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Jacquet–Langlands–Shimizu correspondence for theta lifts to $GSp(2)$ and its inner forms II: An explicit formula for Bessel periods and the non-vanishing of theta lifts

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Abstract. This paper is a continuation of the first paper. The aim of this second paper is to discuss the non-vanishing of the theta lifts to the indefinite symplectic group $GSp(1,1)$, which have been shown to be involved in the Jacquet–Langlands–Shimizu correspondence with some theta lifts to the $\mathbb{Q}$-split symplectic group $GSp(2)$ of degree two. We study an explicit formula for the square norms of the Bessel periods of the theta lifts to $GSp(1,1)$ in terms of central L-values. This study involves two aspects in proving the non-vanishing of the theta lifts. One aspect is to apply the results by Hsieh and Chida–Hsieh on "non-vanishing modulo p" of central L-values for some Rankin L-functions. The other is to relate such non-vanishing with studies on some special values of hypergeometric functions. We also take up the theta lifts to the compact inner form $GSp^*(2)$. We provide examples of the non-vanishing theta lifts to $GSp^*(2)$, which are essentially due to Ibukiyama and Ihara.

1. Introduction.

1.1. The aim of the paper.

Let B be a definite quaternion algebra over $\mathbb{Q}$ with the discriminant d_B and D a divisor of d_B. Let (f, f') be a pair consisting of an elliptic cusp form f of weight κ_1 and level D (cf. [**19**, Section 3.1]) and an automorphic form f' on $B_{\mathbb{A}}^\times$ of weight κ_2 (cf. [**19**, Section 3.2]). For such (f, f') we have introduced the theta lifts to the non-compact inner form $GSp(1,1)$ and the compact inner form $GSp^*(2)$ of the $\mathbb{Q}$-split symplectic group $GSp(2)$ of degree two uniformly denoted by $\mathcal{L}(f, f')$ in the previous paper [**21**]. In this paper we denote the former and the latter by $\mathcal{L}^{\mathrm{nc}}(f, f')$ and $\mathcal{L}^{\mathrm{c}}(f, f')$ respectively. In [**21**], assuming that f and f' are Hecke eigenforms, we have shown that the automorphic representations generated by the theta lifts $\mathcal{L}^{\mathrm{nc}}(f, f')$ and $\mathcal{L}^{\mathrm{c}}(f, f')$ are involved in the Jacquet–Langlands–Shimizu correspondence with the theta lift $\pi'(f, \mathrm{JL}(f'))$ to $GSp(2)$ (for the definition of $\pi'(f, \mathrm{JL}(f'))$ see [**21**, Section 4.4]), where $\mathrm{JL}(f')$ denotes the primitive cusp form corresponding to f'. We have remarked that the Jacquet–Langlands–Shimizu correspondence just mentioned should preserve a correspondence between paramodular level

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structures for $GSp(2)$ and some invariance conditions with respect to maximal open compact subgroups for the inner forms (cf. [**21**, Conjecture 4.2]) and that the automorphic representation generated by $\mathcal{L}^{\mathrm{nc}}(f,f')$ or $\mathcal{L}^{\mathrm{c}}(f,f')$ corresponds to $\pi'(f,\mathrm{JL}(f'))$ in this manner.

We now describe the two theta lifts $\mathcal{L}^{\mathrm{nc}}(f,f')$, $\mathcal{L}^{\mathrm{c}}(f,f')$ by the following:

$$(f,f')\mapsto\begin{cases}\mathcal{L}^{\mathrm{nc}}(f,f') & (1<\kappa_1<\kappa_2+2),\\ \mathcal{L}^{\mathrm{c}}(f,f') & (1<\kappa_2+2<\kappa_1).\end{cases}$$

The inequality of (κ_1,κ_2) is due to the regularity of the Harish-Chandra parameters for the archimedean representation generated by (f,f'). In fact, the archimedean parts of the lifts $\mathcal{L}^{\mathrm{nc}}(f,f')$ and $\mathcal{L}^{\mathrm{c}}(f,f')$ are understood in terms of the theta correspondence of discrete series representations for $(Sp(1,1)(\mathbb{R}),O_4^*(\mathbb{R}))$ and $(Sp^*(2)(\mathbb{R}),O_4^*(\mathbb{R}))$ respectively (see [**15**, Theorem 5.1, Part 1] and [**21**, Remark 3.4 (1)]). We should remark that (f,f') cannot have the non-vanishing theta lifts to $GSp(1,1)$ and $GSp^*(2)$ simultaneously under the same inequality condition on (κ_1,κ_2) (see [**15**, Theorem 5.1, Part 3] and [**21**, Remark 3.4 (2)]). Taking the Jacquet–Langlands correspondence mentioned above into account, some experts would try to understand the two lifts in terms of the global Arthur L-packets of $GSp(2)$ studied by Roberts [**23**]. In fact, the corresponding Galois sides of our lifts belong to “Type (B) parameters” in the sense of [**23**].

The main goal of this paper is to discuss the non-vanishing of the theta lift $\mathcal{L}^{\mathrm{nc}}(f,f')$ to $GSp(1,1)$. For the non-vanishing of theta lifts to classical groups we should remark that Yamana [**29**] has established a general criterion in terms of their local non-vanishing and some analytic properties of the automorphic L-functions. Our method is quite different, i.e. to study the Bessel periods of $\mathcal{L}^{\mathrm{nc}}(f,f')$ explicitly. As we will see, this method yields the positivity of some central L values for $\mathcal{L}^{\mathrm{nc}}(f,f')$s as well as several results on the non-vanishing of $\mathcal{L}^{\mathrm{nc}}(f,f')s$. For this we remark that the non-vanishing of the theta lifts $\pi'(f,\mathrm{JL}(f'))$ is already known as is pointed out in [**21**, Section 4.4]. We also take up the theta lifts $\mathcal{L}^{\mathrm{c}}(f,f')$s to $GSp^*(2)$. For the case of $GSp^*(2)$ we will remark that Ibukiyama and Ihara [**13**] essentially provide examples of the non-vanishing theta lifts. We furthermore remark that the non-vanishing of the theta lifts in our concern is not an immediate consequence from the known general results like Yamana [**29**] since they deal with the cases of the groups of isometry while this paper deals with the cases of groups of similitudes.

1.2. Two aspects on the non-vanishing of the theta lifts.

For the non-vanishing of the theta lift $\mathcal{L}^{\mathrm{nc}}(f,f')$ to $GSp(1,1)$ we verify the existence of a non-vanishing Bessel period $\mathcal{L}^{\mathrm{nc}}(f,f')_{\xi}^{\chi}$ of $\mathcal{L}^{\mathrm{nc}}(f,f')$ (for the definition of the Bessel periods see Section 2.3), where χ denotes a Hecke character of an imaginary quadratic field $\mathbb{Q}(\xi)$ generated by a non-zero pure quaternion ξ in B. We can study the non-vanishing of the Bessel periods by reducing the problem to the simultaneous non-vanishing of two toral integrals of f and f' with respect to χ (cf. Theorem 2.3 and Corollary 2.4). We provide two aspects of studying such non-vanishing of the two integrals, for which we note that there is a general formula by Waldspurger [**27**, Proposition 7] relating the square norms of the toral integrals to the central values of Rankin–Selberg L-functions for $GL(2)$.

One aspect is to use the results by Hsieh [**11**] and Chida–Hsieh [**2**] on "non-vanishing modulo p" of central values of some Rankin–Selberg L-functions for $GL(2)$ (cf. Theorem 3.3 and Theorem 3.5), where p is an appropriate prime number. We remark that the idea of this approach is originally due to Hsieh–Namikawa [**12**]. To explain another aspect we specify B and $\mathbb{Q}(\xi)$ so that both class numbers of these are one. More specifically, we consider B with the prime discriminant p and $\mathbb{Q}(\xi)$ with the discriminant $-p$ or $-4p$. The assumption on the class numbers implies $p = 2,\ 3$ or 7 as we will see later. An essential point for this approach is to relate the non-vanishing of the toral integral for f' to the study of special values of some hypergeometric functions (cf. Lemma 3.9).

For $GSp^*(2)$ we remark that Ibukiyama and Ihara [**13**, Section 3.2] essentially provide examples of non-vanishing $\mathcal{L}^{\mathrm{c}}(f,f')$s (cf. Proposition 3.18). In our previous paper [**21**] we have defined the theta lift $\mathcal{L}^{\mathrm{c}}(f,f')$ to $GSp^*(2)$, modifying the formulation by Löschel [**16**]. The theta lifts are vector-valued and their coefficient functions turn out to be the Petersson inner product of f and a theta series attached to some harmonic polynomial on two copies of the Hamilton quaternion algebra. In Ibukiyama–Ihara [**13**, Section 3.2] we find examples of non-zero such theta series, which lead to the existence of non-zero $\mathcal{L}^{\mathrm{c}}(f,f')$.

For $GSp(2)$ we have remarked in [**21**, Section 4.4] that the lifts $\pi'(f,\mathrm{JL}(f'))$ to $GSp(2)(\mathbb{A})$ are non-vanishing. We remind the readers that we follow the formulation of the theta lifts by Roberts [**23**] and Harris–Kudla [**9**] to define $\pi'(f,\mathrm{JL}(f'))$.

1.3. The main results.

Now let us explain the detail on the case of $GSp(1,1)$. We deal with an explicit formula for the square norm of a Bessel period $\mathcal{L}^{\mathrm{nc}}(f,f')^{\chi}_{\xi}$ in terms of some convolution type L-function $L(\mathcal{L}^{\mathrm{nc}}(f,f'),\chi^{-1},s)$ (cf. Section 4.2). We can define the convolution type L-function $L(\pi'(f,\mathrm{JL}(f')),\chi^{-1},s)$ also for the theta lift $\pi'(f,\mathrm{JL}(f'))$ to $GSp(2)$. We have $L(\mathcal{L}^{\mathrm{nc}}(f,f'),\chi^{-1},s) = L(\pi'(f,\mathrm{JL}(f')),\chi^{-1},s)$ (cf. Proposition 4.3). We now state the following theorem.

THEOREM 1.1 (Theorem 4.4). *Let $\xi \in B \setminus \{0\}$ be a pure quaternion and assume it to be primitive* (*for the definition see Section* 2.4 (2)). *With an explicit constant $C(f,f',\xi,\chi)$ of proportionality and an explicitly given $g_0 \in GSp(1,1)(\mathbb{A}_f)$, we have*

$$\begin{aligned}\frac{\|\mathcal{L}^{\mathrm{nc}}(f,f')^{\chi}_{\xi}(g_0)\|^2}{\langle f,f\rangle\langle f',f'\rangle} &= C(f,f',\xi,\chi)L\left(\mathcal{L}^{\mathrm{nc}}(f,f'),\chi^{-1},\frac{1}{2}\right)\\ &= C(f,f',\xi,\chi)L\left(\pi'(f,\mathrm{JL}(f')),\chi^{-1},\frac{1}{2}\right),\end{aligned}$$

where $\langle f,f\rangle$ and $\langle f',f'\rangle$ denote the norms of f and f' respectively.

This is a generalization of [**20**, Theorem 2.11], in which the weights κ_1 and κ_2 of f and f' are assumed to equal. Such assumption is removed for the theorem. It should be remarked that this result controls the central L-values of the cuspidal representations of $GSp(2)$ (as well as those of $GSp(1,1)$) by means of the Bessel periods of $GSp(1,1)$ and of the explicit Jacquet–Langlands correspondence in the previous paper [**21**].

Theorem 1.1 is useful to show the existence of non-vanishing $\mathcal{L}^{\rm nc}(f,f')$s, and also yields the positivity of the central value of the convolution type L-function for some non-zero $\mathcal{L}^{\rm nc}(f,f')$. The approach we shall first take up is to use the results by Hsieh [**11**] and Chida–Hsieh [**2**], which enable us to discuss the non-vanishing of $\mathcal{L}^{\rm nc}(f,f')$ and the positivity of $L(\mathcal{L}^{\rm nc}(f,f'),\chi^{-1},1/2)=L(\pi'(f,{\rm JL}(f')),\chi^{-1},1/2)$ in a general situation. Let Π and Π' be the base change lifts to $\mathbb{Q}(\xi)$ of automorphic representations generated by f and ${\rm JL}(f')$ respectively. Then we have

$$L(\mathcal{L}^{\rm nc}(f,f'),\chi^{-1},s)=L(\pi'(f,{\rm JL}(f')),\chi^{-1},s)=L(\Pi,\chi^{-1},s)L(\Pi',\chi^{-1},s)$$

(cf. Proposition 4.3), where $L(\Pi,\chi^{-1},s)$ (respectively $L(\Pi',\chi^{-1},s)$) denotes the L-function of Π (respectively Π') twisted by χ^{-1}. With a suitable prime number p Hsieh [**11**] and Chida–Hsieh [**2**] discuss the non-vanishing modulo p of $L(\Pi,\chi^{-1},1/2)$ and $L(\Pi',\chi^{-1},1/2)$ in the spirit of the Iwasawa theory, where the conductor of χ is assumed to be a power of some fixed prime number $l\nmid p$. Let $X_\xi(l)_f$ be the set of Hecke characters on $\mathbb{A}^\times\mathbb{Q}(\xi)^\times\backslash\mathbb{A}^\times_{\mathbb{Q}(\xi)}$ of finite order whose conductor is a power of l. As a result of [**11**, Theorem C], [**2**, Theorem 5.9] and Theorem 1.1 we have the following:

Theorem 1.2 (Theorem 3.5 and Corollary 4.6). *Let f (respectively f') be a non-zero primitive form (respectively a non-zero Hecke eigenform).*

(1) *Assume that f and f' have the same signature of the Atkin–Lehner involution at finite primes p dividing D (cf. Section 2.4). Suppose that the discriminant of $\mathbb{Q}(\xi)$ is divisible by d_B and that $\mathbb{Q}(\xi)$ is not isomorphic to $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$. Fix a prime number l which splits in $\mathbb{Q}(\xi)$, and assume that χ has a power of l as the conductor and the weight $w_\infty(\chi)=-\kappa_1$ at the archimedean place. We furthermore make the local assumptions on (f,f') with respect to χ at finite primes dividing d_B as follows:*

- $S_1=S_2^+(f,\chi)=\emptyset$ *(this is equivalent to* [**2**, (ST)]*),*
- $\pi''_p|_{E_p^\times}=\chi_p$ *for $p|D_B$ (this is equivalent to* [**11**, Hypothesis A]*),*

where see Section 3.1 for the notation.

Then $\mathcal{L}^{\rm nc}(f,f')$ is non-vanishing and satisfies $L(\Pi,(\chi\nu)^{-1},1/2)>0$, $L(\Pi',(\chi\nu)^{-1},1/2)>0$, thus

$$L\left(\mathcal{L}^{\rm nc}(f,f'),(\chi\nu)^{-1},\frac{1}{2}\right)=L\left(\pi'(f,{\rm JL}(f')),(\chi\nu)^{-1},\frac{1}{2}\right)>0$$

for infinitely many $\nu\in X_\xi(l)_f$.

(2) *The theta lift $\mathcal{L}^{\rm nc}(f,f')$ is non-vanishing if and only if the signatures of the Atkin–Lehner involutions of f and f' are the same at p dividing D (for the condition on the signature see* [**21**, Proposition 3.3 (1)] *and Section* 2.4 (1)*). In particular, the non-vanishing of $\mathcal{L}^{\rm nc}(f,f')$ holds without the condition on the signatures when $D=1$.*

For the first assertion of this theorem we note that the local assumptions on (f,f') with respect to χ ensure the non-vanishing of the constant $C(f,f',\xi,\chi)$ in the formula of Theorem 1.1. In fact, as the proof of Theorem 3.5 indicates (see also the remark just before it), we can show that there are infinitely many $\nu\in X_\xi(l)_f$ with $C(f,f',\xi,\chi\nu)\neq 0$.

We remark that the second assertion is what should be called "ϵ-dichotomy" (cf. [**10**]) for our theta lifts $\mathcal{L}^{\mathrm{nc}}(f, f')$s.

To state the result on the second aspect let B be a definite quaternion algebra with the discriminant $d_B = 2$, 3 or 7 and specify the primitive ξ so that $\mathbb{Q}(\xi)$ is isomorphic to $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$ (for $d_B = 2$), $\mathbb{Q}(\sqrt{-3})$ (for $d_B = 3$) or $\mathbb{Q}(\sqrt{-7})$ (for $d_B = 7$) (cf. Section 3.3). Then the class numbers of B and $\mathbb{Q}(\xi)$ are one. Let χ be a Hecke character of $\mathbb{Q}(\xi)$ unramified at every finite place and of weight $w_\infty(\chi) = -\kappa_1$ at the infinite place. These assumptions reduce the toral integrals of f and f' to the special values of f and f' at some CM-points. Under this convenient situation we are able to show the following result, which includes a generalization of [**20**, Proposition 2.12, Theorem 2.13 and Theorem 2.14] without assuming $\kappa_1 = \kappa_2$.

