X(\mathcal{I}_{p,X}, \mathcal{I}_{p,X}) = 1.$$

Thus we have

$$\binom{m}{3} \leq \hom_X(\mathcal{E}_p, \mathcal{E}_p) \leq 1 + \binom{m}{3}.$$

Since $h^0(\mathcal{O}_X) = 1$, we have $\hom_X(\mathcal{A}_i, \mathcal{A}_i) = 1$ for each i. So we get

$$\hom_X(\mathcal{E}_p, \mathcal{E}_p) = 2 + \hom_X(\mathcal{A}_1, \mathcal{A}_2) + \hom_X(\mathcal{A}_2, \mathcal{A}_1).$$

Since X is integral, each $\mathcal{A}_i$ is stable and we get either $\mathcal{A}_1 \cong \mathcal{A}_2$ or $\hom_X(\mathcal{A}_i, \mathcal{A}_{3-i}) = 0$ for each i. In the latter case we have $\hom_X(\mathcal{E}_p, \mathcal{E}_p) = 2 < \binom{m}{3}$, a contradiction. In the former case, we have $\hom_X(\mathcal{E}_p, \mathcal{E}_p) = 4$ and the only possibility is $m = 4$. But this is also impossible, since we would get $\mathcal{A}_1^{\otimes 2} \cong \det(\mathcal{E}_p) \cong \mathcal{O}_X(1)$. $\square$

Proposition 3.4. *Let $X \subset \mathbb{P}^3$ be a surface of degree $m \geq 2$ and let $Z \subset X$ be a zero-dimensional subscheme of degree 3, which is not collinear. Assume that Z is a locally complete intersection inside X, i.e. for each $p \in Z_{\mathrm{red}}$ the ideal sheaf of Z at $\mathcal{O}_{X,p}$ is generated by two elements of $\mathcal{O}_{X,p}$. Then there is a vector bundle $\mathcal{G}$ of rank two fitting* *into an exact sequence*

$$0 \longrightarrow \mathcal{O}_X(m-4) \longrightarrow \mathcal{G} \longrightarrow \mathcal{I}_{Z,X} \longrightarrow 0 \tag{5}$$

with $h^1(\mathcal{G}(t)) = 0$ *for all* $t \neq 0$ *and* $h^1(\mathcal{G}) = 1$*. There is also an exact sequence*

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \xrightarrow{u} \mathcal{O}_X \longrightarrow 0, \tag{6}$$

where $\mathcal{E}$ *is an ACM vector bundle of rank three such that* $\mathcal{E} \not\cong \mathcal{O}_X(a_1) \oplus \mathcal{O}_X(a_2) \oplus \mathcal{O}_X(a_3)$ *for any* $(a_1, a_2, a_3) \in \mathbb{Z}^{\oplus 3}$*. Moreover, if* $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$*, then* $\mathcal{E}$ *is indecomposable.*

Proof. Note that $\omega_X \cong \mathcal{O}_X(m-4)$ and so we have $h^0(\mathcal{I}_{p,X} \otimes \mathcal{O}_X(4-m) \otimes \omega_X) = 0$ for all $p \in Z_{\mathrm{red}}$. Since Z is a locally complete intersection, the Cayley–Bacharach condition is satisfied and so there is a locally free $\mathcal{G}$ fitting into (5); see [**6**]. From (5) we immediately get $h^1(\mathcal{G}(t)) = 0$ for all $t > 0$, because Z is not collinear. Note that $\det(\mathcal{G}) \cong \mathcal{O}_X(m-4)$ and $\mathcal{G}$ is a vector bundle of rank two. This implies $\mathcal{G}^\vee \cong \mathcal{G}(4-m)$. For $t < 0$, we have $h^1(\mathcal{G}(t)) = h^1(\mathcal{G}^\vee(-t) \otimes \omega_X) = h^1(\mathcal{G}(-t)) = 0$ by Serre's duality. Now consider the coboudnary map $\delta_1 : H^1(\mathcal{I}_{Z,X}) \to H^2(\mathcal{O}_X(m-4)) \cong \mathbf{k}$ with $\ker(\delta_1) = H^1(\mathcal{G})$. The dual of δ_1 is the map

$$\mathrm{Hom}_X(\mathcal{O}_X(m-4), \mathcal{O}_X(m-4)) \longrightarrow \mathrm{Ext}^1_X(\mathcal{I}_{Z,X}, \mathcal{O}_X(m-4))$$

sending the identity map to the element corresponding to $\mathcal{G}$. This implies that δ_1 is surjective and $h^1(\mathcal{G}) = 1$.

Now we apply Lemma 2.4 to $\mathcal{G}$ to obtain an ACM vector bundle $\mathcal{E}$ of rank three fitting into (6). Since $h^1(\mathcal{G}) = 1$ and $h^1(\mathcal{E}) = 0$, (5) and (6) give $h^0(\mathcal{E}) = h^0(\mathcal{G}) = \binom{m-1}{3}$. Assume the existence of integers $a_1 \geq a_2 \geq a_3$ such that $\mathcal{E} \cong \bigoplus_{i=1}^3 \mathcal{O}_X(a_i)$. Since $\det(\mathcal{E}) \cong \mathcal{O}_X(m-4)$, we have $a_1 + a_2 + a_3 = m-4$. If $2 \leq m \leq 3$, then we have $a_1 \geq 0$ from $a_1 + a_2 + a_3 = m-4$. This implies that $h^0(\mathcal{O}_X(a_1)) > 0 = \binom{m-1}{3} = h^0(\mathcal{E})$, a contradiction. If $m = 4$, then we have $h^0(\mathcal{E}) = 1$. Since $a_1 + a_2 + a_3 = 0$, we have $\sum_{i=1}^3 h^0(\mathcal{O}_X(a_i)) > 1$, a contradiction. Finally assume $m > 4$. From (5) and (6) we see that $\mathcal{O}_X(m-2)$ is the first non-trivial sheaf in the Harder–Narasimhan filtration of $\mathcal{E}$. Thus $a_1 = m-4$ and $h^0(\mathcal{O}_X(a_1)) = \binom{m-1}{3}$. Since $a_2 + a_3 = 0$, we have $h^0(\mathcal{O}_X(a_2)) > 0$ and so $h^0(\mathcal{E}) > \binom{m-1}{3}$, a contradiction. Hence we get $\mathcal{E} \not\cong \bigoplus_{i=1}^3 \mathcal{O}_X(a_i)$ for any triple of integers (a_1, a_2, a_3).

It remains to show the last assertion. Assume $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$ and that $\mathcal{E}$ is decomposable; by the previous assertion we have $\mathcal{E} \cong \mathcal{A}_1 \oplus \mathcal{A}_2$ with $\mathrm{rank}(\mathcal{A}_i) = i$ for each i and $\mathcal{A}_2$ indecomposable. Set $\mathcal{A}_1 \cong \mathcal{O}_X(a)$ for $a \in \mathbb{Z}$. Since $h^0(\mathcal{E}) = \binom{m-1}{3}$, we have $a \leq m-4$. From (5) and (6) we get the existence of a subsheaf $\mathcal{F} \subset \mathcal{E}$ such that $\mathcal{F} \cong \mathcal{O}_X(m-4)$ and $\mathcal{E}/\mathcal{F}$ is an extension $\mathcal{H}$ of $\mathcal{O}_X$ by $\mathcal{I}_{Z,X}$. Note that $\mathcal{H}$ is not locally free, because $\mathcal{I}_{Z,X}$ has not depth 2. In particular, $\mathcal{H}$ is not isomorphic to $\mathcal{A}_2$ and we get $\mathcal{A}_1 \not\cong \mathcal{F}$; otherwise we would get that $\mathcal{A}_2 \cong \mathcal{E}/\mathcal{A}_1 \cong \mathcal{H}$ is locally free. So we have $a < m-4$. Now consider a restriction map

$$u_{|\{0\} \oplus \mathcal{A}_2} : \{0\} \oplus \mathcal{A}_2 \longrightarrow \mathcal{O}_X.$$

If this restriction map is surjective, then its kernel is a line bundle, say $\mathcal{O}_X(b)$. Since X is ACM, we get $\mathcal{A}_2 \cong \mathcal{O}_X \oplus \mathcal{O}_X(b)$, a contradiction. Thus the restriction map is not surjecitve and so the other restriction map $u_{|\mathcal{A}_1 \oplus \{0\}}$ is not zero. In particular, we get $a \le 0$. If $a = 0$, then we have $\mathcal{A}_1 \cong \mathcal{O}_X$ and the map $u_{|\mathcal{A}_1 \oplus \{0\}}$ is an isomorphism. Thus (6) splits and we get $h^1(\mathcal{E}) \ge h^1(\mathcal{G}) > 0$, a contradiction. So we get $a < 0$. Since there is no nonzero map $\mathcal{F} \to \mathcal{A}_1$ by $a < m-4$, $\mathcal{F}$ is isomorphic to a subsheaf $\mathcal{F}_1$ of $\mathcal{A}_2$ and we get $\mathcal{H} \cong \mathcal{O}_X(a) \oplus \mathcal{A}_2/\mathcal{F}_1$. From $a < 0$ there is no nonzero map $\mathcal{I}_{Z,X} \to \mathcal{O}_X(a)$. Since $\mathcal{H}$ is an extension of $\mathcal{O}_X$ by $\mathcal{I}_{Z,X}$, we get that $\mathcal{I}_{Z,X} \cong \mathcal{A}_2/\mathcal{F}_1$ and so $\mathcal{O}_X(a) \cong \mathcal{O}_X$, a contradiction. □

REMARK 3.5. In case $m = 1$, i.e. $X = \mathbb{P}^2$, we fail in obtaining an indecomposable ACM vector bundle of rank three, using the method in Proposition 3.4. Indeed, we get $\mathcal{G} \cong \Omega^1_{\mathbb{P}^2}$ and the corresponding vector bundle of rank three is $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}$.

