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Height, trunk and representativity of knots

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Abstract. In this paper, we investigate three geometrical invariants of knots, the height, the trunk and the representativity. First, we give a counterexample for the conjecture which states that the height is additive under connected sum of knots. We also define the minimal height of a knot and give a potential example which has a gap between the height and the minimal height. Next, we show that the representativity is bounded above by a half of the trunk. We also define the trunk of a tangle and show that if a knot has an essential tangle decomposition, then the representativity is bounded above by half of the trunk of either of the two tangles. Finally, we remark on the difference among Gabai's thin position, ordered thin position and minimal critical position. We also give an example of a knot which bounds an essential non-orientable spanning surface, but has arbitrarily large representativity.

1. Introduction.

We study a knot in the 3-sphere via a standard Morse function $h : S^3 \to \mathbb{R}$. We derive two geometrical invariants of a knot, one is "height" from the vertical direction of h, and another is "trunk" from the horizontal direction.

Our main results are counterexamples for the additivity of "height" with respect to connected sum (Theorem 2.1), and the inequality between "representativity" and "trunk" (Theorem 3.1). In the following Sections 2 and 3, we explain height of knots, representativity and trunk of knots respectively. In Section 4, we give proofs for results in those sections. Finally, we discuss several versions of thin position, waist and representativity, representativity and non-orientable spanning surfaces in Sections 5, 6 and 7 respectively.

2. Height of knots.

It is often difficult to determine how geometrically defined knot invariants behave with respect to connected sum. Some classical invariants are known to be predictably well-behaved such as genus and bridge number [**34**]. While others are only conjectured to be well-behaved such as crossing number and unknotting number. Still others have been shown to exhibit complicated behavior with respect to connected sum such as tunnel number [**17**] and width [**6**]. In this paper we study the behavior of *height* of a knot with respect to connected sum and show that this invariant best fits in the third category

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by demonstrating that height is not additive with respect to connected sum, giving a counterexample to Conjecture 3.5 of [**25**].

Let K be an ambient isotopy class of knot in S^3 and let $h : S^3 \to \mathbb{R}$ be the standard height function. If γ is a smooth embedding of knot type K, $h|_\gamma$ is Morse and all critical points of $h|_\gamma$ have distinct critical values, then we will write $\gamma \in K$. Though an abuse of notation, we will also let γ denote the image of the embedding.

Then the *bridge number* of $\gamma \in K$ is denoted by $\beta(\gamma)$ and is defined to be the number of maxima of $h|_\gamma$. The *bridge number* of K, $\beta(K)$, is the minimum of $\beta(\gamma)$ over all $\gamma \in K$. Schubert showed that the bridge number of a connected sum $K_1 \# K_2$ always satisfies the equality

$$\beta(K_1 \# K_2) = \beta(K_1) + \beta(K_2) - 1.$$

Bridge number is closely related to the width of a knot which was originally defined by Gabai and used in the proof of the property R conjecture [**10**]. To define width, we first need some additional structure. If t is a regular value of $h|_\gamma$, then $h^{-1}(t)$ is called a *level sphere* with width $w(h^{-1}(t)) = |\gamma \cap h^{-1}(t)|$, where $| * |$ denotes the number of connected components of $*$. If $c_0 < c_1 < \cdots < c_n$ are all the critical values of $h|_\gamma$, choose regular values $r_1, r_2, \dots, r_n$ such that $c_{i-1} < r_i < c_i$. Then the *width of* γ is defined by $w(\gamma) = \sum w(h^{-1}(r_i))$. The *width* of K, $w(K)$, is the minimum of $w(\gamma)$ over all $\gamma \in K$. We say that $\gamma \in K$ is a *thin position* for K if $w(\gamma) = w(K)$ and write $\gamma \in \mathrm{TP}(K)$ where $\mathrm{TP}(K)$ denotes the set of all thin positions of K.

Based in part on Schubert's equality, it was widely conjectured that the width of a connected sum $K_1 \# K_2$ always satisfies the equality

$$w(K_1 \# K_2) = w(K_1) + w(K_2) - 2.$$

Rieck and Sedgwick made progress on this conjecture when they showed that the above equality always holds when K_1 and K_2 are mp-small knots [**28**]. Additionally, Scharlemann and Schultens showed that $w(K_1 \# K_2) \geq \max\{w(K_1), w(K_2)\}$ ([**31**]). However, Scharlemann and Thompson proposed counterexamples to the equality in [**33**] and Blair and Tomova proved that an infinite class of the Scharlemann–Thompson examples were counterexamples [**6**]. However, there are alternative definitions of width for which width is well-behaved with respect to connected sum [**36**]. In general, the best known inequalities for $w(K_1 \# K_2)$ are

$$\max\{w(K_1), w(K_2)\} \leq w(K_1 \# K_2) \leq w(K_1) + w(K_2) - 2$$

with each of these individual inequalities known to be equalities for certain choices of K_1 and K_2.

To define the height of a knot, we first need to introduce the notion of thick and thin level. A level sphere $h^{-1}(t)$ for $\gamma \in K$ is called *thin* if the highest critical point for γ below it is a maximum and the lowest critical point above it is a minimum. If the highest critical point for γ below $h^{-1}(t)$ is a minimum and the lowest critical point above it is a maximum, then the level sphere is called *thick*. As the lowest critical point of K is a minimum and the highest is a maximum, a thick level sphere can always be found.

Note that some embeddings will have no thin spheres. When this occurs the unique thick sphere is called a *bridge sphere* and the embedding is said to be a *bridge position* for K.

Given $\gamma \in K$ such that γ is a thin position, we define the *height* of γ, denoted by $\mathrm{ht}(\gamma)$ to be the number of thick level spheres for γ. Here, thick spheres are considered up to isotopy with respect to γ. Similarly, the *height* of a knot type K is defined in [**19**] as

$$\mathrm{ht}(K) = \max_{\gamma \in \mathrm{TP}(K)} \mathrm{ht}(\gamma).$$

Alternatively, we will define the *min-height* of a knot type K to be

$$\mathrm{ht}_{\min}(K) = \min_{\gamma \in \mathrm{TP}(K)} \mathrm{ht}(\gamma).$$

Clearly, $\mathrm{ht}_{\min}(K) \leq \mathrm{ht}(K)$ for all knots K.