Theorem 1.3 (Theorem 3.14, Corollary 4.5). (1) *In addition to the assumption on χ, ξ and d_B just mentioned assume that $D = 1$ or $D = d_B$ and that $(\kappa_1, \kappa_2) \in (4\mathbb{Z}_{>0})^2$ with $\kappa_2 \geq \kappa_1$. When $d_B = 2$ or 3 we further assume the following*:

$$\binom{(\kappa_1+\kappa_2)/2}{(\kappa_2-\kappa_1)/2}\, {}_2F_1\left(-\frac{\kappa_2-\kappa_1}{2}, -\frac{\kappa_2-\kappa_1}{2}; \kappa_1+1; -1\right) \neq (-1)^{\kappa_1/4+1} 2^{(\kappa_2-4)/2} \ (\textit{for } d_B = 2),$$

$$\binom{(\kappa_1+\kappa_2)/2}{(\kappa_2-\kappa_1)/2}\, {}_2F_1\left(-\frac{\kappa_2-\kappa_1}{2}, -\frac{\kappa_2-\kappa_1}{2}; \kappa_1+1; -3\right) \neq -2^{\kappa_1-1} \ (\textit{for } d_B = 3),$$

where ${}_2F_1$ denotes Gauss's hypergeometric function. Then (*under some extra assumptions on* (κ_1, κ_2)) *there exist Hecke eigenforms (f, f') such that*

1. $\mathcal{L}^{\mathrm{nc}}(f, f')^\chi_\xi \not\equiv 0 (\Rightarrow \mathcal{L}^{\mathrm{nc}}(f, f') \not\equiv 0)$,
2. $L(\Pi, \chi^{-1}, 1/2) > 0$, $L(\Pi', \chi^{-1}, 1/2) > 0$,
   thus $L(\mathcal{L}^{\mathrm{nc}}(f, f'), \chi^{-1}, 1/2) = L(\pi'(f, \mathrm{JL}(f')), \chi^{-1}, 1/2) > 0$.

(2) (*Ebisu, Haagerup–Schlichtkrull*) *When $d_B = 2$ the assumption on the hypergeometric function is satisfied for $(\kappa_1, \kappa_2) \in (4\mathbb{Z}_{>0})^2$ with $\kappa_2 \geq 4\kappa_1$ except for $(\kappa_1, \kappa_2) = (4, 16)$ or for $(\kappa_1, \kappa_2) \in (4\mathbb{Z}_{>0}) \times (2\mathbb{Z}_{>0})$ with $\kappa_2 \geq \kappa_1$ and $\kappa_2 \geq 16^4 \cdot 3^4 - 1$. When $d_B = 3$ this is satisfied for any $(\kappa_1, \kappa_2) \in (4\mathbb{Z}_{>0})^2$ with $\kappa_2 \geq \kappa_1$.*

As a reference relevant to Theorems 1.2, 1.3 we cite Saha–Schmidt [**24**], which provides some quantitative results on simultaneous non-vanishing of central L-values like $L(\Pi, \chi^{-1}, 1/2)$ and $L(\Pi', \chi^{-1}, 1/2)$ by using holomorphic Siegel modular forms given as Yoshida lifts. However, as far as the author knows, it seems that there are only quite a few papers finding automorphic forms satisfying the positivity on the central L-values in a specific manner like ours. We further note that $\mathcal{L}^{\mathrm{nc}}(f, f')$s are non-holomorphic. Maybe the arithmetic studies of non-holomorphic automorphic forms can be said to be quite rare. In fact, it seems that several difficulties caused by the non-holomorphy often give rise to obstructions for such studies to be active.

As we have seen, the first approach leads to a general criterion on the non-vanishing of $\mathcal{L}^{\mathrm{nc}}(f, f')$ as well as the positivity of the central L-values. As for the second approach there seem to be no results controlling the positivity of central L-values of automorphic L-functions by means of special values of the hypergeometric functions. The assertion (2) of the above theorem for the cases of $d_B = 2$ with $\kappa_2 \geq 4\kappa_1$ and $d_B = 3$ is given by Ebisu [**4**] (cf. Proposition 3.11) and the inequality $\kappa_2 \geq 4\kappa_1$ is due to the limitation arising from the technique using the three term relations of the hypergeometric series. The rest of the assertion (2) (for $d_B = 2$) is due to Haagerup–Schlichtkrull [**8**] (cf. Proposition 3.12). The author hopes that the second approach has its own significance drawing the attention of experts not necessarily specialized to the number theory.

1.4. Outline of the paper.

This paper begins with studying an explicit formula for Bessel periods of the theta lifting $\mathcal{L}^{\mathrm{nc}}(f, f')$ in Section 2. As an application of this we discuss the non-vanishing of the theta lifts in the two aspects mentioned above in Section 3. More specifically we review some results on the toral integrals necessary for us in Section 3.1. The Sections 3.2, 3.3 are devoted to studying the non-vanishing of the theta lifts $\mathcal{L}^{\mathrm{nc}}(f, f')$s in terms of the two aspects. Section 3.4 takes up examples of non-vanishing theta lifts for $\mathcal{L}^{\mathrm{c}}(f, f')$s after Ibukiyama–Ihara [**13**]. In Section 4 we provide an explicit formula for the square norms of the Bessel periods of $\mathcal{L}^{\mathrm{nc}}(f, f')$s in terms of central L-values $L(\mathcal{L}^{\mathrm{nc}}(f, f'), \chi^{-1}, 1/2)$s. From this we deduce several consequences on the positivity of the central L-values. Sections 4.1, 4.2 take up these topics. In Section 4.3 we provide examples of Hecke eigenforms (f, f') satisfying $L(\mathcal{L}^{\mathrm{nc}}(f, f'), \chi^{-1}, 1/2) > 0$.

As we have explained, the results on the theta lifts $\mathcal{L}^{\mathrm{nc}}(f, f')$s in this paper are viewed as a generalization of the work [**20**]. This paper often uses results in [**18**] and [**19**]. We therefore require more or less familiarity with these works [**18**], [**19**] and [**20**] as well as the paper [**21**] of the first part.

2. Bessel periods of theta lifts to $GSp(1,1)$.

In the first paper [**21**], we have used the notation $\mathcal{L}(f, f')$ uniformly for the theta lifts to $GSp(1,1)(\mathbb{A})$ and $GSp^*(2)(\mathbb{A})$ from $(f, f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$. Instead we denote it by $\mathcal{L}^{\mathrm{nc}}(f, f')$ for $GSp(1,1)$ and $\mathcal{L}^{\mathrm{c}}(f, f')$ for $GSp^*(2)$ respectively (cf. Sections 2.1, 2.2). In Section 2.6 we state and prove an explicit formula for Bessel periods (or Fourier coefficients) of $\mathcal{L}^{\mathrm{nc}}(f, f')$s on $GSp(1,1)(\mathbb{A})$ in terms of the toral integrals of (f, f') (cf. Theorem 2.3). In Sections 2.1 and 2.2 we first introduce basic notation and then make a review on the theta lifts $\mathcal{L}^{\mathrm{nc}}(f, f')$ and $\mathcal{L}^{\mathrm{c}}(f, f')$ in some detail. After this the explanation of the Bessel periods begins with Section 2.3. The non-archimedean part of the explicit formula is the same as that of the formula in [**19**, Theorem 5.2.1]. For Sections 2.4 to 2.6 we review the notation of [**19**, Theorem 5.2.1], following the arrangement of [**20**, Section 1].

2.1. Basic notation.

In this paper we keep the notation used in [**21**]. Let B be a definite quaternion algebra over $\mathbb{Q}$ and d_B denote the discriminant of B. For $x \in B$ we denote by $\mathrm{tr}(x)$ and $n(x)$ its reduced trace and reduced norm respectively. We write H and H' for $\mathbb{Q}$-algebraic groups GL_2 and $B^\times$ respectively. For a number field F we denote its adele ring by $\mathbb{A}_F$. Following [**21**] we denote $\mathbb{A}_{\mathbb{Q}}$ simply by $\mathbb{A}$ and the ring of finite adeles in $\mathbb{A}$ by $\mathbb{A}_f$. For a divisor D of d_B we let $S_{\kappa_1}(D)$ be the space of elliptic cusp forms f of weight κ_1 and level D (cf. [**19**, Section 3.1]). Given a maximal order $\mathfrak{O}$ of B, $\mathcal{A}_{\kappa_2}$ denotes the space of automorphic forms f' on $H'(\mathbb{A}_{\mathbb{Q}})$ of "weight σ_{κ_2} and level $\mathfrak{O}^\times$"

(cf. [**19**, Section 3.2]), where σ_{κ_2} denotes the κ_2-th symmetric tensor representation of $\mathbb{H}^{(1)} := \{b \in B^\times(\mathbb{R}) \mid n(b) = 1\} \simeq SU(2)$ with the representation space V_{κ_2} (for σ_{κ_2} see [**19**, Section 1.2]). Here note that (f, f') have the trivial central character by the definition of $S_{\kappa_1}(D)$ and $\mathcal{A}_{\kappa_2}$. Therefore κ_1 and κ_2 have to be even.

Let us recall that the non-compact $\mathbb{Q}$-inner form $GSp(1,1)$ and the compact $\mathbb{Q}$-inner form $GSp^*(2)$ of the $\mathbb{Q}$-split symplectic group $GSp(2)$ of degree two have been introduced in [**21**, Section 2.1]. Recall also that $Sp(1,1)$, $Sp^*(2)$ and $Sp(2)$ denote the corresponding isometry groups. For $(\lambda_1, \lambda_2) \in \mathbb{Z}^2_{>0}$ such that $\lambda_1 > \lambda_2$ we have introduced the space $\mathcal{S}^{\mathrm{nc}}_{\tau_\Lambda}(D)$ of cusp forms on $GSp(1,1)(\mathbb{A})$ with $\Lambda = (\lambda_1, \lambda_2 - 1)$ (respectively the space $\mathcal{A}^{\mathrm{c}}_{\tau_\Lambda}(D)$ of automorphic forms on $GSp^*(2)(\mathbb{A})$ with $\Lambda = (\lambda_1 - 2, \lambda_2 - 1)$) generating the square integrable representation (modulo center) of $GSp(1,1)(\mathbb{R})$ (respectively $GSp^*(2)(\mathbb{R})$) parametrized by (λ_1, λ_2) (cf. [**21**, Definitions 3.1, 3.2]). Here τ_Λ denotes the irreducible representation of the maximal compact subgroup with highest weight Λ, which is viewed as the square integrable representation (modulo center) of $GSp^*(2)(\mathbb{R})$ for the case of the compact inner form (respectively which is called the minimal K-type of the square integrable representation for the case of $GSp(1,1)$). According to [**21**, Proposition 3.3 (2)], the theta lift $\mathcal{L}^{\mathrm{nc}}(f, f')$ on $GSp(1,1)(\mathbb{A})$ (respectively $\mathcal{L}^{\mathrm{c}}(f, f')$ on $GSp^*(2)(\mathbb{A})$) belongs to $\mathcal{S}^{\mathrm{nc}}_{\tau_\Lambda}(D)$ (respectively $\mathcal{A}^{\mathrm{c}}_{\tau_\Lambda}(D)$) for $\Lambda = ((\kappa_1 + \kappa_2)/2, (\kappa_2 - \kappa_1)/2)$ with $(\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^2$ such that $1 < \kappa_1 < \kappa_2 + 2$ (respectively $\Lambda = ((\kappa_1 + \kappa_2)/2 - 2, (\kappa_1 - \kappa_2)/2 - 2)$ with $(\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^2$ such that $1 < \kappa_2 + 2 < \kappa_1$). For this we remark that, when $\kappa_1 = \kappa_2$, the square-integrable representations of $GSp(1,1)(\mathbb{R})$ are induced from quaternionic discrete series representations of $Sp(1,1)(\mathbb{R})$. Such cases are taken up in [**18**], [**19**] and [**20**].

2.2. Review on the theta lifts.

We review the theta lifts $\mathcal{L}^{\mathrm{nc}}(f, f')$ and $\mathcal{L}^{\mathrm{c}}(f, f')$ from $(f, f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$ to $GSp(1,1)(\mathbb{A})$ and $GSp^*(2)(\mathbb{A})$ respectively based on [**21**, Section 3.2.2]. Though this paper mainly takes up the case of $GSp(1,1)$ we also review the case of $GSp^*(2)$, which is taken up at Section 3.4.

Let us recall that, at the archimedean place, we have modified the metaplectic representation r introduced in [**18**, Sections 3, 4] to formulate the theta lift $\mathcal{L}^{\mathrm{nc}}(f, f')$ and $\mathcal{L}^{\mathrm{c}}(f, f')$ to $GSp(1,1)(\mathbb{A})$ and $GSp^*(2)(\mathbb{A})$ respectively in our concern. The representation space of r is given as the restricted tensor product $\bigotimes'_{p<\infty} \mathbb{V}_p \times V_\infty$ over places of $\mathbb{Q}$ of the local Schwartz spaces $\mathbb{V}_v$ at $v = p < \infty$ or $v = \infty$ with respect to local Schwartz functions $\{\varphi_{0,p}\}_{p<\infty}$. To explain $\varphi_{0,p}$ for $p < \infty$ let $\mathfrak{O}_p$ be the integer ring of the p-adic completion B_p of B at p and $\mathfrak{P}_p$ the prime ideal of $\mathfrak{O}_p$. We put

$$L_p := \begin{cases} \mathfrak{O}_p \oplus \mathfrak{O}_p & (p \nmid d_B/D), \\ \mathfrak{O}_p \oplus \mathfrak{P}_p^{-1} & (p | d_B/D). \end{cases}$$

Then $\varphi_{0,p}$ is the function on $B_p \times \mathbb{Q}_p^\times$ defined to be the characteristic functions of $L_p \times \mathbb{Z}_p^\times$ for each finite prime $p < \infty$. The local Schwartz space $\mathbb{V}_p$ at a finite prime p is the space of locally constant functions on $B_p \times \mathbb{Q}_p^\times$ with a compact support. This definition is the same as that of [**18**, Section 3].

In [**21**, Section 3.2.2], the Schwartz space $\mathbb{V}_\infty$ is modified for both of $GSp(1,1)$ and

$GSp^*(2)$. For both modifications we need the usual Schwartz space $\mathcal{S}(\mathbb{H}^2)$ on $\mathbb{H}^2$. For the case of $GSp(1,1)(\mathbb{A})$, letting $\kappa_2 \geq \kappa_1$, $\mathbb{V}_\infty$ stands for the space of smooth functions φ on $\mathbb{H}^2 \times \mathbb{R}^\times$ such that, for each fixed $t \in \mathbb{R}^\times$, $\mathbb{H}^2 \ni X \mapsto \varphi(X,t)$ belongs to $\mathcal{S}(\mathbb{H}^2) \otimes \mathrm{End}(V_{(\kappa_1+\kappa_2)/2} \boxtimes V_{(\kappa_2-\kappa_1)/2})$, where V_κ denotes the representation space of the κ-th symmetric tensor representation σ_κ for a non-negative integer κ. For this case we remark that the paper [**18**] considers the case of $\kappa_1 = \kappa_2$. For the case of $GSp^*(2)$ we need the space $\mathcal{H}_{\kappa_1-4}$ of harmonic polynomials on $\mathbb{H}^2$ of degree $\kappa_1 - 4$ with $\kappa_1 \geq 4$. For this case the definition $\mathbb{V}_\infty$ is given similarly with replacing $\mathrm{End}(V_{(\kappa_1+\kappa_2)/2} \boxtimes V_{(\kappa_2-\kappa_1)/2})$ by $\mathcal{H}_{\kappa_1-4}$.

We next review the test functions $\varphi_0^{\mathrm{nc}} = \prod_{p<\infty} \varphi_{0,p} \times \varphi_{0,\infty}^{\mathrm{nc}}$ and $\varphi_0^{\mathrm{c}} = \prod_{p<\infty} \varphi_{0,p} \times \varphi_{0,\infty}^{\mathrm{c}}$ of the theta integral kernel for the theta lifts. For both of $GSp(1,1)$ and $GSp^*(2)$ we put $\varphi_{0,p}$ to be the aforementioned characteristic function of $L_p \times \mathbb{Z}_p^\times$ for each $p < \infty$. For the case of $GSp(1,1)$ the archimedean part $\varphi_{0,\infty}^{\mathrm{nc}}$ is defined as

$$\varphi_{0,\infty}^{\mathrm{nc}}(X,t) := \begin{cases} t^{(\kappa_2+3)/2}\sigma_{(\kappa_1+\kappa_2)/2}(X_1+X_2) \boxtimes \sigma_{(\kappa_2-\kappa_1)/2}(X_1-X_2)\exp(-2\pi {}^t\bar{X}X) \\ \hfill (t>0), \\ 0 \hfill (t<0). \end{cases}$$

On the other hand, for the case of $GSp^*(2)$, we need the reproducing kernel function $\mathbb{H}^2 \ni X \mapsto C_X \in \mathcal{H}_{\kappa-4}$. More specifically, given a fixed inner product $(*,*)$ of $\mathcal{H}_{\kappa_1-4}$, this is characterized by $(C_X, \Phi) = \Phi(X)$ for $\Phi \in \mathcal{H}_{\kappa_1-4}$. Then $\varphi_{0,\infty}^{\mathrm{c}}$ is defined as

$$\varphi_{0,\infty}^{\mathrm{c}}(X,t) := \begin{cases} t^{(\kappa_1-1)/2}\exp(-2\pi t\ {}^t\bar{X}X)C_X & (t>0), \\ 0 & (t<0). \end{cases}$$

Here $X = {}^t(X_1, X_2) \in \mathbb{H}^2$ and ${}^t\bar{X} = (\bar{X_1}, \bar{X_2})$ with $X_1,\ X_2 \in \mathbb{H}$.

For the review about the case of $GSp^*(2)$ we need a further discussion. From [**21**, (3.1)] we recall that $\mathcal{H}_{\kappa_1-4}$ admits a decomposition

$$\mathcal{H}_{\kappa_1-4} = \bigoplus_{\substack{a \geq b \geq 0 \\ a+b=\kappa_1-4}} W_{a,b} \otimes V_{a-b}$$

as a $Sp^*(2)(\mathbb{R}) \times \mathbb{H}^{(1)}$-module, where $W_{a,b}$ (respectively V_{a-b}) denotes the representation space of $\tau_{(a,b)}$ (respectively the representation space of the $(a-b)$-th symmetric tensor representation of $\mathbb{H}^{(1)}$) (for $\tau_{(a,b)}$ see Section 2.1). From this decomposition we know that, up to constant multiples, there is a $Sp^*(2)(\mathbb{R})$-equivariant $W_{(a,b)}$-valued paring $(*,*)_{a,b}$ of $\mathcal{H}_{\kappa_1-4} \times V_{a-b}$.

