

Asymptotic behavior of lifetime sums for random simplicial complex processes

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Abstract. We study the homological properties of random simplicial complexes. In particular, we obtain the asymptotic behavior of lifetime sums for a class of increasing random simplicial complexes; this result is a higher-dimensional counterpart of Frieze’s $\zeta(3)$ -limit theorem for the Erdős–Rényi graph process. The main results include solutions to questions posed in an earlier study by Hiraoka and Shirai about the Linial–Meshulam complex process and the random clique complex process. One of the key elements of the arguments is a new upper bound on the Betti numbers of general simplicial complexes in terms of the number of small eigenvalues of Laplacians on links. This bound can be regarded as a quantitative version of the cohomology vanishing theorem.

1. Introduction.

The Erdős–Rényi $G(n, p)$ model has been extensively studied since the 1960s ([9], [4], [5]). This model, defined as the distribution of random graphs with n vertices where the edge between each pair of vertices is included with probability p independently of any other edge, is one of the most typical models of random graphs. One of the main themes in $G(n, p)$ theory is searching for threshold probabilities. For example, Erdős and Rényi [5] showed that the threshold for graph connectivity of $G(n, p)$ is $p = (\log n)/n$. When we vary p and consider a family of the Erdős–Rényi graphs with parameter p , the following construction is often useful. Let $K_n = V_n \sqcup E_n$ be the complete graph with n given vertices, where V_n and E_n denote the sets of vertices and edges, respectively. We assign an independent random variable u_e to each edge $e \in E_n$ and let u_e be uniformly random on $[0, 1]$. For each $p \in [0, 1]$, a random subgraph $K_n(p)$ of K_n is then defined by

$$K_n(p) := V_n \sqcup \{e \in E_n \mid u_e \leq p\}.$$

This construction, the so-called Erdős–Rényi graph process over n vertices, yields an increasing family $\mathcal{K}_n := \{K_n(t)\}_{t \in [0, 1]}$ of random graphs. This process is closely related to the concept of the minimum weight on K_n , which can be seen as follows. For each spanning tree T in K_n , define its weight as $\sum_{e \in T} u_e$. Let W_n be the minimum weight among all the spanning trees in K_n . Then

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$$W_n = \sum_{i=1}^{n-1} t_i = \int_0^1 \beta_0(K_n(t)) dt, \quad (1.1)$$

where $t_i \in [0, 1]$ is the i -th random time at which the number of connected components of $K_n(t)$ decreases, and $\beta_0(K_n(t))$ denotes the zeroth (reduced) Betti number of $K_n(t)$, that is, the number of connected components of $K_n(t)$ minus one. This type of relation holds for a general increasing family of graphs. Applying this formula and analyzing $\beta_0(K_n(t))$ in detail, Frieze [7] obtained the following significant result about the behavior of W_n .

THEOREM 1.1 ($\zeta(3)$ -limit theorem [7]). *It holds that*

$$\lim_{n \rightarrow \infty} \mathbb{E}[W_n] = \zeta(3) \left(= \sum_{k=1}^{\infty} k^{-3} = 1.202 \dots \right)$$

and for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|W_n - \zeta(3)| > \varepsilon) = 0.$$

Recently, there has been a growing interest in studying random simplicial complexes as a higher-dimensional generalization of random graphs. Since an Erdős–Rényi graph can be regarded as a one-dimensional random simplicial complex, and graph connectivity can be equivalently described as the vanishing of the zeroth (reduced) homology, it is natural to seek a higher-dimensional analogue to the theory of Erdős–Rényi's $G(n, p)$ model. The d -Linial–Meshulam model [14] and the random clique complex model [12] are typical models of this type. The d -Linial–Meshulam model $Y_d(n, p)$ is defined as the distribution of d -dimensional random simplicial complexes with n vertices and the complete $(d-1)$ -dimensional skeleton such that each d -simplex is placed with independent probability p . The random clique complex model $C(n, p)$ is defined as the distribution of the clique complex of the Erdős–Rényi graph that follows $G(n, p)$. Here, given a graph G , its clique complex $\text{Cl}(G)$ is defined as the maximal simplicial complex among those for which the one-dimensional skeletons are equal to G . Linial, Meshulam, and Wallach [14], [16] exhibited the threshold for the vanishing of the $(d-1)$ -th homology for the d -Linial–Meshulam model, which is analogous to the connectivity threshold of the Erdős–Rényi graph. Later, Kahle [13] obtained similar results for the random clique complex model.

Along another line, Hiraoka and Shirai [10] obtained a higher-dimensional analogue of (1.1) in the context of the theory of persistent homology. Persistent homologies can describe the topological features of a filtration (i.e., an increasing family of simplicial complexes; see, e.g., [3], [17]). In particular, these provide rigorous definitions for the concepts of birth and death times of higher-dimensional holes, sometimes called cycles and cavities. The lifetimes, which are defined as the difference between the birth time and death time, measure the persistence of each hole in the filtration. In [10], Hiraoka and Shirai proved that the lifetime sum $L_k(\mathcal{X})$ of the k -th persistent homology associated with a filtration $\mathcal{X} = \{X(t)\}_{0 \leq t \leq 1}$ is equal to $\int_0^1 \beta_k(X(t)) dt$, where $\beta_k(X(t))$ represents the k -th (reduced) Betti number of the simplicial complex $X(t)$ (see Theorem 4.2). When $k = 0$ and \mathcal{X} is the Erdős–Rényi graph process over n vertices, the result is consistent

with the second identity of (1.1). They also obtained a relation analogous to the first identity (see Theorem 1.1 in [10]). Thus, it is natural to seek the asymptotic behavior of the lifetime sum $L_k(\mathcal{X}_n)$ for a random filtration $\mathcal{X}_n = \{X_n(t)\}_{0 \leq t \leq 1}$ or $\mathcal{X}_n = \{X_n(t)\}_{t \geq 0}$ over n vertices as a higher-dimensional generalization of Theorem 1.1. Typical random filtrations include the d -Linial–Meshulam complex process $\mathcal{K}_n^{(d)} = \{K_n^{(d)}(t)\}_{0 \leq t \leq 1}$, where $K_n^{(d)}(t)$ follows $Y_d(n, t)$, and the random clique complex process $\mathcal{C}_n = \{C_n(t)\}_{0 \leq t \leq 1}$, where $C_n(t)$ follows $C(n, t)$ (see Section 4.2 for further details). For these models, the following estimates were proved by Hiraoka and Shirai.

