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L_p regularity theorem for elliptic equations in less smooth domains

By Yoichi Miyazaki

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Abstract. We consider a $2m$th-order strongly elliptic operator A subject to Dirichlet boundary conditions in a domain Ω of $\mathbb{R}^n$, and show the L_p regularity theorem, assuming that the domain has less smooth boundary. We derive the regularity theorem from the following isomorphism theorems in Sobolev spaces. Let k be a nonnegative integer. When A is a divergence form elliptic operator, $A-\lambda$ has a bounded inverse from the Sobolev space $W_p^{k-m}(\Omega)$ into $W_p^{k+m}(\Omega)$ for λ belonging to a suitable sectorial region of the complex plane, if Ω is a uniformly $C^{k,1}$ domain. When A is a non-divergence form elliptic operator, $A-\lambda$ has a bounded inverse from $W_p^k(\Omega)$ into $W_p^{k+2m}(\Omega)$, if Ω is a uniformly $C^{k+m,1}$ domain. Compared with the known results, we weaken the smoothness assumption on the boundary of Ω by $m-1$.

1. Introduction.

Let us consider a $2m$th-order strongly elliptic operator in divergence form

$$A = \sum_{\substack{|\alpha|\le m \\ |\beta|\le m}} D^\alpha(a_{\alpha\beta}D^\beta\,\cdot\,) \tag{1.1}$$

subject to Dirichlet boundary conditions in a domain Ω of $\mathbb{R}^n$ with $n \ge 2$. The well known regularity theorem for the elliptic equation

$$Au = f \tag{1.2}$$

can be stated in terms of the L_p-based Sobolev spaces with $1 < p < \infty$ as follows: If we assume, for some integer k with $k \ge 1$, that the coefficients $a_{\alpha\beta}$ satisfy

$$a_{\alpha\beta} \in C^{|\alpha|+k-m}(\bar{\Omega}), \quad \text{for } |\alpha|+k-m > 0,$$

that Ω is a bounded domain with boundary of class C^{k+m}, and that $f \in W_p^{k-m}(\Omega)$, then a solution u to (1.2) in $W_{p,D}^m(\Omega)$ satisfies $u \in W_p^{k+m}(\Omega)$ and

$$\|u\|_{W_p^{k+m}(\Omega)} \le C(\|f\|_{W_p^{k-m}(\Omega)} + \|u\|_{L_p(\Omega)}).$$

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Here $W^m_{p,D}(\Omega)$ denotes the closure of $C^\infty_0(\Omega)$ in $W^m_p(\Omega)$. Most literature deals with the case $p = 2$. We refer to [**1**, Theorem 14.1], [**6**, Theorem 17.2] for $k \geq m$, [**9**, Theorem 2.5.1.1] for $m = 1$ and general k, and [**23**, Theorem 20.4] for general m and k.

The aim of this paper is to improve the regularity theorem by replacing the smoothness assumption on the boundary $\partial\Omega$ of Ω by the condition that $\partial\Omega$ is of class C^{k+1}. Since $k+m > k+1$ implies $m > 1$, our result is new for higher-order elliptic equations. We derive the regularity theorem from the isomorphism theorem, which states that the operator $A-\lambda$ has a bounded inverse from $W^{k-m}_p(\Omega)$ onto $W^{k+m}_p(\Omega)\cap W^m_{p,D}(\Omega)$ for λ belonging to a suitable sectorial region in the complex plane. Since the regularity theorem is an immediate consequence of the isomorphism theorem, our main task in this paper is to construct the inverse of $A - \lambda$. In [**16**] (see also [**13**], [**14**], [**15**]) we have already done so for $k = 0$ on the basis of the Hardy-type inequality for the Sobolev spaces satisfying Dirichlet boundary conditions.

We also consider a $2m$th-order strongly elliptic operator in non-divergence form

$$\mathcal{A} = \sum_{|\alpha|\leq 2m} a_\alpha D^\alpha \tag{1.3}$$

subject to Dirichlet boundary conditions in a domain Ω of $\mathbb{R}^n$. For this operator we know the isomorphism theorem, which states that the operator $\mathcal{A} - \lambda$ has a bounded inverse from $W^k_p(\Omega)$ onto $W^{k+2m}_p(\Omega) \cap W^m_{p,D}(\Omega)$ with $1 < p < \infty$ for λ belonging to a suitable region, if the coefficients a_α satisfy

$$a_\alpha \in C^k(\bar{\Omega}), \quad \text{for } |\alpha| \leq 2m,$$

and if Ω is a C^{k+2m} domain. For $m = 1$ we refer to [**11**, Chapter 9], and for general m and $k = 0$ we refer to [**3**, Theorem 8.2], [**22**, Chapter 5]. We can also find the regularity theorem for the operator (1.3); we refer to [**8**, Theorem 8.13] for $m = 1$ and $p = 2$, [**18**, Theorem 3.14] for general m and $p = 2$, and [**2**, Theorem 15.2] for general m, k, p. In this paper, we obtain these theorems for non-divergence form operators under the assumption that the boundary of a domain is of class C^{k+m+1}.

2. Main results.

In order to state the main results we define some symbols. Let $\mathrm{i} = \sqrt{-1}$. Let $x = (x_1, \ldots, x_n)$ be a generic point in $\mathbb{R}^n$ and set

$$\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}, \quad D^\alpha = \mathrm{i}^{-|\alpha|}\partial^\alpha$$

for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ of length $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Let $\mathbb{N}$ be the set of positive integers, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $s \in \mathbb{N}_0$ and $1 \leq p \leq \infty$ the Sobolev space $W^s_p(\Omega)$ is the set of functions f whose weak derivatives of order up to s belong to $L_p(\Omega)$. We denote by $W^m_{p,D}(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $W^m_p(\Omega)$, and set

$$W^s_{p,D}(\Omega) = W^s_p(\Omega) \cap W^m_{p,D}(\Omega), \quad \text{for } s \in \mathbb{N} \text{ with } s \geq m.$$

For our purpose it is convenient to introduce the λ-dependent norm

$$\|f\|_{W^s_{p,\lambda}(\Omega)} := \sum_{|\alpha|\le s} |\lambda|^{(s-|\alpha|)/2m} \|D^\alpha f\|_{L_p(\Omega)}$$

for $\lambda \in \mathbb{C} \setminus \{0\}$. This norm is equivalent to the usual norm, which corresponds to the case $\lambda = 1$.

For $s \in \mathbb{N}$ the Sobolev space $W^{-s}_p(\Omega)$ of negative order is the set of functions f which can be written as

$$f = \sum_{|\alpha|\le s} D^\alpha f_\alpha \quad \text{with } f_\alpha \in L_p(\Omega). \tag{2.1}$$

We also define the λ-dependent norm in $W^{-s}_p(\Omega)$ by

$$\|f\|_{W^{-s}_{p,\lambda}(\Omega)} = \inf \sum_{|\alpha|\le s} |\lambda|^{(|\alpha|-s)/2m} \|f_\alpha\|_{L_p(\Omega)},$$

where the infimum is taken over all the expressions of f in (2.1).

Let $s \in \mathbb{N}_0$. By definition we have the inequalities

$$\begin{aligned} &\|D^\alpha f\|_{L_p(\Omega)} \le |\lambda|^{(|\alpha|-s)/2m} \|f\|_{W^s_{p,\lambda}(\Omega)}, \quad \text{for } f \in W^s_p(\Omega) \text{ and } |\alpha| \le s,\\ &\|D^\alpha f\|_{W^{-s}_{p,\lambda}(\Omega)} \le |\lambda|^{(|\alpha|-s)/2m} \|f\|_{L_p(\Omega)}, \quad \text{for } f \in L_p(\Omega) \text{ and } |\alpha| \le s, \end{aligned} \tag{2.2}$$

which will be frequently used.

We first consider an elliptic operator A in divergence form given by (1.1). Let $a_0(x,\xi)$ be the principal symbol of A:

$$a_0(x,\xi) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta}.$$

We fix a nonnegative integer $k \in \mathbb{N}_0$ and assume the following conditions:

(HD1) There exists $\delta > 0$ such that

$$\operatorname{Re} a_0(x,\xi) \ge \delta|\xi|^{2m}, \quad \text{for } x \in \Omega,\ \xi \in \mathbb{R}^n.$$

(HD2)$_k$ All the coefficients $a_{\alpha\beta}$ belong to $L_\infty(\Omega)$. In addition, if $k \ge 1$, then

$$a_{\alpha\beta} \in W^{|\alpha|+k-m}_\infty(\Omega), \quad \text{for } |\alpha| + k - m > 0.$$

If $k = 0$, then the leading coefficients are uniformly continuous.

(HD3)$_k$ The domain Ω is a uniformly $C^{k,1}$ domain if $k \ge 1$, and a uniformly C^1 domain if $k = 0$. Or $\Omega = \mathbb{R}^n$.

We will define a special $C^{k,1}$ domain in Definition 5.1 below. A uniformly $C^{k,1}$ domain is defined in terms of special $C^{k,1}$ domains, as a domain with minimally smooth boundary is defined in terms of special Lipschitz domains (see [**21**, Chapter VI]). We note that a bounded domain with $C^{k,1}$ boundary is a uniformly $C^{k,1}$ domain.

By the extension theorem [**21**, Chapter VI, Theorem 5] and the characterization of L_∞-based Sobolev space (see [**22**, Theorem 3.12]) we know that the Sobolev space $W^s_\infty(\Omega)$ with $s \in \mathbb{N}$ coincides with the set of C^{s-1} functions whose derivatives are bounded and Lipschitz continuous. So $(\mathrm{HD2})_k$ means that $a_{\alpha\beta} \in C^{|\alpha|+k-m-1}(\Omega)$ and the derivatives of $a_{\alpha\beta}$ of order up to $|\alpha|+k-m-1$ are Lipschitz continuous. Thus $(\mathrm{HD2})_k$ is slightly weaker than the condition $a_{\alpha\beta} \in C^{|\alpha|+k-m}(\bar{\Omega})$ if $k \geq 1$.

In order to state clearly the dependency of the constants which will appear in the main results, we use the following symbols:

$$\begin{aligned}
\delta_A &= \max\{\delta > 0 : \text{(HD1) is satisfied.}\},\\
\theta_A &= \sup_{x\in\Omega} \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} |\arg a_0(x,\xi)|,\\
M_A &= \sum_{|\alpha|\leq m,\, |\beta|\leq m} \|a_{\alpha\beta}\|_{L_\infty(\Omega)},\\
M_{k,A} &= \sum_{|\alpha|>m-k} \sum_{|\beta|\leq m} \|a_{\alpha\beta}\|_{W^{|\alpha|+k-m}_\infty(\Omega)} + \sum_{|\alpha|\leq m-k} \sum_{|\beta|\leq m} \|a_{\alpha\beta}\|_{L_\infty(\Omega)}.
\end{aligned} \tag{2.3}$$

Note that the strong ellipticity implies $\theta_A \in [0, \pi/2)$. We also use the function ω_A on $(0,\infty)$ which describes the modulus of continuity of the leading coefficients:

$$\omega_A(t) = \max_{|\alpha|=|\beta|=m} \sup_{\substack{x,y\in\Omega\\ |x-y|\leq t}} |a_{\alpha\beta}(x) - a_{\alpha\beta}(y)|, \quad \text{for } t > 0. \tag{2.4}$$

For $R > 0$ and $\theta \in (0,\pi]$ we set

$$\Sigma(R,\theta) = \{\lambda \in \mathbb{C} : |\lambda| \geq R,\, \theta \leq \arg\lambda \leq 2\pi - \theta\}.$$

Theorem 2.1. *Let* $1 < p < \infty$, $k \in \mathbb{N}_0$, *and assume* (HD1), $(\mathrm{HD2})_k$, $(\mathrm{HD3})_k$. *Then for a given* $\theta \in (\theta_A, \pi]$ *there exist constants*

$$R = R(m,n,p,\theta,\delta_A,\omega_A,M_A,\Omega) \geq 1,\ C = C(k,m,n,p,\theta,\delta_A,\omega_A,M_{k,A},\Omega) > 0$$

such that, for $\lambda \in \Sigma(R,\theta)$, *the operator*

$$A - \lambda : W^{k+m}_{p,D}(\Omega) \to W^{k-m}_p(\Omega)$$

has a bounded inverse and satisfies

$$\|(A-\lambda)^{-1}\|_{W^{k-m}_{p,\lambda}(\Omega)\to W^{k+m}_{p,\lambda}(\Omega)} \leq C. \tag{2.5}$$

Remark 2.2. When $k = 0$, we can rewrite (2.5) as

$$\|D^\alpha(A-\lambda)^{-1}D^\beta f\|_{L_p(\Omega)} \leq C|\lambda|^{-1+(|\alpha|+|\beta|)/2m}\|f\|_{L_p(\Omega)}, \quad \text{for } f \in L_p(\Omega)$$

with $|\alpha| \leq m$, $|\beta| \leq m$. Since these estimates are equivalent to those obtained in [**16**, Theorem 2.1], we know that Theorem 2.1 has been already proved for $k = 0$ in [**16**].