REMARK 3.6. In case $m = 2$, i.e. $X = Q_2$ a smooth quadric surface, there exist exactly three ACM vector bundle up to twist: $\mathcal{O}_{Q_2}$, $\mathcal{O}_{Q_2}(1,0)$ and $\mathcal{O}_{Q_2}(0,1)$. Thus we may set the bundle in Proposition 3.4 is $\mathcal{E} \cong \mathcal{O}_{Q_2}(a,a) \oplus \mathcal{O}_{Q_2}(b_1,b_2) \oplus \mathcal{O}_{Q_2}(c_1,c_2)$; since $c_1(\mathcal{E}) \cong \mathcal{O}_{Q_2}(-2,-2)$ and $h^0(\mathcal{E}) = 0$, there must be exactly one direct summand of the form $\mathcal{O}_{Q_2}(a,a)$. After a simple computation, we see that $\mathcal{E} \cong \mathcal{O}_{Q_2}(-1,-1) \oplus \mathcal{O}_{Q_2}(-1,0) \oplus \mathcal{O}_{Q_2}(0,-1)$.

COROLLARY 3.7. *Let $X \subset \mathbb{P}^3$ be union of multiple planes in which at least one plane occurs with multiplicity* 1. *Then there is an indecomposable ACM vector bundle of rank three on X. If $m > 4$, we have a family of such ACM vector bundles of dimension* 6.

PROOF. Assume that X has one component H with multiplicity 1. In this case we take as Z a set of 3 general points in H. Then the first assertion follows from Proposition 3.4. Note that the set of all such Z has dimension 6. Now assume that X has a component H with multiplicity 3. Fix a general point $p \in H$ and take a general line $L \subset \mathbb{P}^3$ with $p \in L$. Then set Z to be the connected component of the scheme $X \cap L$ with p as its reduction. Then we may get the assertion from Proposition 3.4 and that $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$ by [**2**, Lemma 2.5]. □

PROPOSITION 3.8. *Let $X \subset \mathbb{P}^3$ be a surface of degree $m \ge 4$ with an irreducible component Y appearing with multiplicity* 2 *in X. Fix $p \in Y_{\mathrm{reg}}$ so that Y is the only irreducible component of X containing p. For a general line $L \subset \mathbb{P}^3$ containing p, let $Z \subset X$ be the connected component of $L \cap X$ with p as its reduction. We have* $\deg(Z) = 2$ *and there is an ACM vector bundle $\mathcal{E}_Z$ of rank two fitting into an exact sequence*

$$0 \longrightarrow \mathcal{O}_X(m-4) \longrightarrow \mathcal{E}_Z \longrightarrow \mathcal{I}_{Z,X} \longrightarrow 0. \tag{7}$$

Moreover, there is an integral 4-dimensional variety Δ, a flat family of ACM vector bundles on X such that each isomorphism classes in (7) *appears for a unique element in Δ with the following properties.*

(i) *For any $\mathcal{E}_Z \in \Delta$, there are no integers a, b with $\mathcal{E}_Z \cong \mathcal{O}_X(a) \oplus \mathcal{O}_X(b)$.*

(ii) *A very general $\mathcal{E}_Z \in \Delta$ is indecomposable.*

(iii) *If* $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$, *then each* $\mathcal{E}_Z \in \Delta$ *is indecomposable.*

(iv) *If* $\mathbb{Z}\langle \mathcal{O}_X(1)\rangle$ *are the only ACM line bundles on* X, *then each* $\mathcal{E}_Z \in \Delta$ *is indecomposable.*

PROOF. Since no other component of X but Y contains p and p is a regular point of X, we have $\deg(Z) = 2$; it is enough to take as L any line through p not contained in the tangent plane T_pY of Y.

Since $\omega_X \cong \mathcal{O}_X(m-4)$, we have $h^0(\mathcal{O}_X(4-m)\otimes\omega_X) = 1$ and $\mathcal{O}_X(4-m)\otimes\omega_X$ is globally generated. Thus we have $h^0(\mathcal{I}_{p,X}\otimes\mathcal{O}_X(4-m)\otimes\omega_X) = 0$. Since Z is a locally complete intersection, the Cayley–Bacharach condition is satisfied for (7) and so there is a locally free $\mathcal{E}_Z$ fitting into (7); see [**6**].

Since $\mathcal{O}_X(1)$ is very ample and $\deg(Z) = 2$, we get $h^1(\mathcal{E}_Z(t)) = 0$ for all $t > 0$ by (5). Note that $\det(\mathcal{E}_Z) \cong \mathcal{O}_X(m-4)$ and $\mathcal{E}_Z$ is a vector bundle of rank two. This implies $\mathcal{E}_Z^\vee \cong \mathcal{E}_Z(4-m)$. For $t < 0$, we have $h^1(\mathcal{E}_Z(t)) = h^1(\mathcal{E}_Z^\vee(m-t-4)) = h^1(\mathcal{E}_Z(-t)) = 0$ by Serre's duality. Now consider the coboudary map $\delta_1 : H^1(\mathcal{I}_{Z,X}) \to H^2(\mathcal{O}_X(m-4)) \cong \mathbf{k}$ with $\ker(\delta_1) = H^1(\mathcal{E}_Z)$. The dual of δ_1 is the map

$$\mathrm{Hom}_X(\mathcal{O}_X(m-4), \mathcal{O}_X(m-4)) \longrightarrow \mathrm{Ext}^1_X(\mathcal{I}_{Z,X}, \mathcal{O}_X(m-4))$$

sending the identity map to the element corresponding to $\mathcal{E}_Z$. This implies that δ_1 is non-zero and hence and $h^1(\mathcal{E}_Z) = 0$. Thus $\mathcal{E}_Z$ is ACM.

The set of all $p \in Y_{\mathrm{reg}}$ such that Y is the only irreducible component of X containing p is an irreducible 2-dimensional variety Δ'. For each $p \in \mathbb{P}^3$ the set of all lines through p is a $\mathbb{P}^2$. Define a variety Δ as follows:

$$\Delta := \{(p, L) \mid p \in \Delta' \text{ and } L \text{ a line in } \mathbb{P}^3 \text{ with } p \in L \text{ and } L \nsubseteq T_pY\}.$$

Since $m \geq 4$, we have $h^0(\mathcal{I}_{Z,X}(4-m)) = 0$. Thus (7) gives $h^0(\mathcal{E}_Z(4-m)) = 1$. Thus the isomorphism classes of $\mathcal{E}_Z$ uniquely determines Z, i.e. if $\mathcal{E}_Z \ncong \mathcal{E}_{Z'}$, then we get $Z \neq Z'$. For two elements $(p_1, L_1), (p_2, L_2) \in \Delta$, let Z_i be the subscheme of degree 2 determined by (p_i, L_i) for each $i = 1, 2$. Since each p_i is the reduction of Z_i and L_i is the line spanned by Z_i, the variety Δ uniquely parametrizes the isomorphism classes of the ACM vector bundles $\mathcal{E}_Z$.

Assume $\mathcal{E}_Z \cong \mathcal{O}_X(a) \oplus \mathcal{O}_X(b)$ for some integers a, b with $a \geq b$. Since $\det(\mathcal{E}_Z) \cong \mathcal{O}_X(m-4)$, we have $b = m-4-a$. But since $h^0(\mathcal{E}_Z(4-m)) = 1$, the only possibility is that $a = 4-m$ and $b < 0$, a contradiction. Thus we get (i). We may get (ii) as in the proof of Theorem 2.6. Now assume that $\mathcal{E}_Z$ is decomposable, say $\mathcal{E}_Z \cong \mathcal{A}_1 \oplus \mathcal{A}_2$ with each $\mathcal{A}_i$ a line bundle. Since $\mathcal{E}_Z$ is ACM, each $\mathcal{A}_i$ is also ACM. Thus (iii) and (iv) follow from (i). □

REMARK 3.9. In case $m = 2$, i.e. $X = 2H$ the double plane with a hyperplane $H \subset \mathbb{P}^3$, the vector bundle $\mathcal{E}_Z$ described in Proposition 3.8 is the vector bundle $\mathcal{O}_X(-1)^{\oplus 2}$.

THEOREM 3.10. *Let* $X \subset \mathbb{P}^3$ *be a surface of degree* $m \geq 4$ *with* $X_{\mathrm{reg}} \neq \emptyset$, *i.e.* X *has an irreducible component* Y *appearing with multiplicity* 1. *We further assume that either* $\mathrm{Pic}(X) = \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$ *or* X *is integral. For a fixed integer* $s > 0$ *and a set* $S \subset X_{\mathrm{reg}} \cap Y$

with $\sharp(S) = s$, a general sheaf $\mathcal{E}_S$ fitting into an exact sequence

$$0 \longrightarrow \mathcal{O}_X(m-3)^{\oplus s} \xrightarrow{v} \mathcal{E}_S \longrightarrow \bigoplus_{p \in S} \mathcal{I}_{p,X} \longrightarrow 0, \tag{8}$$

is a locally free, indecomposable and ACM sheaf of rank $2s$. Moreover, if $S' \subset X_{\mathrm{reg}} \cap Y$ is another set with $\sharp(S') = s$ and $S' \neq S$, then we have $\mathcal{E}_{S'} \not\cong \mathcal{E}_S$.

We have $\mathrm{ext}^1_X(\mathcal{I}_{p,X}, \mathcal{O}_X(m-3)) = h^1(\mathcal{I}_{p,X}(-1)) = 1$ for each $p \in X_{\mathrm{reg}}$ by Serre's duality. So the extension $\mathcal{E}_S$ corresponds to an element in a finite dimensional vector space

$$\mathbb{E}(S) := \mathrm{Ext}^1_X\left(\bigoplus_{p \in S} \mathcal{I}_{p,X}, \mathcal{O}_X(m-3)^{\oplus s}\right) \cong \mathbf{k}^{s^2}.$$

If $s = 1$, say $S = \{p\}$, the dimension of $\mathbb{E}(S)$ is one. Thus there exists a unique non-trivial extension. Denote this non-trivial extension simply by $\mathcal{E}_p$.