THEOREM 1.2 ([10, Theorem 1.2]). *Let $d \geq 1$. There exist positive constants c and C such that for sufficiently large n ,*

$$cn^{d-1} \leq \mathbb{E}[L_{d-1}(\mathcal{K}_n^{(d)})] \leq Cn^{d-1}.$$

THEOREM 1.3 ([10, Theorem 6.10]). *Let $k \geq 0$. There exist positive constants c and C such that for sufficiently large n ,*

$$cn^{k/2+1-1/(k+1)} \leq \mathbb{E}[L_k(\mathcal{C}_n)] \leq \begin{cases} Cn^k \log n & (k = 0, 1), \\ Cn^k & (k \geq 2). \end{cases} \tag{1.2}$$

In [10], the asymptotic behavior of $\mathbb{E}[L_{d-1}(\mathcal{K}_n^{(d)})]/n^{d-1}$ as $n \rightarrow \infty$ is also discussed, and a possible limiting constant I_{d-1} is found by a heuristic argument. For the exact value of I_{d-1} , see (4.10) and (4.11) and Section 4.4. The exact growth exponent of $\mathbb{E}[L_k(\mathcal{C}_n)]$, mentioned as a problem in that paper, is considered here.

In this paper, we obtain sharp quantitative estimates of lifetime sums $L_k(\mathcal{X}_n)$ for a large class of random filtrations over n vertices. The main results include a rigorous proof of the convergence of $L_k(\mathcal{K}_n^{(d)})/n^{d-1}$ to I_{d-1} and a determination of the growth exponent of $\mathbb{E}[L_k(\mathcal{C}_n)]$. These results solve problems posed in [10] and are summarized in the following theorems.

THEOREM 1.4. *Let $d \geq 1$. For any $r \in [1, \infty)$,*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left| \frac{L_{d-1}(\mathcal{K}_n^{(d)})}{n^{d-1}} - I_{d-1} \right|^r \right] = 0.$$

In particular, $\mathbb{E}[L_{d-1}(\mathcal{K}_n^{(d)})]/n^{d-1}$ converges to I_{d-1} as $n \rightarrow \infty$.

THEOREM 1.5. *Let $k \geq 0$. There exist positive constants c and C such that, for sufficiently large n ,*

$$cn^{k/2+1-1/(k+1)} \leq \mathbb{E}[L_k(\mathcal{C}_n)] \leq Cn^{k/2+1-1/(k+1)}.$$

As seen from Theorem 1.5, the exponent of the lower estimate in (1.2) is exact. To prove these theorems, we introduce a new upper estimate of the Betti numbers of general simplicial complexes (Theorem 2.5). This estimate is a quantitative version of the cohomology vanishing theorem ([8], [1], see also Theorem 2.3). By applying this theorem and

modified versions of known estimates to a family of multi-parameter random simplicial complexes, including the d -Linial–Meshulam complex and the random clique complex, we obtain several inequalities involving the expectations of Betti numbers. Theorem 3.6, in particular, provides an essentially new upper estimate. By integrating these inequalities with respect to the filtration parameter, we obtain good estimates of $\mathbb{E}[L_k(\mathcal{X}_n)]$ for a class of random filtrations over n vertices (Theorems 4.3 and 4.4). Theorem 1.5 is a special case of these theorems, and Theorem 1.2 also follows from them. The proof of Theorem 1.2 in [10] is based on the monotonicity of $\beta_{d-1}(K_n^{(d)}(t))$ with respect to t . Our approach is different and is applicable to more general random filtrations. For the proof of Theorem 1.4 and an extension (Theorem 4.11), we additionally make use of the result by Linial and Peled [15] on the convergence of $\beta_d(K_n^{(d)}(c/n))/n^d$ as $n \rightarrow \infty$ for each $c \geq 0$.

This paper is organized as follows. In Section 2, we provide fundamental concepts for graphs and simplicial complexes and derive upper estimates of Betti numbers as a quantitative generalization of the cohomology vanishing theorem. In Section 3, we provide several estimates for the expectations of Betti numbers for a class of random simplicial complexes. In Section 4, we introduce the concept of persistent homologies and prove the main theorems about lifetime sums for random simplicial complex processes.

NOTATION. We use the Bachmann–Landau big- O , little- o , and some related notation associated with n (the number of vertices) tending to ∞ . Furthermore, for non-negative functions $f(n)$ and $g(n)$,

- $f(n) = \Omega(g(n))$ means that $g(n) = O(f(n))$; and
- $f(n) \asymp g(n)$ means that $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

The notation $X \sim \nu$ indicates that a random variable X has probability distribution ν . For $a, b \in \mathbb{R}$, $a \vee b$ and $a \wedge b$ denote $\max\{a, b\}$ and $\min\{a, b\}$, respectively.

2. Upper bounds of Betti numbers of simplicial complexes.

2.1. Preliminaries and statement of results.

Let V be a finite set. For $k \geq 0$, $\binom{V}{k}$ denotes the set of all subsets A of V whose cardinalities $\#A$ are k . Note that $\binom{V}{0}$ contains a single element \emptyset . Assume $V \neq \emptyset$ and let $E \subset \binom{V}{2}$. We regard V and E as a vertex set and an edge set, respectively, and call $G = V \sqcup E$ an undirected graph on V . Throughout this article, graphs are simple undirected finite graphs, with no multiple edges and no self-loops.

Saying that $v \in V$ is adjacent to $w \in V$ means $\{v, w\} \in E$. For $v \in V$, the degree of v is defined as $\#\{w \in V \mid \{v, w\} \in E\}$ and is denoted by $\deg(v)$. A vertex $v \in V$ is called isolated if $\deg(v) = 0$. The averaging matrix $A[G] = \{a_{vw}\}_{v, w \in V}$ associated with G is defined by

$$a_{vw} := \begin{cases} 1/\deg(v) & \text{if } w \text{ is adjacent to } v, \\ 1 & \text{if } v \text{ is isolated and } v = w, \\ 0 & \text{otherwise.} \end{cases}$$

The Laplacian $\mathcal{L}[G]$ of a simple random walk on G is defined by $\mathcal{L}[G] := I_V - A[G]$, where I_V is the matrix that acts as the identity operator on V . All the eigenvalues of $\mathcal{L}[G]$ are real and belong to the interval $[0, 2]$ (see Section 2.2 for details). Note that the nonzero vectors that are constant on each connected component are eigenvectors with associated eigenvalue zero, and the number of connected components of G coincides with the multiplicity of the zero eigenvalues. When $\#V \geq 2$, $\lambda_2[G]$ denotes the second smallest eigenvalue of $\mathcal{L}[G]$, counting multiplicities. In particular, $\lambda_2[G] > 0$ if and only if G is connected. We call $\lambda_2[G]$ the spectral gap of G . By convention, $\lambda_2[\emptyset] = 0$, and we set $\lambda_2[G] = 0$ for $\#V = 1$.

We next introduce the concept of simplicial complexes, which are higher-dimensional counterparts of graphs.

DEFINITION 2.1. Let V be a nonempty finite set and X a collection of nonempty subsets of V . X is called an abstract simplicial complex on V if X satisfies the following two conditions.

- (1) $\{v\} \in X$ for all $v \in V$;
- (2) If $\sigma \in X$ and $\emptyset \neq \tau \subset \sigma$, then $\tau \in X$.