We are interested in understanding how height behaves with respect to connected sum. It was remarked in [**25**] that the height is additive with respect to connected sum for meridionally small knots (cf. [**22**, Theorem 1.8]), and conjectured that for non-trivial knots K_1 and K_2, $\mathrm{ht}(K_1 \# K_2) = \mathrm{ht}(K_1) + \mathrm{ht}(K_2)$ always holds. By a similar argument, it follows that min-height is also additive with respect to connected sum for meridionally small knots. Hence, it is natural to ask if min-height is always additive with respect to connected sum. Our first results provide counterexamples to each of these conjectures by defining an infinite class of knots $\mathcal{K}$ such that for every $K \in \mathcal{K}$, $\mathrm{ht}_{\min}(K) = \mathrm{ht}(K) = 3$ and the following theorem holds.

Theorem 2.1. *Let $K \in \mathcal{K}$ and let K_2 be any two-bridge knot, then*

$$\mathrm{ht}(K \# K_2) = \mathrm{ht}(K) = 3,$$
$$\mathrm{ht}_{\min}(K \# K_2) = \mathrm{ht}_{\min}(K) = 3.$$

Since the height of any two-bridge knot is one, Theorem 2.1 gives a counterexample to Conjecture 3.5 of [**25**] that for all knots K_1 and K_2, $\mathrm{ht}(K_1 \# K_2) = \mathrm{ht}(K_1) + \mathrm{ht}(K_2)$. By Theorem 2.1, the additivity of height does not hold with respect to connected sum of knots. At this stage, we expect the following.

Conjecture 2.2. *For any two knots K_1, K_2, it holds that*

$$\max\{\mathrm{ht}(K_1), \mathrm{ht}(K_2)\} \leq \mathrm{ht}(K_1 \# K_2) \leq \mathrm{ht}(K_1) + \mathrm{ht}(K_2),$$
$$\max\{\mathrm{ht}_{\min}(K_1), \mathrm{ht}_{\min}(K_2)\} \leq \mathrm{ht}_{\min}(K_1 \# K_2) \leq \mathrm{ht}_{\min}(K_1) + \mathrm{ht}_{\min}(K_2).$$

For a knot K given in Theorem 2.1, we have $\mathrm{ht}_{\min}(K) = \mathrm{ht}(K)$. But it is natural to think that the gap between the min-height and the height can be arbitrarily large in general.

Conjecture 2.3. *There exists a knot K such that* $\mathrm{ht}_{\min}(K) < \mathrm{ht}(K)$.

We give a potential example K for Conjecture 2.3 in Figure 1. The width of the embedding $\gamma \in K$ on the left is $22^2/2 = 242$ while the width of the embedding on the right $\gamma' \in K$ is $18^2/2 + 14^2/2 - 6^2/2 = 242$.

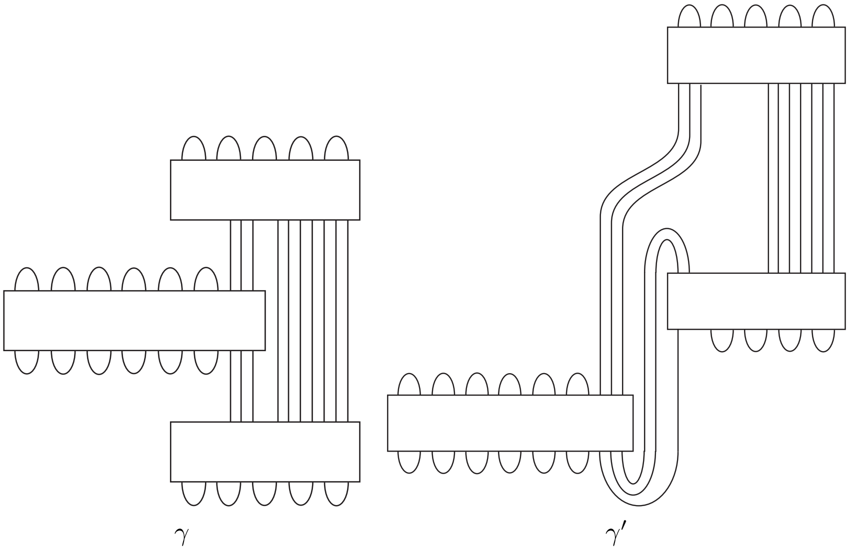

Figure 1. A potential example for Conjecture 2.3, where $w(\gamma) = w(\gamma') = 242$.

3. Representativity and trunk of knots.

To measure the "density" of a graph embedded in a closed surface, Robertson and Vitray introduced the *representativity* in [**29**] as the minimal number of points of intersection between the graph and any essential closed curve on the closed surface. This concept was applied to a knot K in the 3-sphere S^3 in [**20**] and extended to a spatial graph in the 3-sphere in [**23**]. Let F be a closed surface containing the knot K. We define the *representativity* of a pair (F, K) as

$$r(F,K) = \min_{D \in \mathcal{D}_F} |\partial D \cap K|,$$

where $\mathcal{D}_F$ denotes the set of all compressing disks for F. Moreover, we define the *representativity* of a knot K as

$$r(K) = \max_{F \in \mathcal{F}} r(F,K),$$

where $\mathcal{F}$ denotes the set of all closed surfaces containing K.

The representativity measures the "spatial density" of a knot. We summarize the known values of representativity of knots.

(i) $r(K) = 1$ if and only if K is the trivial knot (since the trivial knot can be considered as a torus knot of type $(1,1)$).

(ii) $r(K) = 2$ for composite knots ([**23**, Corollary 9]).

(iii) $r(K) \leq 2n$ for knots with essential n-string tangle decompositions ([**23**, Theorem 8]).

(iv) $r(K) = 2$ for two-bridge knots ([**23**, Theorem 3]).

(v) $r(K) = \min\{p, q\}$ for (p, q)-torus knots ([**23**, Theorem 3]).

