J. Math. Soc. Japan
Vol. 71, No. 4 (2019) pp. 1243–1256
doi: 10.2969/jmsj/80878087

Hardy and Rellich inequalities with exact missing terms on homogeneous groups

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(Received July 14, 2018)

Abstract. We prove several identities on homogeneous groups that imply the Hardy and Rellich inequalities for Bessel pairs. These equalities give a straightforward understanding of some of the Hardy and Rellich inequalities as well as the absence of nontrivial optimizers and the existence/nonexistence of "virtual" extremizers.

1. Introduction.

The well-known Hardy inequality:

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \tag{1.1}$$

has been investigated intensively and extensively in the literature due to its applications and its important roles in several contexts of mathematics. We refer the interested reader to, for instance, [**1**], [**7**], [**13**], [**21**], [**22**], [**30**] which now become standard references on the subject.

Hardy type inequalities have been also studied intensively on homogeneous Carnot groups. These problems are important in the analysis of sub-Laplacian and p-sub-Laplacian on homogeneous Carnot groups. In these situations, the Euclidean norm is usually replaced by the so-called $\mathcal{L}$-gauge N which is a particular quasi-norm obtained from the fundamental solution of the sub-Laplacian [**9**]. For instance, the Hardy inequalities and their extensions were established in the case of the Heisenberg group in [**11**], [**5**], [**28**]. These inequalities were also studied on groups of Heisenberg type by Danielli, Garofalo and Phuc [**6**], on polarizable Carnot groups by Goldstein and Kombe [**15**]: for all $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, with $Q \geq 3$, $1 < p < Q$:

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} f|^p dx \geq \left(\frac{Q-p}{p}\right)^p \int_{\mathbb{G}} \frac{|\nabla_{\mathbb{G}} N|^p}{N^p} |f|^p dx.$$

In [**36**], the authors established the sharp weighted Hardy inequalities on polarizable Carnot groups: for all $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, $1 < p < Q$, $\alpha \in \mathbb{R}$, $\gamma > -1$:

2010 *Mathematics Subject Classification.* Primary 22E30; Secondary 43A80, 26D10, 43A85.

Key Words and Phrases. Hardy inequality, Hardy–Rellich inequality, Bessel pair, remainder term, homogeneous Lie group.

The second author (the corresponding author) was partially supported by a fellowship from the Pacific Institute for the Mathematical Sciences.

$$\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}N|^{\gamma}|\nabla_{\mathbb{G}}f|^p dx \geq \left(\frac{Q+\alpha-p}{p}\right)^p \int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^{p+\gamma}}{N^p}|f|^p dx.$$

Using a special class of weighted p-sub-Laplacian and the corresponding fundamental solution, Jin and Shen [**19**] obtained weighted Hardy type inequalities on general Carnot groups.

To unify many known results about the weighted Hardy type inequalities on Carnot groups, in [**16**], Goldstein, Kombe and Yener set up a constructive approach to derive Hardy type inequalities with different weights. More exactly, they provide a simple sufficient condition on a pair of nonnegative weight functions $V(x)$ and $W(x)$ on a Carnot group so that the following weighted Hardy type inequality holds for any $f \in C_0^{\infty}(\mathbb{G})$:

$$\int_{\mathbb{G}} V(x)|\nabla_{\mathbb{G}}f|^p dx \geq \int_{\mathbb{G}} W(x)|f|^p dx.$$

Their results recover and improve most of the Hardy type inequalities that have known to date.

Pioneered by Ruzhansky and his collaborators, Hardy type and other functional inequalities were also investigated intensively and extensively in the setting of the homogeneous groups. For examples, see [**31**], [**32**], [**33**], [**34**], [**35**]. Some of the results in these papers are already new even in the Euclidean space $\mathbb{R}^n$. We recall here that a homogeneous group $\mathbb{G}$ is a simply connected Lie group where its Lie algebra $\mathfrak{g}$ is equipped with a family of the following dilations:

$$D_{\lambda} = \mathrm{Exp}(A \ln \lambda).$$

Here A is a diagonalizable positive linear operator on $\mathfrak{g}$, and every D_{λ} is a morphism of $\mathfrak{g}$. The exponential mapping $\exp_{\mathbb{G}} : \mathfrak{g} \to \mathbb{G}$ of this group is a global diffeomorphism. Thus, this implies the dilation structure, and this dilation is denoted by $D_{\lambda}x$ or λx. We denote by $Q = \mathrm{Tr}A$ the homogeneous dimension of $\mathbb{G}$. The Haar measure on a homogeneous group $\mathbb{G}$ is the standard Lebesgue measure for $\mathbb{R}^n$.

Let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. We then define the quasi-ball centered at $x \in \mathbb{G}$ by

$$B(x,R) := \left\{y \in \mathbb{G} : \left|x^{-1}y\right| < R\right\}.$$

There is a (unique) positive Borel measure σ on the unit sphere $\mathfrak{S} := \{x \in \mathbb{G} : |x| = 1\}$, such that for every $f \in L^1(\mathbb{G})$, we have

$$\int_{\mathbb{G}} f(x)dx = \int_0^{\infty}\int_{\mathfrak{S}} f(ry)r^{Q-1}d\sigma(y)dr. \tag{1.2}$$