In what follows, we omit the word “abstract” and simply call X a simplicial complex. For $\sigma \in X$, its dimension $\dim \sigma$ is defined to be $\#\sigma - 1$. We call $\sigma \in X$ with $\dim \sigma = k$ a k -dimensional simplex or, equivalently, a k -simplex. The dimension of X is defined as the maximum among the dimensions of the simplices in X . Graphs are regarded as zero- or one-dimensional simplicial complexes in a natural manner. We say that $\tau \in X$ is a face of $\sigma \in X$ whenever $\tau \subset \sigma$. For $k \geq 0$, X_k denotes the set of all k -simplices of X . By convention, we regard \emptyset as a (-1) -simplex and set $X_{-1} = \{\emptyset\}$. The k -dimensional skeleton $X^{(k)}$ of X is defined by $X^{(k)} := \bigsqcup_{j=0}^k X_j$. The simplicial complex X is said to include the complete k -dimensional skeleton if $X_k = \binom{V}{k+1}$.

Given a simplicial complex X on V and $k \geq 0$, an ordered sequence (v_0, v_1, \dots, v_k) consisting of $k + 1$ distinct elements of V is called an ordered (k) -simplex of X if $\{v_0, v_1, \dots, v_k\} \in X_k$. The collection of all ordered k -simplices of X is denoted by $\Sigma(X_k)$, with $\Sigma(X) := \bigsqcup_j \Sigma(X_j)$. By convention, we set $\Sigma(X_{-1}) = \Sigma(\{\emptyset\}) := \{\emptyset\}$. Two ordered simplices are called equivalent if they can be transformed into each other by an even permutation. The equivalence class of an ordered simplex $\sigma = (v_0, v_1, \dots, v_k)$ is denoted by $\langle \sigma \rangle$ or $\langle v_0, v_1, \dots, v_k \rangle$ and is called the oriented simplex generated by σ . Let $C_k(X)$ be the \mathbb{R} -vector space of all linear combinations of oriented k -simplices in X with coefficients in \mathbb{R} under the relation that $\langle v_0, v_1, \dots, v_k \rangle = -\langle v_1, v_0, \dots, v_k \rangle$ for any oriented k -simplices. We set $C_{-1}(X) = \mathbb{R}$ per convention. For $k \geq 1$, the k -th boundary map $\partial_k: C_k(X) \rightarrow C_{k-1}(X)$ is well-defined as a linear extension of

$$\partial_k \langle v_0, v_1, \dots, v_k \rangle := \sum_{i=0}^k (-1)^i \langle v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_k \rangle$$

for $\langle v_0, v_1, \dots, v_k \rangle \in C_k(X)$. We also define a linear map $\partial_0: C_0(X) \rightarrow \mathbb{R}$ such that $\partial_0 \langle v \rangle = 1$ for $v \in V$. For all $k \geq 0$, it holds that $\partial_k \circ \partial_{k+1} = 0$, that is, $\ker \partial_k \supset \text{Im } \partial_{k+1}$.

The k -th homology vector space of X over \mathbb{R} is defined by $H_k(X) := \ker \partial_k / \text{Im } \partial_{k+1}$. The dimension of this space is called the k -th Betti number of X and is denoted by $\beta_k(X)$.

REMARK 2.2. In the usual definitions for homologies, ∂_0 is defined as a zero operator. This difference makes the Betti number $\beta_0(X)$ as defined above smaller by 1 than the conventional Betti number. In this sense, $\beta_k(X)$ in our definition is often called the *reduced* Betti number. For simplicity, we omit the word “reduced” throughout this paper.

Let $k \geq 0$. A real-valued function f on $\Sigma(X_k)$ is called a k -cochain if f is alternating, that is, if $f(v_{\xi(0)}, v_{\xi(1)}, \dots, v_{\xi(k)}) = (\text{sgn } \xi) f(v_0, v_1, \dots, v_k)$ for all $(v_0, v_1, \dots, v_k) \in \Sigma(X_k)$ and all permutations ξ on $\{0, 1, \dots, k\}$. The real vector space $C^k(X)$ formed by all k -cochains is called the k -cochain vector space. We set $C^{-1}(X) = \mathbb{R}$ per convention. The k -th coboundary map $d_k: C^k(X) \rightarrow C^{k+1}(X)$ is defined by the linear extension of

$$d_k \varphi(\sigma) := \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma_i)$$

for $\varphi \in C^k(X)$ and $\sigma = (v_0, \dots, v_{k+1}) \in \Sigma(X_{k+1})$, where

$$\sigma_i := (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_{k+1}) \in \Sigma(X_k). \tag{2.1}$$

By definition, $d_{-1} \varphi$ for $\varphi \in C^{-1}(X) = \mathbb{R}$ is identically φ on $\Sigma(X_0)$. Elements of $\text{Im } d_{k-1}$ (resp., $\ker d_k$) are called k -coboundaries (k -cocycles). That $d_k \circ d_{k-1} = 0$, that is, $\ker d_k \supset \text{Im } d_{k-1}$, can be verified. The k -th cohomology vector space of X is then defined by $H^k(X) := \ker d_k / \text{Im } d_{k-1}$. Note that $H^k(X)$ is isomorphic to $H_k(X)$.

To state the cohomology vanishing theorem and its quantitative generalization, we further introduce the concept of links of simplicial complexes. Given a D -dimensional simplicial complex X and a j -simplex τ in X with $-1 \leq j \leq D$, we define the link $\text{lk}_X(\tau)$ of τ in X by

$$\text{lk}_X(\tau) := \{\sigma \in X \mid \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in X\}.$$

Note that $\text{lk}_X(\tau)$ is either the empty set or a simplicial complex with dimension at most $D - j - 1$. If the codimension of τ in X is no more than 2 (i.e., $D - j \leq 2$), then $\text{lk}_X(\tau)$ is either the empty set or a graph. By definition, $\text{lk}_X(\emptyset)$ is always equal to X .

A simplicial complex X is said to be pure D -dimensional if, for every simplex σ in X , there exists some D -simplex containing σ . Note that a pure D -dimensional simplicial complex is D -dimensional. The following is a special case of Theorem 2.1 of [1].

THEOREM 2.3 (Cohomology vanishing theorem [8], [1]). *Let $D \geq 1$, and let X be a pure D -dimensional simplicial complex such that $\lambda_2[\text{lk}_X(\tau)] > 1 - D^{-1}$ for every $(D - 2)$ -simplex $\tau \in X$. Then, $H^{D-1}(X) = \{0\}$.*

The main purpose of this section is to generalize Theorem 2.3 to an upper estimate of the Betti number. This estimate is one of the key elements of the arguments in the

later sections.

DEFINITION 2.4. Let G be a graph on V and let $\{\lambda_i\}_{i=1}^{\#V}$ be all the not necessarily distinct eigenvalues of $\mathcal{L}[G]$. We define

$$\gamma(G; \alpha) := \#\{i \mid \lambda_i \leq \alpha\} - 1$$

for $\alpha \geq 0$. We also set $\gamma(\emptyset; \alpha) := 0$ per convention.