We will prove Theorem 2.1 in Sections 3 through 5. Theorem 2.1 immediately yields the following corollary.

Corollary 2.3. *Let $1 < p < \infty$, $k \in \mathbb{N}_0$, and assume* (HD1), (HD2)$_k$, (H3)$_k$. *If $u \in W^m_{p,D}(\Omega)$ and $Au \in W^{k-m}_p(\Omega)$, then $u \in W^{k+m}_{p,D}(\Omega)$ and*

$$\|u\|_{W^{k+m}_p(\Omega)} \le C(\|Au\|_{W^{k-m}_p(\Omega)} + \|u\|_{L_p(\Omega)}) \tag{2.6}$$

with $C = C(k, m, n, p, \delta_A, \omega_A, M_{k,A}, \Omega)$.

Proof. We write A_k for A if A is considered as an operator from $W^{k+m}_{p,D}(\Omega)$ to $W^{k-m}_p(\Omega)$. Let R be the constant in Theorem 2.1 for $\theta = \pi$. We first observe that if $v \in W^m_{p,D}(\Omega)$ and $g \in W^{k-m}_p(\Omega)$ satisfies $(A + R)v = g$, then $v = (A_k + R)^{-1}g \in W^{k+m}_{p,D}(\Omega)$. Indeed, if we set $w = (A_k + R)^{-1}g$, then $w \in W^{k+m}_{p,D}(\Omega)$ and $(A_0 + R)v = g = (A_0 + R)w$; hence the existence of $(A_0 + R)^{-1}$ gives $v = w$.

The corollary is proved by induction on k. The assertion for $k = 0$ is obvious. Suppose that the assertion for $k - 1$ is true; we will show the assertion for k. Set $Au = f$. By the assertion for $k - 1$ we know that $u \in W^{(k-1)+m}_{p,D}(\Omega) \subset W^{k-m}_p(\Omega)$. Writing $(A + R)u = f + Ru$ and applying the above observation, we find that $u = (A_k + R)^{-1}(f + Ru) \in W^{k+m}_{p,D}(\Omega)$ and

$$\|u\|_{W^{k+m}_p} \le C\|f + Ru\|_{W^{k-m}_p} \le C\|f\|_{W^{k-m}_p} + C\|u\|_{W^{(k-1)+m}_p}.$$

The interpolation inequality gives (2.6). □

We next consider an elliptic operator $\mathcal{A}$ in non-divergence form given by (1.3). Let $a_0(x, \xi)$ be the principal symbol of $\mathcal{A}$:

$$a_0(x, \xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha.$$

For a fixed $k \in \mathbb{N}_0$ we assume the following conditions:

(HN1) There exists $\delta > 0$ such that

$$\operatorname{Re} a_0(x, \xi) \ge \delta|\xi|^{2m}, \quad \text{for } x \in \Omega,\ \xi \in \mathbb{R}^n.$$

(HN2)$_k$ All the coefficients a_α satisfy

$$a_\alpha \in W^k_\infty(\Omega), \quad \text{for } |\alpha| \le 2m.$$

In addition, the leading coefficients are uniformly continuous if $k = 0$.

(HN3)$_k$ The domain Ω is a uniformly $C^{k+m,1}$ domain or $\mathbb{R}^n$.

We define the following symbols, which are similar to those in (2.3) and (2.4):

$$
\begin{aligned}
\delta_{\mathcal{A}} &= \max\{\delta > 0 : \text{(HN1) is satisfied.}\}, \\
\theta_{\mathcal{A}} &= \sup_{x\in\Omega} \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} |\arg a_0(x,\xi)|, \\
M_{\mathcal{A}} &= \sum_{|\alpha|\le 2m} \|a_\alpha\|_{L_\infty(\Omega)}, \\
M_{k,\mathcal{A}} &= \sum_{|\alpha|\le 2m} \|a_\alpha\|_{W^k_\infty(\Omega)}, \\
\omega_{\mathcal{A}}(t) &= \max_{|\alpha|=2m} \sup_{\substack{x,y\in\Omega \\ |x-y|\le t}} |a_\alpha(x) - a_\alpha(y)|, \quad \text{for } t > 0.
\end{aligned}
$$

Theorem 2.4. *Let $1 < p < \infty$, $k \in \mathbb{N}_0$, and assume* (HN1), $(\mathrm{HN2})_k$, $(\mathrm{HN3})_k$. *Then for a given $\theta \in (\theta_{\mathcal{A}}, \pi]$ there exist constants*

$$
R = R(m,n,p,\theta,\delta_{\mathcal{A}},\omega_{\mathcal{A}},M_{\mathcal{A}},\Omega) \ge 1,\ C = C(k,m,n,p,\theta,\delta_{\mathcal{A}},\omega_{\mathcal{A}},M_{k,\mathcal{A}},\Omega) > 0
$$

such that, for $\lambda \in \Sigma(R,\theta)$, the operator

$$
\mathcal{A} - \lambda : W^{k+2m}_{p,D}(\Omega) \to W^k_p(\Omega)
$$

has a bounded inverse and satisfies

$$
\|(\mathcal{A}-\lambda)^{-1}\|_{W^k_{p,\lambda}(\Omega)\to W^{k+2m}_{p,\lambda}(\Omega)} \le C. \tag{2.7}
$$

We will prove Theorem 2.4 in Sections 6 through 8.

Corollary 2.5. *Let $1 < p < \infty$, $k \in \mathbb{N}_0$, and assume* (HN1), $(\mathrm{HN2})_k$, $(\mathrm{HN3})_k$. *If $u \in W^{2m}_{p,D}(\Omega)$ and $\mathcal{A}u \in W^k_p(\Omega)$, then $u \in W^{k+2m}_{p,D}(\Omega)$ and*

$$
\|u\|_{W^{k+2m}_p(\Omega)} \le C(\|\mathcal{A}u\|_{W^k_p(\Omega)} + \|u\|_{L_p(\Omega)})
$$

with $C = C(k,m,n,p,\delta_{\mathcal{A}},\omega_{\mathcal{A}},M_{k,\mathcal{A}},\Omega)$.

Proof. The corollary can be proved in the same way as Corollary 2.3. □

We conclude this section with some remarks. In the proof of Theorem 2.1 we always assume that $\Omega \ne \mathbb{R}^n$, since the case $\Omega = \mathbb{R}^n$ can be handled by a slight modification of the proof for the case $\Omega = \mathbb{R}^n_+$. We will prove Theorem 2.1 for the half space by the method of difference quotient and then for a special $C^{k,1}$ domain by a method similar to that used in [**14**], [**15**], [**16**]. Once we establish the theorem for a special $C^{k,1}$ domain, we can extend it to a uniformly $C^{k,1}$ domain by a partition of unity. The detailed argument for carrying over the result to a uniformly $C^{k,1}$ domain is found in [**15**], where we derived the result for a uniformly C^m domain under the assumptions (HD1) and $(\mathrm{HD2})_k$ with $k = 0$. We can also make the same remarks for the proof of Theorem 2.4.

For the case $k = 0$ we do not try to fully investigate whether the smoothness conditions on the coefficients can be relaxed or not, since this paper mainly targets the case $k \ge 1$. In some cases Theorems 2.1 and 2.4 also hold for $k = 0$, if the smoothness assumption on the coefficients is weakened to VMO class. Heck and Hieber [**10**] showed this for non-divergence form operators in $\mathbb{R}^n$. Dong and Kim also showed this under the assumption that $\lambda < 0$ in [**4**], and extended the result to the case of variably partially VMO coefficients for a Reifenberg flat domain in [**5**]. We also refer to Maz'ya et al. [**12**] who considered an elliptic equation with VMO coefficients in a domain whose boundary has exterior normal vectors belonging to VMO class.

Throughout this paper we use the abbreviation

$$\lambda_m = |\lambda|^{1/2m}. \tag{2.8}$$

We often write $a^{(\alpha)}$ for the derivative $D^\alpha a$, when a is the coefficient of an elliptic operator:

$$a^{(\alpha)} = D^\alpha a = \mathrm{i}^{-|\alpha|}\partial^\alpha a. \tag{2.9}$$

3. Some reductions for the proof of Theorem 2.1.

In this section we will make some reductions for the proof of Theorem 2.1.

LEMMA 3.1. *Let $k \in \mathbb{N}$. Suppose that $A-\lambda$ is injective as an operator from $W^m_{p,D}(\Omega)$ to $W^{-m}_p(\Omega)$. Then $A-\lambda$ is also injective as an operator from $W^{k+m}_{p,D}(\Omega)$ to $W^{k-m}_p(\Omega)$. Consequently, if $A-\lambda$ has a right inverse as an operator from $W^{k+m}_{p,D}(\Omega)$ to $W^{k-m}_p(\Omega)$, then the right inverse is exactly the inverse of $A-\lambda$.*

PROOF. The lemma is obvious by $W^{k+m}_{p,D}(\Omega) \subset W^m_{p,D}(\Omega)$. □

LEMMA 3.2. *It is sufficient to prove Theorem* 2.1 *with the constant R which may depend on k for each $k \in \mathbb{N}_0$.*

PROOF. Given $\theta \in (\theta_A, \pi]$, suppose that we have proved Theorem 2.1 with the constant $R = R(k, m, n, p, \theta, \delta_A, \omega_A, M_A, \Omega)$, which may depend on k. We simply write $R(k)$ for this constant. We will show by induction that for all $k \in \mathbb{N}$ we can take $R(0)$ as the constant R in Theorem 2.1.

Suppose that the assertion for $k-1$ is true; we will prove the assertion for k. If $R(k) \le R(0)$, then there is nothing to prove. So we may assume $R(k) > R(0)$. We must show that $A - \lambda : W^{k+m}_{p,D}(\Omega) \to W^{k-m}_p(\Omega)$ has a bounded inverse for $\lambda \in \Sigma(R(0), \theta)$ with $|\lambda| < R(k)$. We know that this operator is injective, since the corresponding operator for $k = 0$ is injective. So it remains to show the surjectivity and to evaluate the operator norm of its inverse.

Let $f \in W^{k-m}_p(\Omega)$. By the assertion for $k-1$ we can find $u \in W^{(k-1)+m}_{p,D}(\Omega)$ satisfying $(A-\lambda)u = f$ and $\|u\|_{W^{(k-1)+m}_p} \le C\|f\|_{W^{(k-1)-m}_p} \le C\|f\|_{W^{k-m}_p}$. Writing $(A + R(k))u = f + (R(k) + \lambda)u$, noting $W^{(k-1)+m}_p(\Omega) \subset W^{k-m}_p(\Omega)$, and using the same argument as in the proof of Corollary 2.3, we have $u \in W^{k+m}_{p,D}(\Omega)$ and

$$\|u\|_{W^{k+m}_p} \le C\left(\|f\|_{W^{k-m}_p} + 2R(k)\|u\|_{W^{(k-1)+m}_p}\right) \le C\|f\|_{W^{k-m}_p}.$$

Thus we conclude the assertion for k. □

Lemma 3.3. *Let $s \in \mathbb{N}_0$ and $1 < p < \infty$. If $u \in W^s_\infty(\Omega)$ and $v \in W^s_p(\Omega)$, then $uv \in W^s_p(\Omega)$ and*

$$D^\alpha(uv) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta}u)(D^\beta v), \quad \textit{for } |\alpha| \le s.$$

Proof. Since $W^s_p(\Omega) \cap C^\infty(\Omega)$ is dense in $W^s_p(\Omega)$ by the Meyers–Serrin theorem [**22**, Theorem 3.11], the lemma follows by Leibniz's rule for the product of a C^∞ function and a distribution. ☐

Lemma 3.4. *For the proof of Theorem* 2.1 *we may assume that A has no lower-order term.*

Proof. Let A_0 be the principal part of A, i.e.

$$A_0 = \sum_{|\alpha|=|\beta|=m} D^\alpha(a_{\alpha\beta} D^\beta \cdot\,).$$

Given $\theta \in (\theta_A, \pi]$, suppose that there exist $R_0 \ge 1$ and $C_0 > 0$ such that, for $\lambda \in \Sigma(R_0, \theta)$, the operator $A_0 - \lambda :\ W^{k+m}_{p,D}(\Omega) \to W^{k-m}_p(\Omega)$ has an inverse and satisfies

$$\|(A_0 - \lambda)^{-1}\|_{W^{k-m}_{p,\lambda} \to W^{k+m}_{p,\lambda}} \le C_0. \tag{3.1}$$