In Theorem 3.10, a "general" choice of $\mathcal{E}_S$ means that there exists a non-empty Zariski open subset $\mathbb{U} \subset \mathbb{E}(S)$ such that the middle term of any extension in $\mathbb{U}$ is ACM, locally free and indecomposable.

4. Proof of Theorem 3.10.

Set $\mathbb{E}'(S)$ to be the set of all elements in $\mathbb{E}(S)$ whose corresponding middle term is locally free and ACM.

Lemma 4.1. *$\mathbb{E}'(S)$ is a non-empty open subset of $\mathbb{E}(S)$.*

Proof. Let $\tilde{\mathcal{E}}$ be the universal family over $\mathbb{E}(S)$, i.e. let $\tilde{\mathcal{E}}$ be the coherent sheaf over $X \times \mathbb{E}(S)$ such that $\mathcal{E}_a := \tilde{\mathcal{E}}_{|X \times \{a\}}$ is the sheaf corresponding to $a \in \mathbb{E}(S)$. Let $\pi_2 : X \times \mathbb{E}(S) \to \mathbb{E}(S)$ denote the projection onto the second factor, and set $\Gamma := \{(x,a) \in X \times \mathbb{E}(S) \mid \tilde{\mathcal{E}}$ is not locally free at $(x,a)\}$. Since local freeness is an open condition, Γ is a closed subscheme of $X \times \mathbb{E}(S)$. Since π_2 is proper, $\pi_2(\Gamma)$ is closed in $\mathbb{E}(S)$ and hence $\mathbb{E}(S) \setminus \pi_2(\Gamma)$ is open in $\mathbb{E}(S)$. We have $\mathbb{E}(S) \setminus \pi_2(\Gamma) = \{a \in \mathbb{E}(S) \mid \mathcal{E}_a$ is locally free$\}$.

On the other hand, we check that ACM is an open property for the set of all locally free $\mathcal{E} \in \mathbb{E}(S)$. Note that $h^1(\mathcal{I}_{S,X}(t)) = 0$ for any set $S \subset X$ with cardinality s and all $t \geq s-1$. In particular, we get $h^1(\mathcal{E}(t)) = 0$ for all $t \geq s-1$ by (8). Dualizing (8), or using the relative case of [**25**, Théorème 1 on page 268] with the fact that a locally free $\mathcal{E}$ has depth 2, we get the existence of a negative integer t_1 such that $h^1(\mathcal{E}(t)) = 0$ for all $t < t_1$ and all locally free $\mathcal{E} \in \mathbb{E}(S)$. By the semicontinuity theorem for cohomology in [**15**, Theorem III.12.8] the set of all $\mathcal{E} \in \mathbb{E}(S)$ such that $h^1(\mathcal{E}(t)) = 0$ for all t such that $t_1 \leq t \leq s-2$ is an open subset $\mathcal{U}$ of $\mathbb{E}(S)$. A locally free $\mathcal{E} \in \mathbb{E}(S)$ is ACM if and only if $\mathcal{E} \in \mathcal{U}$.

Now we see that $\mathbb{E}'(S)$ is an open subset of $\mathbb{E}(S)$. Thus it remains to prove that $\mathbb{E}'(S) \neq \emptyset$. Proposition 3.3 gives the case $s = 1$. For $s > 1$, we may find a direct sum of ACM vector bundles of rank two fitting into (8), i.e. take $\bigoplus_{p\in S}\mathcal{E}_p$. This implies $\mathbb{E}'(S)\neq\emptyset$. $\square$

REMARK 4.2. In the set-up of (8) set $\mathcal{A} := v(\mathcal{O}_X(m-3)^{\oplus s})$. By Lemma 3.1 and Remark 3.2 together with the assumption $m\geq 3$, we see that $\mathcal{A}$ is the first term of the Harder–Narasimhan filtration of $\mathcal{E}_S$. Thus we get $f(\mathcal{A})\subseteq\mathcal{A}$ for any $f\in\mathrm{End}(\mathcal{E}_S)$.

LEMMA 4.3. *If $\mathcal{E}$ is the middle term of an extension $\varepsilon\in\mathbb{E}'(S)$, then $\mathcal{E}$ has no line bundle as a factor.*

PROOF. Assume that $\mathcal{L}$ is a line bundle that is a factor of $\mathcal{E}$, i.e. $\mathcal{E}=\mathcal{L}\oplus\mathcal{G}$ for some ACM vector bundle $\mathcal{G}$ of rank $2s-1$. Since $m\geq 3$, we have

$$h^0(\mathcal{L}(3-m))+h^0(\mathcal{G}(3-m))=h^0(\mathcal{E}(3-m))=s.$$

First assume $h^0(\mathcal{L}(3-m))=0$ and $h^0(\mathcal{G}(3-m))=s$. Then we have $\mathcal{A} := v(\mathcal{O}_X(m-3)^{\oplus s})\subset\{0\}\oplus\mathcal{G}$ in (8) and so $\mathcal{L}\cong\mathcal{I}_{p,X}$ for some $p\in S$ by the uniqueness of the Harder–Narasimhan filtration (8), a contradiction. Thus we have $h^0(\mathcal{L}(3-m))>0$ and so $h^0(\mathcal{G}(3-m))<s$. In particular, there is a nonzero map $u:\mathcal{O}_X(m-3)\to\mathcal{L}$. Assume for the moment that $\mathrm{Pic}(X)\cong\mathbb{Z}\langle\mathcal{O}_X(1)\rangle$ and write $\mathcal{L}\cong\mathcal{O}_X(a)$ for some $a\in\mathbb{Z}$. The map u gives $a\geq m-3$. Since $m\geq 3$, (8) is the Harder–Narasimhan filtration of $\mathcal{E}$ and we get $a=m-3$. Thus $\mathcal{G}$ fits into an exact sequence

$$0\longrightarrow\mathcal{O}_X(m-3)^{\oplus(s-1)}\longrightarrow\mathcal{G}\longrightarrow\bigoplus_{p\in S}\mathcal{I}_{p,X}\longrightarrow 0.$$

Then we get $h^1(\mathcal{G}(-1))\geq 1$ from $h^1(\mathcal{I}_{p,X}(-1))=1$ and $h^2(\mathcal{O}_X(m-4))=1$. Thus $\mathcal{G}$ is not ACM, a contradiction. If X is integral, then every line bundle is stable and so (8) is the Harder–Narasimhan filtration of $\mathcal{E}$, we get either $\mathcal{L}\cong\mathcal{O}_X(m-3)$; we get a contradiction as above, or $\mathcal{L}$ is a factor of $\bigoplus_{p\in S}\mathcal{I}_{p,X}$, which is not locally free, a contradiction. $\square$

Let $\mathbb{F}(S)$ (resp. $\mathbb{F}'(S)$) be the set of isomorphism classes of middle terms of extensions in $\mathbb{E}(S)$ (resp. $\mathbb{E}'(S)$). Let us denote by $\mathcal{E}=\mathcal{E}(\varepsilon)$ the middle term of the extension corresponding to $\varepsilon\in\mathbb{E}'(S)$.

LEMMA 4.4. *For two non-empty finite sets $S_1,S_2\subset X_{\mathrm{reg}}$ with $\sharp(S_i)=s_i$, take $\mathcal{E}_i\in\mathbb{F}'(S_i)$ and call $\mathcal{A}_i$ the subsheaf of $\mathcal{E}_i$ isomorphic to $\mathcal{O}_X(m-3)^{\oplus s_i}$ for each $i=1,2$. If there exists a map $f:\mathcal{E}_1\to\mathcal{E}_2$ with $f(\mathcal{E}_1)\not\subset\mathcal{A}_2$, then we have $S_1\cap S_2\neq\emptyset$.*

PROOF. Since $\mathrm{Hom}_X(\mathcal{O}_X(m-3),\mathcal{I}_{p,X})=0$ for all $p\in X$, we have $f(\mathcal{A}_1)\subseteq\mathcal{A}_2$. In particular, f induces a nonzero map $\tilde{f}:\bigoplus_{p\in S_1}\mathcal{I}_{p,X}\to\bigoplus_{q\in S_2}\mathcal{I}_{q,X}$. This implies that $S_1\cap S_2\neq\emptyset$. $\square$

LEMMA 4.5. *Assume that $\mathcal{E}\in\mathbb{F}'(S)$ is decomposable; $\mathcal{E}\cong\mathcal{E}_1\oplus\cdots\oplus\mathcal{E}_h$ with each $\mathcal{E}_i$ indecomposable. Then there is a partition $S=\bigsqcup_{i=1}^h S_i$ with $\mathcal{E}_i\in\mathbb{F}'(S_i)$ for each i. If there is another decomposition $\mathcal{E}\cong\mathcal{E}'_1\oplus\cdots\oplus\mathcal{E}'_k$ with each $\mathcal{E}'_j$ indecomposable, then we* *get $k = h$ and there is a permutation $\sigma : \{1, \dots, h\} \to \{1, \dots, h\}$ such that $\mathcal{E}'_{\sigma(i)} \cong \mathcal{E}_i$ for all i and $\mathcal{E}'_{\sigma(i)} \in \mathbb{F}(S_{\sigma(i)})$.*

PROOF. We use induction on s. The case $s = 1$ is true, because each $\mathcal{E}_p$ for $p \in X_{\mathrm{reg}}$ is indecomposable by Proposition 3.3. Since $\mathcal{E}$ is ACM by the definition of $\mathbb{F}(S)$, each $\mathcal{E}_i$ is also ACM. We consider the subsheaf $\mathcal{A} \cong \mathcal{O}_X(m-3)^{\oplus s} \subset \mathcal{E}$ as in Remark 4.2 and set $\mathcal{G}_i := \mathcal{A} \cap \mathcal{E}_i$. Since the Harder–Narasimhan filtration of $\mathcal{E}$ is obtained from the ones of each factors, we have

$$\mathcal{A} \cong \bigoplus_{i=1}^{h} \mathcal{G}_i \quad \text{and} \quad \bigoplus_{p \in S} \mathcal{I}_{p,X} \cong \bigoplus_{i=1}^{h} \mathcal{E}_i/\mathcal{G}_i.$$