THEOREM 2.5. Let X be a simplicial complex and $D \geq 1$. Then,

$$\beta_{D-1}(X) \leq \sum_{\tau \in X_{D-2}} \gamma(\text{lk}_X(\tau)^{(1)}; 1 - D^{-1}). \tag{2.2}$$

Recall that the graph $\text{lk}_X(\tau)^{(1)}$ is the one-dimensional skeleton of the simplicial complex $\text{lk}_X(\tau)$. This theorem is an extension of Theorem 2.3. In fact, under the assumptions of Theorem 2.3, $\gamma(\text{lk}_X(\tau)^{(1)}; 1 - D^{-1}) = \gamma(\text{lk}_X(\tau); 1 - D^{-1}) = 0$ for every $\tau \in X_{D-2}$, so that $\beta_{D-1}(X) = 0$.

We devote the rest of this section to proving Theorem 2.5. The proof is based on careful modifications to the proof of Theorem 2.3 and a nice transformation of X to remove the assumption of pure dimensionality.

2.2. Auxiliary operators.

Here, we give an overview of some preliminary concepts and facts about simplicial complexes as described in [1] and state them in a way that is useful for this research. Let X be a pure D -dimensional simplicial complex. For simplices σ in X , $m(\sigma)$ denotes the number of D -simplices containing σ . Note that $m(\cdot) \geq 1$ from the assumption of pure D -dimensionality. For ordered simplices $\sigma = (v_0, \dots, v_k) \in \Sigma(X)$, $m(\sigma)$ is defined as $m(\{v_0, \dots, v_k\})$. Simple calculations show that, for all $-1 \leq k \leq D$ and $\tau \in \Sigma(X_k)$,

$$\sum_{\sigma \in \Sigma(X_{k+1}); \sigma \supset \tau} m(\sigma) = (k + 2)! (D - k)m(\tau),$$

where $\sigma \supset \tau$ implies that all vertices of τ are vertices of σ . We equip $C^k(X)$ with an inner product (\cdot, \cdot) defined by

$$(\varphi, \psi) = \frac{1}{(k + 1)!} \sum_{\sigma \in \Sigma(X_k)} m(\sigma)\varphi(\sigma)\psi(\sigma) \quad \text{for } \varphi, \psi \in C^k(X), k \geq 0 \tag{2.3}$$

and

$$(\varphi, \psi) = \#X_D\varphi\psi \quad \text{for } \varphi, \psi \in C^{-1}(X) = \mathbb{R}. \tag{2.4}$$

The induced norm is denoted by $\|\cdot\|$. For $k \geq -1$, denote by $\delta_{k+1}: C^{k+1}(X) \rightarrow C^k(X)$ the adjoint operator of d_k , that is, the unique operator satisfying $(d_k\varphi, \psi) = (\varphi, \delta_{k+1}\psi)$ for all $\varphi \in C^k(X)$ and $\psi \in C^{k+1}(X)$. For ordered simplices $\sigma = (v_0, \dots, v_k) \in \Sigma(X_k)$ and $\tau = (w_0, \dots, w_l) \in \Sigma(X_l)$, the notation $\sigma\tau$ indicates an ordered sequence

$(v_0, \dots, v_k, w_0, \dots, w_l)$. A straightforward calculation gives the following expressions. For $\varphi \in C^k(X)$, $\psi \in C^{k+1}(X)$, and $\sigma \in \Sigma(X_k)$ with $k \geq 0$,

$$\delta_{k+1}\psi(\sigma) = \sum_{\substack{v \in \Sigma(X_0); \\ v\sigma \in \Sigma(X_{k+1})}} \frac{m(v\sigma)}{m(\sigma)} \psi(v\sigma)$$

and

$$\delta_{k+1}d_k\varphi(\sigma) = (D - k)\varphi(\sigma) - \sum_{\substack{v \in \Sigma(X_0); \\ v\sigma \in \Sigma(X_{k+1})}} \sum_{i=0}^k (-1)^i \frac{m(v\sigma)}{m(\sigma)} \varphi(v\sigma_i). \tag{2.5}$$

See (2.1) for the definition of σ_i .

For $k \geq 0$, the down Laplacian and the up Laplacian on $C^k(X)$ are defined by $L_k^{\text{down}} := d_{k-1}\delta_k$ and $L_k^{\text{up}} := \delta_{k+1}d_k$, respectively. The Laplacian L_k on $C^k(X)$ is defined by

$$L_k := L_k^{\text{down}} + L_k^{\text{up}}.$$

Note that L_k^{down} , L_k^{up} , and L_k are self-adjoint and non-negative definite operators with respect to the inner product (2.3) and, further, that the relations

$$\ker L_k^{\text{down}} = \ker \delta_k, \tag{2.6}$$

$$\ker L_k^{\text{up}} = \ker d_k, \tag{2.7}$$

$$H^k(X) \simeq \ker L_k^{\text{down}} \cap \ker L_k^{\text{up}} = \ker L_k \tag{2.8}$$

hold, which can be shown by simple calculations. We also note that if G is a pure one-dimensional simplicial complex on V (i.e., a graph without isolated vertices), then $L_0^{\text{up}} = \mathcal{L}[G]$ from (2.5). By combining the fact that the transpose of $A[G]$ is a stochastic matrix, the eigenvalues of $\mathcal{L}[G]$ are all real and lie between 0 and 2.

2.3. Localization.

Let X be a pure D -dimensional simplicial complex, and let $\tau = (v_0, \dots, v_j) \in \Sigma(X_j)$ be a fixed ordered j -simplex in X with $-1 \leq j \leq D$. We write $\text{lk}_X(\tau)$ for $\text{lk}_X(\{v_0, \dots, v_j\})$ and define $m_\tau(\eta) = m(\tau\eta)$ for $\eta \in \text{lk}_X(\tau)$. In other words, $m_\tau(\eta)$ is the number of $(D - j - 1)$ -simplices in $\text{lk}_X(\tau)$ containing η . For $0 \leq l \leq D - j - 1$, $\Sigma(\text{lk}_X(\tau)_l)$ denotes the set of all ordered l -simplices of $\text{lk}_X(\tau)$. Since $\text{lk}_X(\tau)$ is a pure $(D - j - 1)$ -dimensional simplicial complex, we can define various concepts for $\text{lk}_X(\tau)$, as we did in the previous subsection for X , by replacing X with $\text{lk}_X(\tau)$. We distinguish the concepts from those for X by adding the superscript (or subscript) τ in the notation. For example, the coboundary operator on $C^l(\text{lk}_X(\tau))$ is denoted d_τ^l . Other symbols are indicated by $(\cdot, \cdot)_\tau, \|\cdot\|_\tau, \delta_\tau^l$, and so on.