Case 1. Let $0 \le k < m$. We evaluate $(A - A_0)u$ for $u \in W^{k+m}_{p,D}(\Omega)$ with the symbol λ_m given in (2.8). For $|\alpha| \le m-k$, we have, by (2.2),

$$\begin{aligned}
\|D^\alpha(a_{\alpha\beta}D^\beta u)\|_{W^{k-m}_{p,\lambda}} &\le \lambda_m^{|\alpha|-(m-k)} \|a_{\alpha\beta}\|_{L_\infty} \|D^\beta u\|_{L_p} \\
&\le C\lambda_m^{|\alpha|-(m-k)} \lambda_m^{|\beta|-(k+m)} \|u\|_{W^{k+m}_{p,\lambda}} \\
&\le C\lambda_m^{|\alpha|+|\beta|-2m} \|u\|_{W^{k+m}_{p,\lambda}}.
\end{aligned}$$

For $|\alpha| > m-k$ we take α^0 so that $\alpha^0 \le \alpha$ and $|\alpha^0| = m-k$. We note that $|\alpha - \alpha^0| = |\alpha| + k - m$ and $a_{\alpha\beta} \in W^{|\alpha|+k-m}_\infty(\Omega)$, and that

$$D^\alpha(a_{\alpha\beta}D^\beta u) = \sum_{\gamma \le \alpha-\alpha^0} \binom{\alpha-\alpha^0}{\gamma} D^{\alpha^0}\left(a^{(\gamma)}_{\alpha\beta} D^{\alpha-\alpha^0-\gamma+\beta} u\right),$$

which follows by Lemma 3.3. Here we used the symbol given in (2.9). Then we have

$$\begin{aligned}
\|D^\alpha(a_{\alpha\beta}D^\beta u)\|_{W^{k-m}_{p,\lambda}} &\le C \sum_{\gamma \le \alpha-\alpha^0} \|a^{(\gamma)}_{\alpha\beta}\|_{L_\infty} \|D^{\alpha-\alpha^0-\gamma+\beta}u\|_{L_p} \\
&\le C \sum_{\gamma \le \alpha-\alpha^0} \lambda_m^{|\alpha-\alpha^0-\gamma+\beta|-(k+m)} \|u\|_{W^{k+m}_p} \\
&\le C\lambda_m^{|\alpha|+|\beta|-2m} \|u\|_{W^{k+m}_p}.
\end{aligned}$$

These estimates imply

$$\|(A - A_0)u\|_{W_{p,\lambda}^{k-m}} \le C_1 \lambda_m^{-1} \|u\|_{W_{p,\lambda}^{k+m}}, \quad \text{for } u \in W_{p,D}^{k+m}(\Omega) \tag{3.2}$$

with $C_1 = C_1(k, m, n, p, M_{k,A}, \Omega)$, since $A - A_0$ consists of the terms with $|\alpha| + |\beta| < 2m$. We write

$$(A - \lambda)(A_0 - \lambda)^{-1} = I_{W_p^{k-m}(\Omega)} + Q, \qquad Q = (A - A_0)(A_0 - \lambda)^{-1}.$$

By (3.1) and (3.2) the operator norm of Q is bounded by $C_0 C_1 \lambda_m^{-1}$. If $C_0 C_1 \lambda_m^{-1} \le 2^{-1}$, i.e. $|\lambda| \ge (2C_0C_1)^{2m}$, then $I_{W_p^{k-m}(\Omega)} + Q$ has an inverse which is given by the Neumann series $\sum_{N=0}^{\infty} (-Q)^N$, and the operator norm of $(I_{W_p^{k-m}} + Q)^{-1}$ is bounded by 2. Therefore $A - \lambda$ has a right inverse whose operator norm is bounded by $2C_0$. By Lemma 3.1 this right inverse is the inverse of $A - \lambda$. Summing up, if we set $R_1 = \max\{R_0, (2C_0C_1)^{2m}\}$, then $(A - \lambda)^{-1}$ exists and satisfies

$$\|(A - \lambda)^{-1}\|_{W_{p,\lambda}^{k-m} \to W_{p,\lambda}^{k+m}} \le 2C_0$$

for $\lambda \in \Sigma(R_1, \theta)$.

Case 2. Let $k \ge m$. Using (2.2), we have, for $u \in W_{p,D}^{k+m}(\Omega)$,

$$\begin{aligned}
\|D^\alpha(a_{\alpha\beta} D^\beta u)\|_{W_{p,\lambda}^{k-m}} &= \sum_{|\gamma| \le k-m} \lambda_m^{(k-m)-|\gamma|} \|D^{\alpha+\gamma}(a_{\alpha\beta} D^\beta u)\|_{L_p} \\
&\le \sum_{|\gamma| \le k-m} \sum_{\delta \le \alpha+\gamma} \binom{\alpha+\gamma}{\delta} \lambda_m^{k-m-|\gamma|} \|a_{\alpha\beta}^{(\delta)} D^{\alpha+\gamma-\delta+\beta} u\|_{L_p} \\
&\le C \sum_{|\gamma| \le k-m} \sum_{\delta \le \alpha+\gamma} \lambda_m^{k-m-|\gamma|} \lambda_m^{|\alpha+\beta+\gamma-\delta|-(k+m)} \|u\|_{W_{p,\lambda}^{k+m}} \\
&\le C \lambda_m^{|\alpha+\beta|-2m} \|u\|_{W_{p,\lambda}^{k+m}}.
\end{aligned}$$

The rest of the proof runs as in Case 1. ☐

Lemma 3.5. *The case $k > m$ in Theorem 2.1 for divergence form elliptic operators reduces to Theorem 2.4 for non-divergence form elliptic operators.*

Proof. Let $k > m$. In view of Lemma 3.4 we may assume that A has no lower-order term. Leibniz's rule gives

$$A = \sum_{|\alpha|=|\beta|=m} D^\alpha(a_{\alpha\beta} D^\beta \cdot\,) = \sum_{|\alpha|=|\beta|=m} \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} a_{\alpha\beta}^{(\gamma)} D^{\alpha-\gamma+\beta}.$$

In particular, the leading term of A as a non-divergence form operator is written as

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^{\alpha+\beta}.$$

Observe that $a_{\alpha\beta}^{(\gamma)} \in W_\infty^{|\alpha|+k-m-|\gamma|}(\Omega) \subset W_\infty^{k-m}(\Omega)$ for $|\alpha| = m$ and $\gamma \le \alpha$. Also note that Ω is a uniformly $C^{k,1}$ domain, i.e. a uniformly $C^{(k-m)+m,1}$ domain. Thus A satisfies conditions (HN1), $(\mathrm{HN2})_{k-m}$, $(\mathrm{HN3})_{k-m}$ as a non-divergence form operator. Therefore we can apply Theorem 2.4 with k replaced by $k-m$. $\square$

4. Proof of Theorem 2.1 for the half space.

In this section we will prove Theorem 2.1 when Ω is the half space

$$\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\, x_n > 0\}.$$

To this end we prepare two lemmas. For $h \in \mathbb{R}$ and $1 \le j < n$ we define the translation $\tau_{j,h}$ and the difference operator $\Delta_{j,h}$ by

$$(\tau_{j,h} f)(x) = f(x + he_j), \qquad \Delta_{j,h} f = \tau_{j,h} f - f,$$

where e_j is the unit vector whose jth entry is 1. We note that

$$\Delta_{j,h}(fg) = (\Delta_{j,h} f) \cdot (\tau_{j,h} g) + f \cdot (\Delta_{j,h} g). \tag{4.1}$$

Lemma 4.1. *Let $\Omega = \mathbb{R}^n_+$, $|\lambda| \ge 1$, $h \in \mathbb{R} \setminus \{0\}$ and $1 \le j < n$.*

(i) *Let $f \in L_p(\Omega)$ and $\alpha \in \mathbb{N}^n_0$. Then*

$$h^{-1} \Delta_{j,h} D^\alpha f = \mathrm{i} D^\alpha D_j \left(\int_0^1 f(x + \theta h e_j)\, d\theta \right).$$

(ii) *If $f \in W^{k-m}_p(\Omega)$ with $k \ge 1$, then*

$$\left\| h^{-1} \Delta_{j,h} f \right\|_{W^{(k-1)-m}_{p,\lambda}(\Omega)} \le \|f\|_{W^{k-m}_{p,\lambda}(\Omega)}, \tag{4.2}$$

$$\|f\|_{W^{(k-1)-m}_{p,\lambda}(\Omega)} \le \lambda_m^{-1} \|f\|_{W^{k-m}_{p,\lambda}(\Omega)}. \tag{4.3}$$

(iii) *Let $u \in W^{k-1}_p(\Omega)$ with $1 \le k \le m$. If u satisfies*

$$\|h^{-1} \Delta_{j,h} u\|_{W^{k-1}_{p,\lambda}(\Omega)} \le M$$

with some $M > 0$, then $D_j u \in W^{k-1}_p(\Omega)$ and $\|D_j u\|_{W^{k-1}_{p,\lambda}(\Omega)} \le M$.

Proof. It is easy to see (i) if $f \in C^\infty_0(\Omega)$. The density of $C^\infty_0(\Omega)$ in $L_p(\Omega)$ gives (i) for the general case by taking the limit in distributional sense.

To show (ii) for $1 \le k \le m$ we write $f = \sum_{|\gamma| \le m-k} D^\gamma f_\gamma$ with $f_\gamma \in L_p(\Omega)$. Then by (i) we have

$$h^{-1} \Delta_{j,h} f = \sum_{|\gamma| \le m-l} D^\gamma D_j g_\gamma, \quad g_\gamma(x) = \mathrm{i} \int_0^1 f_\gamma(x + \theta h e_j)\, d\theta.$$

Hence

$$\|h^{-1}\Delta_{j,h}f\|_{W^{(k-1)-m}_{p,\lambda}} \le \sum_{|\gamma|\le m-k} \lambda_m^{|\gamma|+1+(k-1)-m}\|g_\gamma\|_{L_p} \le \sum_{|\gamma|\le m-k} \lambda_m^{|\gamma|+k-m}\|f_\gamma\|_{L_p}.$$

Taking the infimum, we get (4.2). Inequality (4.3) follows by the definition of λ-norm. The case $k \ge m+1$ can be handled similarly.

We can obtain (iii) by adjusting the proof of [**11**, Theorem 9.1.1]. □

LEMMA 4.2. *Let $\Omega = \mathbb{R}^n_+$, $|\lambda| \ge 1$, $m, k \in \mathbb{N}$ and $0 < k \le m$. Suppose that $v \in L_p(\Omega)$ satisfies $D_j^k v \in L_p(\Omega)$ for $1 \le j < n$ and $D_n^m v \in W_p^{k-m}(\Omega)$. Then we have $D_n^k v \in L_p(\Omega)$ and*

$$\|D_n^k v\|_{L_p(\Omega)} \le C\left(\sum_{j=1}^{n-1}\|D_j^k v\|_{L_p(\Omega)} + \|D_n^m v\|_{W^{k-m}_{p,\lambda}(\Omega)} + \lambda_m^k\|v\|_{L_p(\Omega)}\right)$$

with $C = C(k,m,n,p)$.

PROOF. This lemma corresponds to [**1**, Lemma 9.3] and [**6**, Lemma 17.2] which deal with the case $p = 2$. We follow the method which is based on Muramatu's integral formula [**19**] and was suggested by Muramatu [**20**]. We need only modify the argument given in the proof of [**17**, Lemma 10.7], paying attention to the λ-norm. Muramatu's integral formula is written as

$$v = \sum_{j=1}^{n-1}\int_0^R t^k (K_j)_t * D_j^k v\,\frac{dt}{t} + \int_0^R t^m (K_n)_t * D_n^m v\,\frac{dt}{t} + \varphi_R * v \tag{4.4}$$

for $R > 0$ with suitable functions $K_j \in C_0^\infty(\mathbb{R}^n)$ with $j = 1, \dots, n$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$, where K_t is defined by $K_t(x) = t^{-n}K(x/t)$ for a function K and $t > 0$. Applying D_n^k to (4.4) and substituting $D_n^m v = \sum_{|\gamma|\le m-k} D^\gamma f_\gamma$ with $f_\gamma \in L_p$, we have

$$\begin{aligned} D_n^k v = &\sum_{j=1}^{n-1}\int_0^R \bigl(K_j^{(ke_n)}\bigr)_t * D_j^k v\,\frac{dt}{t} + \sum_{|\gamma|\le m-k}\int_0^R t^{m-k-|\gamma|}\bigl(K_n^{(ke_n+\gamma)}\bigr)_t * f_\gamma\,\frac{dt}{t} \\ &+ R^{-k}\bigl(\varphi^{(ke_n)}\bigr)_R * v. \end{aligned}$$

Setting $R = \lambda_m^{-1}$ and using the L_p bounded theorem for Muramatu's formula, we get $D_n^k v \in L_p$ and

$$\|D_n^k v\|_{L_p} \le C\sum_{j=1}^{n-1}\|D_j^k v\|_{L_p} + C\sum_{|\gamma|\le m-k}\lambda_m^{|\gamma|-(m-k)}\|f_\gamma\|_{L_p} + C\lambda_m^k\|v\|_{L_p},$$

which gives the desired inequality. □

PROOF OF THEOREM 2.1 FOR THE HALF SPACE. We prove Theorem 2.1 by induction on k. As stated in Remark 2.2, we know that Theorem 2.1 has been already proved for $k = 0$. In view of Lemmas 3.4 and 3.5 we may assume that A has no lower-order term, and that $1 \le k \le m$.