Now, we fix a basis $\{X_1, \dots, X_n\}$ of a Lie algebra $\mathfrak{g}$ such that $AX_k = \nu_k X_k$ for every k, so that the matrix $A = \mathrm{diag}(\nu_1, \dots, \nu_n)$. Then each X_k is homogeneous of degree ν_k and also $Q = \nu_1 + \cdots + \nu_n$. The decomposition of $\exp_{\mathbb{G}}^{-1}(x)$ in the Lie algebra $\mathfrak{g}$ defines the vector

$$e(x) = (e_1(x), \dots, e_n(x))$$

by the formula

$$\exp_{\mathbb{G}}^{-1}(x) = e(x) \cdot \nabla$$

where $\nabla = (X_1, \ldots, X_n)$. Then, by denoting $x = ry$, $y \in \mathfrak{S}$, we have

$$e(x) = e(ry) = (r^{\nu_1} e_1(y), \ldots, r^{\nu_n} e_n(y))\,.$$

We also define

$$\mathcal{R} f(x) := \frac{d}{d|x|} f(x).$$

The operator $\mathcal{R}$ is homogeneous of order -1 and plays the role of the radial derivative on $\mathbb{G}$. We note that the operator $\mathcal{R}$ has appeared naturally in the literature. Indeed, one of the interesting problems is to investigate the functional and geometric inequalities on general homogeneous groups. However, as mentioned in [**32**], since these spaces do not have to be stratified or even graded, the concept of horizontal gradients does not make sense. Thus, it is logical to work with the full gradient. On the other hand, unless the homogeneous groups are abelian, the full gradient is not homogeneous. Nevertheless, on the homogeneous groups, the operator $\mathcal{R}$ is homogeneous of order -1 and is analogous to the radial derivative $x/|x| \cdot \nabla$ on $\mathbb{R}^n$.

For further details on this topic we refer the interested readers to [**3**], [**8**], [**10**] and the references therein.

In this note, we would like to set up some versions of the two-weight Hardy type inequalities with exact missing terms on homogeneous groups $\mathbb{G}$. Our results are inspired by the developments in [**16**], the events in [**12**], [**14**], where Ghoussoub and Moradifam provided on isotropic Euclidean space the necessary and sufficient criterions on a pair of positive radial functions so that certain two-weight Hardy inequalities hold true, and the equalities in [**17**], [**18**], [**24**], [**26**], [**27**] that provide simple and straightforward understandings of the Hardy and Hardy–Rellich inequalities on Euclidean space as well as the nonexistence of nontrivial optimizers.

Our first main theorem can be stated as follows:

Theorem 1.1. *Let $0 < R \leq \infty$, V and W be positive C^1-functions on $(0, R)$ such that $\int_0^R (1/r^{Q-1}V(r))dr = \infty$ and $\int_0^R r^{Q-1}V(r)dr < \infty$. Then if $(r^{Q-1}V, r^{Q-1}W)$ is a (1-dimensional) Bessel pair on $(0, R)$, that is, if the ordinary differential equation*

$$y''(r) + \left(\frac{Q-1}{r} + \frac{V_r(r)}{V(r)}\right) y'(r) + \frac{W(r)}{V(r)} y(r) = 0$$

has a positive solution $\varphi_{V,W;R}$ on the interval $(0, R)$, then we have

$$\int_{B(0,R)} V(|x|)|\mathcal{R}u|^2 dx = \int_{B(0,R)} W(|x|)|u|^2 dx + \int_{B(0,R)} V(|x|) \left|\mathcal{R}\left(\frac{u}{\varphi_{V,W;R}}\right)\right|^2 \varphi_{V,W;R}^2 dx$$

for all $u \in C_0^\infty(B(0, R))$.

In the same spirit, we also show that

THEOREM 1.2. *Let W be a positive function on $(0,\infty)$ and $\widetilde{W}$ be the antiderivative of $W(r)r^{Q-1}$. Then for all $u \in C_0^\infty(\mathbb{G}\setminus\{0\})$, we have*

$$\int_{\mathbb{G}} \frac{4\widetilde{W}^2(|x|)}{W(|x|)|x|^{2Q-2}}|\mathcal{R}u(x)|^2dx - \int_{\mathbb{G}} W(|x|)|u|^2dx = \int_{\mathbb{G}} \frac{4\big|\widetilde{W}(|x|)\big|}{W(|x|)|x|^{2Q-2}}\left|\mathcal{R}\left(u(x)\sqrt{\big|\widetilde{W}(|x|)\big|}\right)\right|^2 dx.$$

If $\widetilde{W}(0)=0$, then the above identities hold for any $u \in C_0^\infty(\mathbb{G})$.

The higher order Hardy inequalities, namely the Rellich inequalities, were also studied on the stratified Lie groups for arbitrary homogeneous quasinorm in [**4**]. However, the best constant was not investigated there. We also refer to [**20**], [**25**] for the results on the sharp Rellich type inequalities for the sub-Laplacian on homogeneous Carnot groups. In [**33**], Ruzhansky and Suragan set up several sharp versions of the horizontal weighted Hardy, Rellich, Caffarelli–Kohn–Nirenberg inequalities on stratified groups. In [**32**], the authors provide sharp remainder terms of weighted Rellich inequalities on one of most general subclasses of nilpotent Lie groups, namely the class of homogeneous groups. They also studied higher order inequalities of Hardy–Rellich type, all with sharp constants, as well as several identities including weighted and higher order types. Recently, the author in [**29**] derived several interesting equalities for the integrals of higher order derivatives and the sharp Hardy–Rellich type inequalities for higher order derivatives including both the subcritical and critical inequalities on the homogeneous groups.