DEFINITION 2.6. Let $-1 \leq j < k \leq D$ and take $\tau \in \Sigma(X_j)$ and $\varphi \in C^k(X)$. The localization $\varphi_\tau \in C^{k-j-1}(\text{lk}_X(\tau))$ of φ with respect to τ is defined by

$$\varphi_\tau(\eta) = \varphi(\tau\eta) \quad \text{for } \eta \in \Sigma(\text{lk}_X(\tau)_{k-j-1}).$$

A straightforward calculation gives the following identities.

LEMMA 2.7 ([1, Lemmas 1.10 and 1.12, and Corollary 1.13]). *For $D \geq 1$ and $\varphi \in C^{D-1}(X)$, the following identities hold.*

$$D\|\varphi\|^2 = \frac{1}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} \|\varphi_\tau\|_\tau^2,$$

$$\|d_{D-1}\varphi\|^2 - \|\varphi\|^2 = \frac{1}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} (\|d_0^\tau \varphi_\tau\|_\tau^2 - \|\varphi_\tau\|_\tau^2).$$

In particular, we have

$$\|d_{D-1}\varphi\|^2 = \frac{1}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} \{\|d_0^\tau \varphi_\tau\|_\tau^2 - (1 - D^{-1})\|\varphi_\tau\|_\tau^2\}. \tag{2.9}$$

2.4. Upper bounds of Betti numbers.

Let $D \geq 1$ and let X be a pure D -dimensional simplicial complex. Take $\tau \in X_{D-2}$ or $\tau \in \Sigma(X_{D-2})$. Let $N_X(\tau)$ denote the number of vertices in $\text{lk}_X(\tau)$. The eigenvalues of $\mathcal{L}[\text{lk}_X(\tau)]$, including repeated values, are denoted by λ_i^τ and the corresponding eigenvectors by ψ_i^τ ($i = 1, \dots, N_X(\tau)$). Without loss of generality, we may assume that $\lambda_1^\tau = 0$, ψ_1^τ is a constant vector, and $\{\psi_i^\tau\}_{i=1}^{N_X(\tau)}$ is an orthonormal basis of $C^0(\text{lk}_X(\tau))$. We consider the orthogonal decomposition of $C^0(\text{lk}_X(\tau))$:

$$C^0(\text{lk}_X(\tau)) = A_1^\tau \oplus A_2^\tau \oplus A_3^\tau,$$

where

$$A_1^\tau = \text{span}_{\mathbb{R}}\{\psi_1^\tau\},$$

$$A_2^\tau = \text{span}_{\mathbb{R}}\{\psi_i^\tau \mid i \neq 1 \text{ and } \lambda_i^\tau \leq 1 - D^{-1}\},$$

$$A_3^\tau = \text{span}_{\mathbb{R}}\{\psi_i^\tau \mid \lambda_i^\tau > 1 - D^{-1}\}.$$

For $j = 1, 2, 3$, π_j^τ denotes the orthogonal projection of $C^0(\text{lk}_X(\tau))$ onto A_j^τ . We obtain the following formula by direct calculation.

LEMMA 2.8 ([1, Lemma 1.11]). *For $\varphi \in C^{D-1}(X)$ and $\tau \in \Sigma(X_{D-2})$,*

$$\|\pi_1^\tau \varphi_\tau\|_\tau^2 = \frac{m(\tau)}{2} ((\delta_{D-1}\varphi)(\tau))^2.$$

The following lemma is important for proving the cohomology vanishing theorem and its quantitative generalization.

LEMMA 2.9. *Let X be a pure D -dimensional simplicial complex with $D \geq 1$. Then, for each $\tau \in X_{D-2}$, there exists some $i \in \{1, \dots, N_X(\tau)\}$ such that $\lambda_i^\tau \geq 1$. Moreover, for $\varphi \in C^{D-1}(X)$,*

$$\|\varphi\|^2 \leq \frac{1}{\lambda D} \left(\frac{2}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} \|\pi_2^\tau \varphi_\tau\|_\tau^2 + \|\delta_{D-1} \varphi\|^2 + \|d_{D-1} \varphi\|^2 \right), \tag{2.10}$$

where $\lambda := \min\{\lambda_i^\tau - (1 - D^{-1}) \mid \tau \in X_{D-2} \text{ and } \lambda_i^\tau > 1 - D^{-1}\} > 0$. In particular, if $\varphi \in \ker L_{D-1}$, then

$$\|\varphi\|^2 \leq \frac{2}{\lambda \cdot D!} \sum_{\tau \in \Sigma(X_{D-2})} \|\pi_2^\tau \varphi_\tau\|_\tau^2. \tag{2.11}$$

PROOF. Note that the zeroth up Laplacian on $\text{lk}_X(\tau)$ is equal to $\mathcal{L}[\text{lk}_X(\tau)]$. For simplicity, we use \mathcal{L} to denote $\mathcal{L}[\text{lk}_X(\tau)]$ in the following. Since X is pure, by the definition of \mathcal{L} ,

$$\sum_{i=1}^{N_X(\tau)} \lambda_i^\tau = \text{tr}(\mathcal{L}) = N_X(\tau)$$

for each $\tau \in X_{D-2}$. This implies that $\lambda_i^\tau \geq 1$ for some $i \in \{1, \dots, N_X(\tau)\}$.

For an arbitrary $\tau \in \Sigma(X_{D-2})$ and $\varphi \in C^{D-1}(X)$,

$$\begin{aligned} \|d_0^\tau \varphi_\tau\|_\tau^2 &= (\mathcal{L}\varphi_\tau, \varphi_\tau)_\tau = \sum_{i=1}^3 (\mathcal{L}\pi_i^\tau \varphi_\tau, \pi_i^\tau \varphi_\tau)_\tau \\ &\geq (\mathcal{L}\pi_3^\tau \varphi_\tau, \pi_3^\tau \varphi_\tau)_\tau \geq \kappa \|\pi_3^\tau \varphi_\tau\|_\tau^2, \end{aligned}$$

where $\kappa := (1 - D^{-1}) + \lambda \leq 2$. In the above equation, the first inequality follows from the non-negative definiteness of \mathcal{L} . The second equality follows by expressing $\pi_3^\tau \varphi_\tau$ as a linear combination of the eigenvectors $\{\psi_i^\tau\}_i$ generating A_3^τ . From (2.9), we have

$$\begin{aligned} \|d_{D-1} \varphi\|^2 &= \frac{1}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} (\|d_0^\tau \varphi_\tau\|_\tau^2 - (1 - D^{-1}) \|\varphi_\tau\|_\tau^2) \\ &\geq \frac{1}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} (\kappa \|\pi_3^\tau \varphi_\tau\|_\tau^2 - (1 - D^{-1}) \|\varphi_\tau\|_\tau^2) \\ &= \frac{1}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} (\lambda \|\pi_3^\tau \varphi_\tau\|_\tau^2 - (1 - D^{-1}) (\|\pi_1^\tau \varphi_\tau\|_\tau^2 + \|\pi_2^\tau \varphi_\tau\|_\tau^2)). \end{aligned}$$