(vi) $r(K) \leq 3$ for algebraic knots ([**24**, Theorem 1.4] for large case. The small case follows from [**23**, Theorem 2] since small algebraic knots are Montesinos knots with length 3).

(vii) For a (p, q, r)-pretzel knot K, $r(K) = 3$ if and only if $(p, q, r) = \pm(-2, 3, 3)$ or $\pm(-2, 3, 5)$ ([**24**]).

(viii) $r(K) = 2$ for alternating knots ([**23**, Conjecture 4], [**13**]).

(ix) $r(K) = p$ for inconsistent cable knots with index p ([**3**]).

(x) $r(K) \leq \beta(K)$ ([**23**, Theorem 2]).

(xi) $r(K) \leq 160\delta(K)$, where $\delta(K)$ denotes the distortion of K ([**27**]).

We remark that the inequality (x) was used to show the above (i), (iv), (v), (vi) and (vii). In this paper, we refine the inequality (x).

As in [**22**], we define the *trunk* of a knot K as

$$\operatorname{trunk}(K) = \min_{\gamma \in K} \max_{t \in \mathbb{R}} |h^{-1}(t) \cap \gamma|.$$

It follows by the definition that $\operatorname{trunk}(K) \leq 2\beta(K)$.

The bridge number of knots behaves as expected under taking connected sums, that is, Schubert proved that $\beta(K_1 \# K_2) = \beta(K_1) + \beta(K_2) - 1$ ([**34**]). On the other hand, it was naturally expected that $\operatorname{trunk}(K_1 \# K_2) = \max\{\operatorname{trunk}(K_1), \operatorname{trunk}(K_2)\}$ ([**22**, Conjecture 1.7]). Davies and Zupan showed in [**8**] that this is true, namely, for two knots K_1 and K_2,

$$\operatorname{trunk}(K_1 \# K_2) = \max\{\operatorname{trunk}(K_1),\ \operatorname{trunk}(K_2)\}.$$

In several cases, the trunk turned out to be useful. For example, it was shown in [**35**] that $m(K) \geq \operatorname{trunk}(K)/2$, where $m(K)$ denotes the multiplicity index of K. It was also shown in [**11**] that a knot K is embeddable into an $(m \times n)$-tube if and only if $\operatorname{trunk}(K) < (m + 1)(n + 1)$.

The following theorem refines [**23**, Theorem 2].

Theorem 3.1. *For any knot K, we have*

$$r(K) \leq \frac{\operatorname{trunk}(K)}{2}.$$

In the following, we introduce a "local trunk" of a knot, that is, the trunk of a tangle which lies in the pair of the 3-sphere and a knot.

Let (B, T) be a tangle, where B is a 3-ball and T is a proper ambient isotopy class of properly embedded arcs in B. Let $h : B \to \mathbb{R}$ be a standard Morse function with a single maximal point p and $B - p \cong \partial B \times (1, 0]$.

If γ is a smooth embedding in the proper ambient isotopy class of T, $h|_\gamma$ is Morse and all critical points of $h|_\gamma$ in the interior of γ have distinct critical values, then we will write $\gamma \in T$.

We define the *trunk* of a tangle (B, T) as

$$\operatorname{trunk}(B, T) = \min_{\gamma \in T} \max_{t \in \mathbb{R}} |h^{-1}(t) \cap \gamma|.$$

Then we obtain the next theorem which is a local version of Theorem 3.1.

Theorem 3.2. *Let K be a knot admitting an essential tangle decomposition $(S^3, K) = (B_1, T_1) \cup (B_2, T_2)$. Then we have*

$$r(K) \leq \frac{\min\{\operatorname{trunk}(B_1, T_1), \operatorname{trunk}(B_2, T_2)\}}{2}.$$

In some cases, Theorem 3.2 is more useful than Theorem 3.1 and (iii). Indeed, we can reprove (vi) above after Theorem 3.2.

Corollary 3.3 ([**24**, Theorem 1.4]). *For a large algebraic knot K, $r(K) \leq 3$.*

Proof. Let K be a large algebraic knot (i.e. algebraic knot with an essential Conway sphere). Then, K admits an essential tangle decomposition $(S^3, K) = (B_1, T_1) \cup (B_2, T_2)$, where (B_1, T_1) is a union of two rational tangles. It is easy to see that $\operatorname{trunk}(B_1, T_1) = 6$. By Theorem 3.2, we obtain $r(K) \leq 3$. $\square$

Theorem 3.2 can be regarded as a local version of Theorem 3.1, namely, "a local property determines a global property". Such results can be seen in Theorem 3.1 in [**7**] which restates Theorem 4.4 in [**12**] for the bridge number, and in [**14**] and [**30**] for determinants.

4. Proof of theorems.

4.1. Proof of Theorem 2.1.

In this subsection we utilize the results in [**31**] to give a lower bound on the height and min-height of some satellite knots.

The following theorem is Corollary 5.4 in [**31**].

Theorem 4.1. *Suppose $h : S^3 \to \mathbb{R}$ is the standard height function and $H \subset S^3$ is a handlebody for which horizontal circles of ∂H with respect to h constitute a complete collection of meridian disk boundaries. Then there is a reimbedding $f : H \to S^3$ so that*

- *$h = h \circ f$ on H and*
- *$f(H) \cup (S^3 \setminus f(H))$ is a Heegaard splitting of S^3.*

The proof of the following theorem is a slight variation on the proof of Corollary 6.3 of [**31**].

