Some infinitely generated non-projective modules over path algebras and their extensions under Martin's axiom

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Abstract. In this paper it is proved that, when Q is a quiver that admits some closure, for any algebraically closed field K and any finite dimensional K-linear representation $\mathcal{X}$ of Q, if $\operatorname{Ext}^1_{KQ}(\mathcal{X}, KQ) = 0$ then $\mathcal{X}$ is projective. In contrast, we show that if Q is a specific quiver of the type above, then there is an infinitely generated non-projective KQ-module M_{ω_1} such that, when K is a countable field, $\mathbf{MA}_{\aleph_1}$ (Martin's axiom for $\aleph_1$ many dense sets, which is a combinatorial axiom in set theory) implies that $\operatorname{Ext}^1_{KQ}(M_{\omega_1}, KQ) = 0$.

Introduction.

Bound quiver algebras of finite connected quivers strongly influence research on representation theory of Artin algebras. Gabriel found a correspondence between finite dimensional algebras and linear representations of bound quivers ([**13**], [**3**, II]), so it follows that studying modules of finite dimensional algebras is reduced to studying modules of bound quiver algebras. In this paper, we concentrate on the study of path algebras, which is one type of bound quiver algebras.

Nakayama conjecture, Tachikawa conjecture, and Auslander–Reiten conjecture are some major research projects in ring theory that present sufficient conditions for projective modules. Related to this, it has been known the following result for Artin algebras: (∗) *For any finite dimensional algebra*[1] Λ *over an algebraically closed field of finite global dimension and any finitely generated* Λ-*module* M, *if* $\operatorname{Ext}^{\geq 1}_{\Lambda}(M, \Lambda) = 0$, *then* M *is projective* (Theorem 1.2). A typical example of finite dimensional algebras is a path algebra of a finite acyclic quiver over an algebraically closed field. Since any path algebra of a quiver over an algebraically closed field is hereditary (even when the quiver is not finite, see e.g. [**14**, Subsection 8.2]), that is, its global dimension is not larger than 1, the following assertion also holds: *For any algebraically closed field* K, *any finite acyclic quiver* Q *and any finitely generated* KQ-*module* M, *if* $\operatorname{Ext}^1_{KQ}(M, KQ) = 0$, *then* M *is projective*. In Theorem 1.10, it is shown that the above assertion is also true for finite dimensional K-linear representations of *some* infinite quivers, one of which is the following quiver of A_∞ type, denoted by $A_\infty^{\leftarrow}$:

$$0 \longleftarrow 1 \longleftarrow 2 \longleftarrow \cdots \longleftarrow n \longleftarrow n+1 \longleftarrow \cdots.$$

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[1] Any finite dimensional algebra is Artin.

Let $A_\infty^\rightarrow$ be the opposite quiver of $A_\infty^\leftarrow$: the set of the vertices of $A_\infty^\rightarrow$ is the same as $A_\infty^\leftarrow$, but the arrows are reversed, that is, each arrow in $A_\infty^\rightarrow$ is of the form $n \longrightarrow n+1$. The category $\mathrm{Mod}KA_\infty^\rightarrow$ of $KA_\infty^\rightarrow$-modules is known to be somewhat simple, actually is pure semisimple, that is, every $KA_\infty^\rightarrow$-module is a direct sum of finitely presented $KA_\infty^\rightarrow$-modules [**5**, Section 2]. In [**7**, Theorem 3.1], a characterization of projective representations of $A_\infty^\rightarrow$ over a unital ring is given. The category of representations of $A_\infty^\leftarrow$ also has been studied, for example, a characterization of projective representations of $A_\infty^\leftarrow$ over a field is presented in [**4**, p.102, Example], and this is extended to such representations over a unital ring in [**8**, Section 3].

In this paper, we consider some specific quivers Q, as specified in Theorem 2.11, one of which is the quiver $A_\infty^\leftarrow$ to construct an infinitely generated *non-projective KQ-module*, which is denoted by M_{ω_1}. To analyze such a KQ-module M_{ω_1}, $\mathbf{MA}_{\aleph_1}$ (Martin's axiom for $\aleph_1$ many dense sets) is used. $\mathbf{MA}_{\aleph_1}$ is a combinatorial axiom of set theory that can neither be proved nor refuted from Zermelo–Fraenkel axiomatic set theory $\mathbf{ZFC}$ with the axiom of choice [**18**], [**21**]. $\mathbf{MA}_{\aleph_1}$ is applied in many areas of mathematics to show that some mathematical statements cannot be refuted from $\mathbf{ZFC}$ (see e.g. [**12**]). One of such examples is Shelah's solution of Whitehead problem [**20**]. Our main result states that *if K is a countable field and $\mathbf{MA}_{\aleph_1}$ holds, then $\mathrm{Ext}^1_{KQ}(M_{\omega_1}, KQ) = 0$* (Theorems 2.4, 2.9 and Theorem 2.11). Therefore, *under $\mathbf{MA}_{\aleph_1}$ and the assumption that K is a countable field, the above assertion (∗) fails for quivers Q as in Theorem 2.11 and infinitely generated KQ-modules.* Trlifaj's construction is used to build such infinitely generated KQ-modules, which will be presented in Subsection 1.4.

This paper is intended to be fairly self contained, but we will assume some basic knowledge about ordinals (see e.g. [**11**, II.1, II.4] and [**17**, I.7, III.6]). Section 1 provides necessary knowledge, which includes some facts on path algebras and set theory. Section 2 provides the proof of the main result of this paper.

1. Preliminaries.

Throughout this paper, a ring R means a ring with enough idempotents (hence R may not be unital), and an R-module means right R-module. For a ring R, $\mathrm{Mod}R$ denotes the category of the R-modules, and $\mathrm{mod}R$ denotes the category of the finitely generated R-modules. For an R-module M and a subset X of M, $\langle X \rangle_R$ denotes the R-submodule of the module M generated by X. For an R-module M and R-submodules N_i, $i \in I$, of M, $\sum_{i \in I} N_i$ denotes the R-submodule that is the R-linear span of the set $\bigcup_{i \in I} N_i$.

We follow the notation of outer direct sums in [**11**, I.2]. For a family $\{M_i : i \in I\}$ of modules, the product module $\prod_{i \in I} M_i$ is the module whose underling set is the set of functions f with domain I such that for each $i \in I$, $f(i)$ belongs to the set M_i, and the operations are defined coordinate-wise. For a member f of the product $\prod_{i \in I} M_i$, the *support* $\mathrm{supp}(f)$ of f is defined by the set

$$\{i \in I : f(i) \neq 0_{M_i}\}.$$

The outer direct sum $\bigoplus_{i \in I} M_i$ of a family $\{M_i : i \in I\}$ of modules is the submodule of

the product module $\prod_{i \in I} M_i$ which consists of the members of the set $\prod_{i \in I} M_i$ whose supports are finite.

We adopt ordinals as the von Neumann ordinals, that is, an ordinal α means the set of ordinals less than α. So for ordinals α and β, α is less than β if and only if $\alpha \in \beta$. ω is the set of all finite ordinals (non-negative integers), ω_1 is the least uncountable ordinal (which is a cardinal). **Lim** denotes the class of all limit ordinals.

The following is a well-known equivalence about projectivity.

THEOREM 1.1 (e.g. [**1**, Propositions 17.1, 17.2]). *For a ring R (with enough idempotents) and an R-module P, the following statements are equivalent.*

(1) *For every R-epimorphism f from an R-module M onto an R-module N and every R-homomorphism g from P into N, there exists an R-homomorphism h from P into M such that $g = f \circ h$.*

(2) *Every R-epimorphism from an R-module onto P splits, that is, it is right invertible.*

(3) *The functor $\mathrm{Hom}_R(P, -)$ within the category $\mathrm{Mod}R$ is exact, that is, for every R-module M, $\mathrm{Ext}^1_R(P, M) = 0$.*

(4) *P is isomorphic to a direct summand of a free R-module.*

1.1. Path algebras and quiver representations.

This subsection is devoted to the basics of representation theory of rings. The readers can skip this subsection if they are familiar with path algebras and quiver representations. Quivers, path algebras, and linear representations of quivers are some basic concepts of representation theory of Artin algebras. Our notation and terminology are fairly standard, see e.g. [**2**], [**3**]. In the next paragraphs, we refer to [**3**, Chapters II–III] for definitions, notation, and terminology.