Therefore,

$$\frac{\lambda}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} \|\pi_3^\tau \varphi_\tau\|_\tau^2 \leq \|d_{D-1} \varphi\|^2 + \frac{1 - D^{-1}}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} (\|\pi_1^\tau \varphi_\tau\|_\tau^2 + \|\pi_2^\tau \varphi_\tau\|_\tau^2). \tag{2.12}$$

Then,

$$\lambda D \|\varphi\|^2 = \frac{1}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} \lambda \|\varphi_\tau\|_\tau^2 \quad (\text{from Lemma 2.7})$$

$$\begin{aligned}
 &= \frac{1}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} \lambda \sum_{i=1}^3 \|\pi_i^\tau \varphi_\tau\|_\tau^2 \\
 &\leq \frac{\kappa}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} (\|\pi_1^\tau \varphi_\tau\|_\tau^2 + \|\pi_2^\tau \varphi_\tau\|_\tau^2) + \|d_{D-1}\varphi\|^2 \quad (\text{from (2.12)}) \\
 &= \frac{\kappa}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} \frac{m(\tau)}{2} ((\delta_{D-1}\varphi)(\tau))^2 \\
 &\quad + \frac{\kappa}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} \|\pi_2^\tau \varphi_\tau\|_\tau^2 + \|d_{D-1}\varphi\|^2 \quad (\text{from Lemma 2.8}) \\
 &\leq \|\delta_{D-1}\varphi\|^2 + \frac{2}{(D-1)!} \sum_{\tau \in \Sigma(X_{D-2})} \|\pi_2^\tau \varphi_\tau\|_\tau^2 + \|d_{D-1}\varphi\|^2.
 \end{aligned}$$

Thus, (2.10) holds. If $\varphi \in \ker L_{D-1}$, then the second and third terms in the right-hand side of (2.10) vanish because

$$\varphi \in \ker L_{D-1} = \ker L_{D-1}^{\text{down}} \cap \ker L_{D-1}^{\text{up}} = \ker \delta_{D-1} \cap \ker d_{D-1}$$

from (2.6), (2.7), and (2.8). Thus, (2.11) holds. □

Theorem 2.3 can now be proved by combining Lemma 2.9 with (2.8), but we proceed further. First, we prove a variant of Theorem 2.5 with the extra assumption that X is pure D -dimensional.

THEOREM 2.10. *Let X be a pure D -dimensional simplicial complex with $D \geq 1$. Then,*

$$\beta_{D-1}(X) \leq \sum_{\tau \in X_{D-2}} \gamma(\text{lk}_X(\tau); 1 - D^{-1}).$$

PROOF. For each $\tau \in X_{D-2}$, select an arbitrary $\hat{\tau} \in \Sigma(X_{D-2})$ from the ordered sequences of elements of τ . Define a linear map

$$F: \ker L_{D-1} \rightarrow \bigoplus_{\tau \in X_{D-2}} A_2^\tau$$

by $F(\varphi) = (\pi_2^\tau \varphi_{\hat{\tau}})_{\tau \in X_{D-2}}$. From Lemma 2.9, F is injective. Comparing the dimensionalities, we have

$$\beta_{D-1}(X) = \dim(\ker L_{D-1}) \leq \sum_{\tau \in X_{D-2}} \dim A_2^\tau = \sum_{\tau \in X_{D-2}} \gamma(\text{lk}_X(\tau); 1 - D^{-1}). \quad \square$$

We now remove the assumption of pure D -dimensionality in Theorem 2.10. Let $D \geq 1$ and let X be a simplicial complex. We assume $\dim X \geq D - 1$ and consider the $(D - 1)$ -th Betti number. Let $M_{D-1}(X)$ be the set of all maximal $(D - 1)$ -simplices in X , namely, the set of all $(D - 1)$ -simplices that are not contained in any D -simplex. We define a new simplicial complex \overline{X}^D by adding vertices and simplices to X_D as follows.

For each $\sigma \in M_{D-1}(X)$, we construct a D -simplex $s_\sigma := \sigma \sqcup \{v_\sigma\}$ with a new vertex v_σ . We let \overline{X}^D be the simplicial complex generated by $X_D \cup \{s_\sigma\}_{\sigma \in M_{D-1}(X)}$, that is, the smallest simplicial complex that includes $X_D \cup \{s_\sigma\}_{\sigma \in M_{D-1}(X)}$. Clearly, \overline{X}^D is a pure D -dimensional simplicial complex. In addition, the identity

$$\beta_{D-1}(X) = \beta_{D-1}(\overline{X}^D) \tag{2.13}$$

holds. Indeed, let $X^{[D]}$ denote the simplicial complex $X \cap \overline{X}^D$. In other words, $X^{[D]}$ is generated by $X_D \cup X_{D-1}$. Since $\ker d_{D-1}$ and $\text{Im } d_{D-2}$ do not change from replacing X with $X^{[D]}$, we have $\beta_{D-1}(X) = \beta_{D-1}(X^{[D]})$. By construction, \overline{X}^D and $X^{[D]}$ are homotopy equivalent, which implies that $\beta_{D-1}(\overline{X}^D) = \beta_{D-1}(X^{[D]})$. Thus, (2.13) holds.

For the proof of the following lemma, we note a few simple facts about some graphs G . Let $\alpha \in [0, 2)$. If G consists of only one vertex or one edge and its two vertices, then $\gamma(G; \alpha) = 0$. Also, adding an isolated vertex or an isolated edge to a non-empty graph G increases $\gamma(G; \alpha)$ by exactly one.

LEMMA 2.11. *Let D and X be as stated above. For $\alpha \in [0, 2)$,*

$$\sum_{\tau \in X_{D-2}} \gamma(\text{lk}_X(\tau)^{(1)}; \alpha) = \sum_{\tau \in (\overline{X}^D)_{D-2}} \gamma_{D-1}(\text{lk}_{\overline{X}^D}(\tau); \alpha).$$

PROOF. If $\tau \in X_{D-2} \setminus (\overline{X}^D)_{D-2}$, then $\text{lk}_X(\tau) = \emptyset$, which implies that $\gamma(\text{lk}_X(\tau)^{(1)}; \alpha) = 0$. Suppose $\tau \in (\overline{X}^D)_{D-2} \setminus X_{D-2}$. Then there exists some $\sigma \in M_{D-1}(X)$ such that $\tau \subset s_\sigma$ and $\tau \not\subset \sigma$. Since s_σ is the only D -simplex of \overline{X}^D that contains τ , $\text{lk}_{\overline{X}^D}(\tau)$ consists of only the isolated edge $s_\sigma \setminus \tau$. Thus, $\gamma(\text{lk}_{\overline{X}^D}(\tau); \alpha) = 0$. If $\tau \in X_{D-2} \cap (\overline{X}^D)_{D-2}$, then $\gamma(\text{lk}_X(\tau)^{(1)}; \alpha) = \gamma(\text{lk}_{\overline{X}^D}(\tau); \alpha)$ because $\text{lk}_{\overline{X}^D}(\tau)$ is obtained from $\text{lk}_{X^{(D)}}(\tau)$ by replacing its isolated vertices with isolated edges. Therefore,

$$\begin{aligned} \sum_{\tau \in X_{D-2}} \gamma(\text{lk}_X(\tau)^{(1)}; \alpha) &= \sum_{\tau \in (X^{[D]})_{D-2}} \gamma(\text{lk}_{X^{[D]}}(\tau); \alpha) \\ &= \sum_{\tau \in (\overline{X}^D)_{D-2}} \gamma(\text{lk}_{\overline{X}^D}(\tau); \alpha). \end{aligned} \quad \square$$

We can now prove Theorem 2.5.