Inspired by the points discussed above, our second purpose of this article is to set up the Rellich type inequalities using Bessel pairs. More precisely, denote

$$\mathcal{R}_2 := \mathcal{R}^2 + \frac{Q-1}{|x|}\mathcal{R},$$

we will show that

THEOREM 1.3. *Let $0 < R \le \infty$, V and W be positive C^1-functions on $(0,R)$ such that $\int_0^R (1/r^{Q-1}V(r))dr = \infty$, $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < Q-2$ and $(r^{Q-1}V, r^{Q-1}W)$ is a (1-dimensional) Bessel pair on $(0,R)$. Then*

$$\begin{aligned}\int_{B(0,R)} & V(|x|)|\mathcal{R}_2u|^2dx \\ &= \int_{B(0,R)} W(|x|)|\mathcal{R}u|^2dx + (Q-1)\int_{B(0,R)}\left[\frac{V(|x|)}{|x|^2} - \frac{V'(|x|)}{|x|}\right]|\mathcal{R}u|^2dx \\ &\quad + \int_{B(0,R)} V(|x|)\left|\mathcal{R}\left(\frac{\mathcal{R}u}{\varphi_{V,W;R}}\right)\right|^2\varphi_{V,W;R}^2dx\end{aligned}$$

for all $u \in C_0^\infty(B(0,R))$. Here $\varphi_{V,W;R}$ is the positive solution of

$$y''(r) + \left(\frac{Q-1}{r} + \frac{V_r(r)}{V(r)}\right) y'(r) + \frac{W(r)}{V(r)} y(r) = 0$$

on the interval $(0, R)$.

It is worth noting that in the setting of Euclidean spaces and using Δ and ∇ instead of $\mathcal{R}_2$ and $\mathcal{R}$, a similar result as our Theorem 1.3 was set up in [**12**] for radial functions only. To get the same result for nonradial functions as well, the following condition was assumed:

$$W(r) - \frac{2V(r)}{r^2} + 2\frac{V_r(r)}{r} - V_{rr}(r) \geq 0 \quad \text{for} \quad 0 \leq r \leq R. \tag{1.3}$$

As discussed in their book [**13**], this is a sufficient but not necessary condition to make sure that the best constant is the same for the radial and for the nonradial case. Hence, our Theorem 1.3 shows that the above assumption can be removed if we use $\mathcal{R}_2$ and $\mathcal{R}$ to replace for Δ and ∇ in the Hardy–Rellich type inequalities.

We also mention here that in [**23**], the Hardy and Hardy–Rellich type inequalities with Bessel pairs were investigated on homogeneous groups using the spherical average of the test function. Our results in this article exploit further and provide the exact remainders as well as the "virtual" ground states of the Hardy and Hardy–Rellich type inequalities in [**23**].

Finally, we note here that in our main results, the homogeneous norm $|\cdot|$ is arbitrary. Hence, our results are somewhat new even in the setting of anisotropic Euclidean spaces.

2. Some useful lemmata and some consequences of main results.

We list here an important result that will be used to treat the integrations by parts in the following sections. The proof of this result can be found in [**13**].

LEMMA 2.1. *Let $R > 0$ and assume that $\varphi \in C^1(0, R)$ is a positive solution of the ODE*

$$y''(r) + \left(\frac{Q-1}{r} + \frac{V_r(r)}{V(r)}\right) y'(r) + \frac{W(r)}{V(r)} y(r) = 0$$

on $(0, R)$ where V, $W \geq 0$ on $(0, R)$ such that $\int_0^R (1/r^{Q-1}V(r))dr = \infty$ and $\int_0^R r^{Q-1}V(r)dr < \infty$. Setting $\psi(x) = u(x)/\varphi(|x|)$ for any $u \in C_0^\infty(B(0, R))$, we then have the following properties:

(1) $\displaystyle\int_0^R V(r)\left(\frac{\varphi'(r)}{\varphi(r)}\right)^2 r^{Q-1}dr < \infty$ *and* $\displaystyle\lim_{r\to 0} V(r)\frac{\varphi'(r)}{\varphi(r)} r^{Q-1} = 0.$

(2) $\displaystyle\int_{B(0,R)} V(|x|)\varphi'(|x|)^2\psi^2(x)dx < \infty$ *and* $\displaystyle\int_{B(0,R)} V(|x|)\varphi(|x|)^2|\mathcal{R}\psi|^2(x)dx < \infty.$

(3) $\displaystyle\left|\int_{B(0,R)} V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)(\mathcal{R}\psi(x))dx\right| < \infty.$

(4) $\lim_{r\to 0}\left|\int_{\partial B(0,R)} V(|x|)\varphi'(|x|)\varphi(|x|)\psi^2(x)ds\right| = 0.$

We will also need the following lemma that can be found in [**32**]:

Lemma 2.2. *Define the Euler's operator $\mathbb{E}$ on $\mathbb{G}$ by $\mathbb{E} = |x|\mathcal{R}$. If $f : \mathbb{G}\setminus\{0\} \to \mathbb{R}$ is continuously differentiable, then $\mathbb{E}f = \upsilon f$ if and only if $f(\lambda x) = \lambda^{\nu} f(x)$, $\forall\lambda > 0$, $x \neq 0$, i.e., f is positively homogeneous of order υ.*

We now provide here some consequences of our main results.

Example 2.1. Assume $Q \geq 3$ and $0 \leq \lambda \leq Q-2$. Then $(r^{Q-1-\lambda}, ((Q-\lambda-2)/2)^2 r^{Q-1-\lambda-2})$ is a Bessel pair on $(0,\infty)$ with $\varphi_{V,W;\infty}(r) = r^{-(Q-\lambda-2)/2}$. Hence, by Theorem 1.1, we obtain

$$\int_{\mathbb{G}} \frac{|\mathcal{R}u|^2}{|x|^{\lambda}}dx = \frac{(Q-\lambda-2)^2}{4}\int_{\mathbb{G}} \frac{|u|^2}{|x|^{\lambda+2}}dx + \int_{\mathbb{G}} \frac{1}{|x|^{Q-2}}\left|\mathcal{R}\big(u|x|^{(Q-\lambda-2)/2}\big)\right|^2 dx.$$