A *quiver* denotes a directed graph. Any quiver Q consists of a pair of a set Q_0 of vertices and a set Q_1 of arrows. Each arrow a is equipped with its source $s(a)$ and its target $t(a)$. A quiver $Q = (Q_0, Q_1)$ is called *finite* if both Q_0 and Q_1 are finite sets. A *path* of the quiver Q is a finite sequence $a_0 a_1 \cdots a_n$ of arrows of the quiver Q such that, for each i with $0 \leq i < n$, the target of the arrow a_i coincides with the source of the arrow a_{i+1}. The path $a_0 a_1 \cdots a_n$ has length $n + 1$. For each vertex v of the quiver Q, we agree to associate with it a path of length 0, called the *trivial path* or the *stationary path at the vertex v*, which is denoted by e_v. A *cycle* is a non-trivial path whose source and target coincide. A quiver is called *acyclic* if there are no cycles in the quiver. For a quiver Q, $\overline{Q}$ denotes the underlying graph of Q that is obtained from Q by forgetting the orientation of the arrows, and a quiver Q is called *connected* if the graph $\overline{Q}$ is a connected graph. For a field K and a quiver Q, the *path algebra KQ of the quiver Q over the field K* is the K-algebra whose underlying set is the K-vector space whose basis is the set of all the paths of the quiver Q (which includes all the stationary paths) such that the product of two paths $a_0 a_1 \cdots a_{m-1}$ and $b_0 b_1 \cdots b_{n-1}$ is defined as follows:

$$a_0 a_1 \cdots a_{m-1} \cdot b_0 b_1 \cdots b_{n-1} = \begin{cases} a_0 a_1 \cdots a_{m-1} b_0 b_1 \cdots b_{n-1} & \text{if } t(a_{m-1}) = s(b_0) \\ 0_{KQ} & \text{otherwise.} \end{cases}$$

The product of basic elements is extended to arbitrary elements of KQ by distributivity. We note that, for any field K and a quiver Q with Q_0 finite, KQ also has an identity, which is of the form $\sum_{v \in Q_0} e_v$. However, for any quiver Q with infinitely many vertices, KQ does not have an identity. We recall that any path algebra KQ of a quiver Q over an algebraically closed field K is hereditary even when a quiver Q is not finite (see e.g. [**14**, Subsection 8.2]), that is, its global dimension is not larger than 1.

For a quiver $Q = (Q_0, Q_1)$ and a field K, a K-*linear representation* of the quiver Q is a system $\mathcal{X} = \langle \mathcal{X}_v, \mathcal{X}_a : v \in Q_0, a \in Q_1 \rangle$ such that, for each vertex $v \in Q_0$, $\mathcal{X}_v$ is a K-vector space and, for each arrow $a \in Q_1$, $\mathcal{X}_a$ is a K-linear map from the K-vector space $\mathcal{X}_{s(a)}$ into the K-vector space $\mathcal{X}_{t(a)}$. A K-linear representation is called *finite dimensional* if each $\mathcal{X}_v$, $v \in Q_0$, is a finite dimensional K-vector space. For two K-linear representations $\mathcal{X}$ and $\mathcal{Y}$, a *morphism from $\mathcal{X}$ into $\mathcal{Y}$* is a tuple $\varphi = \langle \varphi_v : v \in Q_0 \rangle$ such that, for each $v \in Q_0$, φ_v is a K-linear map from the K-vector space $\mathcal{X}_v$ into the K-vector space $\mathcal{Y}_v$ and, for each arrow $a \in Q_1$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{X}_{s(a)} & \xrightarrow{\;\mathcal{X}_a\;} & \mathcal{X}_{t(a)} \\
{\scriptstyle \varphi_{s(a)}}\downarrow & & \downarrow{\scriptstyle \varphi_{t(a)}} \\
\mathcal{Y}_{s(a)} & \xrightarrow{\;\mathcal{Y}_a\;} & \mathcal{Y}_{t(a)}
\end{array} \quad .
$$

$\mathrm{Rep}_K Q$ denotes the category of the K-linear representations of a quiver Q over a field K, and $\mathrm{rep}_K Q$ denotes the category of the finite dimensional K-linear representations $\mathcal{X}$ of Q over K. In [**3**], these are defined for finite quivers, however, we adopt them for all quivers.

There is a correspondence between KQ-modules and K-linear representation of Q (see e.g. [**3**, Theorem III. 1.6]). For a KQ-module M, define the K-linear representation $F(M)$ of Q such that, for each $v \in Q_0$, $F(M)_v := Me_v = \{me_v : m \in M\}$, and, for each $a \in Q_1$, $F(M)_a$ is the K-homomorphism from $F(M)_{s(a)}$ into $F(M)_{t(a)}$ such that, for each $x \in F(M)_{s(a)}$, $F(M)_a(x) = xa$. For a K-linear representation $\mathcal{X}$ of Q, define the KQ-module $G(\mathcal{X})$ whose underlying set is the direct sum $\bigoplus_{v \in Q_0} \mathcal{X}_v$ such that, for each element $m = \sum_{v \in Q_0} x_v$ of $\bigoplus_{v \in Q_0} \mathcal{X}_v$ (in this notation, for all but finitely many $v \in Q_0$, x_v is the zero of $\mathcal{X}_v$), $w \in Q_0$ and $a \in Q_1$, $me_w := x_w$ and $ma := \mathcal{X}_a(me_{s(a)})$, and the product by any arbitrary element of KQ is extended by distributivity. We notice that, for every K-linear representation $\mathcal{X}$ of Q, $F(G(\mathcal{X})) = \mathcal{X}$, and, for every KQ-module M, if $M = \sum_{m \in M} mKQ$ then $G(F(M)) = M$. Therefore, if Q is a finite connected quiver, then the category $\mathrm{Mod}KQ$ is equivalent to the category $\mathrm{Rep}_K Q$ by the functors F and G [**3**, Theorem III. 1.6] and, for any finite acyclic quiver Q, $\mathrm{mod}KQ$ is equivalent to $\mathrm{rep}_K Q$ [**3**, Theorem III. 1.7].

1.2. Path algebras of infinite quivers.

Throughout this subsection, F denotes the canonical functor from $\mathrm{Mod}KQ$ to $\mathrm{Rep}_K Q$, and G denotes the canonical functor from $\mathrm{Rep}_K Q$ to $\mathrm{Mod}KQ$, for a field K and a quiver Q, as in the last paragraph of the previous subsection. The following theorem gives a sufficient condition for finitely generated projective modules over a finite dimensional algebra. For example, the following is mentioned without proof in the proof

of [**19**, Theorem 4.7].

THEOREM 1.2 (folklore).   *Suppose that Λ is a finite dimensional algebra over an algebraically closed field K with finite global dimension. Then for any finitely generated Λ-module M, if $\operatorname{Ext}_\Lambda^{\geq 1}(M, \Lambda) = 0$ then M is projective.*

PROOF.   Suppose that M is a finitely generated Λ-module and $\operatorname{Ext}_\Lambda^{\geq 1}(M, \Lambda) = 0$. The point of the proof is to show that, *for any finitely generated projective Λ-module P,* $\operatorname{Ext}_\Lambda^{\geq 1}(M, P) = 0$. To see this, let P' be a complementary direct summand of P such that $P \oplus P'$ is isomorphic to a direct sum Λ^n of finitely many copies of Λ. Then for each integer $k \geq 1$,

$$\operatorname{Ext}_\Lambda^k(M, \Lambda^n) = \left(\operatorname{Ext}_\Lambda^k(M, \Lambda)\right)^n = 0,$$

and

$$\operatorname{Ext}_\Lambda^k(M, \Lambda^n) = \operatorname{Ext}_\Lambda^k(M, P) \oplus \operatorname{Ext}_\Lambda^k(M, P').$$

Therefore, $\operatorname{Ext}_\Lambda^k(M, P) = 0$. Hence $\operatorname{Ext}_\Lambda^{\geq 1}(M, P) = 0$.

Let d be the projective dimension pd$M$ of M. Since Λ has finite global dimension, $0 \leq d < \infty$. Assume, towards a contradiction, that $d \geq 1$. Let the sequence

$$0 \longrightarrow P_d \xrightarrow{f_d} P_{d-1} \xrightarrow{f_{d-1}} \cdots \quad \cdots \quad \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$
$$\Omega^{d-1}M$$
$$0$$

be a projective resolution of M of length d such that each P_i is finitely generated, where $\Omega^{d-1}M$ is the $(d-1)$-th syzygy of M. Since pd$M = d$, the projective dimension of the Λ-module $\Omega^{d-1}M$ is exactly 1, in particular, $\Omega^{d-1}M$ is not projective. Applying $\operatorname{Hom}_\Lambda(-, P_d)$ to the short exact sequence $0 \to P_d \to P_{d-1} \to \Omega^{d-1}M \to 0$, we obtain the following exact sequence

$$0 \twoheadrightarrow \operatorname{Hom}_\Lambda(\Omega^{d-1}M, P_d) \twoheadrightarrow$$
$$\operatorname{Hom}_\Lambda(P_{d-1}, P_d) \xrightarrow{\operatorname{Hom}_\Lambda(f_d, P_d)} \operatorname{Hom}_\Lambda(P_d, P_d) \twoheadrightarrow \operatorname{Ext}_\Lambda^1(\Omega^{d-1}M, P_d) .$$

Since P_d is a finitely generated projective Λ-module,

$$\operatorname{Ext}_\Lambda^1(\Omega^{d-1}M, P_d) = \operatorname{Ext}_\Lambda^d(M, P_d) = 0.$$

Thus $\operatorname{Hom}_\Lambda(f_d, P_d)$ is surjective. So there exists a homomorphism g_d from P_{d-1} into P_d such that the composition $g_d \circ f_d$ is the identity on P_d. Therefore the short exact sequence $0 \to P_d \to P_{d-1} \to \Omega^{d-1}M \to 0$ splits, and hence $\Omega^{d-1}M$ is a direct summand of the projective module P_{d-1}, which is a contradiction. $\square$

DEFINITION 1.3.   For a ring R and a subclass $\mathfrak{M}$ of Mod$R$, we define the assertion

$\mathbf{P}_R(\mathfrak{M})$ that means that, for any $M \in \mathfrak{M}$, if $\operatorname{Ext}_R^{\geq 1}(M, R) = 0$ then M is projective.