PROOF OF THEOREM 2.5. We may assume that $\dim X \geq D - 1$, since otherwise both sides of (2.2) vanish. Noting that \overline{X}^D is a pure D -dimensional simplicial complex, we have

$$\begin{aligned} \beta_{D-1}(X) &= \beta_{D-1}(\overline{X}^D) \quad (\text{from (2.13)}) \\ &\leq \sum_{\tau \in (\overline{X}^D)_{D-2}} \gamma(\text{lk}_{\overline{X}^D}(\tau); 1 - D^{-1}) \quad (\text{from Theorem 2.10}) \\ &= \sum_{\tau \in X_{D-2}} \gamma(\text{lk}_X(\tau)^{(1)}; 1 - D^{-1}) \quad (\text{from Lemma 2.11}). \end{aligned} \quad \square$$

As a corollary, the assumption of pure D -dimensionality in Theorem 2.3 can be removed.

COROLLARY 2.12. *Let $D \geq 1$, and let X be a simplicial complex such that $\lambda_2[\text{lk}_X(\tau)] > 1 - D^{-1}$ for every $(D - 2)$ -simplex $\tau \in X$. Then, $H^{D-1}(X) = \{0\}$.*

3. Estimates of Betti numbers of random simplicial complexes.

3.1. Statement of results.

In this section, we consider multi-parameter random simplicial complexes, which were introduced in [2], [6], and give some estimates of their Betti numbers. Linial–Meshulam complexes [14] and random clique complexes [12] are shown as typical examples in this framework.

Let $n \in \mathbb{N}$ and $\mathbf{p} = (p_0, p_1, \dots, p_{n-1})$ be a multi-parameter with $0 \leq p_i \leq 1$ for all $i = 0, 1, \dots, n - 1$. We start with the set V of n vertices and retain each vertex with independent probability p_0 . Next, each edge with both ends retained is added with independent probability p_1 . Iteratively, for $i = 2, 3, \dots, n - 1$, each i -simplex for which all faces were added by this procedure is added to our complex with independent probability p_i . The distribution of the resulting random simplicial complexes X is denoted by $X(n, \mathbf{p})$. We call this model the multi-parameter random complex model with n vertices and multi-parameter \mathbf{p} .

EXAMPLE 3.1. Let $n > d \geq 1$ and let $0 \leq p \leq 1$ be fixed. Define $\mathbf{p} = (p_0, p_1, \dots, p_{n-1})$ by

$$p_i := \begin{cases} 1 & (0 \leq i \leq d - 1), \\ p & (i = d), \\ 0 & (d + 1 \leq i \leq n - 1). \end{cases}$$

The corresponding random simplicial complex follows the d -Linial–Meshulam complex model $Y_d(n, p)$. The Erdős–Rényi graph model $G(n, p)$ is identified with $Y_1(n, p)$.

EXAMPLE 3.2. Let $n > d \geq 1$ and let $0 \leq p \leq 1$ be fixed. Define $\mathbf{p} = (p_0, p_1, \dots, p_{n-1})$ by

$$p_i := \begin{cases} 1 & (0 \leq i \leq d - 1), \\ p & (i = d), \\ 1 & (d + 1 \leq i \leq n - 1). \end{cases}$$

We call the corresponding random simplicial complex $C_n^{(d)}(p)$ the random d -flag complex. The random clique complex $C_n(p)$ is identical to the random 1-flag complex $C_n^{(1)}(p)$.

We state several estimates for the Betti numbers of the multi-parameter random complexes $X(n, \mathbf{p})$ according to the dependence of \mathbf{p} on n . Their proofs are left to subsequent subsections. To state the propositions, we introduce some notation:

$$\begin{aligned}
 q_{-1} &:= 1, & q_k &:= \prod_{i=0}^k p_i^{\binom{k+1}{i+1}} \quad (0 \leq k \leq n-1), \\
 r_k &:= \frac{q_{k+1}}{q_k} = \prod_{i=0}^{k+1} p_i^{\binom{k+1}{i}} \quad (-1 \leq k \leq n-2).
 \end{aligned}
 \tag{3.1}$$

Here, we set $0/0 = 0$ and $\binom{0}{0} = 1$ per convention. Note that $0 \leq q_k \leq 1$, $0 \leq r_k \leq 1$, and $\mathbb{P}(\sigma \in X) = q_k$ for any $\sigma \in \binom{V}{k+1}$. Moreover, both q_k and r_k are nonincreasing with respect to k .

In what follows, $k \geq 0$ is always fixed. The following proposition follows from an easy application of the Morse inequality (see also Section 6 of [6]).

PROPOSITION 3.3. *Let $c_1 > 0$ and $c_2 \geq 0$ satisfy $(k+1)/c_1 + c_2/(k+2) < 1$. Then, there exist some $n_0 \in \mathbb{N}$ and $\varepsilon_0 > 0$, depending only on k , c_1 , and c_2 , such that if $n \geq n_0$, then*

$$r_{k-1} \geq \frac{c_1}{n} \quad \text{and} \quad r_k \leq \frac{c_2}{n} \tag{3.2}$$

together imply $\mathbb{E}[\beta_k(X)] \geq \varepsilon_0 n^{k+1} q_k$.

The next result is an upper estimate for r_k sufficiently large. This is a generalization of the cohomology vanishing theorem to multi-parameter random complexes (see, e.g., [13, Theorem 1.1 (1)] and [6, Theorem 2]).

PROPOSITION 3.4. *Let $\rho \geq 1$ and $\delta > 0$. Then, there exists a $K_0 > 0$, depending only on k , ρ , and δ , such that if*

$$r_{k-1} \geq \frac{K_0}{n} \quad \text{and} \quad r_k \geq \frac{(\rho + \delta) \log(nr_{k-1})}{n}, \tag{3.3}$$

then $\mathbb{P}(\beta_k(X) \neq 0) \leq n^{k+1} q_k (nr_{k-1})^{-\rho}$ and $\mathbb{E}[\beta_k(X)] \leq n^{k+1} q_k (nr_{k-1})^{1-\rho}$.