THEOREM 4.2. *Suppose γ is an embedding of knot-type K in an unknotted solid torus H in S^3. Suppose $f : H \to S^3$ is a knotted embedding of H and $\gamma' = f(\gamma)$ is an embedding of knot-type K'. If $w(K) = w(K')$, then $\mathrm{ht}_{\min}(K') \geq \mathrm{ht}_{\min}(K)$ and $\mathrm{ht}(K') \leq \mathrm{ht}(K)$.*

PROOF. Let $\gamma^* \in \mathrm{TP}(K')$ such that γ^* has $\mathrm{ht}_{\min}(K')$ thick levels with respect to h. Let H^* be the image of $f(H)$ under an isotopy taking γ' to γ^*. We can additionally assume ∂H^* is in a Morse position with respect to h after this isotopy. For every regular value s of $h|_{\partial H^*}$, $(h|_{\partial H^*})^{-1}(s)$ is an unlink in S^3. By standard Morse theory and since ∂H^* is a torus, there exists a regular value s^* such that $(h|_{\partial H^*})^{-1}(s^*)$ has a component c that is an essential loop in ∂H^*. Moreover, since H^* is a knotted solid torus, c is a meridian curve for H^*. By Theorem 4.1, there is a reimbedding $g : H^* \to S^3$ of H^* that preserves height and results in $g(H^*)$ being unknotted. Moreover, after a suitable choice of g, we can assume that $g(\gamma^*) \in K$, see [**31**] for details. Since g is height preserving, γ^* and $g(\gamma^*)$ have the same number of thick levels and $w(\gamma^*) = w(g(\gamma^*))$. Since γ^* is a thin position for K' and $w(K) = w(K')$, then $g(\gamma^*)$ is a thin position for K. Hence, $\mathrm{ht}_{\min}(K') \geq \mathrm{ht}_{\min}(K)$.

Alternatively, let $\gamma^* \in \mathrm{TP}(K')$ such that γ^* has $\mathrm{ht}(K')$ thick levels with respect to h. By the same argument as give above, we can find a height preserving reimbedding g and $g(\gamma^*) \in K$ such that $w(K) = w(K') = w(\gamma^*) = w(g(\gamma^*))$ and $\mathrm{ht}(K') = \mathrm{ht}(\gamma^*) = \mathrm{ht}(g(\gamma^*))$. Hence, $\mathrm{ht}(K') \leq \mathrm{ht}(K)$. $\square$

REMARK 4.3. It is interesting to note that Theorem 4.2 does not hold if the hypothesis of $w(K) = w(K')$ is omitted. For example, if L is a two-bridge knot, it is an easy exercise to show that $\mathrm{ht}(L) = 1$ and $\mathrm{ht}(L\#L) = 2$. However, declaring $L = K$ and $L\#L = K'$ meets all the hypotheses of Theorem 4.2 except $w(K) = w(K')$. Additionally, in Figure 2 we give an example of a thin position γ' for a knot-type K' embedded in a knotted solid torus $f(H)$ together with an embedding γ of knot type K contained in the unknotted solid torus H such that $\gamma' = f(\gamma)$. (K is a Montesinos knot with length 4, and the knot type of $f(H)$ is a trefoil.) The embedding γ' depicted in Figure 2 is a thin position of K' by Lemma 6.0.6 of [**5**]. Hence $\mathrm{ht}_{\min}(K') = 1$. Since $\beta(K) = 4$, then the embedding γ in the figure illustrates that no bridge position for K is a thin position. Hence, $\mathrm{ht}_{\min}(K) \geq 2$ and moreover $\mathrm{ht}_{\min}(K) = 2$ since the maximal number of disjoint, non-parallel, planar, meridional, essential surfaces in the exterior of K is equal to 1 by [**18**]. Thus, $1 = \mathrm{ht}(K') = \mathrm{ht}_{\min}(K') < \mathrm{ht}_{\min}(K) = \mathrm{ht}(K) = 2$ and $w(K') > w(K)$.

In [**6**], Blair and Tomova construct an infinite collection of ambient isotopy classes of knots $\mathcal{K}$ from the schematic depicted in the left-hand side of Figure 3 by inserting suitable braids $B_1, \ldots, B_4$ into the boxes shown. By Theorems 12.1 and 12.2 of [**6**], for all $K \in \mathcal{K}$, $w(K) = 134$ and any thin position for K has exactly three thick levels of width 10 and exactly two thin levels of width 4. Hence, $\mathrm{ht}_{\min}(K) = \mathrm{ht}(K) = 3$ for all $K \in \mathcal{K}$. Note that if we consider the height function to be increasing from the bottom to the top of Figure 3, then, for suitable choices of $B_1, \ldots, B_4$, the left-hand side of the

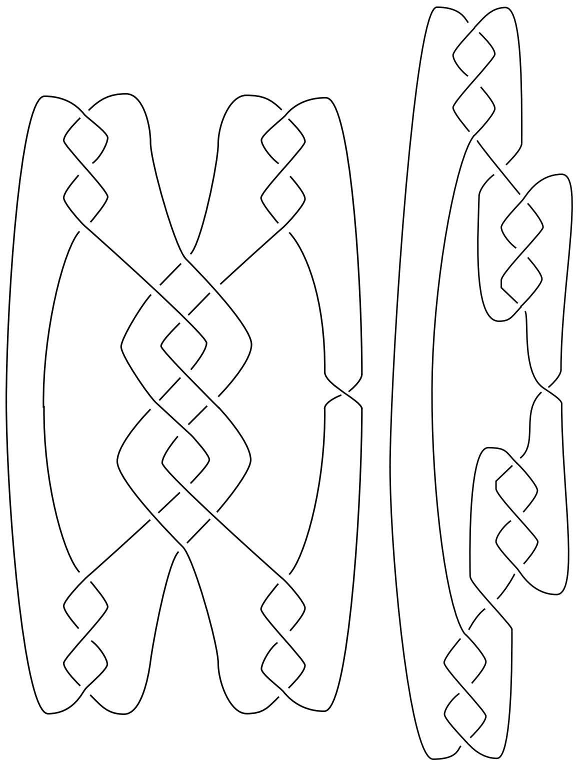

Figure 2. A counterexample for Theorem 4.2 without the condition $w(K) = w(K')$.

figure depicts a thin position for any knot in $\mathcal{K}$.

Proof of Theorem 2.1. By [**6**], $w(K) = 134$ and Figure 3 gives a thin position for K. Figure 3 demonstrates that $w(K_2\#K) \leq w(K) = 134$. By Corollary 6.4 of [**31**], $w(K_2\#K) \geq w(K)$ and, therefore, $w(K_2\#K) = w(K)$. Hence, Figure 3 demonstrates a thin position or $K_2\#K$ with three thick levels. In particular, $\mathrm{ht}(K_2\#K) \geq 3$ and $\mathrm{ht}_{\min}(K_2\#K) \leq 3$.