We are now able to formulate the theta lifts $\mathcal{L}^{\mathrm{nc}}(f,f')$ and $\mathcal{L}^{\mathrm{c}}(f,f')$ for $\mathcal{G} := GSp(1,1)$ and $GSp^*(2)$ as in [**21**, Section 3.2.2]. We introduce the theta integral kernels

$$\begin{cases} \sum_{(X,t)\in B^2\times\mathbb{Q}^\times} r(g,h,h')\varphi_0^{\mathrm{nc}}(X,t) & (\mathcal{G} = GSp(1,1)) \\ \sum_{(X,t)\in B^2\times\mathbb{Q}^\times} r(g,h,h')\varphi_0^{\mathrm{c}}(X,t) & (\mathcal{G} = GSp^*(2)) \end{cases}, \quad \forall (g,h,h') \in \mathcal{G}(\mathbb{A}) \times H(\mathbb{A}) \times H'(\mathbb{A}).$$

We denote these by $\theta_{\kappa_1,\kappa_2}^{\mathrm{nc}}(g,h,h')$ and $\theta_{\kappa_1,\kappa_2}^{\mathrm{c}}(g,h,h')$ for $GSp(1,1)$ and $GSp^*(2)$ respectively. The theta lifts $\mathcal{L}^{\mathrm{nc}}(f,f')$ and $\mathcal{L}^{\mathrm{c}}(f,f')$ are thus defined by

$$\begin{cases} \mathcal{L}^{\mathrm{nc}}(f,f')(g) := \displaystyle\int_{\mathbb{R}_+^2(H\times H')(\mathbb{Q})\backslash(H\times H')(\mathbb{A})} \overline{f(h)}\theta^{\mathrm{nc}}_{\kappa_1,\kappa_2}(g,h,h')f'(h')dhdh', \\ \mathcal{L}^{\mathrm{c}}(f,f')(g) := \displaystyle\int_{\mathbb{R}_+^2(H\times H')(\mathbb{Q})\backslash(H\times H')(\mathbb{A})} \\ \qquad\qquad \overline{f(h)}(\theta^{\mathrm{c}}_{\kappa_1,\kappa_2}(g,h,h'),f'(h'))_{(\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2}dhdh', \end{cases}$$

respectively.

2.3. Fourier expansions of theta lifts to $GSp(1,1)$.

We now review the Fourier expansion of $\mathcal{L}^{\mathrm{nc}}(f,f')$ on $GSp(1,1)(\mathbb{A})$ described in [**19**, Section 1.3] and [**20**, Section 1.2]. We introduce the set $B^- := \{x\in B \mid \mathrm{tr}(x)=0\}$ of pure quaternions in B and have

$$\mathcal{L}^{\mathrm{nc}}(f,f')(g) = \sum_{\xi\in B^-}\mathcal{L}^{\mathrm{nc}}(f,f')_\xi(g),$$

where

$$\mathcal{L}^{\mathrm{nc}}(f,f')_\xi(g) := \int_{B^-\backslash B^-(\mathbb{A})}\mathcal{L}^{\mathrm{nc}}(f,f')\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)\psi(-\mathrm{tr}(\xi x))dx$$

with the standard additive character ψ on $\mathbb{Q}\backslash\mathbb{A}$. Here we normalize the measure dx so that the volume of $B^-\backslash B^-(\mathbb{A})$ is one. For $\xi\in B^-\setminus\{0\}$ we let $E_\xi := \mathbb{Q}(\xi)$, which is isomorphic to an imaginary quadratic field, and X_ξ be the set of unitary characters on $\mathbb{A}^\times E_\xi^\times\backslash\mathbb{A}_{E_\xi}^\times$ with the idele group $\mathbb{A}_{E_\xi}^\times$ for E_ξ, which are called Hecke characters. Note that $\mathcal{L}^{\mathrm{nc}}(f,f')$ is cuspidal, namely $\mathcal{L}^{\mathrm{nc}}(f,f')_0=0$. The Fourier expansion is then refined as follows:

$$\mathcal{L}^{\mathrm{nc}}(f,f')(g) = \sum_{\xi\in B^-\setminus\{0\}}\sum_{\chi\in X_\xi}\mathcal{L}^{\mathrm{nc}}(f,f')^\chi_\xi(g),$$

with

$$\mathcal{L}^{\mathrm{nc}}(f,f')^\chi_\xi(g) := \mathrm{vol}(\mathbb{R}_+^\times\mathbb{A}^\times\backslash\mathbb{A}_{E_\xi}^\times)^{-1}\int_{\mathbb{R}_+^\times E_\xi^\times\backslash\mathbb{A}_{E_\xi}^\times}\mathcal{L}^{\mathrm{nc}}(f,f')_\xi(s1_2\cdot g)\chi(s)^{-1}ds.$$

We call $\mathcal{L}^{\mathrm{nc}}(f,f')^\chi_\xi$ the Fourier coefficient or the Bessel period of $\mathcal{L}^{\mathrm{nc}}(f,f')$ indexed by ξ and χ.

2.4. Working assumptions.

To deduce our formula for $\mathcal{L}^{\mathrm{nc}}(f,f')^\chi_\xi$, we assume the following two for $(f,f')\in S_{\kappa_1}(D)\times\mathcal{A}_{\kappa_2}$:

(1) The two forms f and f' are Hecke eigenforms and have the same eigenvalue for the "Atkin–Lehner involution" (cf. [**19**, Section 5.1], [**20**, Section 1.3 (1)]), namely $\epsilon_p=\epsilon'_p$ holds for (f,f'), where ϵ_p and ϵ'_p denotes the eigenvalue (or signature) for f and f' respectively. Otherwise $\mathcal{L}^{\mathrm{nc}}(f,f')\equiv 0$ (cf. [**21**, Proposition 3.3 (1)]).

(2) We assume that $\xi\in B^-\setminus\{0\}$ is primitive. Namely, for each finite prime p, we let

$$\mathfrak{a}_p := \begin{cases} \mathfrak{O}_p & (p \nmid d_B \text{ or } p|D) \\ \mathfrak{P}_p & (p|d_B/D) \end{cases}, \ (\mathfrak{a}_p^-)^* := \{z \in B_p^- \mid \mathrm{tr}(\bar{z}w) \in \mathbb{Z}_p, \text{ for any } w \in \mathfrak{a}_p \cap B_p^-\}$$

with the p-adic completion B_p^- of B^- and assume that

$$\xi \in (\mathfrak{a}_p^-)^* \setminus p(\mathfrak{a}_p^-)^*,$$

where recall that $\mathfrak{P}_p$ has denoted the maximal ideal of $\mathfrak{O}_p$ for $p|d_B$ (cf. Section 2.2). For this assumption see [**19**, Section 4.1] and [**20**, Section 1.3 (2)]. For the assumption we note that, in general, a Fourier coefficient F_ξ of an automorphic form F on $GSp(1,1)(\mathbb{A})$ satisfies

$$F_\xi\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g\right) = F_{t\xi}(g) \quad (t \in \mathbb{Q}^\times).$$

We then see that the problem determining F_ξ is reduced to the case where ξ is primitive.

2.5. Notations for the quadratic extensions.

In what follows, we denote $E_\xi = \mathbb{Q}(\xi)$ simply by E for $\xi \in B^- \setminus \{0\}$. Towards the statement of the explicit formula for the Bessel periods we prepare notations for this quadratic extension.

Let d_ξ denote the discriminant of $E = E_\xi$. We introduce two rational numbers

$$a := \begin{cases} 2\sqrt{-n(\xi)}\sqrt{d_\xi} & (d_\xi \text{ is odd}) \\ \sqrt{-n(\xi)}\sqrt{d_\xi} & (d_\xi \text{ is even}) \end{cases}, \ b := \xi^2 - \frac{a^2}{4}.$$

With these a and b we define $\iota_\xi : E^\times \hookrightarrow GL_2(\mathbb{Q})$ by

$$\iota_\xi(x + y\xi) = x \cdot 1_2 + y \cdot \begin{pmatrix} a/2 & b \\ 1 & -a/2 \end{pmatrix} \ (x, y \in \mathbb{Q}).$$

Put $r = 2\sqrt{n(\xi)}/\sqrt{d_\xi} \in \mathbb{Q}^\times$ and $\theta := r^{-1}(\xi - a/2)$. Then $\{1, \theta\}$ forms a $\mathbb{Z}$-basis of the integer ring $\mathfrak{O}_E$ of E (cf. [**19**, Lemma 4.3.1 (i)]). We can rewrite ι_ξ as

$$\iota_\xi(x + y\theta) = \begin{pmatrix} x & -rN_{E/\mathbb{Q}}(\theta)y \\ r^{-1}y & x + \mathrm{Tr}_{E/\mathbb{Q}}(\theta)y \end{pmatrix} \quad (x, y \in \mathbb{Q}).$$

The p-adic completion $E_p = \mathbb{Q}_p + \mathbb{Q}_p\theta$ is a field for an inert or ramified p. If p is split, we identify E_p with $\mathbb{Q}_p \times \mathbb{Q}_p$. The p-adic completion $\mathfrak{O}_{E,p}$ of $\mathfrak{O}_E$ is $\mathbb{Z}_p + \mathbb{Z}_p\theta$, which is identified with $\mathbb{Z}_p \times \mathbb{Z}_p$ for a split prime p. The completion E_∞ of E at ∞ is identified with $\mathbb{C}$ by

$$\delta_\xi : E_\infty \ni x + y\xi \mapsto x + y\sqrt{-n(\xi)} \in \mathbb{C} \ (x, y \in \mathbb{R}).$$

For a Hecke character $\chi = \prod_{v \le \infty} \chi_v$ of $\mathbb{R}_+^\times E^\times \backslash \mathbb{A}_E^\times$, we let $w_\infty(\chi) \in \mathbb{Z}$ be such that

$$\chi_\infty(u) = (\delta_\xi(u)/|\delta_\xi(u)|)^{w_\infty(\chi)} \ (u \in E_\infty).$$

We call $w_\infty(\chi)$ the weight of χ. Furthermore, for each prime $v = p < \infty$, we let $p^{i_p(\chi)}$ be the conductor of χ at p and

$$\mu_p := \frac{\mathrm{ord}_p(2\xi)^2 - \mathrm{ord}_p(d_\xi)}{2},$$

which coincides with $\mathrm{ord}_p(r)$.

2.6. An explicit formula for Bessel periods : Further notations and the result.

We need further notations to state our formula for $\mathcal{L}^{\mathrm{nc}}(f,f')^\chi_\xi$.

We define $\gamma_0 = (\gamma_{0,p})_{p\le\infty} \in H(\mathbb{A})$ and $\gamma_0' = (\gamma_{0,p}')_{p<\infty} \in H'(\mathbb{A}_f)$ as follows:

$$\gamma_{0,p} := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & p^{-\mu_p + i_p(\chi)} \end{pmatrix} & (p \nmid D), \\ 1_2 & (p|D \text{ and } p \text{ is inert in } E), \\ \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & (p|D \text{ and } p \text{ ramifies in } E), \\ \begin{pmatrix} 1 & a/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} n(\xi)^{1/4} & 0 \\ 0 & n(\xi)^{-1/4} \end{pmatrix} & (p=\infty), \end{cases}$$

$$\gamma_{0,p}' := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & p^{-\mu_p + i_p(\chi)} \end{pmatrix} & (p \nmid d_B), \\ \varpi_{B,p}^{-1} & (p|d_B). \end{cases}$$

Here $\varpi_{B,p}$ denotes a prime element of B_p.

In addition, we introduce the following local constants:

$$C_p(f,\xi,\chi) := \begin{cases} p^{2\mu_p - i_p(\chi)}(1 - \delta(i_p(\chi) > 0)e_p(E)p^{-1}) & (p \nmid d_B), \\ 1 & \left(p \middle| \dfrac{d_B}{D}\right), \\ 2\epsilon_p & (p|D \text{ and } p \text{ is inert in } E), \\ (p+1)^{-1} & (p|D \text{ and } p \text{ ramifies in } E), \end{cases}$$

where

$$e_p(E) = \begin{cases} -1 & (p \text{ is inert in } E), \\ 0 & (p \text{ ramifies in } E), \\ 1 & (p \text{ splits in } E). \end{cases}$$

For $(f,f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$ we introduce their toral integrals with respect to χ (cf. [**27**]):

$$P_\chi(f;h) := \int_{\mathbb{R}_+^\times E^\times \backslash \mathbb{A}_E} f(\iota_\xi(s)h)\chi(s)^{-1}ds,\ P_\chi(f';h') := \int_{\mathbb{R}_+^\times E^\times \backslash \mathbb{A}_E^\times} f'(sh')\chi(s)^{-1}ds,$$

where $(h,h') \in H(\mathbb{A}) \times H'(\mathbb{A})$. As in [**19**, Section 2.4] and [**20**, Section 1.5] we normalize the measure ds of $\mathbb{A}_E^\times$ so that

$$\mathrm{vol}(\mathfrak{O}_{E,p}^\times)=1 \text{ for any } p<\infty, \quad \mathrm{vol}(E_\infty^1)=1,$$

where $\mathfrak{O}_{E,p}^\times$ (respectively E_∞^1) denotes the unit group of the p-adic completion $\mathfrak{O}_{E,p}$ of $\mathfrak{O}_E$ (respectively the group of complex numbers with the norm one). In addition we normalize the measure of $\mathbb{A}^\times$ so that

$$\mathrm{vol}(\mathbb{Z}_p^\times)=1 \text{ for any } p<\infty.$$

Recall that $(\sigma_\kappa, V_\kappa)$ denotes the κ-th symmetric tensor representation of $\mathbb{H}^{(1)}$ with the representation space V_κ (cf. Section 2.1). For the forthcoming discussion we need to describe weight vectors of V_κ explicitly. We realize V_κ by the $\mathbb{C}$-vector space of degree κ homogeneous polynomials of two variables (X,Y). We fix a unitary inner product $(*,*)_\kappa$ of V_κ. For $v\in V_\kappa$ we denote by v^* the dual vector of V_κ with respect to $(*,*)_\kappa$. For each $0\le i\le\kappa$ let $v_{\kappa,i}:=X^iY^{\kappa-i}$, which is a vector of weight $2i-\kappa$ in V_κ.

For a non-zero pure quaternion $\xi\in\mathbb{H}\setminus\{0\}$, namely $\mathrm{tr}(\xi)=0$, take another non-zero pure quaternion $\rho_\xi\in\mathbb{H}\setminus\{0\}$ so that $\xi\rho_\xi=-\rho_\xi\xi$. By the well known Skolem–Nöther theorem, there is a $u_\xi\in\mathbb{H}^{(1)}$ such that

$$u_\xi\{1,\ i,\ j,\ k\}u_\xi^{-1}=\left\{1,\ \xi/\sqrt{n(\xi)},\ \rho_\xi/\sqrt{n(\rho_\xi)},\ \xi\rho_\xi/\sqrt{n(\xi\rho_\xi)}\right\},$$

where $\{1,\ i,\ j,\ k\}$ denotes the standard basis of $\mathbb{H}$, which is identical to the $\mathbb{Q}$-basis of B for $d_B=2$. With this u_ξ we put $v_{\kappa,i}(\xi):=\sigma_\kappa(u_\xi)v_{\kappa,i}$, which is a weight vector for the $\mathbb{R}(\xi)^\times$-action via σ_κ. We will use these notations in the proof of Theorem 2.3 and Proposition 3.8.

We now recall that we have introduced the local Schwartz–Bruhat space $\mathbb{V}_v$ for each place v of $\mathbb{Q}$ and have put $\mathbb{V}$ to be the restricted tensor product $\bigotimes'_{p<\infty}\mathbb{V}_p\times\mathbb{V}_\infty$ (see Section 2.2 or [**21**, Section 3.2.2]). For each place v let $\mathcal{I}_v\colon\mathbb{V}_v\to\mathbb{V}_v$ be an intertwining operator of $\mathbb{V}_v$ given by the following partial Fourier transform:

$$\mathcal{I}_v\varphi\left(\begin{pmatrix}X_1\\X_2\end{pmatrix},t\right):=\int_{B_v}\psi_v(-t\,\mathrm{tr}(\overline{Y}X_1))\varphi\left(\begin{pmatrix}Y\\X_2\end{pmatrix},t\right)dY\ \ (\varphi\in\mathbb{V}_v, X_1,X_2\in B_v, t\in\mathbb{Q}_v^\times),$$

where dY is the Haar measure on B_v self-dual with respect to the paring $B_v\times B_v\ni(X,Y)\mapsto\psi_v(\mathrm{tr}(\overline{X}Y))$. We put $\mathcal{I}:=\bigotimes_{v\le\infty}\mathcal{I}_v\in\mathrm{End}(\mathbb{V})$. For $\xi\in B^-\setminus\{0\}$ let us introduce a $\mathbb{Q}$-algebraic group $S(\xi)$ by

$$S(\xi)(\mathbb{Q})=\{(\iota_\xi(s),\bar{s}^{-1})\mid s\in\mathbb{Q}(\xi)^\times\}\subset H(\mathbb{Q})\times H'(\mathbb{Q}).$$

We denote by $\mathbf{h}(E)$ and $\mathbf{w}(E)$ the class number of E and the number of the roots of unity in E respectively. By the same reasoning as the proof of [**19**, Proposition 2.4.1] we then verify the following integral formula for the Bessel periods (or Fourier coefficients) of $\mathcal{L}^{\mathrm{nc}}(f,f')$:

Proposition 2.1. *For $\xi\in B^-\setminus\{0\}$ and $\chi\in X_\xi$, we have*

$$\mathcal{L}^{\mathrm{nc}}(f,f')_\xi^\chi(g)=\frac{\mathbf{w}(E)}{\mathbf{h}(E)}\int_{(\mathbb{R}_+^\times)^2S(\xi)(\mathbb{A})\backslash(H\times H')(\mathbb{A})}\Phi_\xi(g,h,h')\overline{P_\chi(f;h)}P_\chi(f';h')dhdh',$$

where

$$\Phi_\xi(g,h,h') := (\mathcal{I}\cdot r(g,h,h')\varphi_0^{\mathrm{nc}})\left(\begin{pmatrix}\xi+a/2\\ 1\end{pmatrix},1\right)$$

(*for* φ_0^{nc} *see Section* 2.2).