By Lemma 4.3 we have $\mathcal{G}_i \subsetneq \mathcal{E}_i$ for all i. By Remark 3.2 we may write $S = \bigsqcup_{i=1}^h S_i$ with $\mathcal{E}_i/\mathcal{G}_i \cong \bigoplus_{p \in S_i} \mathcal{I}_{p,X}$. Since $\mathcal{E}_i/\mathcal{G}_i \neq 0$, we have $S_i \neq \emptyset$ for all i. Thus the set $\{S_1, \dots, S_h\}$ gives a partition of S. To prove the first part of the lemma it is sufficient to prove that $\sharp(S_i) = \mathrm{rank}(\mathcal{E}_i)/2$ for all i. If this is not true, then there is $i \in \{1, \dots, h\}$ with $\sharp(S_i) > \mathrm{rank}(\mathcal{E}_i)/2$, i.e. $\mathrm{rank}(\mathcal{G}_i) < \sharp(S_i)$. The exact sequence

$$0 \longrightarrow \mathcal{G}_i(-1) \longrightarrow \mathcal{E}_i(-1) \longrightarrow \bigoplus_{p \in S_i} \mathcal{I}_{p,X}(-1) \longrightarrow 0$$

gives $h^1(\mathcal{E}_i(-1)) \geq \sharp(S_i) - \mathrm{rank}(\mathcal{G}_i) > 0$. In particular, $\mathcal{E}_i$ is not ACM, a contradiction.

Now we check the last assertion of the lemma. Take two partitions

$$S = S_1 \sqcup \cdots \sqcup S_h = S'_1 \sqcup \cdots \sqcup S'_k$$

such that there is a decomposition

$$\mathcal{E} \cong \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_h \cong \mathcal{E}'_1 \oplus \cdots \oplus \mathcal{E}'_k$$

with $\mathcal{E}_i \in \mathbb{F}'(S_i)$ and $\mathcal{E}'_j \in \mathbb{F}'(S'_j)$ indecomposable. By the Krull–Schmidt theorem in [**1**], we get $h = k$ and there is a permutation $\sigma : \{1, \dots, h\} \to \{1, \dots, h\}$ such that $\mathcal{B}_{\sigma(i)} \cong \mathcal{E}_i$ for all i. By renaming $\{\mathcal{E}'_1, \dots, \mathcal{E}'_h\}$, we may assume that $\mathcal{E}'_i \cong \mathcal{E}_i$ for all i. This implies

$$\sharp(S_i) = \mathrm{rank}(\mathcal{E}_i)/2 = \mathrm{rank}(\mathcal{E}'_i)/2 = \sharp(S'_i).$$

Now fix an isomorphism $f_i : \mathcal{E}_i \to \mathcal{E}'_i$ for each i. Since (8) gives the Harder–Narasimhan filtrations of $\mathcal{E}_i$ and $\mathcal{E}'_i$, the map f_i induces an isomorphism $\tilde{f}_i : \bigoplus_{p \in S_i} \mathcal{I}_{p,X} \to \bigoplus_{p \in S'_i} \mathcal{I}_{p,X}$. Since p is the unique point of X at which $\mathcal{I}_{p,X}$ is not locally free, we get $S_i = S'_i$. For each i, let $\mathcal{A}_i$ be the unique subsheaf of $\mathcal{E}_i$ isomorphic to $\mathcal{O}_X(m-3)^{\sharp(S_i)}$. Then for any embedding $u : \mathcal{E}_i \to \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_h$, the composition $v_j \circ \pi_j \circ u$

$$\mathcal{E}_i \xrightarrow{u} \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_h \xrightarrow{\pi_j} \mathcal{E}_j \xrightarrow{v_j} \bigoplus_{p \in S_j} \mathcal{I}_{p,X}$$

is zero for any $j \neq i$ by Lemma 4.4, where $\pi_j : \mathcal{E} \to \mathcal{E}_j$ is the projection and $v_j : \mathcal{E}_j \to \bigoplus_{p \in S_j} \mathcal{I}_{p,X}$ is the surjection in (8) for S_j. Since u is an embedding, we see that $v_i \circ \pi_i \circ u$

is surjective. Thus $\mathcal{G} := \pi_i(u(\mathcal{E}_i))$ is a subsheaf with $v_i(\mathcal{G}) = \bigoplus_{p\in S_i} \mathcal{I}_{p,X}$. $\square$

LEMMA 4.6. *With the setting as in Theorem* 3.10, *we have* $\mathrm{ext}^1_X(\mathcal{E}_p, \mathcal{E}_q) \geq 2$ *for two points* $p, q \in X_{\mathrm{reg}}$, *possibly* $p = q$.

PROOF. Set $\mathcal{F}_o := \mathcal{E}_o(3-m)$ for $o \in \{p, q\}$. Since $\mathrm{Ext}^i_X(\mathcal{E}_p, \mathcal{E}_q) \cong \mathrm{Ext}^i_X(\mathcal{F}_p, \mathcal{F}_q)$, we have $\chi(\mathcal{E}_p \otimes \mathcal{E}_q^\vee) = \chi(\mathcal{F}_p \otimes \mathcal{F}_q^\vee)$. Since Euler's characteristic is constant in a flat family of vector bundles and $p, q \in X_{\mathrm{reg}}$, it is sufficient to compute $\chi(\mathcal{F}_p \otimes \mathcal{F}_q^\vee)$ when X is smooth. So from now on we assume that X is smooth. Since a smooth surface in $\mathbb{P}^3$ is connected, the same observation applied to a family of vector bundles on X shows $\chi(\mathcal{F}_p \otimes \mathcal{F}_q^\vee) = \chi(\mathcal{F}_p \otimes \mathcal{F}_p^\vee)$.

We have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \overset{v}{\longrightarrow} \mathcal{F}_p \overset{w}{\longrightarrow} \mathcal{I}_{p,X}(3-m) \longrightarrow 0 \tag{9}$$

with $\det(\mathcal{F}_p) \cong \mathcal{O}_X(3-m)$ and $c_2(\mathcal{F}_p) = 1$. Since $X \subset \mathbb{P}^3$ is a surface of degree m, we have $c_1(\mathcal{F}_p)^2 = m(m-3)^2$. By Riemann–Roch for $\mathcal{E}nd(\mathcal{F}_p)$, we have

$$\begin{aligned}\chi(\mathcal{E}nd(\mathcal{F}_p)) &= c_1(\mathcal{F}_p)^2 - 4c_2(\mathcal{F}_p) + 4\chi(\mathcal{O}_X) = m(m-3)^2 - 4 + 4\binom{m-1}{3} + 4 \\ &= \frac{1}{6}\left(10m^3 - 60m^2 + 98m - 24\right).\end{aligned}$$

In particular, we have $\chi \sim (5/3)m^3$ for $m \gg 0$. Note that by Serre's duality we have $h^2(\mathcal{F}_p \otimes \mathcal{F}_p^\vee) = h^0(\mathcal{F}_p \otimes \mathcal{F}_p^\vee(m-4))$.

CLAIM 1. *We have* $\hom_X(\mathcal{F}_p, \mathcal{F}_p) = 1 + \binom{m}{3}$.

PROOF OF CLAIM 1. We have $\hom_X(\mathcal{I}_{p,X}(3-m), \mathcal{O}_X) = h^0(\mathcal{O}_X(m-3)) = \binom{m}{3}$ and any nonzero map $u : \mathcal{I}_{p,X}(3-m) \to \mathcal{O}_X$ induces an element $\tilde{u}$ in $\mathrm{Hom}_X(\mathcal{F}_p, \mathcal{F}_p)$ with rank one as the following composition:

$$\mathcal{F}_p \overset{w}{\longrightarrow} \mathcal{I}_{p,X}(3-m) \longrightarrow \mathcal{O}_X \overset{v}{\longrightarrow} \mathcal{F}_p.$$

This defines a one-to-one map $\mathrm{Hom}_X(\mathcal{I}_{p,X}(3-m), \mathcal{O}_X) \to \mathrm{Hom}_X(\mathcal{F}_p, \mathcal{F}_p)$, because for two $u, u' \in \mathrm{Hom}_X(\mathcal{I}_{p,X}(3-m), \mathcal{O}_X)$ we have $\mathrm{Im}(\tilde{u}) \neq \mathrm{Im}(\tilde{u'})$. The vector space $\mathrm{Hom}_X(\mathcal{F}_p, \mathcal{F}_p)$ also contains the nonzero multiples of the identity map $\mathcal{F}_p \to \mathcal{F}_p$ and these maps have rank two. Thus we get $h^0(\mathcal{F}_p \otimes \mathcal{F}_p^\vee) \geq 1 + \binom{m}{3}$. On the other hand, for any $f \in \mathrm{Hom}_X(\mathcal{F}_p, \mathcal{F}_p)$ we get $w \circ f \circ (v(\mathcal{O}_X)) \subseteq v(\mathcal{O}_X)$ from $h^0(\mathcal{I}_{p,X}(3-m)) = 0$. Thus $w \circ f \circ v$ induces a map $f_1 : \mathcal{O}_X \to \mathcal{O}_X$, which is induced by the multiplication by $c \in \mathbf{k}$. Hence $f - c{\cdot}\mathrm{Id}_{\mathcal{F}_p}$ is induced by a unique $g \in \mathrm{Hom}_X(\mathcal{I}_{p,X}(3-m), \mathcal{F}_p)$. Since $\mathcal{F}_p$ is locally free and X is smooth, we have $\mathrm{Hom}_X(\mathcal{I}_{p,X}(3-m), \mathcal{F}_p) = H^0(\mathcal{F}_p(m-3))$. By (9) we have $h^0(\mathcal{F}_p(m-3)) = \binom{m}{3}$ and so $\hom_X(\mathcal{F}_p, \mathcal{F}_p) \leq 1 + \binom{m}{3}$. $\square$

CLAIM 2. *We have* $\hom_X(\mathcal{F}_p, \mathcal{F}_p(m-4)) \geq \binom{2m-4}{3} + 2\binom{m-1}{3} - \binom{m-4}{3} - 1$.