REMARK 1.4.   For any Noetherian ring Λ with finite global dimension and any finitely generated Λ-module M, there is a projective precover of M which is finitely generated. So the above proof works for any Noetherian ring of finite global dimension. Therefore, for any Noetherian ring Λ of finite global dimension, $\mathbf{P}_\Lambda(\operatorname{mod}\Lambda)$.

It is known that any path algebra KQ over an algebraically closed field K, even when the quiver Q is not finite, is hereditary, that is, its global dimension is not larger than 1 (see e.g. [**14**, Subsection 8.2]). So any path algebra over an algebraically closed field is an algebra with finite global dimension. Therefore Theorem 1.2 implies the following.

COROLLARY 1.5.   *Suppose that K is an algebraically closed field and Q is a finite acyclic quiver. Then $\mathbf{P}_{KQ}(\operatorname{mod}KQ)$. In particular, for any finitely generated KQ-module M, if $\operatorname{Ext}_{KQ}^1(M, KQ) = 0$ then M is projective.*

REMARK 1.6.   A finite quiver of the form
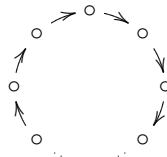
is called a cyclic quiver. Since the path algebra of a cyclic quiver Q over an algebraically closed field K is Noetherian with finite global dimension, it follows from Remark 1.4 that $\mathbf{P}_{KQ}(\operatorname{mod}KQ)$.

We can extend the above corollary to *some* infinite quivers. To introduce such infinite quivers explicitly, we define the following notions.

DEFINITION 1.7.   1. A quiver $P = (P_0, P_1)$ is called a *subquiver* of a quiver $Q = (Q_0, Q_1)$ if P_0 and P_1 are subsets of Q_0 and Q_1 respectively (hence, for any $a \in P_1$, $s(a)$ and $t(a)$ belong to P_0).

2. For a quiver Q, a subquiver P of Q, a field K and a K-linear representation $\mathcal{X}$ of Q, the K-linear representation $\mathcal{X} \upharpoonright P$ of the quiver P is called the *restricted representation of $\mathcal{X}$ by P* if for every $v \in P_0$ and $a \in P_1$, $(\mathcal{X} \upharpoonright P)_v = \mathcal{X}_v$ and $(\mathcal{X} \upharpoonright P)_a = \mathcal{X}_a$.

3. For a quiver $Q = (Q_0, Q_1)$ and a subset P_0' of Q_0, the *closure of P_0' under Q* is the subquiver $\overline{P_0'}^Q = \left(\left(\overline{P_0'}^Q \right)_0, \left(\overline{P_0'}^Q \right)_1 \right)$ of the quiver Q such that

$$\left(\overline{P_0'}^Q \right)_0 := \{ v \in Q_0 : \text{ there exists a path from a member of } P_0'$$
$$\text{to the vertex } v \text{ through the quiver } Q \}$$

and

$$\left(\overline{P_0'}^Q\right)_1 := \left\{a \in Q_1 : s(a) \in \left(\overline{P_0'}^Q\right)_0\right\}.$$

A subquiver P of a quiver Q is called a *closed subquiver* of Q if P is a closure of some subset of Q_0 under Q. A subquiver P of a quiver Q is called a *finite closed subquiver* of Q if P is a closed subquiver of Q and it is also a finite quiver.

PROPOSITION 1.8. *Suppose that K is a field, Q is a quiver, $\mathcal{X}$ is a K-linear representation of Q such that $\mathrm{Ext}_{KQ}^1(G(\mathcal{X}), KQ) = 0$, and P is a closed subquiver of Q. Then $\mathrm{Ext}_{KP}^1(G(\mathcal{X} \restriction P), KP) = 0$.*

PROOF. Let S be the functor from the category $\mathrm{Rep}_K Q$ into the category $\mathrm{Rep}_K P$ such that, for each K-linear representation $\mathcal{Y}$ of Q, $S(\mathcal{Y}) := \mathcal{Y} \restriction P$, and let T be the functor from $\mathrm{Rep}_K P$ into $\mathrm{Rep}_K Q$ such that, for each K-linear representation $\mathcal{Z}$ of P, $T(\mathcal{Z})$ is the K-linear representation of Q such that $T(\mathcal{Z})_v := \mathcal{Z}_v$ for every $v \in P_0$, $T(\mathcal{Z})_v$ is the trivial K-vector space for every $v \in Q_0 \setminus P_0$, $T(\mathcal{Z})_a := \mathcal{Z}_a$ for every $a \in P_1$, and $T(\mathcal{Z})_a$ is the unique K-linear map from the trivial K-vector space $T(\mathcal{Z})_{s(a)}$ into the K-vector space $T(\mathcal{Z})_{t(a)}$ for every $a \in Q_1 \setminus P_1$. We notice that both S and T are exact functors, and the functor T is a right adjoint of the restricted functor S. Moreover, since P is a closed subquiver of Q, T is well-defined, that is, the above $T(\mathcal{Z})$ is certainly a K-representation of Q. We also notice that the composition $S \circ T$ is the identity functor over $\mathrm{Rep}_K P$, and

$$KP = \bigoplus_{v \in P_0} e_v KP = \bigoplus_{v \in P_0} e_v KQ,$$

which implies that $T(F(KP))$ is a direct summand of $F(KQ)$. Note that $G(T(F(KP)))$ is just KP as a KQ-module, so it follows from our assumption that $\mathrm{Ext}_{KQ}^1(G(\mathcal{X}), KP) = 0$.

$\mathrm{Ext}_{KQ}^1(G(\mathcal{X}), KP) = 0$ means that any short exact sequence of K-linear representations of Q of the form

$$0 \longrightarrow T(F(KP)) \longrightarrow \mathcal{E} \overset{\varphi}{\longrightarrow} \mathcal{X} \longrightarrow 0$$

splits. We note that in such a short exact sequence, for any $v \in Q_0 \setminus P_0$, $\mathcal{E}_v = \mathcal{X}_v$ and φ_v is an automorphism of $\mathcal{X}_v$ (because $T(F(KP))_v$ is the trivial K-vector space). So, for any short exact sequence of K-linear representations of P of the form

$$L' : \quad 0 \longrightarrow F(KP) \longrightarrow \mathcal{E}' \longrightarrow \mathcal{X} \restriction P \longrightarrow 0\,,$$

there exists a short exact sequence of K-linear representations of Q of the form

$$L : \quad 0 \longrightarrow T(F(KP)) \longrightarrow \mathcal{E} \longrightarrow \mathcal{X} \longrightarrow 0$$

such that $S(L) = L'$. Therefore, it follows that any short exact sequence of K-linear representations of P of the form

$$0 \longrightarrow F(KP) \longrightarrow \mathcal{E}' \longrightarrow \mathcal{X} \upharpoonright P \longrightarrow 0$$

splits, which is equivalent to say that $\mathrm{Ext}^1_{KP}(G(\mathcal{X} \upharpoonright P), KP) = 0$. $\hfill \square$

Proposition 1.9. *Suppose that K is a field, Q is an acyclic quiver that contains the quiver*

$$v_0 \longleftarrow v_1 \longleftarrow v_2 \longleftarrow \cdots \longleftarrow v_n \longleftarrow v_{n+1} \longleftarrow \cdots$$

as a subquiver, and $\mathcal{X}$ is a finite dimensional K-linear representation of Q such that, for each $n \in \omega$, $\mathcal{X}_{v_n} \neq \{0_K\}$, and $\mathcal{X} \upharpoonright \overline{\{v_n\}}^Q$ is a direct sum of finitely many copies of the corresponding K-linear representation $F(e_{v_n}KQ)$ of $e_{v_n}KQ$. Then $\mathrm{Ext}^1_{KQ}(G(\mathcal{X}), KQ) \neq 0$.