By the same reasoning as in [**19**, Theorem 5.1.1] we see the following:

PROPOSITION 2.2. *For* $\xi\in B^-\setminus\{0\}$, $\mathcal{L}^{\mathrm{nc}}(f,f')^\chi_\xi\equiv 0$ *unless* $i_p(\chi)=0$ *for any* $p|d_B$ *and* $w_\infty(\chi)=-\kappa_1$.

Following the normalization of the measures as in [**19**, Section 3.3], we obtain our formula for $\mathcal{L}^{\mathrm{nc}}(f,f')^\chi_\xi$, which is a generalization of [**19**, Theorem 5.2.1] without assuming $\kappa_1=\kappa_2$.

THEOREM 2.3. *Let* (f,f') *be Hecke eigenforms with the same signature of the Atkin–Lehner involutions and* $\xi\in B^-\setminus\{0\}$ *be primitive* (*cf.* (1) *and* (2) *in Section* 2.4). *Suppose that* χ *satisfies* $i_p(\chi)=0$ *for any* $p|d_B$ *and* $w_\infty(\chi)=-\kappa_1$ (*cf. Proposition* 2.2). *We then have the following formula*:

$$\begin{aligned}&\mathcal{L}^{\mathrm{nc}}(f,f')^\chi_\xi\left(g_{0,f}\begin{pmatrix}\sqrt{\eta_\infty} & 0\\ 0 & \sqrt{\eta_\infty}^{-1}\end{pmatrix}\right)\\ &\quad=\frac{\mathbf{w}(E)}{\mathbf{h}(E)}\left(\prod_{p<\infty}C_p(f,\xi,\chi)\right)W_\xi^{\kappa_1,\kappa_2}(\eta_\infty)\,\overline{P_\chi(f;\gamma_0)}P_\chi(f';\gamma_0').\end{aligned}$$

Here, for $\eta_\infty\in\mathbb{R}_+^\times$,

$$\begin{aligned}W_\xi^{\kappa_1,\kappa_2}(\eta_\infty) &:= 2^{\kappa_1}n(\xi)^{(\kappa_2+2)/4}(-\sqrt{-1})^{\kappa_2-\kappa_1}\eta_\infty^{\kappa_2/2+2}\exp\bigl(-4\pi\eta_\infty\sqrt{n(\xi)}\bigr)\\ &\quad\times\sum_{i=0}^{(\kappa_2-\kappa_1)/2}\sum_{j=0}^{(\kappa_2-\kappa_1)/2-i}\binom{(\kappa_2-\kappa_1)/2}{i}\binom{(\kappa_2-\kappa_1)/2-i}{j}\\ &\qquad\frac{2^{(\kappa_2-\kappa_1)-2(i+j)}\Gamma(i+1/2)\Gamma(j+1/2)}{(2\pi\eta_\infty\sqrt{n(\xi)})^{i+j+1}}\end{aligned}$$

and $g_{0,f}=(g_{0,p})_{p<\infty}\in G_{\mathbb{A}_f}$ *is given by*

$$g_{0,p}:=\begin{cases}\mathrm{diag}(p^{i_p(\chi)-\mu_p},p^{2(i_p(\chi)-\mu_p)},1,p^{i_p(\chi)-\mu_p}) & (p\nmid d_B)\\ 1_2 & (p|d_B)\end{cases}.$$

PROOF. The proof starts with the integral formula in Proposition 2.1. The calculation of the integral is reduced to local ones since the integral is Eulerian. The non-archimedean local calculations are already settled in [**19**, Section 6–Section 11]. The archimedean local calculation is a generalization of [**19**, Proposition 13.2.2]. With the reproducing kernel function w_{κ_1} for $\mathcal{S}_{\kappa_1}(D)$ (cf. [**19**, (12.2)]) the archimedean local part can be written as

$$\int_{(\mathbb{R}_+^\times)^2 S(\xi)(\mathbb{R})\backslash(H\times H')(\mathbb{R})} \Phi_\xi\left(\left(\begin{pmatrix}\sqrt{\eta_\infty} & \\ & \sqrt{\eta_\infty}^{-1}\end{pmatrix}, h_\infty, h'_\infty\right)\overline{w_{\kappa_1}(\gamma_{0,\infty}^{-1}h_\infty)}n(h'_\infty)^{\kappa_1/2}\right.$$
$$\times (\sigma_{(\kappa_1+\kappa_2)/2}\otimes\sigma_{(\kappa_2-\kappa_1)/2}(h'_\infty))^{-1}v_{(\kappa_1+\kappa_2)/2,(\kappa_1+\kappa_2)/2}(\xi)\otimes v_{(\kappa_2-\kappa_1)/2,0}(\xi)dh_\infty dh'_\infty.$$

Here note that σ_κ (or the outer tensor product $\sigma_\kappa \boxtimes \sigma_0$) and $v_{\xi,\kappa}$ in the proof of [**19**, Proposition 13.2.2] are replaced by the outer tensor product $\sigma_{(\kappa_1+\kappa_2)/2}\boxtimes\sigma_{(\kappa_2-\kappa_1)/2}$ and $v_{(\kappa_1+\kappa_2)/2,(\kappa_1+\kappa_2)/2}(\xi)\otimes v_{(\kappa_2-\kappa_1)/2,0}(\xi)$ respectively, and that $\overline{P_\chi(f;\gamma_0)}P_\chi(f';\gamma'_0)$ is a constant multiple of $v_{\kappa_2,(\kappa_1+\kappa_2)/2}(\xi)$ for σ_{κ_2}, which is viewed as $v_{(\kappa_1+\kappa_2)/2,(\kappa_1+\kappa_2)/2}(\xi)\otimes v_{(\kappa_2-\kappa_1)/2,0}(\xi)$ via the unique embedding of σ_{κ_2} into the inner tensor product $\sigma_{(\kappa_1+\kappa_2)/2}\otimes\sigma_{(\kappa_2-\kappa_1)/2}$, where it is a fundamental fact that the inner tensor product $\sigma_{(\kappa_1+\kappa_2)/2}\otimes\sigma_{(\kappa_2-\kappa_1)/2}$ decomposes into irreducible pieces with multiplicity one.

It follows from the argument in [**19**, Sections 13.2, 13.3] that the above integral is verified to coincide with $I(\eta_\infty)v_{(\kappa_1+\kappa_2)/2,(\kappa_1+\kappa_2)/2}(\xi)\otimes v_{(\kappa_2-\kappa_1)/2,0}(\xi)$, where

$$\begin{aligned}I(\eta_\infty) &= \left(2n(\xi)^{1/4}\right)^{\kappa_1}\eta_\infty^{\kappa_2/2+2}(-\sqrt{-1})^{\kappa_2-\kappa_1}\\ &\times\int_0^\infty\int_{-\infty}^\infty y^{(\kappa_1-\kappa_2)/2}\left(x^2+\left(y+\sqrt{n(\xi)}\right)^2\right)^{(\kappa_2-\kappa_1)/2}\\ &\qquad \exp\left(-2\pi\eta_\infty(y^{-1}(x^2+n(\xi))+y)\right)dx\frac{dy}{y}.\end{aligned}$$

We now carry out the integration with respect to x, and then have

$$\begin{aligned}I(\eta_\infty) &= \left(2n(\xi)^{1/4}\right)^{\kappa_1}\eta_\infty^{\kappa_2/2+2}(-\sqrt{-1})^{\kappa_2-\kappa_1}\\ &\cdot\sum_{i=0}^{(\kappa_2-\kappa_1)/2}\binom{(\kappa_2-\kappa_1)/2}{i}\sqrt{n(\xi)}^{(\kappa_2-\kappa_1)/2-i}\frac{\Gamma(i+1/2)}{(2\pi\eta_\infty)^{i+1/2}}\\ &\times\int_0^\infty\sqrt{y}\left(\frac{y}{\sqrt{n(\xi)}}+2+\frac{\sqrt{n(\xi)}}{y}\right)^{(\kappa_2-\kappa_1)/2-i}\\ &\qquad\exp\left(-2\pi\eta_\infty\sqrt{n(\xi)}\left(\frac{y}{\sqrt{n(\xi)}}+\frac{\sqrt{n(\xi)}}{y}\right)\right)\frac{dy}{y}\\ &= \left(2n(\xi)^{1/4}\right)^{\kappa_1}\eta_\infty^{\kappa_2/2+2}(-\sqrt{-1})^{\kappa_2-\kappa_1}\\ &\cdot\sum_{i=0}^{(\kappa_2-\kappa_1)/2}\binom{(\kappa_2-\kappa_1)/2}{i}\frac{\sqrt{n(\xi)}^{(\kappa_2-\kappa_1)/2+1}\Gamma(i+1/2)}{\left(2\pi\eta_\infty\sqrt{n(\xi)}\right)^{i+1/2}}\\ &\times\int_0^\infty\frac{1}{2}(\sqrt{y}+\sqrt{y}^{-1})(y+2+y^{-1})^{(\kappa_2-\kappa_1)/2-i}\\ &\qquad\exp\left(-2\pi\eta_\infty\sqrt{n(\xi)}(y+y^{-1})\right)\frac{dy}{y}.\end{aligned}$$

Putting $t=\sqrt{y}-\sqrt{y}^{-1}$, we rewrite this as

$$\left(2n(\xi)^{1/4}\right)^{\kappa_1}\eta_\infty^{\kappa_2/2+2}(-\sqrt{-1})^{\kappa_2-\kappa_1}\cdot\sum_{i=0}^{(\kappa_2-\kappa_1)/2}\binom{(\kappa_2-\kappa_1)/2}{i}\frac{\sqrt{n(\xi)}^{(\kappa_2-\kappa_1)/2+1}\Gamma(i+1/2)}{(2\pi\eta_\infty\sqrt{n(\xi)})^{i+1/2}}$$

$$\times \int_{-\infty}^{\infty} (t^2+4)^{(\kappa_2-\kappa_1)/2-i} \exp\big(-2\pi\eta_\infty \sqrt{n(\xi)}(t^2+2)\big)\,dt.$$

Carrying out the integration with respect to t we obtain the formula. □

Clearly the constant $\mathbf{w}(E)/\mathbf{h}(E)(\prod_{p<\infty} C_p(f,\xi,\chi))W_\xi^{\kappa_1,\kappa_2}(\eta_\infty)$ is non-zero by definition. As an immediate consequence of the theorem we have the following:

Corollary 2.4. *Under the same assumption in Theorem* 2.3, *the theta lift* $\mathcal{L}^{\mathrm{nc}}(f,f')$ *is non-vanishing if and only if there is a primitive* $\xi \in B^- \setminus \{0\}$ *and a Hecke character* $\chi \in X_\xi$ *of weight* $w_\infty(\chi) = -\kappa_1$ *such that*

$$P_\chi(f;\gamma_0)P_\chi(f';\gamma_0') \neq 0.$$

Remark 2.5. According to Sugano [**25**, Theorem 2-1], the Fourier coefficient $\mathcal{L}^{\mathrm{nc}}(f,f')_\xi^\chi$ is determined by the evaluation at $g_{0,f}\begin{pmatrix} \sqrt{\eta_\infty} & 0 \\ 0 & \sqrt{\eta_\infty}^{-1} \end{pmatrix}$ (see also [**19**, Lemma 1.4.2]).

3. Non-vanishing of the theta lifts.

3.1. Review on the results of the toral integrals.

We are going to take up two aspects on the non-vanishing of $\mathcal{L}^{\mathrm{nc}}(f,f')$ for $GSp(1,1)$. For that purpose we need explicit formulas for the square norms of the toral integrals in terms of central L-values.

Denote by $L(\pi,\mathrm{Ad},s)$ the adjoint L-function of an automorphic representation π of $GL_2(\mathbb{A})$. We recall that, for Hecke eigenforms $(f,f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$, $\pi(f)$ denotes the automorphic representation of $GL_2(\mathbb{A})$ generated by f and $\mathrm{JL}(\pi(f'))$ the Jacquet–Langlands lift of the automorphic representation $\pi(f')$ generated by f' (cf. [**21**, Section 3.2.1]).

We now recall that we have used the notation E (respectively χ) to denote $\mathbb{Q}(\xi)$ for a fixed $\xi \in B^- \setminus \{0\}$ (respectively a Hecke character for E). Let Π (respectively Π') be the base change lift of $\pi(f)$ (resp. $\mathrm{JL}(\pi(f'))$) to $GL_2(\mathbb{A}_E)$ and let $L(\Pi,\chi^{-1},s)$ (respectively $L(\Pi',\chi^{-1},s)$) be the L-function of Π (respectively Π') twisted by χ^{-1}, for which see [**20**, Section 2.2]. We fix a unitary inner product $(*,*)_{\kappa_2}$ of V_{κ_2} and denote by $\|*\|$ the norm of V_{κ_2} induced by this inner product. We denote the norm of f and f' by $\langle f,f\rangle$ and $\langle f',f'\rangle$ respectively (cf. [**20**, Sections 2.3 and 2.4]), where we note that the norm $\langle f',f'\rangle$ is induced by $(*,*)_{\kappa_2}$.

We define f'_∞, $f'_{\infty,\kappa_1} \in V_{\kappa_2}$ as in [**20**, Section 2.4], where note that $w_\infty(\chi)$ is now $-\kappa_1$ instead of $-\kappa$ in [**20**, Section 2.4].

We now introduce two finite subsets of rational primes

$$S_1 = \{p|D : p \text{ is inert in } E\},\ S_2^+(f,\chi) = \{p|D : p \text{ is ramified in } E \text{ and } \chi_p(\varpi_p) = -\epsilon_p\}$$

with a prime element ϖ_p of the p-adic completion E_p of E, where ϵ_p denotes the eigenvalue of the Atkin–Lehner involution of f at $p|D$ (see [**21**, Proposition 3.3 (1)] and Section 2.4 (1)). We put $\delta(D) := \sharp\{p : p|D\}$ and $A(\chi) := \prod_{p<\infty} p^{i_p(\chi)}$, which are also used in [**20**, Section 2.3].

Let r_p be the ramification index of p in E, i.e.

$$r_p := \begin{cases} 1 & (p\text{:non-ramified in } E) \\ 2 & (p\text{:ramified in } E) \end{cases}.$$

For the quadratic character η of $\mathbb{A}^\times/\mathbb{Q}^\times$ attached to the extension $E/\mathbb{Q}$, $L_p(\eta_p, s)$ denotes the p-factor of the L-function $L(\eta, s)$ for η. We denote by π''_p the p-component of $\pi(f')$.

PROPOSITION 3.1. *Suppose that $i_p(\chi) = 0$ for $p|d_B$ and $w_\infty(\chi) = -\kappa_1$.*
(1) *For a non-zero primitive form f, we have*

$$\frac{|P_\chi(f;\gamma_0)|^2}{\langle f, f\rangle} = \begin{cases} C(f,\chi)\, L\left(\Pi, \chi^{-1}, \dfrac{1}{2}\right) & (S_1 = S_2^+(f,\chi) = \emptyset), \\ 0 & (\textit{otherwise}), \end{cases}$$

with

$$C(f,\chi) := \frac{2^{|d(D)|-2}|d_\xi| \prod_{p|A(\chi)} L_p(\eta_p, 1)^2}{D^{3/2} A(\chi) L(\pi(f), \mathrm{Ad}, 1)}.$$

(2) *For a non-zero Hecke eigenform f', we have*

$$\frac{\|P_\chi(f';\gamma'_0)\|^2}{\langle f', f'\rangle} = \begin{cases} C(f',\chi)\, L\left(\Pi', \chi^{-1}, \dfrac{1}{2}\right) \\ \qquad\qquad (\pi''_p|_{E_p^\times} = \chi_p \textit{ when } p \textit{ divides } d_B \textit{ and is ramified in } E), \\ 0 \qquad\qquad (\textit{otherwise}), \end{cases}$$

where

$$C(f',\chi) := \frac{\sqrt{|d_\xi|}(\kappa_2+1)\binom{\kappa_2}{(\kappa_1+\kappa_2)/2}}{4A(\chi)L(\mathrm{JL}(\pi(f')), \mathrm{Ad}, 1)} \cdot \prod_{p|A(\chi)} L_p(\eta_p, 1)^2 \cdot \prod_{p|d_B} r_p p^{-1} \cdot \frac{(f'_{\infty,\kappa_1}, f'_{\infty,\kappa_1})_{\kappa_2}}{(f'_\infty, f'_\infty)_{\kappa_2}}.$$

Both assertions are explicit versions of the general formula by Waldspurger [**27**, Proposition 7]. The first assertion is nothing but [**20**, Proposition 2.6] and the second one is obtained by modifying the formula of [**20**, Proposition 2.7] at the archimedean place. Regarding the modification on the second formula we state the generalization of [**20**, Lemma 3.14] to our case as follows:

LEMMA 3.2. *We have the following formula for special values of the archimedean L-factors*:

$$L_\infty(\pi'_\infty, \mathrm{Ad}, 1) = 2^{-(\kappa_2+1)}\pi^{-(\kappa_2+3)}(\kappa_2+1)!,$$
$$L_\infty\left(\Pi'_\infty, \chi_\infty^{-1}, \frac{1}{2}\right) = 2^{-\kappa_2}\pi^{-(\kappa_2+2)}\left(\frac{\kappa_1+\kappa_2}{2}\right)!\left(\frac{\kappa_2-\kappa_1}{2}\right)!,$$
$$\zeta_\infty(2) = L_\infty(\eta_\infty, 1) = \pi^{-1},$$

where π'_∞, Π'_∞, χ_∞ and η_∞ denote the archimedean factor of $\mathrm{JL}(\pi(f'))$, Π', χ *and* η *respectively, and ζ_∞ the archimedean factor of the Riemann zeta function.*

3.2. Non-vanishing of the theta lifts to $GSp(1,1)$ (I): An application of the non-vanishing modulo p of central L-values.