Let $\Omega = \mathbb{R}^n_+$, $f \in W_p^{k-m}(\Omega)$ and $|\lambda| \geq 1$. We write A_k for A if A is considered as an operator from $W_{p,D}^{k+m}(\Omega)$ to $W_p^{k-m}(\Omega)$. In view of Lemma 3.1 we need only prove that if the operator $A_{k-1} - \lambda$ is invertible and satisfies

$$\|(A_{k-1} - \lambda)^{-1}\|_{W_{p,\lambda}^{(k-1)-m}(\Omega) \to W_{p,\lambda}^{(k-1)+m}(\Omega)} \leq C, \tag{4.5}$$

then the equation $(A - \lambda)u = f$ has a solution u satisfying

$$u \in W_{p,D}^{k+m}(\Omega) \quad \text{and} \quad \|u\|_{W_{p,\lambda}^{k+m}} \leq C\|f\|_{W_{p,\lambda}^{k-m}}. \tag{4.6}$$

Setting $u = (A_{k-1} - \lambda)^{-1} f$, which belongs to $W_{p,D}^{(k-1)+m}(\Omega)$, we will show (4.6) in three steps.

Step 1. We will show that

$$D_j u \in W_p^{(k-1)+m}(\Omega) \quad \text{and} \quad \|D_j u\|_{W_{p,\lambda}^{(k-1)+m}} \leq C\|f\|_{W_{p,\lambda}^{k-m}}, \quad \text{for } 1 \leq j < n. \tag{4.7}$$

We simply write Δ_h and τ_h for $\Delta_{j,h}$ and $\tau_{j,h}$, respectively. Applying Δ_h to $(A-\lambda)u = f$ with the help of (4.1), we have

$$(A - \lambda)(\Delta_h u) = \Delta_h f - \sum_{|\alpha|=|\beta|=m} D^\alpha((\Delta_h a_{\alpha\beta})\tau_h D^\beta u). \tag{4.8}$$

For each α with $|\alpha| = m$ we choose γ so that

$$\gamma \leq \alpha, \quad |\gamma| = k-1,$$

and rewrite (4.8) as

$$(A - \lambda)(\Delta_h u) = \Delta_h f - \sum_{|\alpha|=|\beta|=m} \sum_{\gamma' \leq \gamma} \binom{\gamma}{\gamma'} D^{\alpha-\gamma}\left((\Delta_h a_{\alpha\beta}^{(\gamma-\gamma')})\tau_h D^{\beta+\gamma'} u\right).$$

Since $\Delta_h u \in W_{p,D}^m(\Omega)$, (4.5) gives

$$\begin{aligned}
&\|h^{-1}\Delta_h u\|_{W_{p,\lambda}^{(k-1)+m}} \\
&\quad \leq C\|h^{-1}\Delta_h f\|_{W_{p,\lambda}^{(k-1)-m}} + C \sum_{|\alpha|=|\beta|=m} \sum_{\gamma' \leq \gamma} \|a_{\alpha\beta}^{(\gamma-\gamma'+e_j)}\|_{L_\infty} \|D^{\beta+\gamma'} u\|_{L_p}.
\end{aligned}$$

Since $\lambda_m^{(k-1)+m-|\beta+\gamma'|}\|D^{\beta+\gamma'}u\|_{L_p} \leq \|u\|_{W_{p,\lambda}^{(k-1)+m}} \leq C\|f\|_{W_{p,\lambda}^{(k-1)-m}}$ by (4.5), we have by Lemma 4.1 (ii)

$$\begin{aligned}
\|h^{-1}\Delta_h u\|_{W_{p,\lambda}^{(k-1)+m}} &\leq C\|f\|_{W_{p,\lambda}^{k-m}} + C \sum_{|\alpha|=|\beta|=m} \sum_{\gamma' \leq \gamma} \lambda_m^{|\beta+\gamma'|-(k-1+m)} \|f\|_{W_{p,\lambda}^{(k-1)-m}} \\
&\leq C\|f\|_{W_{p,\lambda}^{k-m}}.
\end{aligned}$$

Then Lemma 4.1 (iii) yields (4.7).

Step 2. Set

$$
\begin{aligned}
&b = a_{me_n, me_n}, \quad v = bD_n^m u, \\
&\mathcal{N} = \{(\alpha, \beta) \in \mathbb{N}_0^n \times \mathbb{N}_0^n : |\alpha| = |\beta| = m\} \setminus \{(me_n, me_n)\}.
\end{aligned}
$$

We will show that

$$
v \in W_p^k(\Omega) \quad \text{and} \quad \|v\|_{W_{p,\lambda}^k} \leq C\|f\|_{W_{p,\lambda}^{k-m}}. \tag{4.9}
$$

By $u \in W_{p,D}^{(k-1)+m}(\Omega)$ and Lemma 4.1 (ii) we know that $v \in L_p(\Omega)$ and

$$
\|v\|_{L_p} \leq C\|D_n^m u\|_{L_p} \leq C\lambda_m^{-(k-1)}\|f\|_{W_{p,\lambda}^{(k-1)-m}} \leq C\lambda_m^{-k}\|f\|_{W_{p,\lambda}^{k-m}}. \tag{4.10}
$$

Let $1 \leq j < n$. We write

$$
D_j^k v = b^{(ke_j)} D_n^m u + \sum_{l=1}^{k} \binom{k}{l} b^{((k-l)e_j)} D_n^m D_j^{l-1}(D_j u).
$$

Using (4.7) and Lemma 4.1 (ii), we find that $D_j^k v \in L_p(\Omega)$ and

$$
\|D_j^k v\|_{L_p} \leq C\|D_n^m u\|_{L_p} + C\|D_j u\|_{W_{p,\lambda}^{(k-1)+m}} \leq C\|f\|_{W_{p,\lambda}^{k-m}}. \tag{4.11}
$$

We now consider $D_n^k v$. For each pair of α and β with $(\alpha, \beta) \in \mathcal{N}$, we choose γ and j with $1 \leq j < n$ so that

$$
\gamma \leq \alpha, \quad |\gamma| = k, \quad e_j \leq \beta + \gamma,
$$

and rewrite $(A - \lambda)u = f$ as

$$
D_n^m v = f + \lambda u - \sum_{(\alpha,\beta)\in\mathcal{N}} \sum_{\gamma' \leq \gamma} \binom{\gamma}{\gamma'} D^{\alpha-\gamma}\left(\alpha_{\alpha\beta}^{(\gamma-\gamma')} D^{\beta+\gamma'} u\right). \tag{4.12}
$$

If $\gamma' = \gamma$, then we get $D^{\beta+\gamma'}u \in L_p$ by $e_j \leq \beta + \gamma'$ and (4.7). If $\gamma' < \gamma$, then (4.5) gives $D^{\beta+\gamma'}u \in L_p$ since $|\beta + \gamma'| \leq (k-1) + m$. Therefore we have $D_n^m v \in W_p^{k-m}(\Omega)$ and

$$
\begin{aligned}
\|D_n^m v\|_{W_{p,\lambda}^{k-m}} &\leq \|f\|_{W_{p,\lambda}^{k-m}} + \lambda_m^{2m}\lambda_m^{k-m}\|u\|_{L_p} + C\sum_{(\alpha,\beta)\in\mathcal{N}} \sum_{\gamma' \leq \gamma} \|D^{\beta+\gamma'}u\|_{L_p} \\
&\leq \|f\|_{W_{p,\lambda}^{k-m}} + \lambda_m\|f\|_{W_{p,\lambda}^{(k-1)-m}} \\
&\qquad + C\sum_{j=1}^{n-1}\|D_j u\|_{W_{p,\lambda}^{(k-1)+m}} + C\|u\|_{W_{p,\lambda}^{(k-1)+m}} \\
&\leq C\|f\|_{W_{p,\lambda}^{k-m}}.
\end{aligned} \tag{4.13}
$$

We can now apply Lemma 4.2 to v with (4.10), (4.11), (4.12) and (4.13). Then $D_n^k v \in L_p(\Omega)$ and

$$\|D_n^k v\|_{L_p} \le C\|f\|_{W^{k-m}_{p,\lambda}} + C\lambda_m^k \|D_n^m u\|_{L_p} \le C\|f\|_{W^{k-m}_{p,\lambda}} + C\lambda_m \|f\|_{W^{(k-1)-m}_{p,\lambda}}$$
$$\le C\|f\|_{W^{k-m}_{p,\lambda}}.$$

The interpolation inequality gives (4.9).

Step 3. It follows from (4.7) that

$$D_j^{k+m} u \in L_p(\Omega) \quad \text{and} \quad \|D_j^{k+m} u\|_{L_p} \le C\|f\|_{W^{k-m}_{p,\lambda}} \tag{4.14}$$

for $1 \le j < n$. Since $b \ge \delta_A$ by (HD1), we also get (4.14) for $j = n$ by (4.9) and $D_n^m u = b^{-1} v$. By (4.5) and Lemma 4.1 (ii) we have

$$\lambda_m^{k+m} \|u\|_{L_p} \le C\lambda_m \|f\|_{W^{(k-1)-m}_{p,\lambda}} \le C\|f\|_{W^{k-m}_{p,\lambda}}.$$

Therefore the interpolation inequality yields (4.6). □

5. Proof of Theorem 2.1 for a special $C^{k,1}$ domain.

Recall that we write a point x in $\mathbb{R}^n$ as $x = (x', x_n) = (x_1, \dots, x_n)$.

Definition 5.1. Let $k \in \mathbb{N}$. We say that Ω is a special $C^{k,1}$ domain if there exists a function $\phi_0 \in C^1(\mathbb{R}^{n-1})$ such that

$$\Omega = \{x \in \mathbb{R}^n : x_n > \phi_0(x')\}, \tag{5.1}$$

and that $\partial_j \phi_0 \in W^k_\infty(\mathbb{R}^{n-1})$ for $1 \le j \le n-1$.

Throughout this section we assume that Ω and ϕ_0 are as in Definition 5.1, and that Ω is written in the form of (5.1).

5.1. C^∞ diffeomorphism on a special domain.

In order to reduce the problem to the half space $\mathbb{R}^n_+$, we construct a suitable C^∞ map $\mathbb{R}^n_+ \to \Omega$. The arguments here are parallel to those in [**16**, Section 3], which treated the case $k = 0$. The idea of constructing a C^∞ map goes back to Gagliardo [**7**].

We take a function $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ satisfying

$$\operatorname{supp}\varphi \subset \{x' \in \mathbb{R}^{n-1} : |x'| < 1\}, \qquad \int_{\mathbb{R}^{n-1}} \varphi(x')\,dx' = 1,$$

and set $\varphi_{x_n}(x') = x_n^{1-n}\varphi(x'/x_n)$. We define $\phi(x) = \phi(x', x_n)$ for $x' \in \mathbb{R}^{n-1}$ and $x_n \ge 0$ by $\phi(x', 0) = \phi_0(x')$ and

$$\phi(x', x_n) = \varphi_{x_n} * \phi_0(x') = x_n^{1-n} \int_{\mathbb{R}^{n-1}} \varphi(x_n^{-1}(x' - z'))\phi_0(z')\,dz', \quad \text{for } x_n > 0.$$

Clearly $\phi \in C^\infty(\mathbb{R}^n_+)$.

Lemma 5.2. *Let $k \in \mathbb{N}$. There exists $C = C(\alpha, k, \Omega)$ such that*

$$|\partial^\alpha \phi(x)| \le C, \quad \text{for } 1 \le |\alpha| \le k+1.$$
$$|\partial^\alpha \phi(x)| \le C x_n^{-|\alpha|+k+1}, \quad \text{for } |\alpha| > k+1.$$

Proof. Simple calculations show that there exist functions $K_{\alpha\beta} \in C_0^\infty(\mathbb{R}^{n-1})$ such that

$$\partial^\alpha \phi(x) = \sum_{|\beta|=|\alpha|} (K_{\alpha\beta})_{x_n} * \partial^\beta \phi_0(x'), \quad \text{for } 1 \le |\alpha| \le k+1,$$
$$\partial^\alpha \phi(x) = x_n^{-|\alpha|+k+1} \sum_{|\beta|=k+1} (K_{\alpha\beta})_{x_n} * \partial^\beta \phi_0(x'), \quad \text{for } |\alpha| > k+1,$$

where $\beta \in \mathbb{N}_0^{n-1}$. The lemma follows from these formulas. □

We define a C^∞ map $\Phi = (\Phi_1, \dots, \Phi_n) : \mathbb{R}_+^n \to \Omega$ by $y = \Phi(x)$ with

$$y' = x', \quad y_n = x_n + \phi(x', \kappa x_n), \tag{5.2}$$

where the constant κ will be specified shortly. We note that

$$\begin{aligned} &\partial_i \Phi_j(x) = \delta_{ij} \quad (1 \le j < n,\ 1 \le i \le n), \\ &\partial_i \Phi_n(x) = \partial_i \phi(x', \kappa x_n) \quad (1 \le i < n), \qquad \partial_n \Phi_n(x) = 1 + \kappa \partial_n \phi(x', \kappa x_n), \end{aligned} \tag{5.3}$$

and that $\det \Phi'(x) = \partial_n \Phi_n(x)$, where $\Phi'(x)$ denotes the Jacobi matrix of Φ. We take $\kappa > 0$ so that $\kappa \|\partial_n \phi\|_{L_\infty(\mathbb{R}_+^n)} \le 2^{-1}$. Then $\Phi : \mathbb{R}_+^n \to \Omega$ is a diffeomorphism. We denote by $\Psi = (\Psi_1, \dots, \Psi_n)$ the inverse function of Φ. Let Φ^* and Ψ^* be the pullbacks by Φ and Ψ, respectively.