If we apply Theorem 4.2 to the embeddings of $K_2\#K$ and K depicted in Figure 3 where $f(H)$ is the knotted "swallow-follow" torus that contains $K_2\#K$, swallows K and follows K_2, then, since $w(K_2\#K) = w(K)$, $\mathrm{ht}(K_2\#K) \leq \mathrm{ht}(K) = 3$ and $\mathrm{ht}_{\min}(K_2\#K) \geq \mathrm{ht}_{\min}(K) = 3$. Hence, $\mathrm{ht}(K_2\#K) = 3$ and $\mathrm{ht}_{\min}(K_2\#K) = 3$. □

4.2. Proof of Theorems 3.1 and 3.2.

Proof of Theorem 3.1. Firstly, if K is the trivial knot, then we have $r(K) = 1$ and $\mathrm{trunk}(K) = 2$, and hence the inequality of Theorem 3.1 holds.

Next, we will show that for a non-trivial knot K, a height function $h : S^3 \to \mathbb{R}$ and a closed surface F containing K,

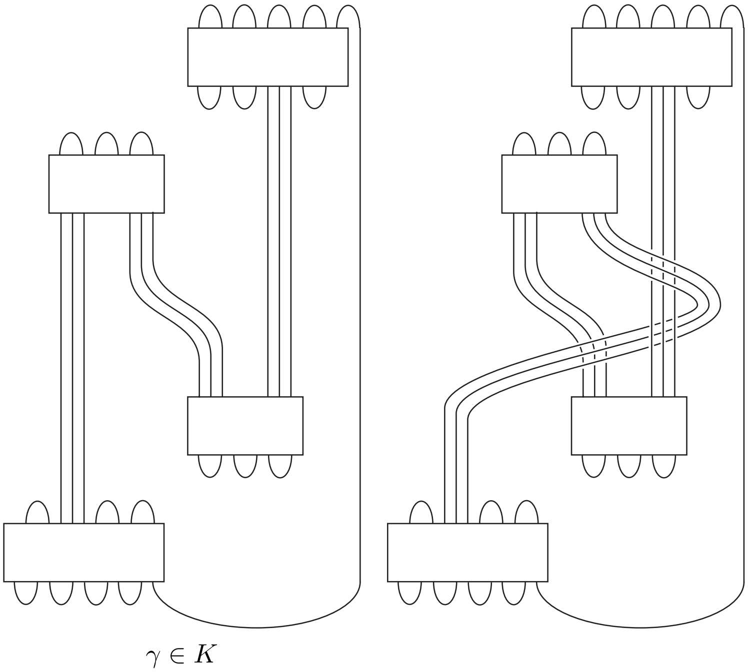

Figure 3. $K \in \mathcal{K}$ and $K \# K_2$.

$$r(F, K) \leq \frac{\max_{t \in \mathbb{R}} |h^{-1}(t) \cap K|}{2}.$$

By taking maximal of the left-hand side and minimal of the right-hand side, we have

$$\max_{F \in \mathcal{F}} r(F, K) \leq \frac{\min_{\gamma \in K} \max_{t \in \mathbb{R}} |h^{-1}(t) \cap \gamma|}{2}.$$

Thus,

$$r(K) \leq \frac{\mathrm{trunk}(K)}{2}.$$

By perturbing F relative to K, we may assume that any critical point of F is not on K and F is also in a Morse position with respect to h. Since the genus of F is greater than 0, there exists a regular value $t \in \mathbb{R}$ for F such that $h^{-1}(t) \cap F$ contains at least two essential loops in F. Take two distinct loops $l_1,\ l_2$ of $h^{-1}(t) \cap F$ which are essential in F and innermost in $h^{-1}(t)$ among all essential loops of $h^{-1}(t) \cap F$. Let $D_1,\ D_2$ be mutually disjoint disks in $h^{-1}(t)$ which are bounded by $l_1,\ l_2$ respectively. Then we have

$$|\partial D_1 \cap K| + |\partial D_2 \cap K| \leq \max_{t \in \mathbb{R}} |h^{-1}(t) \cap K|.$$

Without loss of generality, we may assume that

$$|\partial D_1 \cap K| \leq \frac{\max_{t\in\mathbb{R}} |h^{-1}(t) \cap K|}{2}.$$

By cutting and pasting D_1 if necessary, we may assume that $D_1 \cap F = \partial D_1$. Thus, D_1 is a compressing disk for F and we have

$$r(F, K) \leq |\partial D_1 \cap K|. \qquad \square$$

Let K be a knot admitting an essential tangle decomposition $(S^3, K) = (B_1, T_1) \cup_S (B_2, T_2)$, where S is a tangle decomposing sphere. In the following, we show

$$r(K) \leq \frac{\min\{\mathrm{trunk}(B_1, T_1), \mathrm{trunk}(B_2, T_2)\}}{2}.$$

Proof of Theorem 3.2. Let F be a closed surface containing K. Let $h_i : B_i \to \mathbb{R}$ be a standard Morse function for $i = 1, 2$. It suffices to show that

$$r(F, K) \leq \frac{\max_{t\in\mathbb{R}} |h_i^{-1}(t) \cap T_i|}{2},$$

for $i = 1, 2$.

We remark that K is non-trivial since K admits an essential tangle decomposition. Hence the genus of F is greater than 0 and we may assume that $2 \leq r(F, K)$. Since F and S are essential in the exterior of K, we may assume that each loop of $F \cap S$ is essential in F. If $F \cap S$ consists of a single essential loop, then both $F \cap B_1$ and $F \cap B_2$ are surfaces with strictly positive genus. Then, there exists a regular value $t_i \in \mathbb{R}$ for F such that $h_i^{-1}(t_i) \cap F$ contains at least two essential loops in F for $i = 1, 2$. Otherwise, $F \cap S$ consists of at least two essential loops and $h_i^{-1}(0) \cap F$ contains at least two essential loops in F for $i = 1, 2$. Similarly to Proof of Theorem 3.1, we obtain a compressing disk D for F in B_i such that

$$r(F, K) \leq |\partial D \cap T_i| \leq \frac{\max_{t\in\mathbb{R}} |h^{-1}(t) \cap T_i|}{2}. \qquad \square$$

5. Several versions of thin position.

The bridge number $\beta(K)$, the trunk $\mathrm{trunk}(K)$ and the width $w(K)$ are fundamental geometrical invariants of a knot K. In this section, we consider several versions of thin position $\mathrm{TP}(K)$, $\mathrm{MCP}(K)$ and $\mathrm{OTP}(K)$ which attain $w(K)$, $\beta(K)$ and $\mathrm{trunk}(K)$ respectively.