As an application of the results by Hsieh [**11**] and Chida–Hsieh [**2**] we discuss the non-vanishing of $\mathcal{L}^{\mathrm{nc}}(f,f')$.

Hsieh [**11**, Theorem C] (respectively Chida and Hsieh [**2**, Theorem 5.9]) have studied the "non-vanishing modulo p" of central L-values $L(\Pi,\chi^{-1},1/2)$ (respectively $L(\Pi',\chi^{-1},1/2)$) in the spirit of the Iwasawa theory. We apply their results to our situation and have the following:

THEOREM 3.3. *Let f be a non-zero primitive form and f' be a non-zero Hecke eigenform, and suppose that (f,f') has the same signature of the Atkin–Lehner involutions (cf. Section 2.4 (1)).*

Take a primitive $\xi\in B^-\setminus\{0\}$ so that d_B divides the discriminant of the imaginary quadratic field $E=\mathbb{Q}(\xi)$. Choose a prime l so that l splits in E. In addition to the assumption $w_\infty(\chi)=-\kappa_1$ in Proposition 3.1 we suppose that the conductor of χ is a power of l.

We assume that f (respectively f') satisfies the local condition $S_1=S_2^+(f,\chi)=\emptyset$ (respectively $\pi''_p|_{E_p^\times}=\chi_p$ for $p|d_B$). We furthermore let $X_\xi(l)_f$ be the set of Hecke characters in X_ξ of finite order whose conductors are powers of l.

For all but finitely many $\nu\in X_\xi(l)_f$ we then have

$$L\left(\Pi,(\chi\nu)^{-1},\frac{1}{2}\right)L\left(\Pi',(\chi\nu)^{-1},\frac{1}{2}\right)\neq 0.$$

PROOF. We first explain the existence of $\xi\in B^-\setminus\{0\}$ with the assumption above. For an imaginary quadratic field E with the discriminant divisible by d_B, the p-adic completion E_p of E is a field for every prime $p|d_B$. This implies that E can be regarded as a subfield of B (cf. [**26**, Theoreme 3.8]). As ξ we may take a non-zero primitive element in $E\cap B^-$.

With the prime l in the assertion we next need the existence of an auxiliary prime $p\neq l$ satisfying the assumptions in [**2**, Theorem 5.9] and [**11**, Theorem C], where, to avoid the notational confusion, we should note that p is replaced by l in [**2**]. For [**2**, Theorem 5.9] and [**11**, Theorem C] just mentioned we make remarks as follows:

1. Among those assumptions, there is an assumption on the existence of a prime p at which residual Galois representations for f and f' modulo p are absolutely irreducible. It is proved by Ribet [**22**, Theorem 2.1] that such existence holds for all but finitely many prime numbers.

2. The condition "l splits in E" in the assertion of the theorem is due to the condition "(ord)" in [**11**].

3. Under the assumption that the discriminant of $\mathbb{Q}(\xi)$ is divisible by d_B, the local condition "$\pi''_p|_{E_p^\times}=\chi_p$ for $p|d_B$" (respectively $S_1=S_2^+(f,\chi)=\emptyset$) is equivalent to the local condition "(ST)" (respectively Hypothesis A), which is assumed in [**2**, Theorem 5.9] (respectively [**11**, Theorem C]).

Without difficulty we then verify that we can choose the auxiliary p so that all the assumptions in [**2**, Theorem 5.9] and [**11**, Theorem C] are satisfied. We therefore see that the theorem is a consequence of [**2**, Theorem 5.9] and [**11**, Theorem C]. □

In order to apply this theorem to show the non-vanishing of $\mathcal{L}^{\mathrm{nc}}(f,f')$ we need the lemma as follows:

Lemma 3.4. *Suppose that the imaginary quadratic field $E \subset B$ is not isomorphic to $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, and let ξ be a non-zero element of $E \cap B^-$. For any given $(\epsilon_{0,p})_{p|d_B} \in \prod_{p|d_B}\{\pm 1\}$ and any even integer κ, there exists a Hecke character $\chi \in X_\xi$ satisfying the following condition:*

- *the conductor of χ is a power of a prime number l, where l does not divide the discriminant of E.*
- *$w_\infty(\chi) = \kappa$.*
- *for $p|d_B$, $\chi_p(\varpi_p) = \epsilon_{0,p}$ with a prime element ϖ_p.*

Proof. We can find a Hecke character $\chi_1 \in X_\xi$ (for X_ξ see Section 2.3) satisfying the first and second conditions. In fact, take a Hecke character in X_ξ of finite order with the conductor in the first condition and take a character of $E_\infty^\times/\mathbb{R}_+^\times \simeq \mathbb{C}^\times/\mathbb{R}_+^\times$ of weight κ. Since the cardinalities of unit groups of orders in E are at most two by the assumption on E (saying $E \not\simeq \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$), such character of $E_\infty^\times/\mathbb{R}_+^\times$ is invariant with respect to the unit group of the integer ring for E and can be thus viewed as an element in X_ξ unramified at every finite prime by the pull back via the projection $\mathbb{A}_E^\times/E^\times \cdot \prod_{p<\infty}\mathfrak{O}_{E,p}^\times \cdot \mathbb{A}^\times \to E_\infty^\times/\mathbb{R}_+^\times$. Then χ_1 can be taken as a product of the two characters.

We next consider a character $\chi_2 := \prod_{p|d_B}\chi_{2,p}$ on $\prod_{p|d_B}E_p^\times/\mathbb{Q}_p^\times\mathfrak{O}_{E,p}^\times$ defined by

$$\chi_{2,p}(\varpi_p) = \epsilon_{0,p}\cdot\chi_{1,p}(\varpi_p) \quad \text{for } p|d_B,$$

with the unit group $\mathfrak{O}_{E,p}^\times$ of the integer ring for E_p. Here note that $\chi_{1,p}(\varpi_p) \in \{\pm 1\}$ with the p-component $\chi_{1,p}$ of χ_1. We now recall again that the assumption on E implies that the unit group of the integer ring of E is $\{\pm 1\}$. Taking this into account we can show that the canonical map

$$\prod_{p|d_B} E_p^\times/\mathbb{Q}_p^\times\mathfrak{O}_{E,p}^\times \to \mathbb{A}_E^\times/E^\times \cdot \prod_{p<\infty}\mathfrak{O}_{E,p}^\times \cdot E_\infty^\times \cdot \mathbb{A}^\times$$

is an injection. In fact, assume that the image of $x \in \prod_{p|d_B}E_p^\times$ belongs to $E^\times \cdot \prod_{p<\infty}\mathfrak{O}_{E,p}^\times \cdot E_\infty^\times \cdot \mathbb{A}^\times$. Then the $E^\times$-part of the image of $x/\bar{x}$ with the Galois conjugate $\bar{x}$ of x is proved to be in $\{\pm 1\}$ by the assumption. When it is -1, we can deduce a contradiction that $E_p/\mathbb{Q}_p$ is unramified at a prime $p|d_B$. When it is 1, the $E^\times$-part of x is in $\mathbb{Q}$ and we then see $x \in \prod_{p|d_B}\mathbb{Q}_p\mathfrak{O}_{E,p}^\times$. It is then verified that χ_2 can be extended to a Hecke character in X_ξ unramified at every finite prime.

As a Hecke character χ in the assertion we can take $\chi = \chi_1\cdot\chi_2$. □

This lemma implies that, for a non-zero primitive form f and a non-zero Hecke eigenform f', there is a Hecke character $\chi \in X_\xi$ with a suitable $\xi \in B^- \setminus \{0\}$ satisfying the assumption of Theorem 3.3. With such a χ we see that $C(f,\chi) \neq 0$ and $C(f',\chi) \neq 0$ for f and f' (see Proposition 3.1 for $C(f,\chi)$ and $C(f',\chi)$), for which we note that $L(\pi(f),\mathrm{Ad},1)$ and $L(\mathrm{JL}(\pi(f')),\mathrm{Ad},1)$ are non-zero since we can relate these to the non-zero Petersson norms for the non-zero forms f and $\mathrm{JL}(f')$ (cf. [**20**, Proposition 2.5]). We now obtain the following theorem.

Theorem 3.5. *Let f and f' be a non-zero primitive form and a non-zero Hecke eigenform respectively. Then the theta lift $\mathcal{L}^{\mathrm{nc}}(f,f')$ is non-vanishing if and only if $\epsilon_p = \epsilon'_p$ for $p|D$ (for this condition see* [**21**, Proposition 3.3 (1)] *and Section* 2.4 (1)*), for which we note that the non-vanishing of $\mathcal{L}^{\mathrm{nc}}(f,f')$ holds without the condition on the ϵ-factors when $D=1$.*

Proof. According to [**21**, Proposition 3.3 (1)] the condition "$\epsilon_p = \epsilon'_p$ for $p|D$" is necessary for $\mathcal{L}^{\mathrm{nc}}(f,f') \not\equiv 0$. We are going to show that Theorem 3.3 and Lemma 3.4 imply that this condition is also sufficient in view of Corollary 2.4 and Proposition 3.1. To explain the sufficiency in detail, let us recall that $X_\xi(l)_f$ is introduced in Theorem 3.3 and consider

$$X_\xi(l)'_f := \{\nu \in X_\xi(l)_f : \nu(\varpi_p) = 1 \ \forall p|d_B\},$$

where ϖ_p denotes a prime element of $E = \mathbb{Q}(\xi)$ at $p|d_B$. This set is non-empty by Lemma 3.4. In fact, it is infinite since $X_\xi(l)'_f$ admits infinitely many conductors. Lemma 3.4 also implies that there exists a Hecke character $\chi \in X_\xi$ of weight $-\kappa_1$ such that $\chi(\varpi_p) = \epsilon'_p$ for each $p|d_B$, for which we note that $\chi(\varpi_p) = \epsilon_p = \epsilon'_p$ holds for $p|D$ by the assumption. Such a χ satisfies the assumptions in Theorem 3.3. Given a Hecke character χ with the assumption of Theorem 3.3, $\chi\nu$ also satisfies the same assumption as χ for any $\nu \in X_\xi(l)'_f$. As a result we see that there exists $\nu \in X_\xi(l)$ such that $P_{\chi\nu}(f;\gamma_0)P_{\chi\nu}(f';\gamma'_0) \neq 0$ for such a χ by virtue of Proposition 3.1 and Theorem 3.3, for which note that $C(f,\chi\nu)C(f',\chi\nu) \neq 0$ holds for $\nu \in X_\xi(l)'_f$ as is remarked just before the statement of the theorem. Then Corollary 2.4 yields $\mathcal{L}^{\mathrm{nc}}(f,f') \not\equiv 0$. □

3.3. Non-vanishing of the theta lifts to $GSp(1,1)$ (II): A relation with special values of hypergeometric functions.

Following the approach of [**19**, Section 14] we discuss the existence of a non-vanishing Bessel period $\mathcal{L}^{\mathrm{nc}}(f,f')^\chi_\xi$, which implies $\mathcal{L}^{\mathrm{nc}}(f,f') \not\equiv 0$. The approach explained soon is to reduce the simultaneous non-vanishing of the toral integras (cf. Corollary 2.4) to two simple conditions (cf. Proposition 3.8). What we stress now is that one of them turns out to be controlled by some special values of hypergeometric functions. Now recall that the definition of the hypergeometric function (cf. [**28**, Chapter XIV]) begins with the series

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

where $(a)_n$, $(b)_n$, $(c)_n$ denote the shifted factorials often called the Pochhammer symbols. This series is known to converge for $|z| < 1$. By the hypergeometric function we mean the function obtained by the analytic continuation of the series. What we need is the hypergeometric functions with parameters a, b in negative integers. These are reduced to polynomials. Thus we do not have to worry about the problem of the convergence for such series and can therefore consider their evaluations at any complex numbers.

We let B be the definite quaternion algebra with the discriminant $d_B = 2,\ 3$ or 7. This condition of d_B implies that d_B is prime and the class number of B is equal to one, which is deduced from Eichler's trace formula for Brandt matrices (cf. [**5**, (64)]). The quaternion algebra B can be expressed as $\mathbb{Q}+\mathbb{Q}i+\mathbb{Q}j+\mathbb{Q}k$ with

$$\begin{cases} i^2 = j^2 = -1,\ ij = -ji = k & (d_B = 2), \\ i^2 = -1,\ j^2 = -d_B,\ ij = -ji = k & (d_B = 3,\ 7) \end{cases}$$

(see [**26**, p.79, Exemple]). The maximal order $\mathfrak{O}$ of B is given as

$$\begin{cases} \mathbb{Z}\dfrac{1+i+j+k}{2} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k & (d_B = 2), \\ \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}\dfrac{1+j}{2} + \mathbb{Z}\dfrac{i+k}{2} & (d_B = 3,\ 7) \end{cases}$$

(see [**26**, p.98, Exercise 5.2]). As a primitive $\xi \in B^-$ (cf. Section 2.4) we can take

$$\xi = \begin{cases} i/2 \text{ or } \dfrac{i+j}{2} & (d_B = 2), \\ j/2d_B & (d_B = 3,\ 7). \end{cases}$$

We then have isomorphisms of quadratic extensions as follows:

$$\mathbb{Q}(\xi) \simeq \begin{cases} \mathbb{Q}(\sqrt{-1}) & (\xi = i/2,\ d_B = 2), \\ \mathbb{Q}(\sqrt{-2}) & \left(\xi = \dfrac{i+j}{2},\ d_B = 2\right), \\ \mathbb{Q}(\sqrt{-3}) & (\xi = j/d_B,\ d_B = 3), \\ \mathbb{Q}(\sqrt{-7}) & (\xi = j/d_B,\ d_B = 7). \end{cases}$$

This choice of the primitive elements is justified by the following lemma, which is verified by a direct calculation.

Lemma 3.6. *Let* $(\mathfrak{a}^-)^* := \{x \in B^- \mid \operatorname{tr}(\bar{z}w) \in \mathbb{Z},\ \forall w \in \mathfrak{a} \cap B^-\}$. *We have*

$$(\mathfrak{a}^-)^* = \begin{cases} \mathbb{Z}\dfrac{i}{2} + \mathbb{Z}\dfrac{j}{2} + \mathbb{Z}\dfrac{k}{2} & (d_B = D = 2), \\ \mathbb{Z}\dfrac{i}{2} + \mathbb{Z}\dfrac{i-j+k}{4} + \mathbb{Z}\dfrac{i-j-k}{4} & (D = 1,\ d_B = 2), \\ \mathbb{Z}\dfrac{d_B i - k}{2d_B} + \mathbb{Z}\dfrac{j}{2d_B} + \mathbb{Z}\dfrac{k}{d_B} & (d_B = D = 3,\ 7), \\ \mathbb{Z}\dfrac{i}{d_B} + \mathbb{Z}\dfrac{j}{2d_B} + \mathbb{Z}\left(\dfrac{i}{2d_B} + \dfrac{k}{2d_B}\right) & (D = 1,\ d_B = 3,\ 7), \end{cases}$$

where note that the level D of f is 1 or d_B.

Remark 3.7. This is a generalization of [**19**, Lemma 14.2.1]. The second formula is a correction to the first formula in [**19**, Lemma 14.2.1]. However, the choice of the primitive element $i/2$ in [**19**, Section 14] remains justified.

We now note that the class number of $\mathbb{Q}(\xi)$ is one for all primitive ξs above. In fact, the prime p equals 2, 3 or 7 if and only if the class number of B with $d_B = p$ is one and the class number of the imaginary quadratic field with the discriminant $-p$ or $-4p$ is also one. This is valid for our Bs and $\mathbb{Q}(\xi)$s.

Throughout this section we assume that ξ denotes a primitive element given above and the Hecke character χ of $\mathbb{Q}(\xi)$ is unramified at every finite place. Now recall that we have let $(f, f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$. Together with this assumption, assume that f' has the eigenvalue $\epsilon'_p \in \{\pm 1\}$ with respect to the Atkin–Lehner involution. As in the proof of [**19**, Proposition 14.2.2], we can then say in a rough manner that the toral integrals $P_\chi(f;\gamma_0)$ and $P_\chi(f';\gamma'_0)$ are reduced to the special values $f(\gamma_0)$ and $f'(\gamma'_0 u_\xi) = \epsilon'_p f'(u_\xi)$ ($p = 2,\ 3,\ 7$) respectively (see Section 2.6 for γ_0, γ'_0 and u_ξ) with the above choice of B and ξ, where we note the identification $u_\xi \mathbb{Q}(i) u_\xi^{-1} = \mathbb{Q}(\xi/\sqrt{n(\xi)})$ to verify this. More precisely, from the proof of [**19**, Proposition 14.2.2] we see that $P_\chi(f;\gamma_0) \neq 0$ and $P_\chi(f';\gamma'_0) \neq 0$ are equivalent to $f(\gamma_0) \neq 0$ and $(f'(u_\xi), (\sigma_{\kappa_2}(u_\xi) v_{\kappa_2,(\kappa_1+\kappa_2)/2})^*)_{\kappa_2} = (\sigma_{\kappa_2}(u_\xi)^{-1} f'(1), \sigma_{\kappa_2}(u_\xi)^{-1} v^*_{\kappa_2,(\kappa_1+\kappa_2)/2})_{\kappa_2} = (f'(1), v^*_{\kappa_2,(\kappa_1+\kappa_2)/2})_{\kappa_2} \neq 0$ respectively (see Section 2.6 for the inner product $(*,*)_{\kappa_2}$ and the vector $v^*_{\kappa_2,(\kappa_1+\kappa_2)/2}$).