In [**9**, Definition 3.5], some type of quivers is defined, which is called rooted. It is true that a quiver Q is rooted if and only if Q does not contain the quiver $A_\infty^\leftarrow$ as a subquiver [**9**, Proposition 3.6]. So a quiver that satisfies the assumption of the proposition is not rooted. For example, let Q be the following quiver

$$
\begin{array}{ccccccccc}
w_0 & & w_1 & & w_2 & & w_3 & & w_4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
v_0 & \longleftarrow & v_1 & \longleftarrow & v_2 & \longleftarrow & v_3 & \longleftarrow & v_4 & \longleftarrow & \cdots
\end{array}
$$

and let $P := \overline{\{v_n : n \in \omega\}}^Q$. Then the quiver Q is a non-rooted quiver, and the quiver P is different from Q, in fact, $P_0 = \{v_n : n \in \omega\}$. For another example, let Q be the following quiver

$$
\begin{array}{ccccccccc}
 & & w_1 & & w_2 & & w_3 & & w_4 \\
 & & \uparrow & & \uparrow\uparrow & & \uparrow & & \uparrow\uparrow \\
v_0 & \longleftarrow & v_1 & \longleftarrow & v_2 & \longleftarrow & v_3 & \longleftarrow & v_4 & \longleftarrow & \cdots
\end{array}
$$

and let $P := \overline{\{v_n : n \in \omega\}}^Q$. Then the quiver P is equal to the quiver Q in this case, and $P_0 \setminus \{v_n : n \in \omega\} = \{w_n : n \in \omega\}$.

Proof. Let $P := \overline{\{v_n : n \in \omega\}}^Q$. By Proposition 1.8, it suffices to show that $\mathrm{Ext}^1_{KP}(G(\mathcal{X} \upharpoonright P), KP) \neq 0$. Since

$$KP = \left(\bigoplus_{n \in \omega} e_{v_n} KP\right) \oplus \left(\bigoplus_{v \in P_0 \setminus \{v_n : n \in \omega\}} e_v KP\right)$$

and

$$\mathrm{Ext}^1_{KQ}(M, N_0 \oplus N_1) = \mathrm{Ext}^1_{KQ}(M, N_0) \oplus \mathrm{Ext}^1_{KQ}(M, N_1)$$

in general, it suffices to show that $\mathrm{Ext}^1_{KP}(G(\mathcal{X} \upharpoonright P), \bigoplus_{n \in \omega} e_{v_n} KP) \neq 0$.

For each $n \in \omega$, let

$$d_n := \max_{i \in \omega} |\{p : \ p \text{ is a path from } v_i \text{ to } v_n \text{ on } P\}|,$$

when such maximum exists as a finite number, or $d_n := \infty$ otherwise. Notice that, for each $n \in \omega$, the dimension of the K-vector space $F(e_{v_n}KQ)_{v_0}$ is equal the number of paths from v_n to v_0. So, if infinitely many d_n were larger than 1, then the dimension of $\mathcal{X}_{v_0}$ had to be infinite. Thus, for all but finitely many $n \in \omega$, $d_n = 1$. Therefore, without loss of generality we may assume that, for every $n \in \omega$, $d_n = 1$. Hence there is a $d \in \omega \setminus \{0\}$ such that, for any $n \in \omega$,

$$\mathcal{X} \upharpoonright \overline{\{v_n\}}^P = \bigoplus_d F(e_{v_n}KP),$$

where the last term is the outer direct sum of d many copies of $F(e_{v_n}KP)$. (Notice that d is the dimension of $\mathcal{X}_{v_n}$.) For each $n \in \omega$, let a_n be the unique arrow from v_{n+1} to v_n, and, for each $v \in P_0$, let

$$m(v) := \min\{m \in \omega : \text{ there is a path from } v_m \text{ to } v\}.$$

Then, for any $v \in P_0$ and $n \geq m(v)$, any path from v_n to v is of the form $a_{n-1} \cdots a_{m(v)}p'$, for some path p' from $v_{m(v)}$ to v in P. Thus

$$\mathcal{X} \upharpoonright P = \bigoplus_d \mathcal{X}^0,$$

where $\mathcal{X}^0$ is the K-linear representation of P such that: for each $v \in P_0$, $\mathcal{X}^0_v$ is the K-vector space whose basis is the set of all paths from $v_{m(v)}$ to v; for each $n \in \omega$, $\mathcal{X}^0_{a_n}$ is the K-linear map from $\mathcal{X}^0_{v_{n+1}}$ onto $\mathcal{X}^0_{v_n}$ such that $\mathcal{X}^0_{a_n}(e_{v_{n+1}}) = e_{v_n}$; and, for each $a \in P_1 \setminus \{a_n : n \in \omega\}$, $\mathcal{X}^0_a$ is the K-linear map from $\mathcal{X}^0_{s(a)}$ onto $\mathcal{X}^0_{t(a)}$ such that, for each path p from $v_{m(s(a))}$ to $s(a)$, $\mathcal{X}^0_a(p) := pa$. Since

$$\mathrm{Ext}^1_{KP}\left(\bigoplus_{i \in I} M_i, KP\right) = \prod_{i \in I} \mathrm{Ext}^1_{KP}(M_i, KP)$$

in general, it suffices to show that $\mathrm{Ext}^1_{KP}(G(\mathcal{X}^0), \bigoplus_{n \in \omega} e_{v_n}KP) \neq 0$.

To see this, let π be the canonical KP-epimorphism from $\bigoplus_{n \in \omega} e_{v_n}KP$ onto $G(\mathcal{X}^0)$ such that, for each $n \in \omega$, $\pi(e_{v_n}) := e_{v_n}$, and, for each $v \in P_0$ and each path p in P ending in v of the form $p = a_{n-1} \cdots a_{m(v)}p'$, $\pi(p) := p'$. Then $\mathrm{Ker}(\pi)$ is the KP-submodule of $\bigoplus_{n \in \omega} e_{v_n}KP$ which is generated by the set

$$\{e_{v_m} - a_n \cdots a_m : m, n \in \omega, m \leq n\}.$$

Applying $\mathrm{Hom}_{KP}(-, KP)$ to the exact sequence

$$0 \rightarrow \mathrm{Ker}(\pi) \xrightarrow{\mathrm{id}_{\mathrm{Ker}(\pi)}} \bigoplus_{n \in \omega} e_{v_n}KP \rightarrow G(\mathcal{X}^0) \rightarrow 0 \ ,$$

we obtain the exact sequence

$$0 \twoheadrightarrow \operatorname{Hom}_{KP}(G(\mathcal{X}^0), KP) \rightarrowtail$$

$$\operatorname{Hom}_{KP}\left(\bigoplus_{n \in \omega} e_{v_n} KP, KP\right) \xrightarrow{\operatorname{Hom}_{KP}(\mathrm{id}_{\mathrm{Ker}(\pi)}, KP)} \operatorname{Hom}_{KP}(\mathrm{Ker}(\pi), KP) \ .$$

Then

$$\operatorname{Ext}^1_{KP}\left(G(\mathcal{X}^0), \bigoplus_{n \in \omega} e_{v_n} KP\right) = \operatorname{Hom}_{KP}(\mathrm{Ker}(\pi), KP)\Big/\operatorname{Im}(\operatorname{Hom}_{KP}(\mathrm{id}_{\mathrm{Ker}(\pi)}, KP)).$$

For each non-stationary path $b_n \cdots b_0$ of P, we fix the notation $b_i \cdots b_0$ by induction on $i \leq n$ in such a way that

$$b_0 \cdots b_0 := b_0$$

and, for $i \leq n$,

$$b_{i+1} \cdots b_0 := b_{i+1} b_i \cdots b_0.$$

For each $m, n \in \omega$ with $m \leq n$, define

$$\varphi(e_{v_m} - a_n \cdots a_m) := \sum_{i=0}^{n-m} a_{m+i} \cdots a_m.$$

Then, for each $l, m, n \in \omega$ with $l < m \leq n$,

$$\varphi(e_{v_m} - a_n \cdots a_m) a_{m-1} \cdots a_l = \left(\sum_{i=0}^{n-m} a_{m+i} \cdots a_m\right) a_{m-1} \cdots a_l$$

$$= \left(\sum_{i=0}^{n-l} a_{l+i} \cdots a_l\right) - \left(\sum_{i=0}^{m-1-l} a_{l+i} \cdots a_l\right)$$

$$= \varphi(e_{v_l} - a_n \cdots a_l) - \varphi(e_{v_l} - a_{m-1} \cdots a_l)$$

and

$$(e_{v_m} - a_n \cdots a_m) a_{m-1} \cdots a_l = a_{m-1} \cdots a_l - a_n \cdots a_l$$

$$= (e_{v_l} - a_n \cdots a_l) - (e_{v_l} - a_{m-1} \cdots a_l).$$

Thus, we can extend φ to a KP-homomorphism from $\mathrm{Ker}(\pi)$ into KP. To finish the proof, it is sufficient to show that φ is not in $\operatorname{Im}(\operatorname{Hom}_{KP}(\mathrm{id}_{\mathrm{Ker}(\pi)}, KP))$.