Lemma 5.3. *Let $k \in \mathbb{N}$. Let Φ be the C^∞ map defined by (5.2) that corresponds to a special $C^{k,1}$ domain Ω. Let $x \in \mathbb{R}_+^n$ and $y \in \Omega$ with $y = \Phi(x)$, and let $|\alpha| \ge 1$. Then there exist functions $b_{\alpha\beta} \in C^\infty(\mathbb{R}_+^n)$ and $c_{\alpha\beta} \in C^\infty(\mathbb{R}_+^n)$ such that $\Phi^* D_y^\alpha \Psi^*$ is written as*

$$\Phi^* D_y^\alpha \Psi^* = \sum_{1 \le |\beta| \le |\alpha|} b_{\alpha\beta} D_x^\beta, \qquad \Phi^* D_y^\alpha \Psi^* = \sum_{|\beta| \le |\alpha|} D_x^\beta (c_{\alpha\beta}\, \cdot\,), \tag{5.4}$$

and that for every $\gamma \in \mathbb{N}_0^n$ the derivatives $b_{\alpha\beta}^{(\gamma)} = D^\gamma b_{\alpha\beta}$ and $c_{\alpha\beta}^{(\gamma)} = D^\gamma c_{\alpha\beta}$ are written as

$$b_{\alpha\beta}^{(\gamma)}(x) = \sum_\tau x_n^{-\tau} b_{\alpha\beta\gamma\tau}(x), \qquad c_{\alpha\beta}^{(\gamma)}(x) = \sum_\tau x_n^{-\tau} c_{\alpha\beta\gamma\tau}(x)$$

with $b_{\alpha\beta\gamma\tau} \in L_\infty(\mathbb{R}_+^n)$, $c_{\alpha\beta\gamma\tau} \in L_\infty(\mathbb{R}_+^n)$ satisfying $\|b_{\alpha\beta\gamma\tau}\|_{L_\infty(\mathbb{R}_+^n)} \le C(\gamma, k, m, n, \Omega)$ and $\|c_{\alpha\beta\gamma\tau}\|_{L_\infty(\mathbb{R}_+^n)} \le C(\gamma, k, m, n, \Omega)$, where the sums are taken over integers τ satisfying

$$0 \le \tau \le \max\{|\alpha| - |\beta| + |\gamma| - k, 0\}.$$

Proof. We first consider the statement for $b_{\alpha\beta}$. Let $y = \Phi(x)$. By repeated use of the identity $\Phi^* D_{y_j} \Psi^* = \sum_{l=1}^n (\partial \Psi_l / \partial y_j) D_{x_l}$ we find that the first identity of (5.4) holds with the coefficients $b_{\alpha\beta}$ written in the form

$$b_{\alpha\beta}(x) = \sum \text{const} \times \Psi_{j_1}^{(\delta^1)}(y) \cdots \Psi_{j_l}^{(\delta^l)}(y),$$

where the sum is taken over $j_1, \dots, j_l \in \{1, \dots, n\}$ and $\delta^1, \dots, \delta^l$ such that $|\delta^i| \geq 1$ with $1 \leq i \leq l$ and

$$(|\delta^1| - 1) + \cdots + (|\delta^l| - 1) = |\alpha| - |\beta|. \tag{5.5}$$

Differentiating $\Psi'(y) = \Phi'(x)^{-1}$ in y repeatedly, and using Cramer's formula with $\det \Phi'(x) = \partial_n \Phi_n(x)$, we have

$$b_{\alpha\beta}(x) = \sum \text{const} \times \frac{\Phi_{j_1}^{(\delta^1)}(x) \cdots \Phi_{j_l}^{(\delta^l)}(x)}{\{\partial_n \Phi_n(x)\}^h}, \tag{5.6}$$

where the sum is taken over $h \in \mathbb{N}_0$, $j_1, \dots, j_l \in \{1, \dots, n\}$, and $\delta^1, \dots, \delta^l$ satisfying $|\delta^i| \geq 1$ with $1 \leq i \leq l$ and (5.5). Using Lemma 5.2 and (5.3), and noting $\partial_n \Phi_n(x) \geq 2^{-1}$, we know that the terms in (5.6) are written in the form $x_n^{-\tau} B(x)$, where $B \in L_\infty(\mathbb{R}_+^n)$ and $\tau \in \mathbb{N}_0$ with

$$0 \leq \tau \leq \sum_{i=1}^{l} \max\{(|\delta^i| - 1) - k, 0\}.$$

Using the inequality

$$\max\{t - k, 0\} + \max\{s - k, 0\} \leq \max\{t + s - k, 0\}, \quad \text{for } t \geq 0 \text{ and } s \geq 0,$$

which can be easily proved by considering four cases according to the signs of $t - k$ and $s - k$, we have

$$0 \leq \tau \leq \max\left\{\sum_{i=1}^{l}(|\delta^i| - 1) - k, 0\right\} = \max\{|\alpha| - |\beta| - k, 0\}.$$

We can also claim the statement for the derivative $b_{\alpha\beta}^{(\gamma)}$ by observing that $b_{\alpha\beta}^{(\gamma)}$ is written in the form of the right-hand side of (5.6), and replacing $|\alpha| - |\beta|$ by $|\alpha| - |\beta| + |\gamma|$ in (5.5).

The statement for $c_{\alpha\beta}$ can be shown by observing that a variant of Leibniz's rule gives

$$b_{\alpha\beta} D^\beta = \sum_{\gamma \leq \beta} (-1)^{|\beta - \gamma|} \binom{\beta}{\gamma} D^\gamma \left(b_{\alpha\beta}^{(\beta - \gamma)} \cdot\right),$$

and reducing the problem to the properties of $b_{\alpha\beta}$. □

5.2. Isomorphisms between Sobolev spaces.

LEMMA 5.4. *Let $k \in \mathbb{N}$ and $|\lambda| \geq 1$. Let Φ be the C^∞ map defined by (5.2) that corresponds to a special $C^{k,1}$ domain Ω. Then the map Φ^* induces an isomorphism from $W_{p,D}^{k+m}(\Omega)$ to $W_{p,D}^{k+m}(\mathbb{R}_+^n)$, and satisfies*

$$C^{-1}\|v\|_{W^{k+m}_{p,\lambda}(\Omega)} \le \|\Phi^* v\|_{W^{k+m}_{p,\lambda}(\mathbb{R}^n_+)} \le C\|v\|_{W^{k+m}_{p,\lambda}(\Omega)}, \quad \textit{for } v \in W^{k+m}_{p,D}(\Omega)$$

with $C = C(k,m,n,p,\Omega)$.

Proof. In [**16**, Lemma 4.3] we showed that $\Phi^* : W^m_{p,D}(\Omega) \to W^m_{p,D}(\mathbb{R}^n_+)$ is an isomorphism, using the inequality

$$\|x_n^{-l} f\|_{L_p(\mathbb{R}^n_+)} \le C(n,k,p)\|D_n^l f\|_{L_p(\mathbb{R}^n_+)}, \text{ for } f \in C_0^\infty(\mathbb{R}^n_+),\, l \in \mathbb{N}_0,\, 0 \le l \le m,$$

which follows by applying the Hardy-type inequality to the Taylor formula divided by x_n^{-l}:

$$x_n^{-l} f(x', x_n) = \int_0^{x_n} \frac{\mathrm{i}^l (x_n - t)^{l-1}}{(l-1)!\, x_n^l} D_n^l f(x', t)\, dt.$$

Therefore, we know that if $u \in W^m_{p,D}(\mathbb{R}^n_+)$ and $\tau + |\beta| \le m$ with $\tau \in \mathbb{N}_0$, then $x_n^{-\tau} D^\beta u \in L_p(\mathbb{R}^n_+)$ and

$$\|x_n^{-\tau} D^\beta u\|_{L_p(\mathbb{R}^n_+)} \le C\|D_n^\tau D^\beta u\|_{L_p(\mathbb{R}^n_+)}. \tag{5.7}$$

Let $u \in W^{k+m}_{p,D}(\mathbb{R}^n_+)$ and set $v(y) = u(x)$ with $y = \Phi(x)$. Then $v \in W^m_{p,D}(\Omega)$ since Φ^* is an isomorphism from $W^m_{p,D}(\Omega)$ onto $W^m_{p,D}(\mathbb{R}^n_+)$. To show that $v \in W^{k+m}_p(\Omega)$ we write, by Lemma 5.3,

$$D_y^\alpha v = \sum_{|\beta| \le |\alpha|} \sum_\tau x_n^{-\tau} b_{\alpha\beta\tau} D_x^\beta u$$

with $b_{\alpha\beta\tau} \in L_\infty(\mathbb{R}^n_+)$ for $|\alpha| \le k+m$, where τ ranges over

$$0 \le \tau \le \max\{|\alpha| - |\beta| - k, 0\}. \tag{5.8}$$

If $|\alpha| - |\beta| - k \le 0$, then (5.8) implies $\tau = 0$ and hence

$$\|x_n^{-\tau} b_{\alpha\beta\tau} D^\beta u\|_{L_p} \le C\|D^\beta u\|_{L_p} \le C\lambda_m^{|\beta| - (k+m)} \|u\|_{W^{k+m}_{p,\lambda}}.$$

If $|\alpha| - |\beta| - k > 0$, then (5.8) implies $\tau \le |\alpha| - |\beta| - k$ and hence $\tau + |\beta| \le |\alpha| - k \le m$. This combined with (5.7) gives

$$\begin{aligned}\|x_n^{-\tau} b_{\alpha\beta\tau} D^\beta u\|_{L_p} &\le C\|D_n^\tau D^\beta u\|_{L_p} \le C\lambda_m^{\tau + |\beta| - (k+m)} \|u\|_{W^{k+m}_{p,\lambda}} \\ &\le C\lambda_m^{|\alpha| - k - (k+m)} \|u\|_{W^{k+m}_{p,\lambda}}.\end{aligned}$$

Therefore $v \in W^{k+m}_p(\Omega)$ and

$$\begin{aligned}\|v\|_{W^{k+m}_{p,\lambda}} &\le C \sum_{|\alpha| \le k+m} \sum_{|\beta| \le |\alpha|} \lambda_m^{(k+m) - |\alpha|} \left(\lambda_m^{|\beta| - (k+m)} + \lambda_m^{|\alpha| - 2k - m}\right) \|u\|_{W^{k+m}_{p,\lambda}} \\ &\le C\|u\|_{W^{k+m}_{p,\lambda}},\end{aligned}$$

which gives the first inequality of the lemma.

In the same way we can show that $v \in W^{k+m}_{p,D}(\Omega)$ implies $u \in W^{k+m}_{p,D}(\mathbb{R}^n_+)$ and that $\|u\|_{W^{k+m}_{p,\lambda}} \le C\|v\|_{W^{k+m}_{p,\lambda}}$, using the properties of $\Psi^* D^\alpha_x \Phi^*$ which correspond to those of $\Phi^* D^\alpha_y \Psi^*$ stated in Lemma 5.3. $\square$

Lemma 5.5. *Let $k \in \mathbb{N}$, $1 \le k \le m$ and $|\lambda| \ge 1$. Let Φ be the C^∞ map defined by* (5.2) *that corresponds to a special $C^{k,1}$ domain Ω. Then the map Φ^* induces an isomorphism from $W^{k-m}_p(\Omega)$ to $W^{k-m}_p(\mathbb{R}^n_+)$, and satisfies*

$$C^{-1}\|v\|_{W^{k-m}_{p,\lambda}(\Omega)} \le \|\Phi^* v\|_{W^{k-m}_{p,\lambda}(\mathbb{R}^n_+)} \le C\|v\|_{W^{k-m}_{p,\lambda}(\Omega)}, \quad \textit{for } v \in W^{k-m}_p(\Omega)$$

with $C = C(k,m,n,p,\Omega)$.