In the previous part of this paper, we considered Gabai's thin position $\mathrm{TP}(K)$ ([**10**]), that is, the set of all position γ minimizing the width $w(\gamma) = \sum w(h^{-1}(r_i))$ for chosen regular values $r_1, r_2, \ldots, r_n$. Then we have already established the following.

TP-1 There exists a knot $K = K_\alpha$ in [**6**] such that $\beta(K)$ cannot be obtained in $\mathrm{TP}(K)$.

TP-2 $w(K)$ can be always obtained in $\mathrm{TP}(K)$.

TP-3 There exists a candidate knot $K = K_{4,1,3,3}$ in [**8**] such that $\mathrm{trunk}(K)$ cannot be obtained in $\mathrm{TP}(K)$.

TP-4 There exists a candidate knot K in Figure 1 such that $\mathrm{ht}(K) > \mathrm{ht}_{\min}(K)$.

TP-5 There exist two knots K and K' in Theorem 2.1 such that $\mathrm{ht}(K\#K') < \mathrm{ht}(K) + \mathrm{ht}(K')$.

TP-6 There exists a knot K in [**7**] such that $\gamma \in \mathrm{TP}(K)$ has a compressible thin level sphere.

TP-7 Every thinnest level sphere for $\gamma \in \mathrm{TP}(K)$ is incompressible in the complement of a knot [**37**].

Next, let $\mathrm{MCP}(K)$ be the set of all Morse positions of K which have minimal critical points among all Morse positions. We say that a knot belonging to $\mathrm{MCP}(K)$ is in a *minimal critical position*. Similarly to $\mathrm{TP}(K)$, we can define the *MCP-height* and the *min-MCP-height* of K as

$$\mathrm{ht}^{\mathrm{MCP}}(K) = \max_{\gamma\in \mathrm{MCP}(K)} \mathrm{ht}(\gamma),$$
$$\mathrm{ht}^{\mathrm{MCP}}_{\min}(K) = \min_{\gamma\in \mathrm{MCP}(K)} \mathrm{ht}(\gamma),$$

respectively. Note that $\mathrm{ht}^{\mathrm{MCP}}_{\min}(K) = 1$ for any knot K. Then we have the following.

MCP-1 $\beta(K)$ can be always obtained in $\mathrm{MCP}(K)$.

MCP-2 There exists a knot $K = K_\alpha$ in [**6**] such that $w(K)$ cannot be obtained in $\mathrm{MCP}(K)$.

MCP-3 There exists a candidate knot $K = K_{4,1,3,3}$ in [**8**] such that $\mathrm{trunk}(K)$ cannot be obtained in $\mathrm{MCP}(K)$.

MCP-4 There exists a knot K in Figure 4 such that $\mathrm{ht}^{\mathrm{MCP}}(K) > \mathrm{ht}^{\mathrm{MCP}}_{\min}(K)$.

MCP-5 There exist two knots K and K' in Figure 4 such that $\mathrm{ht}^{\mathrm{MCP}}(K\#K') > \mathrm{ht}^{\mathrm{MCP}}(K) + \mathrm{ht}^{\mathrm{MCP}}(K')$.

MCP-6 There exists a knot K in Figure 4 such that $\gamma \in \mathrm{MCP}(K)$ has a compressible thin level sphere.

MCP-7 There exists a knot K in Figure 4 such that $\gamma \in \mathrm{MCP}(K)$ has a compressible thinnest level sphere.

Note that a knot K in Figure 4 is not in a thin position, thus we have $\gamma \in \mathrm{MCP}(K)\backslash\mathrm{TP}(K)$. Moreover, we remark that there exists a knot $K = K_\alpha$ in [**6**] such that $\mathrm{TP}(K) \cap \mathrm{MCP}(K) = \phi$.

Finally, we define the *ordered thin position* $\mathrm{OTP}(K)$, that is, a Morse position γ of K which minimizes the lexicographical order of monotonically non-increasing ordered sequences $\{w_i\}$, where w_i $(i = 1,\dots,k)$ is the number of points of intersection between thin/thick level spheres and γ, and k is the total number of thin/thick level spheres.

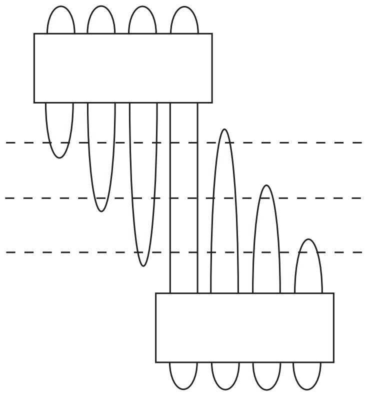

Figure 4. A minimal critical position of $K\#K'$.

For example, the embedding $\gamma \in K\#K'$ in Figure 4 has the complexity $\{10, 10, 10, 8, 8\}$, but it can be reduced to $\{8, 8, 2\}$ and we obtain an embedding $\gamma' \in \mathrm{OTP}(K\#K')$.

Similarly to $\mathrm{TP}(K)$, we define the *OTP-height* and the *min-OTP-height* of K as

$$\mathrm{ht}^{\mathrm{OTP}}(K) = \max_{\gamma\in \mathrm{OTP}(K)} \mathrm{ht}(\gamma),$$
$$\mathrm{ht}^{\mathrm{OTP}}_{\min}(K) = \min_{\gamma\in \mathrm{OTP}(K)} \mathrm{ht}(\gamma),$$

respectively. Then we have the following.