Assume it is, and let $\psi \in \operatorname{Hom}_{KP}(\bigoplus_{n \in \omega} e_{v_n} KP, KP)$ be such that

$$\varphi = \operatorname{Hom}_{KP}(\mathrm{id}_{\mathrm{Ker}(\pi)}, KP)(\psi) = \psi \restriction \mathrm{Ker}(\pi).$$

For each $n \in \omega$,

$$\psi(e_{v_0}) - \psi(e_{v_{n+2}})a_{n+1}\cdots a_0 = \psi(e_{v_0}) - \psi(a_{n+1}\cdots a_0)$$
$$= \psi(e_{v_0} - a_{n+1}\cdots a_0)$$
$$= \sum_{i=0}^{n+1} a_i \cdots a_0.$$

Therefore, for *every* $n \in \omega$, $\psi(e_{v_0})$ belongs to the set

$$\left(\sum_{i=0}^{n} a_i \cdots a_0\right) + KP\left(a_{n+1}\cdots a_0\right).$$

However, this is a contradiction because $\psi(e_{v_0})$ have to belong to KP. $\qquad\square$

THEOREM 1.10. *Suppose that K is an algebraically closed field, and Q is a connected quiver such that, for any finite subset P_0' of Q_0, the closure of P_0' under Q is a finite acyclic quiver. Then $\mathbf{P}_{KQ}(\mathrm{rep}_K Q)$.*

For example, the following quivers satisfy the assumption of the theorem:
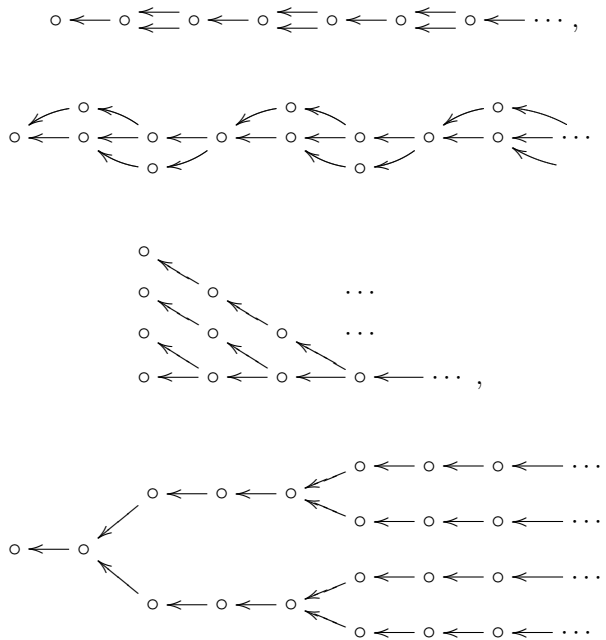
Note that any infinite quiver as in the assumption of the theorem contains at least one of the following quivers as a subquiver:
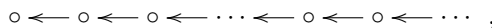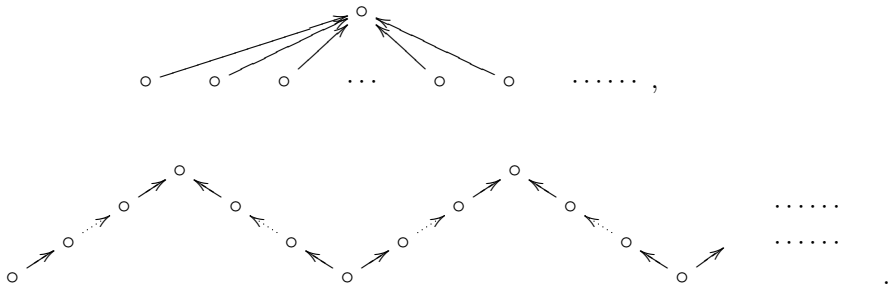
PROOF.   This theorem has been proved when Q is a finite quiver in Corollary 1.5. Suppose that Q is an infinite quiver, and $\mathcal{X}$ is a finite dimensional K-linear representation of Q such that $\mathrm{Ext}^1_{KQ}(G(\mathcal{X}), KQ) = 0$. We show that $\mathcal{X}$ is projective. Since

$$\mathrm{Ext}^1_{KQ}\left(\bigoplus_{i \in I} M_i, KQ\right) = \prod_{i \in I} \mathrm{Ext}^1_{KQ}(M_i, KQ)$$

in general, without loss of generality we may assume that $\mathcal{X}$ is indecomposable.

By Proposition 1.8, for every finite closed subquiver P of Q, $\mathrm{Ext}^1_{KP}(G(\mathcal{X} \restriction P), KP) = 0$. Therefore, by $\mathbf{P}_{KP}(\mathrm{mod} KP)$, $\mathcal{X} \restriction P$ is projective. It is known that any indecomposable projective KP-module is of the form $e_v KP$ for some $v \in P_0$ [**3**, Section III. 2]. Since P is a closed subquiver of Q, the underlying set of $e_v KP$ is equal to $e_v KQ$, and $F(e_v KQ) \restriction P = F(e_v KP)$. So, since $\mathcal{X}$ is finite dimensional, $\mathcal{X} \restriction P$ is isomorphic to a direct sum of finitely many K-linear representations of the form $F(e_v KQ)$ for $v \in P_0$. Therefore, since $\mathcal{X}$ is indecomposable, only one of the following statements holds:

1. $\mathcal{X}$ is isomorphic to $F(e_v KQ)$ for some $v \in Q_0$, or

2. Q contains the quiver

$$v_0 \longleftarrow v_1 \longleftarrow v_2 \longleftarrow \cdots \longleftarrow v_n \longleftarrow v_{n+1} \longleftarrow \cdots$$

   as a subquiver such that, for each $n \in \omega$, $\mathcal{X}_{v_n} \neq \{0_K\}$, and $\mathcal{X} \restriction \overline{\{v_n\}}^Q$ is a direct sum of finitely many copies of $F(e_{v_n} KQ)$.

By Proposition 1.9 and the assumption that $\mathrm{Ext}^1_{KQ}(G(\mathcal{X}), KQ) = 0$, $\mathcal{X}$ is isomorphic to $F(e_v KQ)$ for some $v \in Q_0$. By our assumption, $\overline{\{v\}}^Q$ is a finite acyclic quiver, so e_v is an idempotent of KQ. Hence $e_v KQ$ is projective, so is $\mathcal{X}$.   □

1.3.   Martin's axiom.

Martin's axiom was introduced by Martin and Solovay [**18**]. This axiom can neither be proved nor refuted from axiomatic set theory **ZFC**, so it is consistent with **ZFC**. Martin's axiom can be considered as a generalization of the Baire category theorem (see e.g. [**17**, Theorem III. 4.7]). $\mathbf{MA}_{\aleph_1}$ denotes Martin's axiom for $\aleph_1$ many dense sets.

$\mathbf{MA}_{\aleph_1}$ implies that $2^{\aleph_0} \geq \aleph_2$. In this paper we use $\mathbf{UP}^2$, which is one combinatorial consequence from $\mathbf{MA}_{\aleph_1}$.

DEFINITION 1.11.   1. A *ladder system* ($on \ \omega_1$) is a sequence $\langle C_\alpha : \alpha \in \omega_1 \cap \mathbf{Lim} \rangle$ such that

- for each $\alpha \in \omega_1 \cap \mathbf{Lim}$, C_α is a cofinal subset of α, that is, for any $\xi \in \alpha$, there is $\eta \in C_\alpha$ such that $\xi \in \eta$, and
- C_α is of order type ω, that is, the elements of C_α can be enumerated as $\{\zeta_n^\alpha : n \in \omega\}$ increasingly, that is, for every $m, n \in \omega$, if $m \in n$, then $\zeta_m^\alpha \in \zeta_n^\alpha$.

2. A *coloring* $\langle d_\alpha : \alpha \in \omega_1 \cap \mathbf{Lim} \rangle$ of a ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \mathbf{Lim} \rangle$ is a sequence of functions such that the domain of each d_α is C_α.

3. We say that a function f with domain ω_1 *uniformizes* a coloring $\langle d_\alpha : \alpha \in \omega_1 \cap \mathbf{Lim} \rangle$ of a ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \mathbf{Lim} \rangle$, $C_\alpha = \{\zeta_n^\alpha : n \in \omega\}$, if for every $\alpha \in \omega_1 \cap \mathbf{Lim}$, the restricted function $f \upharpoonright C_\alpha$ of f by C_α is equal to the function d_α for all but finitely many points, that is, there exists an $N \in \omega$ such that, for any $n \in \omega \setminus N$, $f(\zeta_n^\alpha) = d_\alpha(\zeta_n^\alpha)$.

4. The assertion $\mathbf{UP}$ means that, for any sequence $\langle X_\beta : \beta \in \omega_1 \rangle$ of countable sets and any coloring $\langle d_\alpha : \alpha \in \omega_1 \cap \mathbf{Lim} \rangle$ of a ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \mathbf{Lim} \rangle$, whenever $d_\alpha(\zeta_n^\alpha)$ belongs to $X_{\zeta_n^\alpha}$ for any $\alpha \in \omega_1 \cap \mathbf{Lim}$ and $n \in \omega$, there exists a function with domain ω_1 which uniformizes the coloring $\langle d_\alpha : \alpha \in \omega_1 \cap \mathbf{Lim} \rangle$.