Proof. The proof is based on [**16**, Lemma 4.4], which states that the operator

$$T_l f(x', x_n) = \int_{x_n}^\infty \frac{\mathrm{i}^{-l}(t-x_n)^{l-1}}{(l-1)!\, t^l} f(x',t)\,dt, \quad \text{for } f \in L_p(\mathbb{R}^n_+),\ l \in \mathbb{N} \tag{5.9}$$

is a bounded operator in $L_p(\mathbb{R}^n_+)$ and satisfies $D^l_n(T_l f) = x_n^{-l} f$.

Let $y = \Phi(x)$ with $x \in \mathbb{R}^n_+$ and $y \in \Omega$. Write $v \in W^{k-m}_p(\Omega)$ as

$$v = \sum_{|\alpha| \le m-k} D^\alpha_y v_\alpha, \quad v_\alpha \in L_p(\Omega),$$

and set $u_\alpha = \Phi^* v_\alpha \in L_p(\mathbb{R}^n_+)$. By Lemma 5.3

$$\Phi^* v = \sum_{|\alpha| \le m-k} \sum_{|\beta| \le |\alpha|} \sum_\tau D^\beta_x (x_n^{-\tau} c_{\alpha\beta\tau} u_\alpha)$$

with $c_{\alpha\beta\tau} \in L_\infty(\mathbb{R}^n_+)$, where τ ranges over $0 \le \tau \le \max\{|\alpha| - |\beta| - k, 0\}$.

If $|\alpha| - |\beta| - k \le 0$, then $\tau = 0$ and

$$\|D^\beta(x_n^{-\tau} c_{\alpha\beta\tau} u_\alpha)\|_{W^{k-m}_{p,\lambda}} \le C\lambda_m^{|\beta|+k-m}\|u_\alpha\|_{L_p} \le C\lambda_m^{|\alpha|+k-m}\|v_\alpha\|_{L_p}.$$

If $|\alpha| - |\beta| - k > 0$, then $\tau \le |\alpha| - |\beta| - k$, which implies $\tau + |\beta| \le m - k$. So (5.9) gives

$$D^\beta(x_n^{-\tau} c_{\alpha\beta\tau} u_\alpha) = D^\beta D^\tau_n (T_\tau(c_{\alpha\beta\tau} u_\alpha))$$

and

$$\|D^\beta(x_n^{-\tau} c_{\alpha\beta\tau} u_\alpha)\|_{W^{k-m}_{p,\lambda}} \le \lambda_m^{|\beta|+\tau+k-m}\|T_\tau(c_{\alpha\beta\tau} u_\alpha)\|_{L_p} \le \lambda_m^{|\alpha|-m}\|v_\alpha\|_{L_p}.$$

Hence

$$\|\Phi^* v\|_{W^{k-m}_{p,\lambda}} \le \sum_{|\alpha| \le m-k} \lambda_m^{|\alpha|+k-m}\|v_\alpha\|_{L_p}.$$

Taking the infimum of the right-hand side, we get the second inequality of the lemma.

The first inequality can be proved similarly. □

5.3. Elliptic operators in a special $C^{k,1}$ domain.

We will prove Theorem 2.1 for a special $C^{k,1}$ domain. Let Ω be as in Definition 5.1. In view of Lemma 3.4 we may assume that A has no lower-order term. In addition, by Remark 2.2 and Lemma 3.5 we need only consider the case $1 \le k \le m$. We set $\tilde{A} = \Phi^* A \Psi^*$ and $\tilde{a}_{\alpha\beta} = a_{\alpha\beta} \circ \Phi$. Then $\tilde{A}$ is a differential operator on $\mathbb{R}^n_+$ and it is written as

$$\tilde{A} = A_0 + A_1$$

with

$$\begin{aligned} A_0 &= \sum_{|\alpha|=|\beta|=m} \sum_{|\alpha'|=|\beta'|=m} D^{\alpha'}(c_{\alpha\alpha'} \tilde{a}_{\alpha\beta} b_{\beta\beta'} D^{\beta'} \cdot\,), \\ A_1 &= \sum_{|\alpha|=|\beta|=m} \sum_{\substack{|\alpha'|\le m \\ |\beta'|\le m \\ |\alpha'|+|\beta'|<2m}} D^{\alpha'}(c_{\alpha\alpha'} \tilde{a}_{\alpha\beta} b_{\beta\beta'} D^{\beta'} \cdot\,) \end{aligned}$$

by Lemma 5.3. Since the principal symbol of A_0 is $a_0(\Phi(x), {}^t\Phi'(x)^{-1}\xi)$, we see that A_0 satisfies the ellipticity condition corresponding to (HD1). Since $a_{\alpha\beta} \in W^k_\infty(\Omega)$ for $|\alpha| = |\beta| = m$, we have $\tilde{a}_{\alpha\beta} \in W^k_\infty(\mathbb{R}^n_+)$ by Lemma 5.2 and (5.2). We also see by Lemma 5.3 that $c_{\alpha\alpha'}$ and $b_{\beta\beta'}$ belong to $W^k_\infty(\mathbb{R}^n_+) \cap C^\infty(\mathbb{R}^n_+)$ if $|\alpha| = |\beta| = |\alpha'| = |\beta'| = m$. Hence A_0 satisfies (HD2)$_k$. Therefore we can apply Theorem 2.1 for $\mathbb{R}^n_+$ to A_0.

On the other hand, the next lemma shows that A_1 is viewed as a perturbation.

Lemma 5.6. *Let $1 < p < \infty$, $k \in \mathbb{N}$, $1 \le k \le m$ and $|\lambda| \ge 1$. Then*

$$\|A_1 u\|_{W^{k-m}_{p,\lambda}(\mathbb{R}^n_+)} \le C\lambda_m^{-1} \|u\|_{W^{k+m}_{p,\lambda}(\mathbb{R}^n_+)}, \quad \text{for } u \in W^{k+m}_{p,D}(\mathbb{R}^n_+)$$

with $C = C(k, m, n, p, M_{k,A}, \Omega)$.

Proof. For $u \in W^{k+m}_{p,D}(\mathbb{R}^n_+)$ we wish to evaluate $A_1 u$. To this end we set

$$v = b_{\beta\beta'} D^{\beta'} u, \quad w = \tilde{a}_{\alpha\beta} v, \quad f = D^{\alpha'}(c_{\alpha\alpha'} w)$$

for α, β, α', β' with $|\alpha| = |\beta| = m$, $|\alpha'| \le m$, $|\beta'| \le m$, $|\alpha'| + |\beta'| < 2m$, and evaluate them respectively.

Step 1. For $|\gamma| \le k$ we write

$$D^\gamma v = \sum_{\gamma^0+\gamma^1=\gamma} \binom{\gamma}{\gamma^0} b^{(\gamma^0)}_{\beta\beta'} D^{\beta'+\gamma^1} u = \sum_{\gamma^0+\gamma^1=\gamma} \sum_{\tau} \binom{\gamma}{\gamma^0} b_{\beta\beta'\gamma^0\tau} x_n^{-\tau} D^{\beta'+\gamma^1} u$$

with $b_{\beta\beta'\gamma^0\tau} \in L_\infty$ by Lemma 5.3, where τ ranges over

$$0 \le \tau \le \max\{|\beta| - |\beta'| + |\gamma^0| - k, 0\}.$$

If $|\beta|-|\beta'|+|\gamma^0|-k\le 0$, then $\tau=0$ and hence

$$\|b_{\beta\beta'\gamma^0\tau}x_n^{-\tau}D^{\beta'+\gamma^1}u\|_{L_p}\le \|D^{\beta'+\gamma^1}u\|_{L_p}\le \lambda_m^{|\beta'|+|\gamma^1|-(k+m)}\|u\|_{W^{k+m}_{p,\lambda}}.$$

If $|\beta|-|\beta'|+|\gamma^0|-k>0$, then $\tau\le|\beta|-|\beta'|+|\gamma^0|-k$ and hence

$$\tau+|\beta'|+|\gamma^1|\le(|\beta|-|\beta'|+|\gamma^0|-k)+|\beta'|+|\gamma^1|=|\beta|+|\gamma|-k\le m.$$

So it follows by the Hardy-type inequality (5.7) that

$$\begin{aligned}\|b_{\beta\beta'\gamma^0\tau}x_n^{-\tau}D^{\beta'+\gamma^1}u\|_{L_p}&\le C\|D_n^\tau D^{\beta'+\gamma^1}u\|_{L_p}\le C\lambda_m^{\tau+|\beta'|+|\gamma^1|-(k+m)}\|u\|_{W^{k+m}_{p,\lambda}}\\&\le C\lambda_m^{|\beta|+|\gamma|-k-(k+m)}\|u\|_{W^{k+m}_{p,\lambda}}=C\lambda_m^{|\gamma|-2k}\|u\|_{W^{k+m}_{p,\lambda}}.\end{aligned}$$

Therefore

$$\begin{aligned}\|v\|_{W^k_{p,\lambda}}&\le C\sum_{|\gamma|\le k}\sum_{\gamma^0+\gamma^1=\gamma}\lambda_m^{k-|\gamma|}\left(\lambda_m^{|\beta'|+|\gamma^1|-(k+m)}+\lambda_m^{|\gamma|-2k}\right)\|u\|_{W^{k+m}_{p,\lambda}}\\&\le C\left(\lambda_m^{|\beta'|-m}+\lambda_m^{-k}\right)\|u\|_{W^{k+m}_{p,\lambda}}.\end{aligned}$$

Step 2. As stated prior to Lemma 5.6, the coefficients $\tilde a_{\alpha\beta}$ belong to $W^k_\infty(\mathbb{R}^n_+)$. So Lemma 3.3 gives $w\in W^k_p(\mathbb{R}^n_+)$ and

$$\|w\|_{W^k_{p,\lambda}}\le C\sum_{|\gamma|\le k}\sum_{\gamma^0+\gamma^1=\gamma}\lambda_m^{k-|\gamma|}\|\tilde a^{(\gamma^0)}_{\alpha\beta}\|_{L_\infty}\|D^{\gamma^1}v\|_{L_p}\le C\|a_{\alpha\beta}\|_{W^k_\infty}\|v\|_{W^k_{p,\lambda}}.$$

Step 3. We will show that

$$\|f\|_{W^{k-m}_{p,\lambda}}\le C\left(\lambda_m^{|\alpha'|-m}+\lambda_m^{-k}\right)\|w\|_{W^k_{p,\lambda}}. \tag{5.10}$$

Let $|\alpha'|\ge m-k$. We take α^0 and γ so that

$$\alpha'=\alpha^0+\gamma,\quad |\alpha^0|=m-k,$$

and write

$$f=\sum_{\gamma^0+\gamma^1=\gamma}\binom{\gamma}{\gamma^0}D^{\alpha^0}\left(c^{(\gamma^0)}_{\alpha\alpha'}D^{\gamma^1}w\right).$$

Since $|\alpha|-|\alpha'|+|\gamma^0|-k\le m-(|\alpha^0|+|\gamma|)+|\gamma|-k=0$, we have $c^{(\gamma^0)}_{\alpha\alpha'}\in L_\infty$ by Lemma 5.3. So

$$\|f\|_{W^{k-m}_{p,\lambda}}\le C\sum_{\gamma^0+\gamma^1=\gamma}\|D^{\gamma^1}w\|_{L_p}\le C\sum_{\gamma^0+\gamma^1=\gamma}\lambda_m^{|\gamma^1|-k}\|w\|_{W^k_{p,\lambda}}.$$

Since $|\gamma^1|-k\le|\gamma|-k=|\alpha'|-m$, we get (5.10).

Let $|\alpha'|<m-k$. Noting that $|\alpha|-|\alpha'|-k=m-k-|\alpha'|>0$, and using Lemma 5.3

and (5.9), we write

$$f = \sum_{0 \le \tau \le |\alpha| - |\alpha'| - k} D^{\alpha'} \left(x_n^{-\tau} c_{\alpha\alpha'\tau} w \right) = \sum_{0 \le \tau \le |\alpha| - |\alpha'| - k} D^{\alpha'} D_n^{\tau} \left(T_\tau (c_{\alpha\alpha'\tau} w) \right)$$

with $c_{\alpha\alpha'\tau} \in L_\infty$. Since $|\alpha'| + \tau \le m - k$, and since T_τ is a bounded operator in $L_p(\mathbb{R}^n_+)$, we have

$$\begin{aligned} \|f\|_{W^{k-m}_{p,\lambda}} &\le C \sum_{\tau} \lambda_m^{|\alpha'| + \tau + k - m} \|T_\tau (c_{\alpha\alpha'\tau} w)\|_{L_p} \le C \sum_{\tau} \lambda_m^{|\alpha'| + \tau + k - m} \|w\|_{L_p} \\ &\le C \sum_{\tau} \lambda_m^{|\alpha'| + \tau - m} \|w\|_{W^k_{p,\lambda}} \le C \lambda_m^{-k} \|w\|_{W^k_{p,\lambda}}, \end{aligned}$$

which gives (5.10).