Now, noting that

$$\frac{V(|x|)}{|x|^2} - \frac{V'(|x|)}{|x|} = \frac{1+\lambda}{|x|^{\lambda+2}},$$

we have by Theorem 1.3

$$\begin{aligned}\int_{\mathbb{G}} \frac{|\mathcal{R}_2 u|^2}{|x|^{\lambda}}dx &= \frac{(Q-\lambda-2)^2}{4}\int_{\mathbb{G}} \frac{|\mathcal{R}u|^2}{|x|^{\lambda+2}}dx + (Q-1)(1+\lambda)\int_{\mathbb{G}} \frac{|\mathcal{R}u|^2}{|x|^{\lambda+2}}dx \\ &\quad + \int_{\mathbb{G}} \frac{1}{|x|^{Q-2}}\left|\mathcal{R}\big(|x|^{(Q-\lambda-2)/2}\mathcal{R}u\big)\right|^2 dx \\ &= \frac{(Q+\lambda)^2}{4}\int_{\mathbb{G}} \frac{|\mathcal{R}u|^2}{|x|^{\lambda+2}}dx + \int_{\mathbb{G}} \frac{1}{|x|^{Q-2}}\left|\mathcal{R}\big(|x|^{(Q-\lambda-2)/2}\mathcal{R}u\big)\right|^2 dx.\end{aligned}$$

Noting that $(r^{Q-1}r^{-\lambda-2}, ((Q-\lambda-4)^2/4)r^{Q-1}r^{-\lambda-4})$, $0 \leq \lambda \leq Q-4$, is a Bessel pair on $(0,\infty)$ with $\varphi_{V,W;\infty}(r) = r^{-(Q-\lambda-4)/2}$, we obtain by Theorem 1.1:

$$\int_{\mathbb{G}} \frac{|\mathcal{R}u|^2}{|x|^{\lambda+2}}dx = \frac{(Q-\lambda-4)^2}{4}\int_{\mathbb{G}} \frac{|u|^2}{|x|^{\lambda+4}}dx + \int_{\mathbb{G}} \frac{1}{|x|^{Q-2}}\left|\mathcal{R}\big(u|x|^{(Q-\lambda-4)/2}\big)\right|^2 dx.$$

Thus

$$\begin{aligned}\int_{\mathbb{G}} \frac{|\mathcal{R}_2 u|^2}{|x|^{\lambda}}dx &= \frac{(Q+\lambda)^2}{4}\frac{(Q-\lambda-4)^2}{4}\int_{\mathbb{G}} \frac{|u|^2}{|x|^{\lambda+4}}dx \\ &+ \frac{(Q+\lambda)^2}{4}\int_{\mathbb{G}} \frac{1}{|x|^{Q-2}}\left|\mathcal{R}\big(u|x|^{(Q-\lambda-4)/2}\big)\right|^2 dx + \int_{\mathbb{G}} \frac{1}{|x|^{Q-2}}\left|\mathcal{R}\big(|x|^{(Q-\lambda-2)/2}\mathcal{R}u\big)\right|^2 dx.\end{aligned}$$

As a consequence

$$\int_{\mathbb{G}} \frac{|\mathcal{R}u|^2}{|x|^{\lambda}}dx \geq \frac{(Q-\lambda-2)^2}{4}\int_{\mathbb{G}} \frac{|u|^2}{|x|^{\lambda+2}}dx \tag{2.1}$$

and

$$\int_{\mathbb{G}} \frac{|\mathcal{R}_2 u|^2}{|x|^\lambda} dx \geq \frac{(Q+\lambda)^2}{4} \frac{(Q-\lambda-4)^2}{4} \int_{\mathbb{G}} \frac{|u|^2}{|x|^{\lambda+4}} dx. \tag{2.2}$$

The equality happens in the (2.1) if and only if $\mathcal{R}(u|x|^{(Q-\lambda-2)/2}) = 0$. That is $\mathbb{E}u = -((Q-\lambda-2)/2)u$. By Lemma 2.2, u is positively homogeneous of order $-(Q-\lambda-2)/2$. Hence, we can find some function ϕ on $\mathfrak{S}$ such that $u(x) = |x|^{-(Q-\lambda-2)/2}\phi(x/|x|)$. However, in this situation

$$\int_{\mathbb{G}} \frac{|u|^2}{|x|^{\lambda+2}} dx = \int_{\mathfrak{S}} \phi^2(y) d\sigma(y) \int_0^\infty \frac{1}{r} dr.$$

Thus $\int_{\mathbb{G}} |u|^2/|x|^{\lambda+2} dx$ is finite if and only if $u = 0$. However, we can say that $|x|^{-(Q-\lambda-2)/2}\phi(x/|x|)$ is the "virtual" optimizer for (2.1). As pointed out by Brezis and Vázquez in [**2**], this phenomenon happens due to the lack of a proper function space setting.

Similarly, the equality occurs in (2.2) if and only if $u = 0$. But again we can say that (2.2) receives "virtual" optimizer of the form $|x|^{-(Q-\lambda-4)/2}\phi(x/|x|)$.

EXAMPLE 2.2. $(r^{Q-1-\lambda}, ((Q-\lambda-2)/2)^2 r^{Q-1-\lambda-2} + (z_0^2/R^2) r^{Q-1-\lambda})$, z_0 is the first zero of the Bessel function J_0, is a Bessel pair on $(0,R)$ with $\varphi_{V,W;R}(r) = r^{-(Q-\lambda-2)/2} J_0(rz_0/R) = r^{-(Q-\lambda-2)/2} J_{0;R}(r)$. Then by Theorem 1.1, we get

$$\begin{aligned}\int_{B(0,R)} \frac{|\mathcal{R}u|^2}{|x|^\lambda} dx &= \frac{(Q-\lambda-2)^2}{4} \int_{B(0,R)} \frac{|u|^2}{|x|^{\lambda+2}} dx + \frac{z_0^2}{R^2} \int_{B(0,R)} \frac{|u|^2}{|x|^\lambda} dx \\ &\quad + \int_{B(0,R)} \frac{1}{|x|^{Q-2}} J_{0;R}^2(|x|) \left| \mathcal{R}\left(\frac{u|x|^{(Q-\lambda-2)/2}}{J_{0;R}} \right) \right|^2 dx.\end{aligned}$$