PROOF OF CLAIM 2. For any $f \in \mathrm{Hom}_X(\mathcal{F}_p, \mathcal{F}_p(4-m))$, set $f_1 := f_{|v(\mathcal{O}_X)}$.

Since $h^0(\mathcal{O}_X(-1)) = 0$, we have $w \circ f_1 = 0$ and so $f_1(v(\mathcal{O}_X)) \subset v(\mathcal{O}_X(m-4))$. Take f with $f_1 \equiv 0$. Such a map f is uniquely determined by an element in $\mathrm{Hom}_X(\mathcal{I}_{p,X}(3-m), \mathcal{F}_p(m-4))$ and the converse also holds. Since $\mathcal{F}_p(m-4)$ is locally free and X is smooth at p, we have $\mathrm{Hom}_X(\mathcal{I}_{p,X}(3-m), \mathcal{F}_p(m-4)) = \mathrm{Hom}_X(\mathcal{O}_X(3-m), \mathcal{F}_p(m-4)) = H^0(\mathcal{F}_p(2m-7))$. Since $h^1(\mathcal{O}_X(t)) = 0$ for any $t \in \mathbb{Z}$, (9) gives

$$\begin{aligned} h^0(\mathcal{F}_p(2m-7)) &= h^0(\mathcal{O}_X(2m-7)) + h^0(\mathcal{O}_X(m-4)) - 1 \\ &= \binom{2m-4}{3} - \binom{m-4}{3} + \binom{m-1}{3} - 1. \end{aligned}$$

Note that a map f obtained by a composition

$$\mathcal{F}_p \xrightarrow{w} \mathcal{I}_{p,X}(3-m) \longrightarrow \mathcal{O}_X(m-4) \xrightarrow{v} \mathcal{F}_p(m-4)$$

has $f_1 \equiv 0$. Now for any linear subspace $W \subset \mathrm{Hom}_X(\mathcal{F}_p, \mathcal{F}_p(m-4))$ such that $f_1 \not\equiv 0$ for any $f \in W \setminus \{0\}$, we would get

$$\hom_X(\mathcal{F}_p, \mathcal{F}_p(m-4)) \geq \binom{2m-4}{3} - \binom{m-4}{3} + \binom{m-1}{3} - 1 + \dim W.$$

We may choose W to consist of the compositions of the identity map $\mathcal{F}_p \to \mathcal{F}_p$ with the multiplication by an element of $H^0(\mathcal{O}_X(m-4))$. Then we have $\dim W = \binom{m-1}{3}$. □

Combining Claims 1 and 2, we get

$$\begin{aligned} h^0(\mathcal{F}_p \otimes \mathcal{F}_p^\vee) + h^2(\mathcal{F}_p \otimes \mathcal{F}_p^\vee) &\geq \binom{2m-4}{3} + \binom{m}{3} + 2\binom{m-1}{3} - \binom{m-4}{3} \\ &= \frac{1}{6}\left(10m^3 - 60m^2 + 98m - 12\right). \end{aligned}$$

Thus we have

$$h^1(\mathcal{F}_p \otimes \mathcal{F}_p^\vee) = h^0(\mathcal{F}_p \otimes \mathcal{F}_p^\vee) + h^2(\mathcal{F}_p \otimes \mathcal{F}_p^\vee) - \chi(\mathcal{E}nd(\mathcal{F}_p)) \geq 2$$

and so we get the assertion. □

Proof of Theorem 3.10. By Remark 4.2 (8) is the Harder–Narasimhan filtration of $\mathcal{E}_S$. Proposition 3.3 gives the case $s = 1$. For $s > 1$, we may find a direct sum of s vector bundles of rank 2 from the case $s = 1$, fitting into (8): just take $\bigoplus_{p \in S} \mathcal{E}_p$. So a general extension in $\mathbb{E}(S)$ has a locally free and ACM middle term, because being local free and ACM are both open conditions.

Note that $h^0(\mathcal{E}_S(3-m)) = s$ from (8). In particular there is a unique subsheaf $\mathcal{A} \subset \mathcal{E}_S$ isomorphic to $\mathcal{O}_X(m-3)^{\oplus s}$ and for each $f \in \mathrm{Hom}(\mathcal{O}_X(m-3), \mathcal{E}_S)$ we have $f(\mathcal{O}_X(m-3)) \subseteq \mathcal{A}$. Now by Lemma 3.1 and Remark 3.2, the extension (8) is the Harder–Narasimhan filtration of $\mathcal{E}_S$. By uniqueness of the Harder–Narasimhan filtration, we get $\mathcal{E}_S \not\cong \mathcal{E}_{S'}$ for $S \neq S'$.

Now it remains to show the indecomposability of $\mathcal{E}_S$. By Lemma 4.3, there is no rank one factor of $\mathcal{E}_S$.

Claim 1. *For two distinct points* p, q *in* X_{reg}*, we have*

$$\mathrm{Hom}_X(\mathcal{I}_{p,X}, \mathcal{I}_{q,X}) = 0, \mathrm{Hom}_X(\mathcal{E}_p, \mathcal{I}_{q,X}) = 0 \quad \textit{and} \quad \mathrm{Ext}^1_X(\mathcal{I}_{p,X}, \mathcal{I}_{q,X}) = 0.$$

Proof of Claim 1. By an extension theorem for locally free sheaves in [**15**, Exercise I.3.20], we have $\mathrm{Hom}_X(\mathcal{I}_{p,X}, \mathcal{I}_{q,X}) = \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{I}_{q,X}) = 0$. The second vanishing is obtained from the first vanishing and $\mathrm{Hom}_X(\mathcal{O}_X(m-3), \mathcal{I}_{q,X}) = 0$. For the last vanishing, we apply the functor $\mathrm{Hom}_X(\mathcal{I}_{p,X}, -)$ to the standard exact sequence for $\mathcal{I}_{q,X} \subset \mathcal{O}_X$ and obtain an exact sequence

$$0 \longrightarrow \mathrm{Hom}_X(\mathcal{I}_{p,X}, \mathcal{O}_X) \longrightarrow \mathrm{Hom}_X(\mathcal{I}_{p,X}, \mathcal{O}_q) \longrightarrow \mathrm{Ext}^1_X(\mathcal{I}_{p,X}, \mathcal{I}_{q,X}) \longrightarrow \mathrm{Ext}^1_X(\mathcal{I}_{p,X}, \mathcal{O}_X)$$

by the first vanishing in the Claim. Here we have

$$\mathrm{Hom}_X(\mathcal{I}_{p,X}, \mathcal{O}_X) \cong \mathrm{Hom}_X(\mathcal{I}_{p,X}, \mathcal{O}_q) \cong \mathbf{k}$$

and $\mathrm{Ext}^1_X(\mathcal{I}_{p,X}, \mathcal{O}_X) \cong H^1(\mathcal{I}_{p,X}(m-4))^\vee$ by Serre's duality. Then we get the assertion from the assumption that $m \geq 4$. □

(a) First assume $s = 2$ and take two distinct points p, q in X_{reg}.

Claim 2. *If there exists a sheaf* $\mathcal{G} \ncong \mathcal{E}_p \oplus \mathcal{E}_q$ *fitting into the exact sequence*

$$0 \longrightarrow \mathcal{E}_p \xrightarrow{u} \mathcal{G} \xrightarrow{v} \mathcal{E}_q \longrightarrow 0, \tag{10}$$

then the case $s = 2$ *is true.*

Proof of Claim 2. Such a sheaf $\mathcal{G}$ would be locally free and ACM with rank 4. Since $h^1(\mathcal{O}_X) = 0$ and (8) gives the Harder–Narasimhan filtrations of $\mathcal{E}_p$ and $\mathcal{E}_q$ by Lemma 3.1 and Remark 3.2, $\mathcal{G}$ has a subsheaf $\mathcal{F} \cong \mathcal{O}_X(m-3)^{\oplus 2}$ such that $\mathcal{G}/\mathcal{F}$ is an extension of $\mathcal{I}_{q,X}(1)$ by $\mathcal{I}_{p,X}(1)$. Claim 1 gives $\mathcal{G}/\mathcal{F} \cong \mathcal{I}_{p,X} \oplus \mathcal{I}_{q,X}$ and so we get $\mathcal{G} \cong \mathcal{E}_S$ with $S = \{p, q\}$. □

Claim 3. *If* $\mathcal{G} \cong \mathcal{E}_p \oplus \mathcal{E}_q$ *for all* $\mathcal{G}$ *in* (10)*, then we have* $\mathrm{Ext}^1_X(\mathcal{E}_q, \mathcal{E}_p) = 0$.