THEOREM 1.12 (Devlin–Shelah [**6**, Theorem 5.2]). $\mathbf{MA}_{\aleph_1}$ *implies* $\mathbf{UP}$.

The assertion $\mathbf{UP}$ was inspired by Shelah's proof that $\mathbf{MA}_{\aleph_1}$ implies the existence of a non-free Whitehead group [**20**, Theorem 3.5] (see also [**10**]).

1.4.   Trlifaj's construction.

In this paper, our modules are built by modifying Trlifaj's construction. As every proof in Section 2 is fairly self-contained, the reader does not need to be familiar with this construction. Trlifaj's construction is a quotient module of the outer direct sum $\bigoplus_{\xi \in \omega_1} F^\xi$ of some sequence $\langle F^\xi : \xi \in \omega_1 \rangle$ of modules, defined in [**22**, Definition 1.1] and [**15**, Notation 5.3], which seems to be inspired by Shelah's solution of Whitehead problem [**20**]. To fix our notation and understand our construction better, Trlifaj's construction is presented as follows.

Let R be a ring with identity and let

$$F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} F_n \xrightarrow{f_n} F_{n+1} \xrightarrow{f_{n+1}} \cdots$$

be a countable direct system of R-modules. Let $\langle C_\alpha : \alpha \in \omega_1 \cap \mathbf{Lim} \rangle$ be a ladder system such that

$$C_\alpha = \{\zeta_n^\alpha : n \in \omega\}$$

---

[2]This notation follows [**11**] but it is not that common in set theory.

is an increasing enumeration and assume that, for each $n \in \omega$, ζ_n^α is of the form $\delta + n + 1$ for some $\delta \in \alpha \cap (\{0\} \cup \mathbf{Lim})$ (there is such a ladder system). Define $F^0 := \{0_R\}$; for each $\gamma \in \omega_1 \setminus \mathbf{Lim}$ with $\gamma = \delta + n_\gamma + 1$ for some $\delta \in \gamma \cap (\{0\} \cup \mathbf{Lim})$ and $n_\gamma \in \omega$, define $F^\gamma := F_{n_\gamma}$; and, for each $\delta \in \omega_1 \cap \mathbf{Lim}$, define $F^\delta := \bigoplus_{n \in \omega} F_n$. So, for each member x of the outer direct sum $\bigoplus_{\xi \in \omega_1} F^\xi$, x forms a finite support function with domain included in ω_1 and, for each $\alpha \in \omega_1$, $x(\alpha)$ belongs to F^α. Hence, if $\delta \in \omega_1 \cap \mathbf{Lim}$, then $x(\delta)$ belongs to the outer direct sum $F^\delta := \bigoplus_{n \in \omega} F_n$, which also forms a finite support function with domain included in ω. For each $\delta \in \omega_1 \cap \mathbf{Lim}$, define the R-submodule

$$
G_\delta := \left\langle \left\{ x \in \bigoplus_{\xi \in \omega_1} F^\xi : \text{ for some } n \in \omega, \text{supp}(x) = \left\{\zeta_n^\delta, \delta\right\}, \right. \right.
$$
$$
\text{supp}(x(\delta)) = \{n, n+1\}, x(\zeta_n^\delta) = x(\delta)(n),
$$
$$
\left. \left. \text{and } x(\delta)(n+1) = f_n(x(\delta)(n)) \right\} \right\rangle_R
$$

of the R-module $\bigoplus_{\xi \in \omega_1} F^\xi$, and define

$$
I_{\omega_1} := \sum_{\delta \in \omega_1 \cap \mathbf{Lim}} G_\delta,
$$

which is an R-submodule of the R-module $\bigoplus_{\xi \in \omega_1} F^\xi$. Trlifaj's construction is the quotient R-module $\bigoplus_{\xi \in \omega_1} F^\xi / I_{\omega_1}$ of the R-module $\bigoplus_{\xi \in \omega_1} F^\xi$ by the R-submodule I_{ω_1}. Trlifaj applied this construction for a non-left perfect ring [22], and Herbera–Trlifaj applied it to analyze some classes of modules called Kaplansky classes or deconstructible classes [15]. For further properties of this module, see [15, Section 5].

2. Some infinitely generated modules of path algebras.

Throughout this section, we fix a ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \mathbf{Lim} \rangle$ such that

$$
C_\alpha = \{\zeta_n^\alpha : n \in \omega\}
$$

is an increasing enumeration and, for each $n \in \omega$, ζ_n^α is of the form $\delta + n + 1$ for some $\delta \in \alpha \cap (\{0\} \cup \mathbf{Lim})$. We note that, for any $\alpha, \beta \in \omega_1 \cap \mathbf{Lim}$ and $m, n \in \omega$, if $\zeta_m^\alpha = \zeta_n^\beta$ then $m = n$. For $\gamma \in \omega_1 \setminus (\{0\} \cup \mathbf{Lim})$, let $n_\gamma \in \omega$ be the unique integer such that $\gamma = \delta + n_\gamma + 1$ for some (unique) $\delta \in \omega_1 \cap (\{0\} \cup \mathbf{Lim})$.

For each subsection of this section, we deal with some quiver Q and build a non-projective KQ-module M_{ω_1}. For each quiver Q in each subsection, we use the following notation. For each $v \in Q_0$, e_v denotes the path of length 0 from the vertex v (to itself). For $\gamma \in \omega_1 \setminus \mathbf{Lim}$, $\alpha \in \omega_1 \cap \mathbf{Lim}$ and $n \in \omega$, let $F^\gamma = F^{\alpha,n} := KQ$, and let F^α be the outer direct sum $\bigoplus_{n \in \omega} F^{\alpha,n}$. For $\gamma \in \omega_1 \setminus \mathbf{Lim}$ and $v \in Q_0$, let e_v^γ be the member of the outer direct sum $\bigoplus_{\xi \in \omega_1} F^\xi$ of KQ-modules such that

$$
\text{supp}(e_v^\gamma) = \{\gamma\}, \quad e_v^\gamma(\gamma) = e_v.
$$

For $\alpha \in \omega_1 \cap \mathbf{Lim}$, $n \in \omega$ and $v \in Q_0$, let $e_v^{\alpha,n} \in \bigoplus_{\xi \in \omega_1} F^\xi$ be such that

$$\mathrm{supp}(e_v^{\alpha,n}) = \{\alpha\}, \quad \mathrm{supp}(e_v^{\alpha,n}(\alpha)) = \{n\}, \quad e_v^{\alpha,n}(\alpha)(n) = e_v.$$

REMARK 2.1. The set

$$\{e_v^\gamma, e_v^{\alpha,n} : \gamma \in \omega_1 \setminus \mathbf{Lim}, v \in Q_0, \alpha \in \omega_1 \cap \mathbf{Lim}, n \in \omega\}$$

is linearly independent with respect to KQ in $\bigoplus_{\xi \in \omega_1} F^\xi$.

2.1. On a quiver of A_∞ type.

Throughout this subsection, let K be a field, and Q the quiver $A_\infty^\leftarrow$ as follows:

$$0 \xleftarrow{\ a_0\ } 1 \xleftarrow{\ a_1\ } \cdots \xleftarrow{\ a_{n-1}\ } n \xleftarrow{\ a_n\ } n+1 \xleftarrow{\ a_{n+1}\ } \cdots \ ,$$

that is, the set Q_0 of vertices is the set of all non-negative integers and the set Q_1 of arrows is defined by

$$\left\{ n \xleftarrow{\ a_n\ } n+1 : n \in \omega \right\}.$$

Since Q_0 is infinite, KQ does not have an identity. By simplifying the notation in this subsection, for each $\alpha \in \omega_1 \cap \mathbf{Lim}$ and $n \in \omega$,

$$e_n^\alpha := e_n^{\alpha,0}.$$

For each $\alpha \in \omega_1 \cap \mathbf{Lim}$, define

$$G_\alpha := \left\langle \left\{ e_n^{\zeta_n^\alpha} - e_n^\alpha + e_{n+1}^\alpha a_n : n \in \omega \right\} \right\rangle_{KQ},$$

$$I_{\omega_1} := \sum_{\xi \in \omega_1 \cap \mathbf{Lim}} G_\xi.$$

For each $x \in \bigoplus_{\xi \in \omega_1} F^\xi$, $x + I_{\omega_1}$ denotes the equivalence class of x in the quotient module $\bigoplus_{\xi \in \omega_1} F^\xi / I_{\omega_1}$. For each $\xi \in \omega_1 + 1$, define the KQ-module M_ξ by

$$\left\langle \left\{ e_{n_\gamma}^\gamma + I_{\omega_1} : \gamma \in \xi \setminus \mathbf{Lim} \right\} \cup \{ e_n^\alpha + I_{\omega_1} : \alpha \in \xi \cap \mathbf{Lim}, n \in \omega \} \right\rangle_{KQ},$$

which is considered as a KQ-submodule of the quotient module $\bigoplus_{\xi \in \omega_1} F^\xi / I_{\omega_1}$.