Also, we can deduce from Theorem 1.3 that

$$\begin{aligned}\int_{B(0,R)} \frac{|\mathcal{R}_2 u|^2}{|x|^\lambda} dx &= \frac{(Q-\lambda-2)^2}{4} \int_{B(0,R)} \frac{|\mathcal{R}u|^2}{|x|^{\lambda+2}} dx + \frac{z_0^2}{R^2} \int_{B(0,R)} \frac{|\mathcal{R}u|^2}{|x|^\lambda} dx \\ &\quad + (Q-1)(1+\lambda) \int_{B(0,R)} \frac{|\mathcal{R}u|^2}{|x|^{\lambda+2}} dx \\ &\quad + \int_{B(0,R)} \frac{1}{|x|^{Q-2}} \left| \mathcal{R}\left(\frac{|x|^{(Q-\lambda-2)/2} \mathcal{R}u}{J_{0;R}(|x|)} \right) \right|^2 J_{0;R}^2(|x|) dx \\ &= \frac{(Q+\lambda)^2}{4} \int_{B(0,R)} \frac{|\mathcal{R}u|^2}{|x|^{\lambda+2}} dx + \frac{z_0^2}{R^2} \int_{B(0,R)} \frac{|\mathcal{R}u|^2}{|x|^\lambda} dx \\ &\quad + \int_{B(0,R)} \frac{1}{|x|^{Q-2}} \left| \mathcal{R}\left(\frac{|x|^{(Q-\lambda-2)/2} \mathcal{R}u}{J_{0;R}(|x|)} \right) \right|^2 J_{0;R}^2(|x|) dx.\end{aligned}$$

By Theorem 1.1 and by noting that $(r^{Q-1} r^{-\lambda-2}, ((Q-\lambda-4)^2/4) r^{Q-1} r^{-\lambda-4} + (z_0^2/R^2) r^{Q-1} r^{-\lambda-2})$, $0 \leq \lambda \leq Q-4$, is a Bessel pair on $(0,R)$ with $\varphi_{V,W;R}(r) = r^{-(Q-\lambda-4)/2} J_{0;R}(r)$, we have

$$\int_{B(0,R)} \frac{|\mathcal{R}u|^2}{|x|^{\lambda+2}}dx = \frac{(Q-\lambda-4)^2}{4}\int_{B(0,R)} \frac{|u|^2}{|x|^{\lambda+4}}dx + \frac{z_0^2}{R^2}\int_{B(0,R)} \frac{|u|^2}{|x|^{\lambda+2}}dx + \int_{B(0,R)} \frac{1}{|x|^{Q-2}} J_{0;R}^2(|x|)\left|\mathcal{R}\left(\frac{u|x|^{(Q-\lambda-4)/2}}{J_{0;R}}\right)\right|^2 dx.$$

Hence

$$\begin{aligned}\int_{B(0,R)} \frac{|\mathcal{R}_2u|^2}{|x|^{\lambda}}dx &= \frac{(Q+\lambda)^2}{4}\frac{(Q-\lambda-4)^2}{4}\int_{B(0,R)} \frac{|u|^2}{|x|^{\lambda+4}}dx\\ &+ \left[\frac{(Q+\lambda)^2}{4}\frac{z_0^2}{R^2} + \frac{(Q-\lambda-2)^2}{4}\frac{z_0^2}{R^2}\right]\int_{B(0,R)} \frac{|u|^2}{|x|^{\lambda+2}}dx + \left(\frac{z_0^2}{R^2}\right)^2\int_{B(0,R)} \frac{|u|^2}{|x|^{\lambda}}dx\\ &+ \frac{(Q+\lambda)^2}{4}\int_{B(0,R)} \frac{1}{|x|^{Q-2}} J_{0;R}^2(|x|)\left|\mathcal{R}\left(\frac{u|x|^{(Q-\lambda-4)/2}}{J_{0;R}}\right)\right|^2 dx\\ &+ \frac{z_0^2}{R^2}\int_{B(0,R)} \frac{1}{|x|^{Q-2}} J_{0;R}^2(|x|)\left|\mathcal{R}\left(\frac{u|x|^{(Q-\lambda-2)/2}}{J_{0;R}}\right)\right|^2 dx\\ &+ \int_{B(0,R)} \frac{1}{|x|^{Q-2}}\left|\mathcal{R}\left(\frac{|x|^{(Q-\lambda-2)/2}\mathcal{R}u}{J_{0;R}}\right)\right|^2 J_{0;R}^2(|x|)dx.\end{aligned}$$

By dropping nonnegative terms, we get

$$\int_{B(0,R)} \frac{|\mathcal{R}u|^2}{|x|^{\lambda}}dx \geq \frac{(Q-\lambda-2)^2}{4}\int_{B(0,R)} \frac{|u|^2}{|x|^{\lambda+2}}dx + \frac{z_0^2}{R^2}\int_{B(0,R)} \frac{|u|^2}{|x|^{\lambda}}dx \tag{2.3}$$

and

$$\begin{aligned}\int_{B(0,R)} \frac{|\mathcal{R}_2u|^2}{|x|^{\lambda}}dx \geq{}& \frac{(Q+\lambda)^2}{4}\frac{(Q-\lambda-4)^2}{4}\int_{B(0,R)} \frac{|u|^2}{|x|^{\lambda+4}}dx\\ &+ \left[\frac{(Q+\lambda)^2}{4}\frac{z_0^2}{R^2} + \frac{(Q-\lambda-2)^2}{4}\frac{z_0^2}{R^2}\right]\int_{B(0,R)} \frac{|u|^2}{|x|^{\lambda+2}}dx\\ &+ \left(\frac{z_0^2}{R^2}\right)^2\int_{B(0,R)} \frac{|u|^2}{|x|^{\lambda}}dx.\end{aligned} \tag{2.4}$$

There is no nontrivial extremizer for (2.3). But, we again can say that (2.3) has "virtual" optimizer of the form $J_{0;R}(|x|)|x|^{-(Q-\lambda-4)/2}\phi(x/|x|)$. Similarly, (2.4) has no nontrivial optimizer. It is also interesting that no function can play the role of "virtual" optimizer for (2.4).