Proof of Claim 3. Let $\mathcal{G} \cong \mathcal{E}_p \oplus \mathcal{E}_q$ fitting into (10) correspond to $\varepsilon \in \mathrm{Ext}^1_X(\mathcal{E}_q, \mathcal{E}_p)$. Then it is sufficient to prove that $\varepsilon = 0$, or $\ker(v) \cong \mathcal{E}_p \oplus \{0\}$. But since $\ker(v) \cong \mathcal{E}_p$, it is sufficient to prove that either $\mathcal{E}_p \oplus \{0\} \supseteq \ker(v)$ or $\mathcal{E}_p \oplus \{0\} \subseteq \ker(v)$. Assume $v(\mathcal{E}_p \oplus \{0\}) \neq 0$. Since $\mathrm{Hom}_X(\mathcal{E}_p, \mathcal{I}_{q,X}) = 0$ by Claim 1, we have $v(\mathcal{E}_p \oplus \{0\}) \subseteq \mathcal{O}_X(m-3)$. This implies that the restriction of the surjection $\mathcal{E}_q \to \mathcal{I}_{q,X}$ to $v(\{0\} \oplus \mathcal{E}_q)$ is surjective. Since $h^0(\mathcal{O}_X) = 1$ and $\mathrm{Hom}_X(\mathcal{O}_X(m-3), \mathcal{I}_{q,X}) = 0$, we get either $v(\{0\} \oplus \mathcal{O}_X(m-3)) = 0$ or v induces an isomorphism $\{0\} \oplus \mathcal{O}_X(m-3) \to \mathcal{O}_X(m-3)$. Assume for the moment $v(\{0\} \oplus \mathcal{O}_X(m-3)) = 0$. Since $v(\mathcal{E}_p \oplus \{0\})$ maps to 0 in $\mathcal{I}_{q,X}$, we get that $v(\{0\} \oplus \mathcal{E}_q)$ is a subsheaf of $\mathcal{E}_q$ which maps isomorphically onto $\mathcal{I}_{q,X}$. So we get $\mathcal{E}_q \cong \mathcal{O}_X(m-3) \oplus \mathcal{I}_{q,X}$, a contradiction. Now assume $v(\{0\} \oplus \mathcal{O}_X(m-3)) = \mathcal{O}_X(m-3)$. Since $v(\{0\} \oplus \mathcal{E}_q)$ maps surjectively onto $\mathcal{I}_{q,X}$, the surjection v induces an isomorphism $\{0\} \oplus \mathcal{E}_q \to \mathcal{E}_q$. Hence we get $\mathcal{E}_p \oplus \{0\} \subseteq \ker(v)$. □

Since $\mathrm{Ext}^1_X(\mathcal{E}_q, \mathcal{E}_p) \neq 0$ by Lemma 4.6, Claim 3 concludes the proof of the case $s = 2$.

(b) Assume $s > 2$ and that Theorem 3.10 holds for smaller numbers. On $\mathbb{E}(S)$ there is a universal family of extensions, i.e. a coherent sheaf $\mathcal{V}$ over $\mathbb{E}(S) \times X$ such that for each $\varepsilon \in \mathbb{E}(S)$ the sheaf $\mathcal{V}_{|\{\varepsilon\} \times X}$ is the middle term $\mathcal{E}(\varepsilon)$ of the extension corresponding to ε. We call $\mathcal{V}'$ the restriction of $\mathcal{V}$ to $\mathbb{E}'(S) \times X$; we thus consider the family of ACM vector bundles induced from the extensions in $\mathbb{E}'(S)$. Call $\pi_1 : \mathbb{E}'(S) \times X \longrightarrow \mathbb{E}'(S)$ the projection onto the first factor, and set $\mathcal{A}_S := \pi_{1*}\mathcal{H}om(\mathcal{V}', \mathcal{V}')$. Since π_1 is a proper morphism, $\mathcal{A}_S$ is a coherent sheaf on $\mathbb{E}'(S)$. This sheaf has $\mathbb{E}'(S)$ as its support, because every vector bundle has the identity map. Since $\mathbb{E}'(S)$ is an integral variety, there is a non-empty open subset $\mathbb{E}(S)_0 \subseteq \mathbb{E}'(S)$ such that $(\mathcal{A}_S)_{|\mathbb{E}(S)_0}$ is locally free. Set $\mathcal{V}_0 := (\mathcal{V}')_{|\mathbb{E}(S)_0 \times X}$. Note that for each $\varepsilon \in \mathbb{E}(S)_0$ the fiber of $\mathcal{A}_S$ at ε is the vector space $\mathrm{End}(\mathcal{E}(\varepsilon))$.

Define $\Gamma(S)$ as a subset of the total space of $\mathcal{A}_S$ as follows:

$$\Gamma(S) := \left\{(\varepsilon, \varphi) \mid \varepsilon \in \mathbb{E}(S)_0 \text{ and } \varphi \in \mathrm{End}(\mathcal{E}(\varepsilon)) \text{ with } \varphi^2 = \varphi\right\}.$$

Note that φ is a projection of $\mathcal{E}(\varepsilon)$ onto a factor of $\mathcal{E}(\varepsilon)$, with the exception when $\varphi = \mathrm{Id}_{\mathcal{E}(\varepsilon)}$ or $\varphi \equiv 0$; if $\mathcal{E}(\varepsilon)$ is indecomposable, only $(\varepsilon, \mathrm{Id}_{\mathcal{E}(\varepsilon)})$ and $(\varepsilon, 0)$ are contained in $\Gamma(S)$. Indeed, for any vector bundle $\mathcal{G}$, there exists a one-to-one correspondence:

$$\{\varphi \in \mathrm{End}(\mathcal{G}) \mid \varphi^2 = \varphi\} \leftrightarrow \{\text{factors of } \mathcal{G}\}$$

via $\varphi \mapsto \mathrm{Im}(\varphi) = \ker(\mathrm{Id}_{\mathcal{G}} - \varphi)$, with $\mathcal{G}$ being associated to $\mathrm{Id}_{\mathcal{G}}$ and 0 associated to the zero map. Thus $\mathcal{G}$ is decomposable if and only if $\mathrm{End}(\mathcal{G})$ has a non-trivial idempotent. Note that $\Gamma(S)$ is closed in the total space of the vector bundle $\pi_{1*}\mathcal{H}om(\mathcal{V}_0, \mathcal{V}_0)$ over $\mathbb{E}(S)_0$. By Lemma 4.5, for each $\mathcal{E}(\varepsilon)$ there is a unique partition of S associated to any decomposition of $\mathcal{E}(\varepsilon)$ with only finitely many indecomposable factors by the Krull–Schmidt theorem in [**1**]. By Lemma 4.5 for each $\mathcal{E} \in \mathbb{F}'(S)$ each isomorphism class of factors of $\mathcal{E}$ corresponds to a unique subset of S; $\mathcal{E}$ and 0 correspond to S and $\emptyset$, respectively. For each $(\varepsilon, \varphi) \in \Gamma(S)$, let $S(\varphi)$ be the subset of S associated to $\mathrm{Im}(\varphi)$ by Lemma 4.5. Set

$$\Gamma_0(S) := \left\{(\varepsilon, \varphi) \in \Gamma(S) \;\middle|\; \varphi \neq 0 \text{ and } \varphi \neq \mathrm{Id}_{|\mathcal{E}(\varepsilon)}\right\}.$$

The goal is to show that $\Gamma_0(S)$ is not dominant over $\mathbb{F}(S)$ for a general S.

Note that up to now we did not use that S is contained in the same connected component $Y \cap X_{\mathrm{reg}}$ of X_{reg}. In particular the case $s = 2$ holds even if X has more than one irreducible components with multiplicity one and the two points of S belong to different connected components of X_{reg}.

Now we use a monodromy argument, which requires that S is contained in a connected component of $T := X_{\mathrm{reg}} \cap Y$ and that S is general in Y. Set $S = \{p_1, \dots, p_s\}$ and fix an ordering of the points in S, along which we get an ordering of the indecomposable factors of the sheaf $\bigoplus_{p \in S} \mathcal{I}_{p,X}$. Together with the usual ordering on the factors of $\mathcal{O}_X(m-3)^{\oplus s}$, we may see any $\varepsilon \in \mathbb{E}(S)$ as an $(s \times s)$-square matrix, say $\varepsilon = (\varepsilon_{ij})$ with $1 \leq i, j \leq s$, where ε_{ij} is an element of the 1-dimensional vector space $\mathrm{Ext}^1_X(\mathcal{I}_{p_j,X}, \mathcal{O}_X(m-3))$. In particular, if $\varepsilon \in \mathbb{E}(S)$ is general, then the associated $(s \times s)$-

square matrix is also general in the space of all such $(s\times s)$-square matrices. Note that for a fixed integer j, each ε_{ij} with $i=1,\dots,s$, is an element of the same 1-dimensional vector space. We write $\mathcal{O}_X(m-3)^{\oplus s}=\mathbb{C}^s\otimes\mathcal{O}_X(m-3)$. In Claim 4 below, we assume that S is general in T, so that we may use the inductive assumption for all proper subsets of S.

Claim 4. *$\mathcal{E}=\mathcal{E}(\varepsilon)$ has two indecomposable factors, one of them being* $\mathrm{Im}(\varphi)$ *and the other one being* $\ker(\varphi)$.

Proof of Claim 4. Since $\varphi^2=\varphi$, we have $\mathcal{E}\cong\mathcal{F}_1\oplus\mathcal{F}_2$ with $\mathcal{F}_1:=\mathrm{Im}(\varphi)$ and $\mathcal{F}_2=\ker(\varphi)$. By the definition of A, we get an exact sequence

$$0\longrightarrow\mathcal{O}_X(m-3)^{\oplus k}\longrightarrow\mathcal{F}_1\longrightarrow\bigoplus_{p\in A}\mathcal{I}_{p,X}\longrightarrow 0, \tag{11}$$

with $k:=\sharp(A)$. Since neither $\varphi\equiv 0$ nor $\varphi=\mathrm{Id}_{\mathcal{E}}$, we have $0<k<s$. Then by Lemma 4.5 we get an exact sequence

$$0\longrightarrow\mathcal{O}_X(m-3)^{\oplus(s-k)}\longrightarrow\mathcal{F}_2\longrightarrow\bigoplus_{p\in S\setminus A}\mathcal{I}_{p,X}\longrightarrow 0. \tag{12}$$