Step 4. Combining the above estimates for v, w and f, we get

$$\begin{aligned} \|A_1 u\|_{W^{k-m}_{p,\lambda}} &\le C \sum_{|\alpha'| + |\beta'| < 2m} \left(\lambda_m^{|\alpha'| - m} + \lambda_m^{-k} \right) \left(\lambda_m^{|\beta'| - m} + \lambda_m^{-k} \right) \|u\|_{W^{k+m}_{p,\lambda}} \\ &\le C \sum_{|\alpha'| + |\beta'| < 2m} \left(\lambda_m^{|\alpha'| + |\beta'| - 2m} + \lambda_m^{-k} \right) \|u\|_{W^{k+m}_{p,\lambda}}. \end{aligned}$$

Since $|\alpha'| + |\beta'| < 2m$ and $k \ge 1$, we get the lemma. □

Proof of Theorem 2.1 for a special $C^{k,1}$ domain. We continue to assume that $1 \le k \le m$. Let $\tilde{A}$, A_0 and A_1 be as above. As already stated, we can apply the result for $\mathbb{R}^n_+$ to A_0. Therefore for a given $\theta \in (\theta_A, \pi]$ there exist $R_0 \ge 1$ and $C_0 > 0$ such that, for $\lambda \in \Sigma(R_0, \theta)$, the inverse of $A_0 - \lambda$ exists and satisfies

$$\|(A_0 - \lambda)^{-1}\|_{W^{k-m}_{p,\lambda} \to W^{k+m}_{p,\lambda}} \le C_0.$$

By Lemma 5.6 there exists C_1 such that $\|A_1\|_{W^{k+m}_{p,\lambda} \to W^{k-m}_{p,\lambda}} \le C_1 \lambda_m^{-1}$, which corresponds to (3.2). Then the same argument as in the proof of Lemma 3.4 shows that $\tilde{A} - \lambda$ has a right inverse, which we denote by $\mathcal{R}_\lambda$, if $|\lambda| \ge \max\{R_0, (2C_0C_1)^{2m}\}$.

It follows from Lemmas 5.4 and 5.5 that $\Psi^* \mathcal{R}_\lambda \Phi^*$ is a right inverse of $A - \lambda$ and that $\|\Psi^* \mathcal{R}_\lambda \Phi^*\|_{W^{k-m}_{p,\lambda} \to W^{k+m}_{p,\lambda}}$ is bounded by a constant. Therefore by invoking Lemmas 3.1, 3.2 and 3.4 we complete the proof of Theorem 2.1 for a special $C^{k,1}$ domain. □

6. Preliminaries for Theorem 2.4.

In the rest of this paper, we will prove Theorem 2.4 for a non-divergence elliptic operator $\mathcal{A}$, which is written in the form (1.3), and satisfies (HN1) and $(\mathrm{HN2})_k$ for a fixed $k \in \mathbb{N}_0$.

6.1. Some reductions for non-divergence form elliptic operators.

Lemma 6.1. *In the proof of Theorem* 2.4

(i) *if we can show the existence of a right inverse of* $\mathcal{A}-\lambda$ *and obtain the norm estimate corresponding to* (2.7), *then the right inverse is exactly the inverse*;

(ii) *it is sufficient to show Theorem* 2.4 *with the constant* R *which may depend on* k;

(iii) *we may assume that* $\mathcal{A}$ *has no lower-order term.*

Proof. To show (i) we regard $\mathcal{A}-\lambda$ as an operator from $W^{k+2m}_{p,D}(\Omega)$ to $W^k_p(\Omega)$, and suppose that $A-\lambda$ has a right inverse P_λ which satisfies the estimate corresponding to (2.7). Let B be the elliptic operator obtained by freezing the coefficients of $\mathcal{A}$ at $x=0$:

$$B=\sum_{|\alpha|\le 2m} a_\alpha(0)D^\alpha.$$

We may also suppose that $B-\lambda$ has a right inverse Q_λ. It follows from $\delta_B\ge\delta_{\mathcal{A}}$, $M_{k,B}\le M_{k,\mathcal{A}}$ and $\omega_B=0$ that the operator norm of Q_λ has the same bound as that of P_λ.

Since B is also regarded as a divergence form elliptic operator from $W^m_{p,D}(\Omega)$ to $W^{-m}_p(\Omega)$, we find by Theorem 2.1 that $B-\lambda$ is injective on $W^m_{p,D}(\Omega)$ and hence on $W^{k+2m}_{p,D}(\Omega)$. Therefore $Q_\lambda=(B-\lambda)^{-1}$. Considering a family of operators

$$B_t=t\mathcal{A}+(1-t)B$$

with parameter $t\in[0,1]$, and using the method of continuity, we can derive the injectivity of $\mathcal{A}-\lambda$ from that of $B-\lambda$. Therefore $P_\lambda=(\mathrm{A}-\lambda)^{-1}$.

We can show (ii) in the same way as Lemma 3.2.

For (iii) set

$$\mathcal{A}_0=\sum_{|\alpha|=2m} a_\alpha D^\alpha.$$

Let $u\in W^{k+2m}_{p,D}(\Omega)$. Since $a_\alpha\in W^k_\infty(\Omega)$, we have

$$\begin{aligned}
\|(\mathcal{A}-A_0)u\|_{W^k_{p,\lambda}} &\le \sum_{|\alpha|<2m}\sum_{|\beta|\le k}\lambda_m^{k-|\beta|}\|D^\beta(a_\alpha D^\alpha u)\|_{L_p}\\
&\le \sum_{|\alpha|<2m}\sum_{|\beta|\le k}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\lambda_m^{k-|\beta|}\|a_\alpha^{(\beta-\gamma)}\|_{L_\infty}\|D^{\alpha+\gamma}u\|_{L_p}\\
&\le C\sum_{|\alpha|<2m}\sum_{|\beta|\le k}\sum_{\gamma\le\beta}\lambda_m^{k-|\beta|}\lambda_m^{|\alpha|+|\gamma|-(k+2m)}\|u\|_{W^{k+2m}_{p,\lambda}}\\
&\le C\sum_{|\alpha|<2m}\lambda_m^{|\alpha|-2m}\|u\|_{W^{k+2m}_{p,\lambda}}\\
&\le C\lambda_m^{-1}\|u\|_{W^{k+2m}_{p,\lambda}}.
\end{aligned}$$

Then the rest of the proof runs as in the proof of Lemma 3.4. □

7. Proof of Theorem 2.4 for the half space.

In this section we will prove Theorem 2.4 for the half space by induction on k. The case $k = 0$ is treated in [**10**, Theorem 5.7] under the assumption that $\lim_{|x|\to\infty} a_\alpha(x)$ exist for all the coefficients a_α. This extra assumption can be removed by the partition of unity, which was used for divergence form elliptic operators in [**14**].

We consider the case $k \geq 1$. In view of Lemma 6.1 we may assume that $\mathcal{A}$ has no lower-order term:

$$\mathcal{A} = \sum_{|\alpha|=2m} a_\alpha D^\alpha.$$

Let $\Omega = \mathbb{R}^n_+$, $f \in W^k_p(\Omega)$ and $|\lambda| \geq 1$. We write $\mathcal{A}_k$ if $\mathcal{A}$ is considered as an operator from $W^{k+2m}_{p,D}(\Omega)$ to $W^k_p(\Omega)$. In the same way as for the case of divergence form elliptic operators in Section 4 we need only prove that if the operator $\mathcal{A}_{k-1} - \lambda$ is invertible and satisfies

$$\|(\mathcal{A}_{k-1} - \lambda)^{-1}\|_{W^{k-1}_{p,\lambda}(\Omega)\to W^{(k-1)+2m}_{p,\lambda}(\Omega)} \leq C, \tag{7.1}$$

then the equation $(A - \lambda)u = f$ has a solution u satisfying

$$u \in W^{k+2m}_{p,D}(\Omega) \quad \text{and} \quad \|u\|_{W^{k+2m}_{p,\lambda}(\Omega)} \leq C\|f\|_{W^k_{p,\lambda}(\Omega)}. \tag{7.2}$$

Setting $u = (\mathcal{A}_{k-1} - \lambda)^{-1} f$, which belongs to $W^{(k-1)+2m}_{p,D}(\Omega)$, we will prove (7.2) in three steps.

Step1. We will show that

$$D_j u \in W^{(k-1)+2m}_p(\Omega) \quad \text{and} \quad \|D_j u\|_{W^{(k-1)+2m}_{p,\lambda}} \leq C\|f\|_{W^k_{p,\lambda}}, \quad \text{for } 1 \leq j < n. \tag{7.3}$$

We simply write Δ_h and τ_h for the difference operator $\Delta_{j,h}$ and the translation $\tau_{j,h}$, respectively, defined in Section 4. Applying Δ_h to $(A - \lambda)u = f$, we have

$$(A - \lambda)(\Delta_h u) = \Delta_h f - \sum_{|\alpha|=2m} (\Delta_h a_\alpha)\tau_h D^\alpha u.$$

By (7.1) and Lemma 4.1 (ii) we have

$$\begin{aligned}
&\|h^{-1}\Delta_h u\|_{W^{(k-1)+2m}_{p,\lambda}} \\
&\quad\leq C\|h^{-1}\Delta_h f\|_{W^{k-1}_{p,\lambda}} \\
&\qquad + C\sum_{|\alpha|=2m}\ \sum_{|\beta|\leq k-1}\ \sum_{\gamma\leq\beta} \lambda_m^{k-1-|\beta|}\binom{\beta}{\gamma}\|h^{-1}(\Delta_h a_\alpha^{(\beta-\gamma)})\tau_h D^{\alpha+\gamma}u\|_{L_p} \\
&\quad\leq C\|f\|_{W^k_{p,\lambda}} \\
&\qquad + C\sum_{|\alpha|=2m}\ \sum_{|\beta|\leq k-1}\ \sum_{\gamma\leq\beta} \|a_\alpha^{(\beta-\gamma+e_j)}\|_{L_\infty}\lambda_m^{k-1-|\beta|}\lambda_m^{|\alpha+\gamma|-(k-1)-2m}\|u\|_{W^{(k-1)+2m}_{p,\lambda}}
\end{aligned}$$

$$\begin{aligned}&\le C\|f\|_{W^k_{p,\lambda}} + C\|f\|_{W^{k-1}_{p,\lambda}}\\ &\le C\|f\|_{W^k_{p,\lambda}}.\end{aligned}$$

By Lemma 4.1 (iii) we get (7.3).

Step 2. Set

$$b = a_{2me_n}, \quad v = bD_n^{2m}u, \quad \mathcal{N} = \{\alpha \in \mathbb{N}_0^n : |\alpha| = 2m,\, \alpha_n < 2m\}.$$

We will show that

$$v \in W_p^k(\Omega) \quad \text{and} \quad \|v\|_{W^k_{p,\lambda}} \le C\|f\|_{W^k_{p,\lambda}}. \tag{7.4}$$

Rewrite $(A-\lambda)u = f$ as

$$v = f + \lambda u - \sum_{\alpha\in\mathcal{N}} a_\alpha D^\alpha u,$$

and note that for $\alpha \in \mathcal{N}$ there exits $1 \le j < n$ such that $e_j \le \alpha$. By (7.1), (7.3) and Lemma 4.1 (ii) we find that $v \in W_p^k(\Omega)$ and

$$\begin{aligned}\|v\|_{W^k_{p,\lambda}} &\le \|f\|_{W^k_{p,\lambda}} + \lambda_m^{2m}\|u\|_{W^k_{p,\lambda}}\\ &\quad + \sum_{\alpha\in\mathcal{N}}\sum_{|\beta|\le k}\sum_{\gamma\le\beta}\lambda_m^{k-|\beta|}\binom{\beta}{\gamma}\|a_\alpha^{(\beta-\gamma)}\|_{L_\infty}\|D^{(\alpha-e_j)+\gamma}D_ju\|_{L_p}\\ &\le \|f\|_{W^k_{p,\lambda}} + \lambda_m\|u\|_{W^{(k-1)+2m}_{p,\lambda}}\\ &\quad + C\sum_{\alpha\in\mathcal{N}}\sum_{|\beta|\le k}\sum_{\gamma\le\beta}\lambda_m^{k-|\beta|}\lambda_m^{|\alpha-e_j+\gamma|-(k-1)-2m}\|D_ju\|_{W^{(k-1)+2m}_{p,\lambda}}\\ &\le C\|f\|_{W^k_{p,\lambda}} + C\lambda_m\|f\|_{W^{k-1}_{p,\lambda}} \le C\|f\|_{W^k_{p,\lambda}}.\end{aligned}$$

Thus we get (7.4).