Proposition 3.8. *Let $(f, f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$ be Hecke eigenforms with the working assumption on the signature (cf. Section 2.4 (1)), and let χ as above satisfy $w_\infty(\chi) = -\kappa_1$. Suppose that $f(\gamma_0) \neq 0$ and $(f'(1), v^*_{\kappa_2,(\kappa_1+\kappa_2)/2})_{\kappa_2} \neq 0$ (see Section 2.6 for $v^*_{\kappa_2,(\kappa_1+\kappa_2)/2}$). Then we have*

1. $\mathcal{L}^{\mathrm{nc}}(f, f')^\chi_\xi \neq 0$*, thus* $\mathcal{L}^{\mathrm{nc}}(f, f') \not\equiv 0$*,*
2. $L\left(\Pi, \chi^{-1}, \frac{1}{2}\right) > 0$ *and* $L\left(\Pi', \chi^{-1}, \frac{1}{2}\right) > 0$.

Proof. The first assertion is verified by the same reasoning as in [**19**, Proposition 14.2.2], for which we can take $u_\xi = 1$ for the case of $\xi = i/2$ and $d_B = 2$.

The second assertion follows from Propositions 3.1. In fact, the constants $C(f,\chi)$ and $C(f',\chi)$ are verified to be positive. Therefore $f(\gamma_0) \neq 0$ and $(f'(1), v_{\kappa_2,(\kappa_1+\kappa_2)/2})_{\kappa_2} \neq 0$ meaning $P_\chi(f;\gamma_0) \neq 0$ and $P_\chi(f', \gamma'_0) \neq 0$ (see the remark just before the proposition) imply $L(\Pi, \chi^{-1}, 1/2) > 0$ and $L(\Pi', \chi^{-1}, 1/2) > 0$ respectively. □

This is applied to finding examples of non-vanishing $\mathcal{L}^{\mathrm{nc}}(f, f')$s, which is stated in Theorem 3.14. Towards the theorem we begin with the following lemma:

Lemma 3.9. *Let $(\kappa_1, \kappa_2) \in (4\mathbb{Z}_{>0}) \times (2\mathbb{Z}_{>0})$ and $\kappa_2 \geq \kappa_1$.*

(1) *Suppose that $d_B = 2$ or 3. Let ${}_2F_1(\alpha, \beta; \gamma; x)$ be the hypergeometric functions with parameters α, β, γ. Suppose that (κ_1, κ_2) satisfies $C^{(d_B)}_{\kappa_1,\kappa_2} \neq 0$, where*

$$C^{(d_B)}_{\kappa_1,\kappa_2} := \begin{cases} 1+2^{-(\kappa_2-4)/2}(-1)^{\kappa_1/4}\binom{(\kappa_1+\kappa_2)/2}{(\kappa_2-\kappa_1)/2}\,{}_2F_1\Big(-\frac{\kappa_2-\kappa_1}{2},-\frac{\kappa_2-\kappa_1}{2};\kappa_1+1;-1\Big) & \\ & (d_B=2), \\ 1+2^{-(\kappa_1-1)}\binom{(\kappa_1+\kappa_2)/2}{(\kappa_2-\kappa_1)/2}\,{}_2F_1\Big(-\frac{\kappa_2-\kappa_1}{2},-\frac{\kappa_2-\kappa_1}{2};\kappa_1+1;-3\Big) & (d_B=3). \end{cases}$$

Then there is a Hecke eigenform $f' \in \mathcal{A}_{\kappa_2}$ such that $(f'(1), v^*_{\kappa_2,(\kappa_1+\kappa_2)/2})_{\kappa_2} \neq 0$.

(2) *Suppose that $d_B = 7$. Then the same assertion holds (without the condition as above on the hypergeometric functions).*

Proof. We first note that, for all the d_Bs above, the class number of B is one, which implies the isomorphism

$$\mathcal{A}_{\kappa_2} \simeq V_{\kappa_2}^{\mathfrak{O}^\times} := \{v \in V_{\kappa_2} \mid \sigma_{\kappa_2}(u)v = v \quad \forall u \in \mathfrak{O}^\times\}.$$

We put $v' := \sum_{u\in\mathfrak{O}^\times} \sigma_{\kappa_2}(u) v_{\kappa_2,(\kappa_1+\kappa_2)/2} \in V_{\kappa_2}^{\mathfrak{O}^\times}$. The condition $(v', v^*_{\kappa_2,(\kappa_1+\kappa_2)/2})_{\kappa_2} \neq 0$ implies the existence of a Hecke eigenform $f' \in \mathcal{A}_{\kappa_2}$ such that $(f'(1), v^*_{\kappa_2,(\kappa_1+\kappa_2)/2})_{\kappa_2} \neq 0$. To consider $(v', v^*_{\kappa_2,(\kappa_1+\kappa_2)/2})_{\kappa_2}$ we note that

$$R = \begin{cases} \Big\{1,\ j,\ \frac{1+i+j+k}{2},\ \frac{1+i-j+k}{2},\ \frac{1+i+j-k}{2},\ \frac{1+i-j-k}{2}\Big\} & (d_B=2) \\ \Big\{1,\ \frac{1+j}{2},\ \frac{-1+j}{2}\Big\} & (d_B=3) \\ \{1\} & (d_B=7) \end{cases}$$

forms a complete set of representatives for $\mathfrak{O}^\times/\{\pm 1, \pm i\}$. This can be referred to as a well known fact or can be checked by a direct calculation. Since each element of the subgroup $\{\pm 1, \pm i\}$ of $\mathfrak{O}^\times$ acts on $v_{\kappa_2,(\kappa_1+\kappa_2)/2}$ trivially, the problem is thus reduced to considering

$$\Bigg(\sum_{u\in R} \sigma_{\kappa_2}(u) v_{\kappa_2,(\kappa_1+\kappa_2)/2}, v^*_{\kappa_2,(\kappa_1+\kappa_2)/2}\Bigg)_{\kappa_2}.$$

We only deal with the case of $d_B = 2$ since the case of $d_B = 3$ is considered similarly and the case of $d_B = 7$ is obvious. By a direct calculation we have the following:

$$\begin{aligned} &(\sigma_{\kappa_2}(u) v_{\kappa_2,(\kappa_1+\kappa_2)/2}, v^*_{\kappa_2,(\kappa_1+\kappa_2)/2})_{\kappa_2} \\ &\quad = \begin{cases} 1 & (u=1) \\ 0 & (u=j) \\ 2^{-\kappa_2/2}(-1)^{\kappa_1/4} \displaystyle\sum_{i=0}^{(\kappa_2-\kappa_1)/2} (-1)^i \binom{(\kappa_1+\kappa_2)/2}{i}\binom{(\kappa_2-\kappa_1)/2}{i} & (u \in R\setminus\{1,\ j\}) \end{cases}, \end{aligned}$$

for which see the explanation of σ_{κ_2} in [**19**, Section 1.2]. We thereby obtain

$$
\begin{aligned}
&\frac{1}{4}(v', v^*_{\kappa_2,(\kappa_1+\kappa_2)/2})_{\kappa_2} \\
&\quad = 1 + 2^{-(\kappa_2-4)/2}(-1)^{\kappa_1/4} \cdot \sum_{i=0}^{(\kappa_2-\kappa_1)/2} (-1)^i \binom{(\kappa_1+\kappa_2)/2}{i}\binom{(\kappa_2-\kappa_1)/2}{i}.
\end{aligned}
$$

Recalling the definition of the hypergeometric functions ${}_2F_1$, we have

$$
\begin{aligned}
&{}_2F_1\left(-\frac{\kappa_2-\kappa_1}{2}, -\frac{\kappa_2-\kappa_1}{2}; \kappa_1+1; -1\right) \\
&\quad = \sum_{i=0}^{(\kappa_2-\kappa_1)/2} (-1)^i \frac{(-(\kappa_2-\kappa_1)/2)_i(-(\kappa_2-\kappa_1)/2)_i}{i!(\kappa_1+1)_i}.
\end{aligned}
$$

We then verify by a direct calculation that $1/4(v', v^*_{\kappa_2,(\kappa_1+\kappa_2)/2})_{\kappa_2}$ coincides with

$$
1 + 2^{-(\kappa_2-4)/2}(-1)^{\kappa_1/4}\binom{(\kappa_1+\kappa_2)/2}{(\kappa_2-\kappa_1)/2}\,{}_2F_1\left(-\frac{\kappa_2-\kappa_1}{2}, -\frac{\kappa_2-\kappa_1}{2}; \kappa_1+1; -1\right).
$$

As a result we verify the assertion. □

REMARK 3.10. For the proof above note that $\{\pm 1, \pm i\}$ acts on $v_{\kappa_2,(\kappa_1+\kappa_2)/2}$ by a non-trivial character of the cyclic group of order 4 when $4 \nmid \kappa_1$. This leads to

$$
\left(\sum_{u\in\mathfrak{O}^\times} \sigma_{\kappa_2}(u)v_{\kappa_2,(\kappa_1+\kappa_2)/2}, v^*_{\kappa_2,(\kappa_1+\kappa_2)/2}\right)_{\kappa_2} = 0.
$$

With our method we cannot therefore show the existence of a Hecke eigenform f' such that $(f'(1), v^*_{\kappa_2,(\kappa_1+\kappa_2)/2})_{\kappa_2} \neq 0$ for $(\kappa_1,\kappa_2) \in (2\mathbb{Z}_{>0})^2$ such that $4 \nmid \kappa_1$.

We now cite the two results by Ebisu [**4**, Propositions 3.2 and 4.4] and Haagerup–Schlichtkrull [**8**, Theorem 1.1] in order to study the non-vanishing of $C^{(d_B)}_{\kappa_1,\kappa_2}$. We first cite Ebisu's result:

PROPOSITION 3.11 (Ebisu). *Let $(\kappa_1,\kappa_2) \in (4\mathbb{Z}_{>0})^2$. When $\kappa_2 \geq 4\kappa_1$ and $(\kappa_1,\kappa_2) \neq (4,16)$ (respectively $\kappa_2 \geq \kappa_1$) we have $C^{(d_B)}_{\kappa_1,\kappa_2} \neq 0$ for $d_B = 2$ (respectively $d_B = 3$).*

A natural question is how much we can improve the condition $\kappa_2 \geq 4\kappa_1$ for the case of $d_B = 2$, which is due to the limitation of the technique of [**4**] by means of the three term relations of the hypergeometric series. We next review the result by Haagerup–Schlichtkrull [**8**] to make some improvement of Ebisu's result above for $d_B = 2$. For that purpose we need to introduce the Jacobi polynomials:

$$
P_n^{(\alpha,\beta)}(x) := \left(\frac{1+x}{2}\right)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}\,{}_2F_1\left(-n, -n-\beta; \alpha+1; \frac{x-1}{x+1}\right)
$$

for a non-negative integer n and two real numbers (α,β) with $\alpha > -1$, $\beta > -1$. By Pfaff's transformation formula (cf. [**1**, Theorem 2.2.5]) this coincides with

$$\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}\,{}_2F_1\left(-n,n+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}\right),$$

which is the definition in [**8**]. From [**8**, Theorem 1.1] we deduce the following formula:

Proposition 3.12. *Let $(\kappa_1,\kappa_2)\in(2\mathbb{Z}_{>0})^2$ satisfy $\kappa_2\geq\kappa_1$.*

(1) *We have*

$$\left|P^{(\kappa_1,0)}_{(\kappa_2-\kappa_1)/2}(0)\right|<\frac{3\cdot 2^{\kappa_1/2+2}}{\sqrt[4]{\kappa_2+1}}.$$

(2) *Suppose that $4|\kappa_1$. Let $\kappa_2\geq 16^4\cdot 3^4-1$. We have $C^{(d_B)}_{\kappa_1,\kappa_2}\neq 0$ for $d_B=2$.*

Proof. The first assertion is a consequence of the formula in [**8**, Theorem 1.1] for $\alpha=\kappa_1$, $\beta=0$ and $n=(\kappa_2-\kappa_1)/2$, where note that the constant C in [**8**, Theorem 1.1] is estimated as $C<12$ as is remarked in [**8**] just after the theorem. To verify the second assertion, we note that $C^{(2)}_{\kappa_1,\kappa_2}\neq 0$ is equivalent to $P^{(\kappa_1,0)}_{(\kappa_2-\kappa_1)/2}(0)\neq(-1)^{\kappa_1/4+1}2^{(\kappa_1-4)/2}$. For the assumption $4|\kappa_1$ see Remark 3.10. We then see by computing the inequality $(3\cdot 2^{\kappa_1/2+2})/\sqrt[4]{\kappa_2+1}\leq 2^{(\kappa_1-4)/2}$ that the second assertion is a consequence of the first assertion. □

We next show the lemma on the existence of Hecke-eigen cusp forms or primitive cusp forms with non-zero toral integrals, which have been proved to be reduced to the condition $f(\gamma_0)\neq 0$ mentioned above. It is not easy to find such Hecke-eigen cusp forms in general. In our setting we are however able to show their existence by finding cusp forms with non-zero toral integrals whose Hecke-eigen property are known only for cases of small weights.

Lemma 3.13. (1) *Let $\kappa\geq 12$ be divisible by 4. There is a Hecke eigenform $f\in S_\kappa(1)$ such that $f(\gamma_0)\neq 0$.*

(2) *Let $\kappa\geq 8$ be divisible by 8. There is a primitive form $f\in S_\kappa(2)$ such that $f(\gamma_0)\neq 0$ and $\epsilon_2=1$.*

(3) *Let $D=3$ or 7 and let $\kappa\geq 6$ be divisible by 12. There is a primitive form $f\in S_\kappa(D)$ such that $f(\gamma_0)\neq 0$ and $\epsilon_D=1$.*

Proof. For the proof we view elements in $S_\kappa(D)$ as elliptic cusp forms on the complex upper half plane $\mathfrak{h}$ in the usual manner. The first and second assertions are already proved in [**19**, Proposition 14.4.1]. We only consider the third assertion, which is also proved similarly. Let $S^{\mathrm{prim}}_\kappa(D)$ denote the subspace of $S_\kappa(D)$ spanned by primitive forms. For $D=3$ or 7 it suffices to find $\varphi\in S^{\mathrm{prim}}_\kappa(D)$ such that φ is non-zero at the CM-point of $\mathfrak{h}$ corresponding to γ_0. Let us introduce

$$(\eta(z)\eta(Dz))^{6l}\quad(l\in\mathbb{Z}_{>0}),$$

where $\eta(z)$ denotes the Dedekind eta function on $\mathfrak{h}$. According to [**3**, Proposition 3.2.2], [**14**, Chapter III, Section 3, Problems 13] and [**17**, Numerical Tables, Table A] we know that $S_6(3)=S^{\mathrm{prim}}_6(3)=\mathbb{C}(\eta(z)\eta(3z))^6$ and $S_3(7,(\frac{*}{7}))=S^{\mathrm{prim}}_3(7,(\frac{*}{7}))=\mathbb{C}(\eta(z)\eta(7z))^3$, where $S_3(7,(\frac{*}{7}))$ denotes the space of the elliptic cusp forms with Neben-type character $(\frac{*}{7})$. If we take $(\eta(z)\eta(Dz))^{6l}$ as φ, we verify that $\varphi \in S_{6l}^{\mathrm{prim}}(D)$ for $D = 3$ or 7 and that it is non-zero at the CM-point. Furthermore assume that l is even. We can then prove $\epsilon_D = 1$ by calculating the signature of the automorphy factor for $\left(\left(\begin{smallmatrix} 0 & 1 \\ -D & 0 \end{smallmatrix}\right), \sqrt{-1}\right)$, as we did in the proof of [**19**, Proposition 14.4.1]. □

The argument so far leads to a result on the non-vanishing of $\mathcal{L}^{\mathrm{nc}}(f,f')^{\chi}_{\xi}$ together with the positivity of the central L-values $L(\Pi,\chi^{-1},1/2)$ and $L(\Pi',\chi^{-1},1/2)$ as follows:

Theorem 3.14. *As we have assumed, let χ be unramified at every finite place, and let $(\kappa_1,\kappa_2) \in (4\mathbb{Z}_{>0})^2$. Let (κ_1,κ_2) satisfy $\kappa_2 \geq 4\kappa_1$, $\kappa_1 = \kappa_2$ or $\kappa_2 \geq 16^4 \cdot 3^4 - 1$ but suppose $(\kappa_1,\kappa_2) \neq (4,16)$ when $d_B = 2$ (respectively $\kappa_2 \geq \kappa_1$ when $d_B = 3$ or $d_B = 7$). We make further assumptions on (κ_1,κ_2) as follows:*

1. *When $D = 1$, assume that $\kappa_1 \geq 12$.*
2. *When $D = 2$, assume that $\kappa_1 \geq 8$ holds together with $8|\kappa_1$.*
3. *When $D = 3$ or $D = 7$, assume that κ_1 is divisible by 12.*

Then there are Hecke eigenforms $(f,f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$ such that

1. $\mathcal{L}^{\mathrm{nc}}(f,f')^{\chi}_{\xi} \not\equiv 0$, *thus* $\mathcal{L}^{\mathrm{nc}}(f,f') \not\equiv 0$,
2. $L\left(\Pi,\chi^{-1},\frac{1}{2}\right) > 0$ *and* $L\left(\Pi',\chi^{-1},\frac{1}{2}\right) > 0$.