Now we need to prove that each $\mathcal{F}_i$ is indecomposable. By the inductive assumption it is sufficient to prove that $\mathcal{F}_1$ and $\mathcal{F}_2$ are the middle terms of general extensions (11) and (12), respectively. Since (8) gives the Harder–Narasimhan filtration of each $\mathcal{F}_i$, there are linear subspaces $V_1,V_2\subset\mathbb{C}^s$ such that $\dim V_1=k$, $\dim V_2=s-k$ and

$$v(\mathbb{C}^s\otimes\mathcal{O}_X(m-3))\cap\mathcal{F}_i=V_i\otimes\mathcal{O}_X(m-3)$$

for each i. From $\mathcal{E}\cong\mathcal{F}_1\oplus\mathcal{F}_2$ we see that $\mathbb{C}^s=V_1\oplus V_2$. Now we reorder the points in S so that all points of A are smaller than any points of $S\setminus A$. Then ε can be understood as an $(s\times s)$-square matrix in a block form:

$$\varepsilon=\left[\begin{array}{c|c}B_{11}&B_{12}\\\hline B_{21}&B_{22}\end{array}\right].$$

Here the $(k\times k)$-matrix B_{11} in the upper left corner, is associated to the extension (11) and similarly the $((s-k)\times(s-k))$-matrix B_{22} in the lower right corner, is associated to the extension (12). The matrix of ε also has a $(k\times(s-k))$-submatrix B_{12} and an $((s-k)\times k)$-submatrix B_{21}. By assumption the $(s\times s)$-square matrix corresponding to ε is general, and this implies that each block matrix B_{ij} is also general in the space of the corresponding sized matrices. In particular, B_{11} and B_{22} are general and this implies that each $\mathcal{F}_i$ is general. The inductive assumption gives that each $\mathcal{F}_i$ is indecomposable. □

Assume that a general $\mathcal{E}=\mathcal{E}(\varepsilon)$ has two indecomposable factors, i.e. the set $\Gamma_0(S)$ is dominant over $\mathbb{F}(S)$. Let $\Gamma'(S)$ be an irreducible component of $\Gamma_0(S)$ dominant over $\mathbb{F}(S)$ and set $A:=S(\varphi)$, where (ε,φ) is any element of $\Gamma'(S)$. Now assume that (ε,φ) is general in $\Gamma'(S)$ and set $\mathcal{E}:=\mathcal{E}(\varepsilon)$. Note that the subset $A\subset S$ is invariant as (ε,φ) varies in $\Gamma_0(S)$, due to the irreducibility of $\Gamma_0(S)$. Let $\mathrm{Sym}^s(T)_0$ denote the set of all subsets of

$T \cap X_{\rm reg}$ with cardinality s. Below we find a contradiction under the assumptions that $\mathcal{E}$ is decomposable and that S is general in $\mathrm{Sym}^s(T)_0$. On $\mathrm{Sym}^s(T)_0 \times X$, we have a family $\mathbb{E}_T$ of relative Ext^1-group, whose fibre over $S \in \mathrm{Sym}^s(T)_0$ is $\mathbb{E}(S)$. Denoting its universal family by $\mathcal{V}'_T$, choose a non-empty subset $\mathcal{V}_T \subset \mathcal{V}'_T$ corresponding to the locally free ACM extensions. For $\pi_2 : \mathbb{E}_T \times X \to X$ the second projection, set $\mathcal{A}_T := \pi_{2*}\mathcal{H}om(\mathcal{V}_T, \mathcal{V}_T)$. Note that an element of $\mathcal{A}_T$ represents a triple $(S, \varepsilon, \varphi)$ with $(S, \varepsilon) \in \Delta_0$ and $\varphi : \mathcal{E}(\varepsilon) \to \mathcal{E}(\varepsilon)$ an endomorphism. Since $\mathbb{E}_T$ is an integral variety, there is a non-empty open subset $\Delta_0 \subset \mathbb{E}_T$ such that $\mathcal{A}_{T|\Delta_0}$ is locally free. Then by restricting Δ_0 and $\mathbb{E}(S)_0$, we may assume that $\mathcal{A}_T$ is an algebraic subset whose fibre over $S \in \mathrm{Sym}^s(T)_0$ is $\Gamma_0(S)$, with a projection map $u : \mathcal{A}_T \to \mathrm{Sym}^s(T)_0$. If u is not dominant, then it would imply that there exists a $2s$-dimensional family of pairwise not isomorphic indecomposable ACM vector bundles of rank $2s$ on X. Thus we may assume that u is dominant. We fix a general $S \in \mathrm{Sym}^s(T)$ and fix an irreducible component $\Gamma'(S)$ of $\Gamma(S)$ to which we apply the previous construction with the partition $A \sqcup (S \setminus A)$ of S attached to $\Gamma'(S)$. Let $\mathcal{A}'_T$ be any irreducible component of $\mathcal{A}_T$ containing $\Gamma'(S)$ such that $u_{|\mathcal{A}'_T}$ is dominant.

Let $\mathcal{U}$ denote a non-empty Zariski open subset of $\mathrm{Sym}^s(T)$ containing S with $A = S(\varphi)$ such that for every $S' \in \mathcal{U}$ a general $\mathcal{E}_{S'} \in \mathbb{E}(S')$ has exactly two indecomposable factors, one associated to a subset F of S' with $|F| = |A| = k$ and the other one associated to $S' \setminus F$. Now we fix $p \in A$ and $q \in S \setminus A$. Since $Y_{\rm reg}$ is a connected manifold and $p, q \in Y_{\rm reg}$, there exists a non-empty Zariski open subset $U \subset \mathbb{A}^1(\mathbf{k})$ with a map $\varphi : U \to Y_{\rm reg}$ such that $\varphi(t_0) = p$ and $\varphi(t_1) = q$ for some $t_0, t_1 \in U$, and $\varphi(U)$ passes no other points of S. Similarly we may consider a map $\varphi' : U \to Y_{\rm reg}$ with $\varphi'(t_1) = p$ and $\varphi'(t_0) = q$ such that $\varphi(t) \neq \varphi'(t)$ for any $t \in U$. For each $t \in U$, set

$$A_t := (A \setminus \{p\}) \cup \{\varphi(t)\}, \quad S_t := (S \setminus \{p, q\}) \cup \{\varphi(t), \varphi'(t)\},$$

e.g. $(A_{t_0}, S_{t_0}) = (A_{t_1}, S_{t_1}) = (A, S)$. Restricting U to an open neighborhood of $\{t_0, t_1\}$, we may assume that $S_t \in \mathcal{V}$ for all $t \in U$. Then for each $t \in U$ we have a partition $S_t = A_t \sqcup (S_t \setminus A_t)$ such that a general $\mathcal{E}_{S_t} \in \Gamma'(S_t)$ has exactly two indecomposable factors, one associated to A_t and the other associated to $S_t \setminus A_t$, due to the choice of $\mathcal{A}'_T$.

We start from $t = t_0$ and vary t in U to arrive at $t = t_1$, where we have $S_{t_1} = S = A_q \sqcup (S \setminus A_q)$ with $A_q = (A \setminus \{p\}) \cup \{q\}$. Since $s > 2$, we have $\{A, S \setminus A\} \neq \{A_q, S \setminus A_q\}$, contradicting the assumption that $\mathcal{E}_S$ has exactly two indecomposable factors. $\square$

Proof of Theorem 1.1. The family Σ of all $S \subset X_{\rm reg}$ with $\sharp(S) = s$ clearly has dimension $2s$. By Theorem 3.10, if S and S' are two distinct sets in Σ, then we get $\mathcal{E}_S \not\cong \mathcal{E}_{S'}$. Now there is a universal family on any Ext^1-group of families of sheaves with $\Sigma \times X$ as its base; see [**19**, Proposition 3.1]. Thus, we get a family of ACM locally free and indecomposable vector bundles with as a parameter space a rank s^2 vector bundle over Σ; the fibre of this vector bundle over $S \in \Sigma$ is $\mathbb{E}(S)$, corresponding to S. Choose a non-empty open subset V of Σ on which this vector bundle is trivial. Then a non-zero section of this bundle over V parametrizes pairwise non-isomophic, ACM and indecomposable vector bundles. $\square$

Remark 4.7. We start with an observation by Matsumura and Monsky. Let $Y \subset \mathbb{P}^{n+1}$ with $n \geq 2$ be a smooth hypersurface of degree $d \geq 3$. Then the set of all

$f \in \mathrm{Aut}(\mathbb{P}^{n+1})$ such that $f(Y) = Y$ is finite by [**21**, Theorem 1]. For any projective scheme X, A. Grothendieck proved that the set $\mathrm{Aut}(X)$ of all automorphisms of X is locally algebraic, i.e. it is a countable disjoint union of algebraic schemes; see [**22**, Theorem 5.23 and Exercise on page 133]. The connected component $\mathrm{Aut}^0(X)$ of $\mathrm{Aut}(X)$ containing the identity map is thus a finite-dimensional algebraic group, but it may have infinitely many (countable) connected components and even modulo the connected component of the identity it may not be finitely presented. However, for a smooth surface $X \subset \mathbb{P}^3$ with $m := \deg(X) > 4$, the situation is simpler for the following reason, as explained in [**21**] in general; see also [**24**, Theorem 1.8]. Every automorphism f of X preserves $\omega_X \cong \mathcal{O}_X(m-4)$ and hence induces a linear isomorphism $H^0(\mathcal{O}_X(m-4)) \to H^0(\mathcal{O}_X(m-4))$. In particular, it also induces an automorphism of $H^0(\mathcal{O}_X(1))$ and so a projective linear transformation of X, because we have

$$H^0(\mathcal{O}_X(m-4)) \cong H^0(\mathcal{O}_{\mathbb{P}^3}(m-4)) \cong \mathrm{Sym}^{m-4} H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \cong \mathrm{Sym}^{m-4} H^0(\mathcal{O}_X(1)).$$

Thus [**21**, Theorem 1] gives that $\mathrm{Aut}(X)$ is finite. For $m = 4$ the situation is different. There are smooth quartic surfaces $X \subset \mathbb{P}^3$ with discrete automorphism group and with an automorphism of infinite order; refer to [**23**, Part (2) of Theorem 1]. See [**9**], [**24**] and references therein for many other very interesting automorphism groups of K3 surfaces. Obviously since each automorphism of X preserves the singular locus we know that $\mathrm{Aut}(X)$ is small for singular surfaces X.