REMARK 2.2. The set $\left\{ e_n^{\zeta_n^\alpha} - e_n^\alpha + e_{n+1}^\alpha a_n : \alpha \in \omega_1 \cap \mathbf{Lim}, n \in \omega \right\}$ is linearly independent with respect to KQ in $\bigoplus_{\xi \in \omega_1} F^\xi$.

In this paper, $\bigoplus_{\omega_1} KQ$ denotes the outer direct sum of ω_1 many copies of KQ, which is considered as a KQ-module.

CLAIM 2.3. $\mathrm{Ext}^1_{KQ}(M_{\omega_1}, \bigoplus_{\omega_1} KQ) \neq 0$. *In particular, M_{ω_1} is not a projective*

KQ-module.

Proof. F_{ω_1} denotes the KQ-module

$$\left\langle \left\{ e_{n_\gamma}^\gamma : \gamma \in \omega_1 \setminus \mathbf{Lim} \right\} \cup \{e_n^\alpha : \alpha \in \omega_1 \cap \mathbf{Lim}, n \in \omega\} \right\rangle_{KQ}.$$

Applying $\mathrm{Hom}_{KQ}(-, \bigoplus_{\omega_1} KQ)$ to the exact sequence

$$0 \rightarrow I_{\omega_1} \xrightarrow{\mathrm{id}_{I_{\omega_1}}} F_{\omega_1} \twoheadrightarrow M_{\omega_1} \twoheadrightarrow 0 \ ,$$

we obtain the exact sequence

$$0 \twoheadrightarrow \mathrm{Hom}_{KQ}\left(M_{\omega_1}, \bigoplus_{\omega_1} KQ\right) \rightarrow$$

$$\mathrm{Hom}_{KQ}\left(F_{\omega_1}, \bigoplus_{\omega_1} KQ\right) \xrightarrow{\mathrm{Hom}_{KQ}\left(\mathrm{id}_{I_{\omega_1}}, \bigoplus_{\omega_1} KQ\right)} \mathrm{Hom}_{KQ}\left(I_{\omega_1}, \bigoplus_{\omega_1} KQ\right).$$

Then

$$\mathrm{Ext}^1_{KQ}\left(M_{\omega_1}, \bigoplus_{\omega_1} KQ\right) = \mathrm{Hom}_{KQ}\left(I_{\omega_1}, \bigoplus_{\omega_1} KQ\right) \bigg/ \mathrm{Im}\left(\mathrm{Hom}_{KQ}\left(\mathrm{id}_{I_{\omega_1}}, \bigoplus_{\omega_1} KQ\right)\right).$$

By Remark 2.2, we can find a KQ-homomorphism φ in $\mathrm{Hom}_{KQ}(I_{\omega_1}, \bigoplus_{\omega_1} KQ)$ such that for each $\alpha \in \omega_1 \cap \mathbf{Lim}$ and $n \in \omega$,

$$\varphi\left(e_n^{\zeta_n^\alpha} - e_n^\alpha + e_{n+1}^\alpha a_n\right) := e_n^\alpha.$$

We show that φ does not belong to $\mathrm{Im}(\mathrm{Hom}_{KQ}(\mathrm{id}_{I_{\omega_1}}, \bigoplus_{\omega_1} KQ))$.

Assume that $\varphi \in \mathrm{Im}(\mathrm{Hom}_{KQ}(\mathrm{id}_{I_{\omega_1}}, \bigoplus_{\omega_1} KQ))$, and let $\psi \in \mathrm{Hom}_{KQ}(F_{\omega_1}, \bigoplus_{\omega_1} KQ)$ such that

$$\mathrm{Hom}_{KQ}\left(\mathrm{id}_{I_{\omega_1}}, \bigoplus_{\omega_1} KQ\right)(\psi) = \psi \circ \mathrm{id}_{I_{\omega_1}} = \psi \restriction I_{\omega_1} = \varphi.$$

We note that for each $\gamma \in \omega_1$ and $n \in \omega$, $\mathrm{supp}(\psi(e_n^\gamma))$ is a finite subset of ω_1. So we can take an $\alpha \in \omega_1 \cap \mathbf{Lim}$ such that, for every $\gamma \in \alpha$ and $n \in \omega$, $\mathrm{supp}(\psi(e_n^\gamma))$ is a finite subset of α[3]. For each $n \in \omega$,

$$\psi\left(e_n^{\zeta_n^\alpha}\right) - \psi(e_n^\alpha) + \psi(e_{n+1}^\alpha)a_n = \psi\left(e_n^{\zeta_n^\alpha} - e_n^\alpha + e_{n+1}^\alpha a_n\right) = e_n^\alpha.$$

Therefore, by induction on $n \in \omega$,

---

[3]This can be done by e.g. [**17**, Exercise III. 6.20]

$$\psi(e_0^\alpha) = \psi\left(e_0^{\zeta_0^\alpha}\right) - e_0^\alpha + \psi(e_1^\alpha)a_0$$

$$= \psi\left(e_0^{\zeta_0^\alpha}\right) - e_0^\alpha + \left(\psi\left(e_1^{\zeta_1^\alpha}\right) - e_1^\alpha + \psi(e_2^\alpha)a_1\right)a_0$$

$$= \psi\left(e_0^{\zeta_0^\alpha}\right) - e_0^\alpha + \psi\left(e_1^{\zeta_1^\alpha}\right)a_0 - e_1^\alpha a_0 + \psi(e_2^\alpha)a_1 a_0$$

$$= \cdots$$

$$= \psi\left(e_0^{\zeta_0^\alpha}\right) - e_0^\alpha + \sum_{i=1}^n \psi\left(e_i^{\zeta_i^\alpha}\right)a_{i-1}\cdots a_0 - \sum_{i=1}^n e_i^\alpha a_{i-1}\cdots a_0$$

$$+ \psi(e_{n+1}^\alpha)a_n \cdots a_0.$$

Hence, for *every* $n \in \omega$, since each supp $\left(\psi\left(e_i^{\zeta_i^\alpha}\right)\right)$ does not contain α as a member,

$$\psi(e_0^\alpha)(\alpha) \notin KQ_{\le n},$$

where $KQ_{\le n}$ is the K-subspace of KQ generated by all paths of length $\le n$. This is a contradiction. $\qquad\square$

The following is similar to [**11**, Theorem XII 2.2, Proposition XIII 0.2].

THEOREM 2.4. *Suppose that K is a countable field. Then* **UP** *implies that* $\mathrm{Ext}^1_{KQ}(M_{\omega_1}, KQ) = 0$. *In particular,* $\mathbf{P}_{KQ}(\mathrm{Mod}KQ)$ *fails.*

PROOF. Applying $\mathrm{Hom}_{KQ}(-, KQ)$ to the exact sequence

$$0 \rightarrow I_{\omega_1} \xrightarrow{\mathrm{id}_{I_{\omega_1}}} F_{\omega_1} \rightarrow M_{\omega_1} \rightarrow 0\,,$$

we obtain the exact sequence

$$0 \rightarrow \mathrm{Hom}_{KQ}(M_{\omega_1}, KQ) \rightarrow \mathrm{Hom}_{KQ}(F_{\omega_1}, KQ) \xrightarrow{\mathrm{Hom}_{KQ}(\mathrm{id}_{I_{\omega_1}}, KQ)} \mathrm{Hom}_{KQ}(I_{\omega_1}, KQ)\,.$$

Then

$$\mathrm{Ext}^1_{KQ}(M_{\omega_1}, KQ) = \mathrm{Hom}_{KQ}(I_{\omega_1}, KQ)/\mathrm{Im}(\mathrm{Hom}_{KQ}(\mathrm{id}_{I_{\omega_1}}, KQ)).$$

Let $\varphi \in \mathrm{Hom}_{KQ}(I_{\omega_1}, KQ)$. We show that φ belongs to $\mathrm{Im}(\mathrm{Hom}_{KQ}(\mathrm{id}_{I_{\omega_1}}, KQ))$. For each $\alpha \in \omega_1 \cap \mathbf{Lim}$ and $n \in \omega$, define

$$d_\alpha(\zeta_n^\alpha) := \varphi\left(e_n^{\zeta_n^\alpha} - e_n^\alpha + e_{n+1}^\alpha a_n\right).$$

We notice that, for each $n \in \omega$,

$$\varphi\left(e_n^{\zeta_n^\alpha} - e_n^\alpha + e_{n+1}^\alpha a_n\right)e_n = \varphi\left(e_n^{\zeta_n^\alpha} - e_n^\alpha + e_{n+1}^\alpha a_n\right),$$

and, for any $m \in Q_0 \setminus \{n\}$,

$$\varphi\left(e_n^{\zeta_n^\alpha} - e_n^\alpha + e_{n+1}^\alpha a_n\right)e_m = \varphi\left(0 \bigoplus_{\xi \in \omega_1} F\xi\right) = 0_{KQ}.$$

Thus, for each $n \in \omega$, $d_\alpha(\zeta_n^\alpha)$ belongs to the countable set

$$\sum_{p \,:\, \text{path in } Q \text{ ending in } n} Kp.$$

Therefore, by **UP**, we can find a uniformization f of the ladder system coloring $\langle d_\alpha : \alpha \in \omega_1 \cap \mathbf{Lim}\rangle$, that is, for each $\alpha \in \omega_1 \cap \mathbf{Lim}$, there is an $N_\alpha \in \omega$ such that, for every $n \geq N_\alpha$, $f(\zeta_n^\alpha) = d_\alpha(\zeta_n^\alpha)$.