EXAMPLE 2.3. $(r^{Q-1}, ((Q-2)/2)^2(r^{Q-1}/r^2(1-(R/r)^{2-Q})^2))$ is a Bessel pair on $(0,R)$ with $\varphi_{V,W;R}(r) = R^{-(Q-2)/2}\sqrt{(R/r)^{Q-2}-1}$. Hence, as a consequence of Theorem 1.1, we receive

$$\int_{B(0,R)} |\mathcal{R}u|^2 dx - \left(\frac{Q-2}{2}\right)^2\int_{B(0,R)} \frac{|u|^2}{|x|^2\left(1-(R/|x|)^{2-Q}\right)^2}dx$$

$$= \int_{B(0,R)} \left| \mathcal{R}\left(\frac{u}{\sqrt{(R/|x|)^{Q-2}-1}} \right) \right|^2 \left[\left(\frac{R}{|x|} \right)^{Q-2} - 1 \right] dx$$

and

$$\int_{B(0,R)} |\mathcal{R}u|^2 dx \geq \left(\frac{Q-2}{2} \right)^2 \int_{B(0,R)} \frac{|u|^2}{|x|^2 \left(1-(R/|x|)^{2-Q}\right)^2} dx.$$

Obviously, $((Q-2)/2)^2$ is sharp and is not attainable by nontrivial extremizers. Also, the "virtual" optimizer of the above Hardy type inequality is of the form $\sqrt{(|x|/R)^{2-Q}-1}\,\phi(x/|x|)$.

We note that the book [**13**] provides various examples and properties about Bessel pairs. Hence we can deduce as many Hardy and Rellich type inequalities as we can form Bessel pairs.

Example 2.4. In the critical case, $W(r) = 1/r^Q$, then $\widetilde{W}(r) = \ln r$ and $4\widetilde{W}^2(|x|)/W(|x|)|x|^{2Q-2} = 4|\ln|x||^2/|x|^{Q-2}$. Hence by Theorem 1.2, we have for $u \in C_0^\infty(\mathbb{G}\setminus\{0\})$:

$$4\int_{\mathbb{G}} \frac{|\ln|x||^2}{|x|^{Q-2}} |\mathcal{R}u(x)|^2 dx \geq \int_{\mathbb{G}} \frac{|u|^2}{|x|^Q} dx.$$

We also note that our results imply a version of the Heisenberg–Pauli–Weyl type uncertainly principle on homogeneous groups: If $(r^{Q-1}V, r^{Q-1}W)$ is a (1-dimensional) Bessel pair on $(0,\infty)$, that is, if the ordinary differential equation

$$y''(r) + \left(\frac{Q-1}{r} + \frac{V_r(r)}{V(r)} \right) y'(r) + \frac{W(r)}{V(r)} y(r) = 0$$

has a positive solution $\varphi_{V,W;\infty}$ on the interval $(0,\infty)$, then we have

$$\begin{aligned} &\left(\int_{\mathbb{G}} |u|^2 dx \right)^2 \\ &\quad \leq \left(\int_{\mathbb{G}} W(|x|)|u|^2 dx \right) \left(\int_{\mathbb{G}} \frac{1}{W(|x|)} |u|^2 dx \right) \\ &\quad = \left(\int_{\mathbb{G}} V(|x|)|\mathcal{R}u|^2 dx - \int_{\mathbb{G}} V(|x|) \left| \mathcal{R}\left(\frac{u}{\varphi_{V,W;\infty}} \right) \right|^2 \varphi_{V,W;\infty}^2 dx \right) \left(\int_{\mathbb{G}} \frac{1}{W(|x|)} |u|^2 dx \right) \\ &\quad \leq \left(\int_{\mathbb{G}} V(|x|)|\mathcal{R}u|^2 dx \right) \left(\int_{\mathbb{G}} \frac{1}{W(|x|)} |u|^2 dx \right). \end{aligned}$$

This covers the classical Heisenberg–Pauli–Weyl uncertainty principle on $\mathbb{R}^N$. Indeed, in this case, we note that $Q = N$ and also $(r^{N-1}, ((N-2)/2)^2 r^{N-1} r^{-2})$ is a Bessel pair on $(0,\infty)$ with $\varphi_{V,W;\infty}(r) = r^{-(N-2)/2}$. Hence

$$\left(\int_{\mathbb{R}^N} |u|^2 dx \right)^2 \leq \left(\int_{\mathbb{R}^N} \frac{1}{|x|^2} |u|^2 dx \right) \left(\int_{\mathbb{R}^N} |x|^2 |u|^2 dx \right)$$

$$
\begin{aligned}
&= \left(\frac{2}{N-2}\right)^2 \left(\int_{\mathbb{R}^N} |\mathcal{R}u|^2 dx - \int_{\mathbb{R}^N} \left|\frac{\mathcal{R}\left(|x|^{\frac{N-2}{2}}u\right)}{|x|^{\frac{N-2}{2}}}\right|^2 dx\right) \left(\int_{\mathbb{R}^N} |x|^2|u|^2 dx\right) \\
&\leq \left(\frac{2}{N-2}\right)^2 \left(\int_{\mathbb{R}^N} |\mathcal{R}u|^2 dx\right) \left(\int_{\mathbb{R}^N} |x|^2|u|^2 dx\right) \\
&\leq \left(\frac{2}{N-2}\right)^2 \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \left(\int_{\mathbb{R}^N} |x|^2|u|^2 dx\right).
\end{aligned}
$$