Hence over any uncountable algebraically closed field, there is an integer t_0 such that for every even integer r, X has a family of dimension at least $r - t_0$, consisting of indecomposable ACM vector bundles of rank r on X with each isomorphism class of vector bundles appearing at most countably many times in this family. If $m > 4$ we may drop the assumption that the base field is uncountable and find a family such that each isomorphism class only appears finitely many times in the family.

5. Non-locally free ACM sheaf.

In this section, we let $X \subset \mathbb{P}^N$ be a closed subscheme with pure dimension n at least two. Assume that X is ACM with respect to $\mathcal{O}_X(1)$, i.e. $h^i(\mathcal{I}_{X,\mathbb{P}^N}(t)) = 0$ for all $t \in \mathbb{Z}$ and all $1 \le i \le n$, and that each local ring $\mathcal{O}_{X,x}$ with $x \in X$ has positive depth. The exact sequence

$$0 \longrightarrow \mathcal{I}_{X,\mathbb{P}^N}(t) \longrightarrow \mathcal{O}_{\mathbb{P}^N}(t) \longrightarrow \mathcal{O}_X(t) \longrightarrow 0$$

shows that $h^i(\mathcal{I}_{X,\mathbb{P}^N}(t)) = h^{i-1}(\mathcal{O}_X(t))$ for all $i \ge 2$. Hence we may restate our assumption as $h^1(\mathcal{I}_{X,\mathbb{P}^N}(t)) = 0$ and $h^i(\mathcal{O}_X(t)) = 0$ for all $t \in \mathbb{Z}$ and $i = 1, \ldots, n-1$. By [**25**, Théorème 1 on page 268] the condition that $h^i(\mathcal{O}_X(-x)) = 0$ for $x \gg 0$ and $i = 1, \ldots, n-1$, plus having positive depth at each $x \in X$, is equivalent to all $\mathcal{O}_{X,x}$ having depth n. Since $h^1(\mathcal{I}_{X,\mathbb{P}^N}) = 0$, we have $h^0(\mathcal{O}_X) = 1$ and in particular X is connected. Since $h^1(\mathcal{I}_{X,\mathbb{P}^N}(1)) = 0$, X is linearly normal in the linear subspace of $\mathbb{P}^N$ spanned by X. Since $n \ge 2$ we have $h^1(\mathcal{O}_X) = 0$ an so $\mathrm{Pic}(X)$ is a finitely generated abelian group.

Lemma 5.1. *Assume X ACM. Let $C \subset X$ be a reduced ACM subvariety of pure* *dimension $n-1$. Then its ideal sheaf $\mathcal{I}_{C,X}$ is an ACM $\mathcal{O}_X$-sheaf such that*

- *it is locally free outside C and*
- *for any closed subscheme $Y \subsetneq X$, it is not an $\mathcal{O}_Y$-sheaf.*

PROOF. Since C is ACM as a closed subscheme of $\mathbb{P}^N$ and C has pure dimension $n-1$, we have $h^1(\mathcal{I}_{C,\mathbb{P}^N}(t)) = 0$ for all $t \in \mathbb{Z}$. Thus the restriction map $\rho_t : H^0(\mathcal{O}_{\mathbb{P}^N}(t)) \to H^0(\mathcal{O}_C(t))$ is surjective for any $t \in \mathbb{Z}$. Since ρ_t factors through the restriction map $\eta_t : H^0(\mathcal{O}_X(t)) \to H^0(\mathcal{O}_C(t))$, η_t is surjective. Since η_t is surjective and $h^1(\mathcal{O}_X(t)) = 0$, we have $h^1(\mathcal{I}_{C,X}(t)) = 0$. This implies that $\mathcal{I}_{C,X}$ is ACM. From $\mathcal{I}_{C,X\setminus C} \cong \mathcal{O}_{X\setminus C}$, we see that $\mathcal{I}_{C,X}$ is locally free and of rank 1 outside C. Since C is not an irreducible component of X_{red} and $\mathcal{I}_{C,X}$ is locally free of positive rank outside C, there is no closed subscheme $Y \subsetneq X$ with $\mathcal{I}_{C,X}$ an $\mathcal{O}_Y$-sheaf. □

LEMMA 5.2. *Assume that $X \subset \mathbb{P}^N$ is an ACM close subscheme with an ACM irreducible component Y of X_{red}. For a fixed integer $e > 0$ and any integral divisor $C \in |\mathcal{O}_Y(e)|$, define*

$$\Sigma_C := \{p \in Y \mid \mathcal{I}_{C,X} \text{ is not locally free at } p\}.$$

(i) *If X is not reduced at a general point of X, then we have $\Sigma_C = C$, i.e. for all $p \in C$ the sheaf $\mathcal{I}_{C,X}$ is not locally free at p. For any two integral curves $C_1, C_2 \in |\mathcal{O}_Y(e)|$, we have $\mathcal{I}_{C_1,X} \cong \mathcal{I}_{C_2,X}$ if and only if $C_1 = C_2$.*

(ii) *Assume that X is reduced at a general point of Y and that X is not integral. Let F be the intersection of Y with the other irreducible components of X. Then we have $F \neq \emptyset$ and F has pure dimension $n-1$. Moreover, we have $\Sigma_C = (F \cap C)_{\mathrm{red}}$ and $F \cap C \neq \emptyset$.*

(iii) *For any two integral divisors $C_1, C_2 \in |\mathcal{O}_Y(e)|$ such that $\mathcal{I}_{C_1,X} \cong \mathcal{I}_{C_2,X}$, we have $\Sigma_{C_1} = \Sigma_{C_2}$; in case* (i) *we have the converse.*

PROOF. By Lemma 5.1 the sheaf $\mathcal{I}_{C,X}$ is ACM and locally free with rank 1 at all $p \in X\setminus C$. Fix $p \in C$ and assume that $\mathcal{I}_{C,X}$ is locally free at p. Then there is $w \in (\mathcal{I}_{C,X})_p$ such that w is not a zero-divisor of $\mathcal{O}_{X,p}$ and $(\mathcal{I}_{C,X})_p \cong w\mathcal{O}_{X,p}$ as a module over the local ring $\mathcal{O}_{X,p}$. We get that in a neighborhood of p the divisor C is a Cartier divisor of X. Let $I \subset \mathcal{O}_{X,p}$ be the ideal of Y and $J \subset \mathcal{O}_{X,p}$ the ideal of C. We have $I \subset J$. First assume that X is not reduced at a general point of X. Since the support of the nilradical $\eta \subset \mathcal{O}_X$ of the structural sheaf $\mathcal{O}_Y$ is a closed subset of X_{red}, X is not reduced at any point of Y and in particular it is not reduced at p. Thus there is a nonzero $h \in I$ such that $h^m = 0$ for some $m > 0$. Since $I \subset J$, we have $h \in J$ and so h is divided by w. Thus we get $w^m = 0$ and so w is a zero-divisor, a contradiction.

Now assume that X is reduced at a general point of Y. Obviously $\mathcal{I}_{C,X}$ is locally free outside the support of C. Since X is ACM, it is connected and so $F \neq \emptyset$. More precisely, since all local rings $\mathcal{O}_{X,x}$ have depth n, X_{red} is locally connected in dimension $\leq n-1$ and so F has pure dimension $n-1$. Since $C \in |\mathcal{O}_Y(e)|$, C is a Cartier divisor of Y. Thus C is a Cartier divisor of X at all points of $C \setminus (C \cap F)$. Since $e > 0$, C is an ample divisor of Y. In particular, we get $F \cap C \neq \emptyset$. Fix $p \in F \cap C$. Any local equation w of C at p vanishes on each irreducible component of X_{red} containing p, because w is assumed to be a non-zero divisor of $\mathcal{O}_{X,p}$. There is at least one another irreducible component of X_{red} containing p, because $p \in F$.

Part (iii) is obvious. □

As a corollary of Lemma 5.2 we get the following result, which shows that X is of wild representation type in a very strong form.

PROPOSITION 5.3. *Let $X \subset \mathbb{P}^N$ be a non-integral closed ACM subscheme with pure dimension at least two such that there exists an ACM irreducible component Y of X_{red}. For a fixed integer $w > 0$, there is an integral quasi-projective variety Δ and a flat family $\{\mathcal{F}_a\}_{a\in\Delta}$ of ACM sheaves of rank one on X with each $\mathcal{F}_a$ locally free outside a one-codimensional subscheme C_a and for each $a \in \Delta$ the set of all $b \in \Delta$ such that $\mathcal{F}_b \cong \mathcal{F}_a$ is contained in an algebraic subscheme $\Delta_a \subset \Delta$ with $\dim \Delta - \dim \Delta_a \geq w$.*

PROOF. First assume that Y has the multiplicity at least two. Fix a positive integer e such that $\dim |\mathcal{O}_Y(e)| \geq w$ and take as Δ the family of all integral $C \in |\mathcal{O}_Y(e)|$. Then we may apply (i) of Lemma 5.2. In this case we may find Δ with the additional condition that for all $a, b \in \Delta$ we have $\mathcal{F}_a \cong \mathcal{F}_b$ if and only if $a = b$.

Now assume that the multiplicity of Y in X is one. Write $F \subset Y$ as in (ii) of Lemma 5.2. Fix an integer $e > 0$ such that $h^0(\mathcal{O}_X(e)) - h^0(\mathcal{O}_X(e)(-F)) > w$ and let Δ be the set of all integral divisors $C \in |\mathcal{O}_X(e)|$ not contained in F and such that the scheme $F \cap C$ is reduced. Since F has pure dimension $n-1$ and C is an ample divisor, the set $(F \cap C)_{\mathrm{red}}$ has pure codimension 2. Fix any finite set $B \subset F$. For $e \gg 0$ we may find $C \in |\mathcal{O}_Y(e)|$ containing no irreducible component of F and with $B \subset C$. Take $|B| \geq w$. Then we may apply (ii) of Lemma 5.2. □

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