Proof. This is a consequence of Propositions 3.8, 3.11, 3.12 and Lemmas 3.9, 3.13 above. For the case of $d_B = 2$ and $\kappa_1 = \kappa_2$ we remark that the assertion is included in [**19**, Theorem 14.1.1]. We explain the case of $D = 2$ in detail since the proof is settled for other Ds similarly. When κ_1 is divisible by 8 the proof of [**19**, Proposition 14.4.1] implies that there is a primitive form f of weight κ_1 and level two with the signature $\epsilon_2 = 1$. In a manner quite similar to the proof of [**19**, Lemma 14.3.4 (ii)], we see that there is a Hecke eigenform f' with $\epsilon_2' = \epsilon_2 = 1$ under the assumption on κ_1 and κ_2 in the assertion. For this we note that the condition $\epsilon_2' = \epsilon_2$ is necessary for $\mathcal{L}^{\mathrm{nc}}(f,f')$ to be non-vanishing. This is nothing but one of the working assumptions in Section 2.4. Under the assumption in the assertion, the three propositions and the two lemmas then yield the theorem for the case of $D = 2$. □

Remark 3.15. As we have seen, the results above on $C^{(2)}_{\kappa_1,\kappa_2}$ does not fully cover the remaining cases for $\kappa_2 < 16^4 \cdot 3^4 - 1$. A natural approach to settle them would be a computer calculation. However, it seems to require an extremely long time to carry out such calculation for all the remaining cases. In fact, as κ_2 becomes larger, the total amount of the calculation for all κ_1 not larger than κ_2 grows enormously. Within author's observation by Magma $C^{(2)}_{\kappa_1,\kappa_2} = 0$ is checked for $(\kappa_1,\kappa_2) = (4,4)$, $(4,8)$, $(4,16)$ and $(12,16)$ but any other example of $C^{(2)}_{\kappa_1,\kappa_2} = 0$ has not been found yet. However, author's ability on the computer calculation have not resolved the problem completely.

In addition, we remark that $C^{(2)}_{\kappa_1,\kappa_2}$s and $C^{(3)}_{\kappa_1,\kappa_2}$s for $(\kappa_1,\kappa_2)\in(4\mathbb{Z}_{>0})\times(2\mathbb{Z}_{>0})$ with $4\nmid\kappa_2$ are out of Ebisu's work [**4**]. The author does not know how to deal with these remaining cases without a computer.

3.4. Non-vanishing theta lifts to $GSp^*(2)$.

Using the notation of Section 2.2 (see also [**21**, Section 3.2.2]) we provide examples of non-vanishing $\mathcal{L}^{\mathrm{c}}(f,f')$s for $GSp^*(2)(\mathbb{A})$, which are essentially due to Ibukiyama and Ihara [**13**, Section 3.2]. Here recall that $(f,f')\in S_{\kappa_1}(D)\times\mathcal{A}_{\kappa_2}$, where $1<\kappa_2+2<\kappa_1$ by the assumption.

Examples of the non-vanishing lifts by Ibukiyama and Ihara.

For our purpose we consider

$$\theta^{\mathrm{c}}_{\kappa_1,\kappa_2}(f')(h):=\int_{\mathbb{R}_+H'(\mathbb{Q})\backslash H'(\mathbb{A})}(\theta^{\mathrm{c}}_{\kappa_1,\kappa_2}(1,h,h'),f'(h'))_{(\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2}dh' \qquad (h\in H(\mathbb{A}))$$

for $f'\in\mathcal{A}_{\kappa_2}$, where see Section 2.2 for $\theta^{\mathrm{c}}_{\kappa_1,\kappa_2}$ and the paring $(*,*)_{(\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2}$. We have seen that this is a $W_{(\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2}$-valued function, where recall that $W_{(\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2}$ denotes the representation space of τ_Λ with $\Lambda=((\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2)$ (for τ_Λ see Section 2.1). We now provide the transformation formula satisfied by local test functions $\varphi^{\mathrm{c}}_{0,\infty}$ and $\varphi_{0,p}$ for all $p<\infty$. Since the proof is quite similar to [**18**, Lemma 3.1, Lemma 3.2] we only state the formula:

LEMMA 3.16. (1) *For a finite prime p let $K_p:=\{k\in GSp^*(2)(\mathbb{Q}_p)\mid gL_p=L_p\}$, $U_p:=\{u=(u_{ij})\in GL_2(\mathbb{Z}_p)\mid u_{21}\in D\mathbb{Z}_p\}$ and $U'_p:=\mathfrak{O}_p^\times$. We have*

$$r(k_p,u_p,u'_p)\varphi_{0,p}=\varphi_{0,p}$$

for any $(k_p,u_p,u'_p)\in K_p\times U_p\times U'_p$.

(2) *For the infinite place ∞ let $K_\infty:=Sp^*(2)(\mathbb{R})$, $U_\infty=SO(2)(\mathbb{R})$ and $U'_\infty=\mathbb{H}^{(1)}$. We have*

$$r(k_\infty,u_\infty,u'_\infty)\varphi^{\mathrm{c}}_{0,\infty}=j(u_\infty,\sqrt{-1})^{-\kappa_1}\tau_\Lambda(k_\infty)^{-1}\cdot\varphi^{\mathrm{c}}_{0,\infty}\cdot\sigma_{\kappa_2}(u'_\infty)$$

for any $(k_\infty,u_\infty,u'_\infty)\in K_\infty\times U_\infty\times U'_\infty$. Here $j(h,z)$ denotes the usual automorphy factor for $(h,z)\in SL_2(\mathbb{R})\times\mathfrak{h}$ with the complex upper half plane $\mathfrak{h}$.

The theta function $\theta^{\mathrm{c}}_{\kappa_1,\kappa_2}$ satisfies the transformation formula exactly as in the above lemma for each place. From this we can verify the following:

LEMMA 3.17. *Coefficient functions of $\theta^{\mathrm{c}}_{\kappa_1,\kappa_2}(f')(h)$ are (adelizations of) elliptic modular forms of weight κ_1 and level D given by theta series attached to harmonic polynomials in $\mathcal{H}_{\kappa_1-4}$.*

PROOF. First recall that the representation space $W_{(\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2}\boxtimes V_{\kappa_2}$ occurs in $\mathcal{H}_{\kappa_1-4}$ with non-zero multiplicity, in fact, multiplicity one (see Section 2.2, [**13**, Section 1.2], [**21**, Section 3.2.2 (3.1)]). Thus $w\otimes f'(h')\in W_{(\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2}\boxtimes V_{\kappa_2}$

with $w \in W_{(\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2}$ and $h' \in H'(\mathbb{A})$ can be regarded as a harmonic polynomial in $\mathcal{H}_{\kappa_1-4}$. By $\Phi_{w,f'(h')}$ we denote the corresponding harmonic polynomial. We put $L_{h'} := B^2 \cap (\mathbb{H}^2 \times \prod_{p<\infty} L_p h'_p)$, with $h' := (h'_v)_{v\leq\infty} \in H'(\mathbb{A})$. This depends only on the coset of $H'(\mathbb{Q})\backslash H'(\mathbb{A})/U'_\infty \times \prod_{p<\infty} U'_p$ represented by h', where note that $\sharp(H'(\mathbb{Q})\backslash H'(\mathbb{A})/U'_\infty \times \prod_{p<\infty} U'_p)$ is the finite number called the class number of B. In addition let $h_z \in SL_2(\mathbb{R})$ such that $h_z \cdot \sqrt{-1} = z$. Given a suitably normalized inner product $(*,*)$ of $W_{(\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2}$, we have $\Phi_{w,f'(h)}(X) = ((C_X, f'(h))_{(\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2}, w)$ for $X \in \mathbb{H}^2$, where recall that $\mathbb{H}^2 \ni X \mapsto C_X \in \mathcal{H}_{\kappa_1-4}$ denotes the reproducing kernel function for $\mathcal{H}_{\kappa_1-4}$ (cf. Section 2.2).

We can thus write down

$$j(h_z,\sqrt{-1})^{\kappa_1}(\theta^{\mathrm{c}}_{\kappa_1,\kappa_2}(f')(h_z),w) = \sum_{h'}\sum_{\lambda\in L_{h'}} \Phi_{w,f'(h')}(\lambda)\exp\left(2\pi\sqrt{-1}\,\frac{n(\lambda)}{n(h')}z\right)$$

as a function in $z \in \mathfrak{h} \simeq SL_2(\mathbb{R})/U_\infty$. For the right hand side, the first sum runs over a complete set of representatives for $H'(\mathbb{Q})\backslash H'(\mathbb{A})/U'_\infty \times \prod_{p<\infty} U'_p$ and by $n(h')$ we denote the positive generator of the $\mathbb{Z}$ ideal generated by the reduced norms of $B \cap (\mathbb{H} \times \prod_{p<\infty} \mathfrak{O}_p h_p)$. The left hand side defines a well-defined function in z since it is right U_∞-invariant with respect to h_z and the right hand side, which is nothing but a sum of theta series attached to harmonic polynomials, does not depend on the choice of representatives of $H'(\mathbb{Q})\backslash H'(\mathbb{A})/U'_\infty \times \prod_{p<\infty} U'_p$. Now let us review the strong approximation theorem $\Gamma_0(D)\backslash\mathfrak{h} \simeq GL(2)(\mathbb{Q})\mathbb{R}_+^\times\backslash GL(2)(\mathbb{A})/U_\infty \times \prod_{p<\infty} U_p$, where $\Gamma_0(D)$ denotes the congruence subgroup of level D and by $\mathbb{R}_+^\times$ we mean the connected component of the center of $GL(2)(\mathbb{R})$. We then see from Lemma 3.16 that this is of weight κ_1 and level D. □

Ibukiyama and Ihara [**13**, Section 3.2 (2)] gave examples of non-zero coefficients of $\theta^{\mathrm{c}}_{\kappa_1,\kappa_2}(f')$ for the following cases:

- $(d_B, D) = (2,1),\ (\kappa_1,\kappa_2) = (12,0)$,
- $d_B = D = 2,\ (\kappa_1,\kappa_2) = (12,0)$,
- $d_B = D = 3,\ (\kappa_1,\kappa_2) = (8,4),\ (10,4),\ (10,6),\ (12,6)$.

These coefficients are verified to be cusp forms. In fact, for each of these examples, we have $\theta^{\mathrm{c}}_{\kappa_1,\kappa_2}(f')(h) \in \mathbb{C}w$ with a non-zero vector $w \in W_{(\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2}$ independent of $h \in H(\mathbb{A})$ (cf. [**13**, Section 3.2 (2)]). As an elliptic cusp form f let us take the non-zero coefficient of $\theta^{\mathrm{c}}_{\kappa_1,\kappa_2}(f')$ in these examples. We then see that the non-vanishing of $\mathcal{L}^{\mathrm{c}}(f,f')(1)$ with the identity element $1 \in GSp^*(2)(\mathbb{A})$ is reduced to that of the Petersson inner product of f, from which the existence of non-zero $\mathcal{L}^{\mathrm{c}}(f,f')$s follows immediately.

Proposition 3.18. *For the cases above, there are (f,f') such that $\mathcal{L}^{\mathrm{c}}(f,f') \not\equiv 0$ for $GSp^*(2)$.*

A remark in terms of the Jacquet–Langlands correspondence.

For all the seven cases above $\dim \mathcal{A}_{\kappa_2} = 1$ as is pointed out in [**13**, Section 3.2 (2)]. Hence $\mathcal{A}_{\kappa_2}$ for each case has a unique Hecke eigenbasis, say f', up to constant multiples. Together with a primitive form $f \in S_{\kappa_1}(D)$ we can consider the automorphic representation $\pi^c(f, f')$ of $GSp^*(2)(\mathbb{A})$ generated by $\mathcal{L}^c(f, f')$ and the cuspidal representation $\pi'(f, \mathrm{JL}(f'))$ of $GSp(2)(\mathbb{A})$ given by the theta lift from the cuspidal representation $\sigma(f, \mathrm{JL}(f'))$ of $GO(2,2)(\mathbb{A})$ (for $\sigma(f, \mathrm{JL}(f'))$ see [**21**, Section 4.4]). As we have shown in [**21**, Propositions 4.4, 4.7, Theorem 4.13] these two automorphic representations are irreducible and are involved in the global Jacquet–Langlands correspondence in the sense of [**21**, Conjecture 4.2]. We can understand this correspondence explicitly for the seven cases mentioned above. The archimedean components of $\pi'(f, \mathrm{JL}(f'))$s are square-integrable representations (modulo center) of $GSp(2)(\mathbb{R})$ induced from non-holomorphic discrete series representation of $Sp(2)(\mathbb{R})$ and these have the same L-parameters with those of the archimedean components of $\pi^c(f, f')$. The archimedean L-parameters are given by Harish-Chandra parameters (cf. [**21**, Section 2.3]). Corresponding to the seven cases, the list of the Harish-Chandra parameters λ is given as follows:

- for $(d_B, D) = (2,1)$, $\lambda = (6,5)$,
- for $d_B = D = 2$, $\lambda = (6,5)$,
- for $d_B = D = 3$, $\lambda = (6,1),\ (7,2),\ (8,1),\ (9,2)$.

As the non-archimedean aspect of this we note that the aforementioned global Jacquet–Langlands correspondence has the level preserving property meaning that the correspondence should send an automorphic representation of $GSp^*(2)$ (or $GSp^*(1,1)$) with a $K_f(D)$-invariant vector to an automorphic representation of $GSp(2)$ with a paramodular new vector of level $d_B D$, where recall that $K_f(D) := \prod_{p \nmid d_B/D} K_1 \times \prod_{p | d_B/D} K_2$ with $K_1 := \{g \in GSp^*(2)(\mathbb{Q}_p) \mid g(\mathfrak{O}_p^2) = \mathfrak{O}_p^2\}$ and $K_2 := \{g \in GSp^*(2)(\mathbb{Q}_p) \mid g(\mathfrak{O}_p \oplus \mathfrak{P}_p^{-1}) = \mathfrak{O}_p \oplus \mathfrak{P}_p^{-1}\}$ (cf. [**21**, Section 2.2]). We remark that, in the three cases above, $\mathcal{L}^c(f, f')$ is $K_f(D)$-invariant with $D = 2,\ 2,\ 3$ and $\pi'(f, \mathrm{JL}(\mathrm{f}'))$ has a unique paramodular new vector of level $d_B D = 2,\ 2^2,\ 3^2$ respectively.

4. A relation with central L-values.

This section deals with an explicit relation between the square norms of the Bessel periods and the central values of some convolution type L-functions for $\mathcal{L}^{\mathrm{nc}}(f, f')$s, which leads to the existence of positive central values of the L-functions. Our results here are the generalizations of Theorem 2.8, Theorem 2.11 and Theorem 2.14 in [**20**].

4.1. A relation with central L-values I.

By virtue of Theorem 2.3 and Proposition 3.1 we can explicitly relate the square norm of $\mathcal{L}^{\mathrm{nc}}(f, f')^{\chi}_{\xi}$ to the product of two central L-values for the two χ^{-1}-twisted L-functions for Π and Π'.

Theorem 4.1. *Under the assumption in Theorem* 2.3 *we have*

$$\frac{\|\mathcal{L}^{\mathrm{nc}}(f,f')^{\chi}_{\xi}(g_0)\|^2}{\langle f,f\rangle\langle f',f'\rangle} = C(f,f',\xi,\chi)\, L\left(\Pi,\chi^{-1},\frac{1}{2}\right) L\left(\Pi',\chi^{-1},\frac{1}{2}\right),$$

where, if $\pi(f')_p|_{E_p^\times} = \chi_p$ *for* $p|d_B$ *ramified in* E *and* $S_1 = S_2^+(f,\chi) = \emptyset$,

$$\begin{aligned} C(f,f',\xi,\chi) = {} & \frac{2^{|\delta(D)|-4}(\kappa_2+1)\binom{\kappa_2}{(\kappa_1+\kappa_2)/2}|d_\xi|^{3/2}\mathbf{w}(E)^2}{\mathbf{h}(E)^2\, A(\chi)^4\, D^{3/2}\, L(\pi(f),\mathrm{Ad},1)\, L(\mathrm{JL}(\pi(f')),\mathrm{Ad},1)} \\ & \times \prod_{p\nmid d_B} p^{4\mu_p}(1-\delta(i_p(\chi)>0)e_p(E)p^{-1})^{-2} \prod_{p|d_B} r_p p^{-1} \prod_{p|D}(p+1)^{-2} \\ & \cdot W_\xi^{\kappa_1,\kappa_2}(1)^2 \cdot \frac{(f'_{\infty,\kappa_1}, f'_{\infty,\kappa_1})_{\kappa_2}}{(f'_\infty, f'_\infty)_{\kappa_2}}, \end{aligned}$$

and $C(f,f',\xi,\chi) = 0$ *otherwise.*

Remark 4.2. In [**20**, Theorem 2.11] the exponent of $1-\delta(i_p(\chi)>0)e_p(E)p^{-1}$ is denoted mistakenly by 2, but this should be replaced by -2 as in the theorem above. For this we remark that $1-\delta(i_p(\chi)>0)e_p(E)p^{-1}$ is nothing but $L_p(\eta_p,1)^{-1}$ (for $L_p(\eta_p,1)$ see Section 3.1).