3. Proof of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Let $u(x) = \varphi_{V,W;R}(|x|)\psi(x)$. By polar coordinate and (1.2), we have that

$$
\begin{aligned}
\int_{B(0,R)} W(|x|)|u|^2 dx &= \int_{\mathfrak{S}} \int_0^R W(r)|u(ry)|^2 r^{Q-1} dr d\sigma(y) \\
&= \int_{\mathfrak{S}} \int_0^R W(r)(\varphi_{V,W;R}(r))^2 |\psi(ry)|^2 r^{Q-1} dr d\sigma(y).
\end{aligned}
$$

Noting that

$$
\frac{d}{dr}\left(V(r)r^{Q-1}\frac{d}{dr}\varphi_{V,W;R}(r)\right) + W(r)r^{Q-1}\varphi_{V,W;R}(r) = 0,
$$

we get

$$
\int_{B(0,R)} W(|x|)|u|^2 dx = -\int_{\mathfrak{S}} \int_0^R \frac{d}{dr}\left(V(r)r^{Q-1}\frac{d}{dr}\varphi_{V,W;R}(r)\right)\varphi_{V,W;R}(r)|\psi(ry)|^2 dr d\sigma(y).
$$

Using Lemma 2.1 to treat the integrations by parts, we obtain for a.e. $y \in \mathfrak{S}$:

$$
\begin{aligned}
&-\int_0^R \frac{d}{dr}\left(V(r)r^{Q-1}\frac{d}{dr}\varphi_{V,W;R}(r)\right)\varphi_{V,W;R}(r)|\psi(ry)|^2 dr \\
&\quad = \int_0^R V(r)r^{Q-1}\frac{d}{dr}\varphi_{V,W;R}(r)\frac{d}{dr}(\varphi_{V,W;R}(r)|\psi(ry)|^2) dr \\
&\quad = \int_0^R V(r)r^{Q-1}\left(\frac{d}{dr}\varphi_{V,W;R}(r)\right)^2 |\psi(ry)|^2 dr \\
&\qquad + 2\,\mathrm{Re}\int_0^R V(r)r^{Q-1}\frac{d}{dr}\varphi_{V,W;R}(r)\varphi_{V,W;R}(r)\psi(ry)\overline{\frac{d}{dr}\psi(ry)} dr.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_{B(0,R)} W(|x|)|u|^2 dx \\
&\quad = \int_{\mathfrak{S}} \int_0^R V(r)r^{Q-1}\left|\frac{d}{dr}\varphi_{V,W;R}(r)\psi(ry) + \varphi_{V,W;R}(r)\frac{d}{dr}\psi(ry)\right|^2 dr d\sigma(y)
\end{aligned}
$$

$$- \int_{\mathfrak{S}} \int_0^R V(r) r^{Q-1} \left| \varphi_{V,W;R}(r) \frac{d}{dr} \psi(ry) \right|^2 dr d\sigma(y)$$
$$= \int_{B(0,R)} V(|x|) |\mathcal{R}u|^2 dx - \int_{B(0,R)} V(|x|) \varphi^2_{V,W;R}(|x|) \left| \mathcal{R} \left(\frac{u}{\varphi_{V,W;R}} \right) \right|^2 dx. \qquad \square$$

Proof of Theorem 1.2. By using the polar coordinates $(r, y) = (|x|, x/|x|) \in (0, \infty) \times \mathfrak{S}$, (1.2) and integrations by parts, we have

$$\begin{aligned}
\int_{\mathbb{G}} W(|x|) |u|^2 dx &= \int_0^\infty W(r) r^{Q-1} \int_{\mathfrak{S}} |u(ry)|^2 d\sigma(y) dr \\
&= -\operatorname{Re} \int_0^\infty \widetilde{W}(r) 2 \int_{\mathfrak{S}} u(ry) \overline{\frac{d}{dr} u(ry)} d\sigma(y) dr \\
&= -2 \operatorname{Re} \int_0^\infty \frac{\widetilde{W}(r)}{\sqrt{W(r)} r^{Q-1}} \int_{\mathfrak{S}} \sqrt{W(r)} u(ry) \overline{\frac{d}{dr} u(ry)} r^{Q-1} d\sigma(y) dr \\
&= -2 \operatorname{Re} \int_{\mathbb{G}} \sqrt{W(|x|)} u(x) \frac{\widetilde{W}(|x|)}{\sqrt{W(|x|)} |x|^{Q-1}} \overline{\mathcal{R}u(x)} dx.
\end{aligned}$$

Hence

$$2 \int_{\mathbb{G}} W(|x|) |u|^2 dx = -4 \operatorname{Re} \int_{\mathbb{G}} \sqrt{W(|x|)} u(x) \frac{\widetilde{W}(|x|)}{\sqrt{W(|x|)} |x|^{Q-1}} \overline{\mathcal{R}u(x)} dx$$

and so

$$\begin{aligned}
&\int_{\mathbb{G}} W(|x|) |u|^2 dx \\
&= -\int_{\mathbb{G}} W(|x|) |u|^2 dx - 2 \operatorname{Re} \int_{\mathbb{G}} \sqrt{W(|x|)} u(x) 2 \frac{\widetilde{W}(|x|)}{\sqrt{W(|x|)} |x|^{Q-1}} \overline{\mathcal{R}u(x)} dx \\
&= -\int_{\mathbb{G}} \left| \sqrt{W(|x|)} u(x) + 2 \frac{\widetilde{W}(|x|)}{\sqrt{W(|x|)} |x|^{Q-1}} \mathcal{R}u(x) \right|^2 dx + 4 \int_{\mathbb{G}} \frac{\widetilde{W}^2(x)}{W(|x|) |x|^{2Q-2}} |\mathcal{R}u(x)|^2 dx.
\end{aligned}$$