OTP-1 There exists a candidate knot $K = K_{4,1,3,3}$ in [**8**] such that $\beta(K)$ cannot be obtained in $\mathrm{OTP}(K)$.

OTP-2 There exists a candidate knot $K = K_{4,1,3,3}$ in [**8**] such that $w(K)$ cannot be obtained in $\mathrm{OTP}(K)$.

OTP-3 $\mathrm{trunk}(K)$ can be always obtained in $\mathrm{OTP}(K)$, as the first term of the monotonically non-increasing ordered set $\{w_i\}$.

OTP-4 For any knot K, $\mathrm{ht}^{\mathrm{OTP}}(K) = \mathrm{ht}^{\mathrm{OTP}}_{\min}(K)$.

OTP-5 There exist a candidate knot $K_{4,1,3,3}$ in [**8**] and a two-bridge knot K_2 such that $\mathrm{ht}^{\mathrm{OTP}}(K_{4,1,3,3}\#K_2) < \mathrm{ht}^{\mathrm{OTP}}(K_{4,1,3,3}) + \mathrm{ht}^{\mathrm{OTP}}(K_2)$ as in Theorem 2.1.

OTP-6 There exists a candidate embedding $\gamma \in K$ in [**7**] such that $\gamma \in \mathrm{OTP}(K)$ and γ has a compressible thin level sphere.

OTP-7 Every thinnest level sphere for $\gamma \in \mathrm{OTP}(K)$ is incompressible in the complement of a knot (by a similar argument to [**37**]).

In Figure 5, we summarize a relation on several versions of thin position. For each region, we give an example of an embedding in the corresponding subset of Morse embeddings. Each of these examples is conjectural with the exception of the two-bridge embedding and the embedding from Figure 4, which can easily be verified. Potential examples of embeddings $k_{2,1,3,7}$, $k'_{2,1,3,7}$, $k_{4,1,3,3}$ and $k'_{4,1,3,3}$ are referred from [**8**]. We have a potential example $\gamma' \in (\mathrm{TP}(K) \cap \mathrm{OTP}(K)) \backslash \mathrm{MCP}(K)$ from Figure 1.

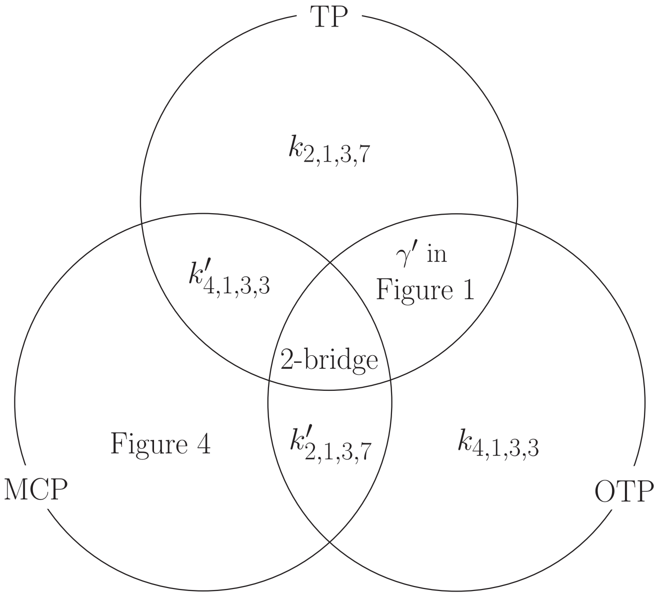

Figure 5. Venn diagram for TP, MCP and OTP.

PROPOSITION 5.1. *The following statements hold true.*

- *For any two-bridge knot type K and a two-bridge embedding $\gamma \in K$, we have $\gamma \in \mathrm{TP}(K) \cap \mathrm{MCP}(K) \cap \mathrm{OTP}(K)$.*
- *For any composite knot type $K\#K'$ and an embedding $\gamma \in K\#K'$ given in Figure 4, we have $\gamma \in \mathrm{MCP}(K\#K') \backslash (\mathrm{TP}(K\#K') \cup \mathrm{OTP}(K\#K'))$.*

CONJECTURE 5.2. *The following statements hold true.*

- *We have $k_{2,1,3,7} \in \mathrm{TP}(K_{2,1,3,7}) \backslash (\mathrm{MCP}(K_{2,1,3,7}) \cup \mathrm{OTP}(K_{2,1,3,7}))$, where $k_{2,1,3,7}$ and $K_{2,1,3,7}$ are given in [**8**].*
- *We have $k'_{2,1,3,7} \in (\mathrm{MCP}(K_{2,1,3,7}) \cap \mathrm{OTP}(K_{2,1,3,7})) \backslash \mathrm{TP}(K_{2,1,3,7})$, where $k'_{2,1,3,7}$ and $K_{2,1,3,7}$ are given in [**8**].*
- *We have $k_{4,1,3,3} \in \mathrm{OTP}(K_{4,1,3,3}) \backslash (\mathrm{TP}(K_{4,1,3,3}) \cup \mathrm{MCP}(K_{4,1,3,3}))$, where $k_{4,1,3,3}$ and $K_{4,1,3,3}$ are given in [**8**].*
- *We have $k'_{4,1,3,3} \in (\mathrm{TP}(K_{4,1,3,3}) \cap \mathrm{MCP}(K_{4,1,3,3})) \backslash \mathrm{OTP}(K_{4,1,3,3})$, where $k'_{4,1,3,3}$ and $K_{4,1,3,3}$ are given in [**8**].*
- *We have $\gamma' \in (\mathrm{TP}(K) \cap \mathrm{OTP}(K)) \backslash \mathrm{MCP}(K)$, where γ' and K are given in Figure 1.*

6. Waist and representativity.

Theorem 3.1 is compared with the inequality between the waist and trunk of knots. We define the *waist* of a knot K as

$$\mathrm{waist}(K) = \max_{F \in \mathcal{F}} \min_{D \in \mathcal{D}_F} |D \cap K|,$$

where $\mathcal{F}$ denotes the set of all closed surfaces in $S^3 - K$, and $\mathcal{D}_F$ denotes the set of all compressing disks for F in S^3 ([**22**]). Then, we have $\mathrm{waist}(K) = 0$ for the trivial knot K since any closed surface in $S^3 - K$ is compressible, and by considering the peripheral torus $\partial N(K)$, $\mathrm{waist}(K) \geq 1$ for non-trivial knots. It is known that $\mathrm{waist}(K) = 1$ for 3-braid knots ([**15**]), alternating knots ([**16**]), almost alternating knots ([**1**]), Montesinos knots ([**18**]), toroidally alternating knots ([**2**]), algebraically alternating knots ([**21**]), and that $\mathrm{waist}(K) = p \cdot \mathrm{waist}(J)$ for inconsistent cable knots with index p, where J is a companion knot for K ([**3**]).