4.2. A relation with central L-values II.

Let us introduce the global L-function

$$L(F,\chi^{-1},s) := L_\infty(F,\chi^{-1},s) \times \prod_{p<\infty} L_p(F,\chi^{-1},s)$$

for a Hecke eigenform $F \in \mathcal{S}^{\mathrm{nc}}_{\tau_\Lambda}(D)$ (cf. Section 2.1) and χ. Here, with the polynomial $Q_{F,p}$ introduced in [**21**, Section 4.5.1] for each finite prime $p<\infty$, the local factors $L_p(F,\chi^{-1},s)$ and $L_\infty(F,\chi^{-1},s)$ are defined for $p<\infty$ and ∞ as follows:

$$L_p(F,\chi^{-1},s) := \begin{cases} Q_{F,p}(\alpha_p^\chi p^{-s})^{-1} Q_{F,p}(\beta_p^\chi p^{-s})^{-1} & (\chi \text{ is unramified at } p<\infty), \\ 1 & (\chi \text{ is ramified at } p<\infty), \end{cases}$$

$$L_\infty(F,\chi^{-1},s) := \Gamma_{\mathbb{C}}\left(s+\kappa_1-\frac{1}{2}\right)\Gamma_{\mathbb{C}}\left(s+\frac{1}{2}\right)\Gamma_{\mathbb{C}}\left(s+\frac{\kappa_1+\kappa_2+1}{2}\right)\Gamma_{\mathbb{C}}\left(s+\frac{\kappa_2-\kappa_1+1}{2}\right),$$

where, when χ is unramified at $p<\infty$,

$$(\alpha_p^\chi,\beta_p^\chi) := \begin{cases} (\chi_p(\varpi_{p,1})^{-1}, \chi_p(\varpi_{p,2})^{-1}) & (p\text{: split}), \\ (\chi_p(p)^{-1}, -\chi_p(p)^{-1}) = (1,-1) & (p\text{: inert}), \\ (\chi_p(\varpi_p)^{-1}, 0) & (p\text{: ramified}) \end{cases}$$

with a prime element ϖ_p for a ramified prime p and prime elements $\varpi_{p,1}$, $\varpi_{p,2}$ for a split prime p such that $p = \varpi_{p,1}\varpi_{p,2}$. This L-function can be viewed as a convolution type L-function of $GSp(2)\times GL(2)$ for the cuspidal representation generated by a Hecke eigenform F and the dihedral representation attached of χ^{-1}.

To explain the archimedean factor we remark that the archimedean local L-function of $GSp(2)\times GL(2)$ is defined as

$$\Gamma_{\mathbb{C}}\left(s+\frac{\lambda_1-\lambda_2}{2}+|l|\right)\Gamma_{\mathbb{C}}\left(s+\frac{\lambda_1+\lambda_2}{2}+|l|\right)$$
$$\Gamma_{\mathbb{C}}\left(s+\left|\frac{\lambda_1-\lambda_2}{2}-|l|\right|\right)\Gamma_{\mathbb{C}}\left(s+\left|\frac{\lambda_1+\lambda_2}{2}-|l|\right|\right)$$

for a general Harish-Chandra parameter $(\lambda_1,\lambda_2)\in\Xi_I$ (for the notation Ξ_I see [**21**, Section 2.3]) and for a general χ of weight $w_\infty(\chi)=2l$ (for $w_\infty(\chi)$ see Section 2.5). Putting $(\lambda_1,\lambda_2)=((\kappa_1+\kappa_2)/2,(\kappa_2-\kappa_1)/2+1)$ and $l=-\kappa_1/2$, we have the definition of the archimedean factor $L_\infty(F,\chi^{-1},s)$, which coincides with that of $L(\Pi,\chi^{-1},s)L(\Pi',\chi^{-1},s)$.

On the other hand, according to [**21**, Proposition 4.9], we know that, at each finite prime p, the polynomial $Q_{F,p}$ defines the local spinor L-functions for all spherical representations of $GSp(1,1)(\mathbb{Q}_p)$, which are (irreducible constituents of) unramified principal series representations of $GSp(2)(\mathbb{Q}_p)$ for $p\nmid d_B$. This definition is justified by the explicit classification of irreducible admissible representations of $GSp(2)(\mathbb{Q}_p)$ involved in the local Jacquet–Langlands correspondence with the spherical representations (cf. [**21**, Appendix, A4]). Now recall that $\pi'(f,\mathrm{JL}(f'))$ denotes the cuspidal representation of $GSp(2)(\mathbb{A})$ given by the theta lift from $\sigma(f,\mathrm{JL}(f'))$ (cf. [**21**, Section 4.4]) and that all of its local components fit into the local Jacqeut–Langlands correspondence with the local components of the cuspidal representation $\pi(f,f')$ generated by $\mathcal{L}^{\mathrm{nc}}(f,f')$ (cf. [**21**, Theorem 4.13]). We can thereby similarly define the L-function $L(\pi'(f,\mathrm{JL}(f')),\chi^{-1},s)$ of convolution type for $\pi'(f,\mathrm{JL}(f'))$ with the global spinor L-function $L(\pi'(f,\mathrm{JL}(f')),\mathrm{spin},s)$ (which coincides with $L(\mathcal{L}^{\mathrm{nc}}(f,f'),\mathrm{spin},s)$) and the Hecke character χ^{-1}.

Proposition 4.3. *We have*

$$L(\mathcal{L}^{\mathrm{nc}}(f,f'),\chi^{-1},s)=L(\pi'(f,\mathrm{JL}(f')),\chi^{-1},s)=L(\Pi,\chi^{-1},s)L(\Pi',\chi^{-1},s).$$

This is an entire function of s and satisfies the functional equation

$$L(\mathcal{L}^{\mathrm{nc}}(f,f'),\chi^{-1},s)=\epsilon(\Pi,\chi^{-1})\epsilon(\Pi',\chi^{-1})L(\mathcal{L}^{\mathrm{nc}}(f,f'),\chi^{-1},1-s),$$

where $\epsilon(\Pi,\chi^{-1})$ (resp. $\epsilon(\Pi,\chi^{-1})$ denotes the ϵ-factor of $L(\Pi,\chi^{-1},s)$ (resp. $L(\Pi',\chi^{-1},s)$).

This is a generalization of [**20**, Proposition 2.10] for our situation. The L-function $L(\mathcal{L}^{\mathrm{nc}}(f,f'),\chi^{-1},s)=L(\pi'(f,\mathrm{JL}(f')),\chi^{-1},s)$ is obviously regular at $s=1/2$. We are now able to reformulate Theorem 4.1 as follows:

Theorem 4.4. *Let the assumption and the notation be as in Theorem* 2.3. *We have*

$$\frac{\|\mathcal{L}^{\mathrm{nc}}(f,f')^{\chi}_{\xi}(g_0)\|^2}{\langle f,f\rangle\langle f',f'\rangle}=C(f,f',\xi,\chi)\,L\left(\mathcal{L}^{\mathrm{nc}}(f,f'),\chi^{-1},\frac{1}{2}\right)$$
$$=C(f,f',\xi,\chi)\,L\left(\pi'(f,\mathrm{JL}(f')),\chi^{-1},\frac{1}{2}\right).$$

Now we take B, $\mathfrak{O}$ and $\xi\in B^-$ as in Section 3.3. Suppose that $\chi\in X_\xi$ is unramified at every finite place and of weight $w_\infty(\chi) = -\kappa_1$ at the infinite place. As a consequence of Theorem 3.14, Theorem 4.4 and the positivity of $C(f, f', \xi, \chi)$ we have the following.

COROLLARY 4.5. *Let the assumption be as in Theorem* 3.14. *There exist Hecke eigenforms* (f, f') *such that* $L(\Pi, \chi^{-1}, 1/2) > 0$ *and* $L(\Pi', \chi^{-1}, 1/2) > 0$ *hold simultaneously, and thus*

$$L\left(\mathcal{L}^{\mathrm{nc}}(f, f'), \chi^{-1}, \frac{1}{2}\right) = L\left(\pi'(f, f'), \chi^{-1}, \frac{1}{2}\right) > 0.$$

On the other hand, from Theorem 3.3, Proposition 4.3, Theorem 4.4 and the positivity of $C(f, f', \xi, \chi)$, we know that the proof of Theorem 3.5 leads to another consequence as follows:

COROLLARY 4.6. *Under the assumption of Theorem* 3.3, *there exist infinitely many* $\nu \in X_\xi(l)_f$ *such that* $L(\Pi, (\chi\nu)^{-1}, 1/2) > 0$ *and* $L(\Pi', (\chi\nu)^{-1}, 1/2) > 0$ *hold simultaneously, and thus*

$$L\left(\mathcal{L}^{\mathrm{nc}}(f, f'), (\chi\nu)^{-1}, \frac{1}{2}\right) = L\left(\pi'(f, f'), (\chi\nu)^{-1}, \frac{1}{2}\right) > 0.$$

REMARK 4.7. The non-archimedean local factors of the convolution type L-functions are defined also by the local Langlands correspondence of $GSp(2)$ (or $GSp(4)$). As is well known, it was established by Gan–Takeda [**7**]. The "Type I (or Type A)" case in [**7**, Section 7] is valid for our case. We have defined non-archimedean local factors of the spinor L-function of $GSp(2)$ by the polynomial $Q_{F,p}$ in the spirit of the classical Hecke theory (cf. [**21**, Section 4.5.1]). The L-parameter defined by [**7**] certainly enables us to reproduce the non-archimedean local spinor L-functions, and thus also the convolution type L-functions. In fact, our definition of the non-archimedean local spinor L-functions are justified by the representation theory in [**21**, Section 4.5.1], as is explained already.

4.3. Examples of positive central *L*-values.

To provide examples of $\mathcal{L}^{\mathrm{nc}}(f, f')$ with the positive central L-values above, we need examples of Hecke eigenforms $(f, f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$ with non-zero toral integrals, which are not easy to find in general as we have remarked just before Lemma 3.13. In order to find examples of Hecke eigenforms in $\mathcal{A}_{\kappa_2}$ we provide the formula for $\dim \mathcal{A}_{\kappa_2}$ with $d_B = 2,\ 3$ and 7 as follows:

LEMMA 4.8. (1) *Let* $d_B = 2$.

1. *For* κ_2 *divisible by* 4 *we have*

$$\dim \mathcal{A}_{\kappa_2} = \begin{cases} (\kappa_2 + 12)/12 & (\kappa_2 \equiv 0 \mod 12), \\ (\kappa_2 - 4)/12 & (\kappa_2 \equiv 4 \mod 12), \\ (\kappa_2 + 4)/12 & (\kappa_2 \equiv 8 \mod 12). \end{cases}$$

2. *For κ_2 not divisible by 4 we have*

$$\dim \mathcal{A}_{\kappa_2} = \begin{cases} (\kappa_2+6)/12 & (\kappa_2 \equiv 0 \mod 6), \\ (\kappa_2-2)/12 & (\kappa_2 \equiv 2 \mod 6), \\ (\kappa_2-10)/12 & (\kappa_2 \equiv 4 \mod 6). \end{cases}$$

(2) *Let $d_B = 3$.*

1. *For κ_2 divisible by 4 we have*

$$\dim \mathcal{A}_{\kappa_2} = \begin{cases} (\kappa_2+6)/6 & (\kappa_2 \equiv 0 \mod 12), \\ (\kappa_2+2)/6 & (\kappa_2 \equiv 4 \mod 12), \\ (\kappa_2+4)/6 & (\kappa_2 \equiv 8 \mod 12). \end{cases}$$

2. *For κ_2 not divisible by 4 we have*

$$\dim \mathcal{A}_{\kappa_2} = \begin{cases} \kappa_2/6 & (\kappa_2 \equiv 0 \mod 6), \\ (\kappa_2-2)/6 & (\kappa_2 \equiv 2 \mod 6), \\ (\kappa_2-4)/6 & (\kappa_2 \equiv 4 \mod 6). \end{cases}$$

(3) *Let $d_B = 7$. We have*

$$\dim \mathcal{A}_{\kappa_2} = \begin{cases} (\kappa_2+2)/2 & (4|\kappa_2), \\ \kappa_2/2 & (4 \nmid \kappa_2). \end{cases}$$

Here note that, for the formulas above, κ_2 is non-zero and even when $4 \nmid \kappa_2$.

Proof. The formula in the part 1 of (1) is already stated in the proof of [**19**, Proposition 14.3.1]. The other formulas in (1), (2) and (3) are proved similarly. These are obtained by calculating Eichler's trace formula for Brandt matrices [**6**, Theorem 5] (see also[**5**, (63)]), for which note that $\dim \mathcal{A}_{\kappa_2} = s_{\kappa_2}(1; d_B, 1)$ for $d_B = 2,\ 3$ and 7 in the notation of Eichler. □

Let $d_B = 2$ or 3. Then $D = 1,\ 2$ or 3, for which note that $D|d_B$. For $d_B = 2$ (respectively $d_B = 3$) we denote by χ a Hecke character of $\mathbb{Q}(i) \simeq \mathbb{Q}(\sqrt{-1})$ (respectively $\mathbb{Q}(j) \simeq \mathbb{Q}(\sqrt{-3})$) unramified at every finite place with weight $w_\infty(\chi) = -\kappa_1$. We provide examples of Hecke eigenforms (f, f')s for small (κ_1, κ_2)s with non-zero toral integrals with respect to χ. Without one exception they yield examples of (f, f') with $L(\mathcal{L}^{\mathrm{nc}}(f, f'), \chi^{-1}, 1/2) > 0$. In the following examples, the Hecke eigenforms denoted by $f'_0,\ f'_1,\ f'_2,\ f'_3$ are realized as $\sum_{u \in \mathfrak{O}^\times} \sigma_{\kappa_2}(u) v_{\kappa,(\kappa_1+\kappa_2)/2} \in V^{\mathfrak{O}}_{\kappa_2} (\simeq \mathcal{A}_{\kappa_2})$ (see the proof of Lemma 3.9).

Example 1 ($(d_B, D) = (2, 1)$). The case of $D = 1$ can be said to be the easiest to find a Hecke eigenform $f \in S_{\kappa_1}(D)$ with the desired properties. In fact, the non-archimedean local assumptions saying $S_1 = S_2^+(f, \chi) = \emptyset$ in Theorem 3.3 or Proposition 3.1 can be removed. As such f, for example, we have cusp forms $\Delta,\ \Delta E_4,\ \Delta E_4^2$

of weight 12, 16, 20, with the Ramanujan delta function Δ and the Eisenstein series E_4 of weight 4. These three have non-zero toral integrals with respect to χ with $w_\infty(\chi) = -12,\ -16,\ -20$ respectively (see the proof of [**19**, Proposition 14.4.1]), and are Hecke eigen cusp forms since $\dim S_{\kappa_1}(1) = 1$ for $\kappa_1 = 12,\ 16,\ 20$.

On the other hand, from Lemma 4.8, we deduce $\dim \mathcal{A}_{14} = \dim \mathcal{A}_{16} = 1$. Thus, up to scalars, each of $\mathcal{A}_{14}$ and $\mathcal{A}_{16}$ has a unique Hecke eigenbasis, say f'_0 and f'_1 respectively. We can verify that the Hecke eigenform f'_0 (respectively f'_1) has the non-zero toral integral with respect to χ of weight $w_\infty(\chi) = -12$ (respectively $w_\infty(\chi) = -16$) since $C^{(2)}_{12,14} = 5/8(\neq 0)$ is checked by a direct calculation (respectively due to [**19**, Lemma 14.3.4]). When $(f, f') = (\Delta, f'_0),\ (\Delta E_4, f'_1)$ we have $L(\mathcal{L}^{\mathrm{nc}}(f, f'), \chi^{-1}, 1/2) > 0$ and Corollary 4.6 holds for the unramified Hecke character χ of $\mathbb{Q}(i)$ just mentioned. As a further example we note that, since $C^{(2)}_{12,16} = 0$ (cf. Remark 3.15), $L(\mathcal{L}^{\mathrm{nc}}(f, f'), \chi^{-1}, 1/2) = 0$ for $(f, f') = (\Delta, f'_1)$ and χ above with $w_\infty(\chi) = -12$ though $\mathcal{L}^{\mathrm{nc}}(f, f')$ is non-vanishing in view of Theorem 3.5.

For the case of $D = 1$ we remark that there is no further example of $\dim \mathcal{A}_{\kappa_2} = 1$ for $d_B = 3,\ 7$ and $\kappa_2 \geq 12$ as the formulas in Lemma 4.8 indicate. Now note that $\kappa_2 \geq \kappa_1$ and we are taking κ_1 as 12, 16, 20. By the approach above we are not thus able to provide examples when $d_B = 3,\ 7$.

Example 2 ($(d_B, D) = (2, 2)$). We know that $\mathcal{A}_8$ is one dimensional (for instance see [**19**, Lemma 14.3.2]) and has a unique Hecke eigenform f'_2, up to scalars. Consider the eta product $(\eta(z)\eta(2z))^8 \in S_8(2)$, which is a Hecke eigenform since $\dim S_8(2) = 1$. In [**19**, Lemma 14.3.4, Proposition 14.4.1] and their proofs we have seen that $(f, f') = ((\eta(z)\eta(2z))^8, f'_2)$ and the unramified Hecke character χ of $\mathbb{Q}(i)$ with $w_\infty(\chi) = -8$ satisfy the assumption of Theorem 3.3. With these (f, f') and χ the positivity $L(\mathcal{L}^{\mathrm{nc}}(f, f'), \chi^{-1}, 1/2) > 0$ and Corollary 4.6 hold.

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