We note that

$$\begin{aligned}
&\int_{\mathbb{G}} \left| \sqrt{W(|x|)} u(x) + 2 \frac{\widetilde{W}(|x|)}{\sqrt{W(|x|)} |x|^{Q-1}} \mathcal{R}u(x) \right|^2 dx \\
&= \int_{\mathbb{G}} \left[\frac{2\sqrt{|\widetilde{W}(|x|)|}}{\sqrt{W(|x|)} |x|^{Q-1}} \frac{\widetilde{W}(|x|)}{|\widetilde{W}(|x|)|} \left| \mathcal{R}u(x) \sqrt{|\widetilde{W}(|x|)|} + \frac{W(|x|) |x|^{Q-1}}{2\sqrt{|\widetilde{W}(|x|)|}} \frac{\widetilde{W}(|x|)}{|\widetilde{W}(|x|)|} u(x) \right| \right]^2 dx \\
&= \int_{\mathbb{G}} \frac{4 |\widetilde{W}(|x|)|}{W(|x|) |x|^{2Q-2}} \left| \mathcal{R} \left(u(x) \sqrt{|\widetilde{W}(|x|)|} \right) \right|^2 dx. \qquad \square
\end{aligned}$$

4. Proof of Theorem 1.3.

Proof of Theorem 1.3. We have

$$\begin{aligned}
&\int_{B(0,R)} V(|x|)\left|\mathcal{R}^2 u + \frac{Q-1}{|x|}\mathcal{R}u\right|^2 dx \\
&= \int_{\mathfrak{S}}\int_0^R V(r)\left|\frac{d}{dr}\left(\frac{d}{dr}u(ry)\right) + \frac{Q-1}{r}\frac{d}{dr}u(ry)\right|^2 r^{Q-1}drd\sigma(y) \\
&= \int_{\mathfrak{S}}\int_0^R V(r)|\partial_{rr}u(ry)|^2 r^{Q-1}drd\sigma(y) + (Q-1)^2\int_{\mathfrak{S}}\int_0^R V(r)|\partial_r u(ry)|^2 r^{Q-3}drd\sigma(y) \\
&\quad + 2(Q-1)\operatorname{Re}\int_{\mathfrak{S}}\int_0^R V(r)\partial_r u(ry)\overline{\partial_{rr}u(ry)}r^{Q-2}drd\sigma(y).
\end{aligned}$$

Using integrations by parts, we get

$$\begin{aligned}
&2\operatorname{Re}\int_0^R V(r)\partial_r u(ry)\overline{\partial_{rr}u(ry)}r^{Q-2}dr \\
&\quad = -\int_0^R |\partial_r u(ry)|^2\frac{d}{dr}[V(r)r^{Q-2}]dr \\
&\quad = -\int_0^R |\partial_r u(ry)|^2 V_r(r)r^{Q-2}dr - (Q-2)\int_0^R |\partial_r u(ry)|^2 V(r)r^{Q-3}dr.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{B(0,R)} V(|x|)\left|\mathcal{R}^2 u + \frac{Q-1}{|x|}\mathcal{R}u\right|^2 dx \\
&\quad = \int_{\mathfrak{S}}\int_0^R V(r)|\partial_{rr}u(ry)|^2 r^{Q-1}drd\sigma(y) \\
&\quad\quad + (Q-1)\int_{\mathfrak{S}}\int_0^R \left[\frac{V(r)}{r^2} - \frac{V_r(r)}{r}\right]|\partial_r u(ry)|^2 r^{Q-1}drd\sigma(y) \\
&\quad = \int_{B(0,R)} V(|x|)\left|\mathcal{R}^2 u\right|^2 dx + (Q-1)\int_{B(0,R)}\left[\frac{V(|x|)}{|x|^2} - \frac{V'(|x|)}{|x|}\right]|\mathcal{R}u|^2 dx.
\end{aligned}$$

Noting that by Theorem 1.1

$$\begin{aligned}
&\int_{B(0,R)} V(|x|)\left|\mathcal{R}^2 u\right|^2 dx \\
&\quad = \int_{B(0,R)} V(|x|)|\mathcal{R}(\mathcal{R}u)|^2 dx \\
&\quad = \int_{B(0,R)} W(|x|)|\mathcal{R}u|^2 dx + \int_{B(0,R)} V(|x|)\left|\mathcal{R}\left(\frac{\mathcal{R}u}{\varphi_{V,W;R}}\right)\right|^2 \varphi_{V,W;R}^2 dx,
\end{aligned}$$

we obtain

$$\int_{B(0,R)} V(|x|) \left| \mathcal{R}^2 u + \frac{Q-1}{|x|} \mathcal{R}u \right|^2 dx - \int_{B(0,R)} W(|x|) |\mathcal{R}u|^2 dx$$
$$- (Q-1) \int_{B(0,R)} \left[\frac{V(|x|)}{|x|^2} - \frac{V'(|x|)}{|x|} \right] |\mathcal{R}u|^2 dx$$
$$= \int_{B(0,R)} V(|x|) \left| \mathcal{R} \left(\frac{\mathcal{R}u}{\varphi_{V,W;R}} \right) \right|^2 \varphi^2_{V,W;R} dx. \qquad \square$$

Acknowledgments. The authors would like to thank the anonymous referees for their helpful and constructive comments that greatly contributed to improving the final version of the paper.

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