Step 3. It follows from (7.3) that

$$D_j^{k+2m}u \in L_p(\Omega) \quad \text{and} \quad \|D_j^{k+2m}u\|_{L_p} \le C\|f\|_{W^k_{p,\lambda}} \tag{7.5}$$

for $1 \le j < n$. Since $b \ge \delta_A > 0$ by (HN1), we also get (7.5) for $j = n$ by (7.4) and $D_n^{2m}u = b^{-1}v$. By (7.1) and Lemma 4.1 (ii) we have

$$\lambda_m^{k+2m}\|u\|_{L_p} \le \lambda_m\|u\|_{W^{(k-1)+2m}_{p,\lambda}} \le C\lambda_m\|f\|_{W^{(k-1)}_{p,\lambda}} \le C\|f\|_{W^k_{p,\lambda}}.$$

Then the interpolation inequality yields (7.2). Thus we complete the proof of Theorem 2.4 for the half space.

8. Proof of Theorem 2.4 for a special $C^{k+m,1}$ domain.

In this section we will prove Theorem 2.4 for a special $C^{k+m,1}$ domain with $k \in \mathbb{N}_0$. In view of Lemma 6.1 we may assume that $\mathcal{A}$ has no lower-order term.

Let Ω be a special $C^{k+m,1}$ domain, which is written as

$$\Omega = \{x \in \mathbb{R}^n : x_n > \phi_0(x')\},$$

where $\phi_0 \in C^1(\mathbb{R}^{n-1})$ satisfies $\partial_j \phi_0 \in W^{k+m}_\infty(\mathbb{R}^{n-1})$ for $1 \le j \le n-1$. Let Φ and Ψ be as in Section 5, and set $\tilde{a}_\alpha = a_\alpha \circ \Phi$. It follows from $(\mathrm{HN})_k$ that $\tilde{a}_\alpha \in W^k_\infty(\mathbb{R}^n_+)$ for $k \in \mathbb{N}_0$, and that $\tilde{a}_\alpha$ is uniformly continuous if $k = 0$. By Lemma 5.3 with k replaced by $k+m$ we can write $\Phi^* \mathcal{A} \Psi^*$, which is an operator on $\mathbb{R}^n_+$, as

$$\Phi^* \mathcal{A} \Psi^* = A_0 + A_1$$

with

$$A_0 = \sum_{|\alpha|=2m} \sum_{|\beta|=2m} \tilde{a}_\alpha b_{\alpha\beta} D^\beta,$$
$$A_1 = \sum_{|\alpha|=2m} \sum_{|\beta|<2m} \tilde{a}_\alpha b_{\alpha\beta} D^\beta.$$

We note that $b_{\alpha\beta} \in C^\infty(\mathbb{R}^n_+)$, and that the derivatives $b^{(\gamma)}_{\alpha\beta} = D^\gamma b_{\alpha\beta}$ are written as

$$b^{(\gamma)}_{\alpha\beta}(x) = \sum_\tau x_n^{-\tau} b_{\alpha\beta\gamma\tau}(x) \tag{8.1}$$

with $b_{\alpha\beta\gamma\tau} \in L_\infty(\mathbb{R}^n_+)$ and $\|b_{\alpha\beta\gamma\tau}\|_{L_\infty(\mathbb{R}^n_+)} \le C(\gamma, k, m, n, \Omega)$, where τ ranges over

$$0 \le \tau \le \max\{|\alpha| - |\beta| + |\gamma| - k - m, 0\}. \tag{8.2}$$

In particular, $b_{\alpha\beta} \in W^{k+m}_\infty(\mathbb{R}^n_+)$ for $|\alpha| = |\beta| = 2m$.

Lemma 8.1. *Let $1 < p < \infty$, $k \in \mathbb{N}_0$, and $|\lambda| \ge 1$. Then*

$$\|A_1 u\|_{W^k_{p,\lambda}(\mathbb{R}^n_+)} \le C \lambda_m^{-1} \|u\|_{W^{k+2m}_{p,\lambda}(\mathbb{R}^n_+)}, \quad \textit{for } u \in W^{k+2m}_{p,D}(\mathbb{R}^n_+)$$

with $C = C(k, m, n, p, M_{k,\mathcal{A}}, \Omega)$.

Proof. For $u \in W^{k+2m}_{p,D}(\mathbb{R}^n_+)$ and $|\gamma| \le k$ we write

$$D^\gamma(A_1 u) = \sum_{|\alpha|=2m} \sum_{|\beta|<2m} \sum_{\gamma^0+\gamma^1+\gamma^2=\gamma} \frac{\gamma!}{\gamma^0! \gamma^1! \gamma^2!} \tilde{a}^{(\gamma^0)}_\alpha b^{(\gamma^1)}_{\alpha\beta} D^{\beta+\gamma^2} u,$$

and set

$$g = \tilde{a}^{(\gamma^0)}_\alpha b^{(\gamma^1)}_{\alpha\beta} D^{\beta+\gamma^2} u.$$

Let $|\alpha| - |\beta| + |\gamma^1| \le k + m$. Then (8.1) and (8.2) give $b^{(\gamma^1)}_{\alpha\beta} \in L_\infty$ and hence

$$\lambda_m^{k-|\gamma|} \|g\|_{L_p} \le C \lambda_m^{k-|\gamma|} \|D^{\beta+\gamma^2} u\|_{L_p} \le C \lambda_m^{k-|\gamma|} \lambda_m^{|\beta|+|\gamma^2|-(k+2m)} \|u\|_{W^{k+2m}_p}$$

$$\leq C\lambda_m^{|\beta|-2m}\|u\|_{W_p^{k+2m}} \leq C\lambda_m^{-1}\|u\|_{W_p^{k+2m}}.$$

Let $|\alpha|-|\beta|+|\gamma^1|>k+m$. Then (8.1) and (8.2) give

$$g=\sum_{\tau}\tilde{a}_{\alpha}^{(\gamma^0)}x_n^{-\tau}b_{\alpha\beta\gamma^1\tau}D^{\beta+\gamma^2}u,$$

where τ ranges over $0\leq\tau\leq|\alpha|-|\beta|+|\gamma^1|-k-m$. Noting that

$$\tau+|\beta|+|\gamma^2|\leq|\alpha|+|\gamma^1+\gamma^2|-k-m\leq m,$$

and using the Hardy-type inequality (5.7), we have

$$\begin{aligned}\lambda_m^{k-|\gamma|}\|g\|_{L_p} &\leq C\sum_{\tau}\lambda_m^{k-|\gamma|}\|D_n^{\tau}D^{\beta+\gamma^2}u\|_{L_p}\\ &\leq C\sum_{\tau}\lambda_m^{k-|\gamma|}\lambda_m^{\tau+|\beta|+|\gamma^2|-(k+2m)}\|u\|_{W_{p,\lambda}^{k+2m}}\\ &\leq C\sum_{\tau}\lambda_m^{k-|\gamma|+|\alpha|+|\gamma^1+\gamma^2|-k-m-(k+2m)}\|u\|_{W_{p,\lambda}^{k+2m}}\\ &\leq C\lambda_m^{-k-m}\|u\|_{W_{p,\lambda}^{k+2m}}\leq C\lambda_m^{-1}\|u\|_{W_{p\,\lambda}^{k+2m}}.\end{aligned}$$

From the above estimates for g we get the lemma. □

Lemma 8.2. *Let $1<p<\infty$, $k\in\mathbb{N}_0$ and $|\lambda|\geq 1$. Let Φ be the C^∞ map defined by (5.2) that corresponds to a special $C^{k+m,1}$ domain Ω. Then the map Φ^* induces an isomorphism from $W_{p,D}^{k+2m}(\Omega)$ to $W_{p,D}^{k+2m}(\mathbb{R}_+^n)$, and satisfies*

$$C^{-1}\|v\|_{W_{p,\lambda}^{k+2m}(\Omega)}\leq\|\Phi^*v\|_{W_{p,\lambda}^{k+2m}(\mathbb{R}_+^n)}\leq C\|v\|_{W_{p,\lambda}^{k+2m}(\Omega)},\quad \text{for } v\in W_{p,D}^{k+2m}(\Omega)$$

with $C=C(k,m,n,p,\Omega)$. Also, Φ^* induces an isomorphism from $W_p^k(\Omega)$ to $W_p^k(\mathbb{R}_+^n)$ and satisfies

$$C^{-1}\|v\|_{W_{p,\lambda}^{k}(\Omega)}\leq\|\Phi^*v\|_{W_{p,\lambda}^{k}(\mathbb{R}_+^n)}\leq C\|v\|_{W_{p,\lambda}^{k}(\Omega)},\quad \text{for } v\in W_{p}^{k}(\Omega)$$

with $C=C(k,m,n,p,\Omega)$.

Proof. If we replace k by $k+m$ in Lemma 5.4, then we conclude the first assertion of the lemma. The proof of the second assertion is not difficult, since Φ is sufficiently smooth, namely $\partial_j\Phi_l\in W_\infty^{k+m}(\mathbb{R}_+^n)$ for $j,l\in\{1,\dots,n\}$. □

Proof of Theorem 2.4 for a special $C^{k+m,1}$ domain. It is easy to see that A_0 satisfies the conditions corresponding to (HN1) and (HN2)$_k$. So we can apply the result for $\mathbb{R}_+^n$ to A_0. The rest of the proof runs in the same way as in the case of divergence form elliptic operators, using Lemmas 6.1, 8.1 and 8.2. □

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References

[1] S. Agmon, Lectures on Elliptic Boundary Value Problems, Amer. Math. Soc., Providence, RI, 2010.

[2] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Commun. Pure Appl. Math., **12** (1959), 623–727.

[3] R. Denk, M. Hieber and J. Prüss, $\mathcal{R}$-Boundedness, Fourier multipliers and problems of elliptic and parabolic type, Mem. Amer. Math. Soc., **166** (2003).

[4] H. Dong and D. Kim, On the L_p-solvability of higher order parabolic and elliptic systems with BMO coefficients, Arch. Ration. Mech. Anal., **199** (2011), 889–941.

[5] H. Dong and D. Kim, Higher order elliptic and parabolic systems with variably partially BMO coefficients in regular and irregular domains, J. Funct. Anal., **261** (2011), 3279–3327.

[6] A. Friedman, Partial Differential Equations, Holt, Rinehart and Winston, New York, 1969.

[7] E. Gagliardo, Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili, Rend. Sem. Mat. Univ. Padova, **27** (1957), 284–305.

[8] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, reprint of the 1998 edition, Springer-Verlag, Berlin, 2001.

[9] P. Grisvard, Elliptic Problems in Nonsmooth Domains, In: Monographs and Studies in Mathematics, **24**, Pitman, Boston, MA, 1985.

[10] H. Heck and M. Hieber, Maximal L^p-regularity for elliptic operators with VMO-coefficients, J. Evol. Equ., **3** (2003), 332–359.

[11] N. V. Krylov, Lectures on Elliptic and Parabolic Equations in Sobolev Spaces, In: Graduate Studies in Mathematics, **96**, Amer. Math. Soc., Providence, RI, 2008.

[12] V. Maz'ya, M. Mitrea and T. Shaposhnikova, The Dirichlet problem in Lipschitz domains for higher order elliptic systems with rough coefficients, J. Anal. Math., **110** (2010), 167–239.

[13] Y. Miyazaki, The L^p resolvents of elliptic operators with uniformly continuous coefficients, J. Differential Equations, **188** (2003), 555–568.

[14] Y. Miyazaki, The L^p resolvents of second-order elliptic operators of divergence form under the Dirichlet condition, J. Differential Equations, **206** (2004), 353–372.

[15] Y. Miyazaki, The L^p theory of divergence form elliptic operators under the Dirichlet condition, J. Differential Equations, **215** (2005), 320–356.

[16] Y. Miyazaki, Higher order elliptic operators of divergence form in C^1 or Lipschitz domains, J. Differential Equations, **230** (2006), 174–195.

[17] Y. Miyazaki, Introduction to L_p Sobolev spaces via Muramatu's integral formula, Milan J. Math., **85** (2017), 103–148.

[18] S. Mizohata, The Theory of Partial Differential Equations, Cambridge Univ. Press, New York, 1973.

[19] T. Muramatu, On Besov spaces and Sobolev spaces of generalized functions defined on a general region, Publ. Res. Inst. Math. Sci., **9** (1973/74), 325–396.

[20] T. Muramatu, On Sobolev spaces and Besov spaces (in Japanese), Sūgaku, **27** (1975), 142–157.

[21] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, 1970.

[22] H. Tanabe, Functional Analytic Methods for Partial Differential Equations, Marcel Dekker, New York, 1997.

[23] J. Wloka, Partial Differential Equations, Cambridge University Press, Cambridge, 1987.

Yoichi Miyazaki
Nihon University
School of Dentistry
1-8-13 Kanda-Surugadai,
Chiyoda-ku
Tokyo 101-8310, Japan
E-mail: miyazaki.yoichi86@nihon-u.ac.jp