For each $\alpha \in \omega_1 \cap \mathbf{Lim}$ and $n \in \omega$, define

- $\psi\left(e_n^{\zeta_n^\alpha}\right) := f(\zeta_n^\alpha)$,

- $\psi(e_n^\alpha) := 0_{KQ}$ when $n \geq N_\alpha$, and

- by downward induction on $n < N_\alpha$, define

$$\psi(e_n^\alpha) := \psi\left(e_n^{\zeta_n^\alpha}\right) + \psi(e_{n+1}^\alpha)a_n - \varphi\left(e_n^{\zeta_n^\alpha} - e_n^\alpha + e_{n+1}^\alpha a_n\right).$$

By Remark 2.1, ψ can be extended to a KQ-homomorphism from F_{ω_1} into KQ. Therefore

$$\psi \upharpoonright I_{\omega_1} = \psi \circ \mathrm{id}_{I_{\omega_1}} = \mathrm{Hom}_{KQ}(\mathrm{id}_{I_{\omega_1}}, KQ)(\psi) = \varphi,$$

which finishes the proof.   □

REMARK 2.5.   By a similar argument to the one in the previous theorem, it can be proved that *if K is a countable field and **UP** holds, then $\mathrm{Ext}^1_{KQ}(M_{\omega_1}, \bigoplus_\omega KQ) = 0$.*

REMARK 2.6.   Jensen introduced the assertion $\diamondsuit$ which is true in Gödel's constructible universe. So $\diamondsuit$ is one set theoretic axiom consistent with **ZFC** (see e.g. [**17**]). For a stationary set S, $\diamondsuit_S$ is a variation of $\diamondsuit$, which is also true in Gödel's constructible universe. Contrary to $\mathbf{MA}_{\aleph_1}$, $\diamondsuit$ (and $\diamondsuit_S$) implies that $2^{\aleph_0} = \aleph_1$. By a similar argument as in [**10**, Lemma 4.3], we can show that *if K is a countable field, then $\diamondsuit$ implies $\mathrm{Ext}^1_{KQ}(M_{\omega_1}, KQ) \neq 0$.* The main ingredient to prove this is the following fact.

CLAIM 2.7.   *Suppose that $\mathrm{Ext}_{KQ}(M_{\alpha+1}/M_\alpha, KQ) \neq 0$, and let*

$$0 \longrightarrow KQ \longrightarrow C_\alpha \stackrel{\pi}{\longrightarrow} M_\alpha \longrightarrow 0$$

be a short exact sequence that splits, that is, there exists a homomorphism ρ from M_α into C_α such that $\pi \circ \rho = \mathrm{id}_{M_\alpha}$. Then there exists a short exact sequence

$$0 \longrightarrow KQ \longrightarrow C_{\alpha+1} \stackrel{\pi'}{\longrightarrow} M_{\alpha+1} \longrightarrow 0$$

*such that*

$$\pi' \upharpoonright C_\alpha = \pi$$

and there is no homomorphism ρ' from $M_{\alpha+1}$ into $C_{\alpha+1}$ such that

$$\pi' \circ \rho' = \mathrm{id}_{M_\alpha+1} \quad \text{and} \quad \rho' \upharpoonright M_\alpha = \rho.$$

Since KQ is countable and Claim 2.3 holds, a similar argument as in [**10**, Theorem 6.3] works well to show that $\mathrm{Ext}_{KQ}(M_{\omega_1}, KQ) \neq 0$. Moreover, by a similar argument as in [**16**], we can show that, *if K is a countable field and there is a set $\{S_\alpha : \alpha \in \omega_1\}$ of pairwise disjoint stationary subsets of ω_1 such that $\diamondsuit_{S_\alpha}$ holds for each $\alpha \in \omega_1$, then the cardinality of $\mathrm{Ext}_{KQ}(M_{\omega_1}, KQ)$ is greater than $\aleph_1$.*

2.2. On a circular quiver.

In this subsection, let K be a field and Q the following quiver.
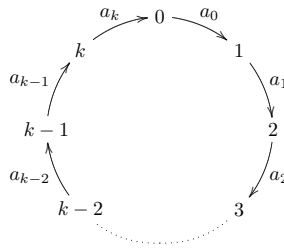
Then the path $a_0 a_1 \cdots a_k$ is a path in Q whose source and target are both the vertex 0. We denote the path

$$(a_0 a_1 \cdots a_k)^0 = e_0,$$

and, for each $n \in \omega$, define the path

$$(a_0 a_1 \cdots a_k)^{n+1} = (a_0 a_1 \cdots a_k)^n \, a_0 a_1 \cdots a_k.$$

Recall that $\sum_{v \in Q_0} e_v$ is the identity of KQ. For each $\alpha \in \omega_1 \cap \mathbf{Lim}$, define

$$G_\alpha := \left\langle \left\{ e_0^{\zeta_n^\alpha} - e_0^{\alpha,n} + e_0^{\alpha,n+1} a_0 a_1 \cdots a_k : n \in \omega \right\} \right\rangle_{KQ},$$

$$I_{\omega_1} := \sum_{\xi \in \omega_1 \cap \mathbf{Lim}} G_\xi,$$

and, for each $\xi \in \omega_1 + 1$, define the KQ-module M_ξ by

$$\left\langle \{ e_0^\gamma + I_{\omega_1} : \gamma \in \xi \setminus \mathbf{Lim} \} \cup \{ e_0^{\alpha,n} + I_{\omega_1} : \alpha \in \xi \cap \mathbf{Lim}, n \in \omega \} \right\rangle_{KQ},$$

which is considered as a KQ-submodule of the quotient module $\bigoplus_{\xi \in \omega_1} F^\xi / I_{\omega_1}$.

CLAIM 2.8. $\mathrm{Ext}^1_{KQ}(M_{\omega_1}, \bigoplus_{\omega_1} KQ) \neq 0$. *Therefore, M_{ω_1} is not projective.*

PROOF. This can be proved in a similar way as in Claim 2.3. To see this, it suffices to replace the formula

$$\varphi\left(e_n^{\zeta_n^\alpha} - e_n^\alpha + e_{n+1}^\alpha a_n\right) := e_n^\alpha$$

by the formula

$$\varphi\left(e_0^{\zeta_n^\alpha} - e_0^{\alpha,n} + e_0^{\alpha,n+1} a_0 a_1 \cdots a_k\right) := e_0^{\alpha,n}$$

in the proof of Claim 2.3. $\hfill\square$

Moreover, by a similar proof as Theorem 2.4, the following theorem can be proved.

THEOREM 2.9. *Suppose that K is a countable field. Then* **UP** *implies that* $\mathrm{Ext}^1_{KQ}(M_{\omega_1}, KQ) = 0$. *In particular,* $\mathbf{P}_{KQ}(\mathrm{Mod}KQ)$ *fails.*

REMARK 2.10.   As in Remark 2.6, if K is a countable field and $\diamondsuit$ holds, then $\mathrm{Ext}^1_{KQ}(M_{\omega_1}, KQ) \neq 0$.

2.3.   Generalizations.

THEOREM 2.11.   *Suppose that K is a countable field and Q' is a quiver that contains a subquiver Q of one of the following types*
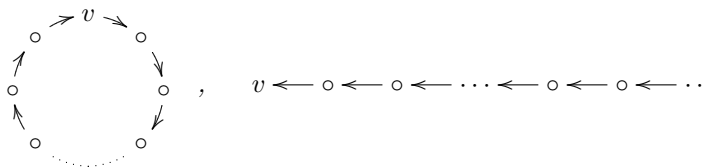
in such a way that the set of all paths in Q' ending in v is countable. Then **UP** *implies the failure of* $\mathbf{P}_{KQ'}(\mathrm{Mod}KQ')$.

PROOF.   Let M_{ω_1} be one of the KQ-modules constructed before. Then, M_{ω_1} can be considered as a KQ'-module and, by a similar argument as before, it can be proved that $\mathrm{Ext}^1_{KQ'}(M_{\omega_1}, \bigoplus_{\omega_1} KQ') \neq 0$, and that **UP** implies $\mathrm{Ext}^1_{KQ'}(M_{\omega_1}, KQ') = 0$. $\hfill\square$

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