Theorem 6.1 ([**22**, Theorem 1.9]). *For any knot K, we have*

$$\mathrm{waist}(K) \leq \frac{\mathrm{trunk}(K)}{3}.$$

Theorems 3.1 and 6.1 bear a close resemblance to each other. We expected in [**23**, Problem 26] that $\mathrm{waist}(K) \leq r(K)$ for any knot K. For example, any alternating knots satisfy this inequality since $\mathrm{waist}(K) = 1$ ([**16**]) and $r(K) = 2$ ([**13**]). However, it does not hold for composite knots in general. The waist behaves as expected under taking connected sums, that is, $\mathrm{waist}(K_1 \# K_2) = \max\{\mathrm{waist}(K_1), \mathrm{waist}(K_2)\}$ ([**22**, Proposition 1.2]). On the other hand, we have $r(K_1 \# K_2) = 2$ whenever K_1 and K_2 are non-trivial. This shows that the representativity of knots behaves dissimilarly to other geometric knot invariants.

7. Representativity and non-orientable spanning surfaces.

Aumann proved that any alternating knot bounds an essential non-orientable spanning surface ([**4**]). Indeed, he showed that both checkerboard surfaces for a reduced alternating diagram are essential. Recently, Kindred proved in [**13**] that $r(K) = 2$ for any non-trivial alternating knot K, which confirmed Conjecture 4 in [**23**]. From these results to extend the result in [**13**], one might expect that if a knot K bounds an essential non-orientable spanning surface, then $r(K) = 2$. However, we have the next theorem.

Theorem 7.1. *For any integer $n \geq 2$, there exists a knot with $r(K) \geq n$ which bounds an essential once punctured Klein bottle.*

Proof. Let $V \cup_F W$ be a genus two Heegaard splitting of S^3. Take a loop C on ∂V as shown in Figure 6. Note that C bounds a Möbius band M properly embedded in V which is formed by a non-separating disk and a band. Let A be a loop obtained from a train track T on ∂V as shown in Figure 6, where $m,\ n \geq 1$. By adding a band B along A to M, we obtain a once punctured Klein bottle $F = M \cup B$ properly embedded in V

and a knot $K = \partial F$.

It is easy to see that K is $\min\{2m, 2n+2\}$-*seamed* with respect to a complete set of essential disks $\{D_1, D_2, D_3\}$ in V, that is, K has been isotoped to intersect $\bigcup \partial D_i$ minimally and for each pair of pants P obtained from ∂V by cutting along $\bigcup \partial D_i$, and for each pair of two boundary components of P, there exist at least $\min\{2m, 2n+2\}$ arcs of intersection in $K \cap P$ that connect that pair of boundary components. The following lemma can be proved by an elementary cut and paste argument.

Lemma 7.2. *If $K \subset \partial V$ is k-seamed with respect to a complete set of meridian disks $\{D_i\}$ in V and Δ is an essential disk in V, then $|K \cap \partial\Delta| \geq 2k$.*

By this lemma, for any compressing disk Δ for F in V, $\partial\Delta$ intersects K at least $2\min\{2m, 2n+2\}$ points.

Finally, to obtain a knot K with $r(K) \geq 2\min\{2m, 2n+2\}$, we re-embed V in S^3 so that $S^3 - \operatorname{int} V$ is boundary-irreducible. Then there exists no compressing disk for ∂V in $S^3 - \operatorname{int} V$, and we have $r(F, K) \geq 2\min\{2m, 2n+2\}$. □

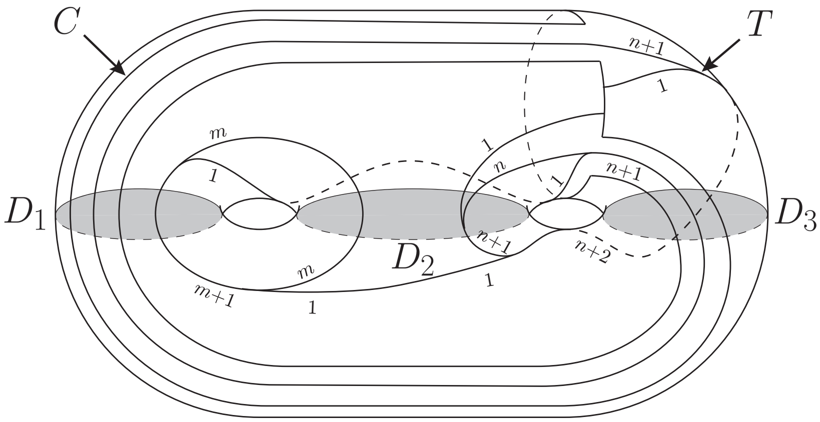

Figure 6. Constructing a knot K with $r(F, K) \geq n$ which bounds an essential once punctured Klein bottle.

Remark 7.3. We remark that there exists a knot which does not bound an essential non-orientable spanning surface ([**9**]), that denies the strong Neuwirth conjecture ([**26**]). Each of our knots produced in Theorem 7.1 admits two distinct "Neuwirth surfaces", ∂V and $\partial N(F)$, where ∂V is the boundary of the genus two handlebody V and $\partial N(F)$ is the boundary of a regular neighborhood of an essential once punctured Klein bottle F. It is conjectured that for any non-trivial knot K, there exists a closed surface (*Neuwirth surface*) S containing K such that $S - K$ is connected and $S \cap E(K)$ is essential, that is, the Neuwirth conjecture ([**26**]).

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