COROLLARY 3.5. *Let $\theta \in (0, 1]$, $\nu \geq \theta$, $\delta > 0$, and $M > 0$. Then, for sufficiently large n ,*

$$r_k \geq \frac{(\nu + \delta) \log n}{n} \quad \text{and} \quad Mr_k^{1-\theta} \geq r_{k-1} \tag{3.4}$$

imply

$$\mathbb{P}(\beta_k(X) \neq 0) \leq n^{k+1} q_k (nr_{k-1})^{-\nu/\theta} \quad \text{and} \quad \mathbb{E}[\beta_k(X)] \leq n^{k+1} q_k (nr_{k-1})^{1-\nu/\theta}.$$

The last result is a general upper estimate, which is the main result of this section.

THEOREM 3.6. *Let $l \in \mathbb{N}$. There exists a constant $C \geq 0$ depending only on k and l such that for all $n \in \mathbb{N}$,*

$$\mathbb{E}[\beta_k(X)] \leq n^{k+1} q_k \{1 \wedge C(nr_k)^{-l}\}.$$

Here, if $r_k = 0$, then the right-hand side is interpreted as $n^{k+1}q_k$.

We apply these results to typical examples.

EXAMPLE 3.7 (Linial–Meshulam complex). Consider the d -Linial–Meshulam complex $X \sim Y_d(n, p)$, as in Example 3.1. Letting $k = d - 1$, we obtain $q_k = 1$, $r_{k-1} = 1$, and $r_k = p$. Theorem 3.6, Proposition 3.3, and Corollary 3.5 with $\theta = M = 1$ together imply that for given $0 < c < k + 2$, $l \in \mathbb{N}$, and $\nu' > \nu \geq 1$, there exist $\varepsilon_0 > 0$ and $C \geq 0$ such that the following hold.

- (1) For every $n \in \mathbb{N}$,

$$\mathbb{E}[\beta_{d-1}(X)] \leq n^d \{1 \wedge C(np)^{-l}\}.$$

- (2) For sufficiently large n , if $p \leq c/n$, then

$$\mathbb{E}[\beta_{d-1}(X)] \geq \varepsilon_0 n^d.$$

- (3) For sufficiently large n , if $p \geq (\nu' \log n)/n$, then

$$\mathbb{P}(\beta_{d-1}(X) \neq 0) \leq n^{d-\nu} \quad \text{and} \quad \mathbb{E}[\beta_{d-1}(X)] \leq n^{d+1-\nu}.$$

EXAMPLE 3.8 (Random clique complex). Consider the random clique complex $X \sim C(n, p)$, as in Example 3.2. For $0 \leq k \leq n - 1$, we obtain $q_k = p^{\binom{k+1}{2}}$, $r_{k-1} = p^k$, and $r_k = p^{k+1}$. Here, $\binom{1}{2} = 0$ per convention. Theorem 3.6, Proposition 3.3, and Corollary 3.5 with $\theta = 1/(k + 1)$ and $M = 1$ together imply that, for given $c_1 > 0$ and $c_2 > 0$ with $(k + 1)/c_1 + c_2/(k + 2) < 1$, $l \in \mathbb{N}$, and $\nu' > \nu \geq 1/(k + 1)$, there exist $\varepsilon_0 > 0$ and $C \geq 0$ such that the following hold.

- (1) For every $n \in \mathbb{N}$,

$$\mathbb{E}[\beta_k(X)] \leq n^{k+1} p^{\binom{k+1}{2}} \{1 \wedge C(np^{k+1})^{-l}\}.$$

- (2) For sufficiently large n , if $(c_1/n)^{1/k} \leq p \leq (c_2/n)^{1/(k+1)}$, then

$$\mathbb{E}[\beta_k(X)] \geq \varepsilon_0 n^{k+1} p^{\binom{k+1}{2}}.$$

- (3) For sufficiently large n , if $p \geq \{(\nu' \log n)/n\}^{1/(k+1)}$, then

$$\begin{aligned} \mathbb{P}(\beta_k(X) \neq 0) &\leq n^{k+1} p^{\binom{k+1}{2}} (np^k)^{-(k+1)\nu} \\ &= n^{k/2+1-\nu} (np^{k+1})^{k/2-k\nu} \end{aligned}$$

and

$$\mathbb{E}[\beta_k(X)] \leq n^{k+1} p^{\binom{k+1}{2}} (np^k)^{1-(k+1)\nu}.$$

Note that when $\nu' > k/2 + 1 + \eta$ for some $\eta \geq 0$ in (3), the first conclusion implies $\mathbb{P}(\beta_k(X) \neq 0) = o(n^{-\eta})$. When $\eta = 0$, this claim is consistent with Theorem 1.1 (1) in [13].

3.2. Proofs of Propositions 3.3 and 3.4.

We follow [12, Section 7] for the proof of Proposition 3.3. For $k \geq 0$, let $f_k(X)$ denote the cardinality of X_k , the set of all k -simplices of X . We set $f_{-1}(X) = 1$ per convention. The following inequality holds for arbitrary simplicial complexes.

LEMMA 3.9 (A version of the Morse inequality). *Let X be a simplicial complex. For every $k \geq 0$,*

$$f_k(X) - f_{k+1}(X) - f_{k-1}(X) \leq \beta_k(X) \leq f_k(X). \tag{3.5}$$

PROOF. Since $f_k(X) = \dim \ker d_k + \text{rank } d_k$, we have

$$\begin{aligned} \beta_k(X) &= \dim \ker d_k - \text{rank } d_{k-1} \\ &= (f_k(X) - \text{rank } d_k) - \text{rank } d_{k-1} \\ &\geq f_k(X) - f_{k+1}(X) - f_{k-1}(X) \end{aligned}$$

and

$$\beta_k(X) \leq \dim \ker d_k \leq f_k(X). \tag{□}$$

PROOF OF PROPOSITION 3.3. Choose $n_0 > k$ such that

$$\frac{n_0(k+1)}{(n_0-k)c_1} + \frac{c_2}{k+2} < 1.$$

Then, (3.2) implies that, for $n \geq n_0$,

$$\mathbb{E}[f_{k+1}(X)] = \binom{n}{k+2} q_{k+1} = \frac{(n-k-1)r_k}{k+2} \mathbb{E}[f_k(X)] \leq \frac{c_2}{k+2} \mathbb{E}[f_k(X)]$$

and

$$\mathbb{E}[f_{k-1}(X)] = \binom{n}{k} q_{k-1} = \frac{k+1}{(n-k)r_{k-1}} \mathbb{E}[f_k(X)] \leq \frac{n_0}{n_0-k} \frac{k+1}{c_1} \mathbb{E}[f_k(X)].$$

Combining these estimates with the first inequality of (3.5) yields the desired inequality. □

We now turn to proving Proposition 3.4. The following theorem states that the spectral gap of the Erdős–Rényi graph $G \sim G(n, p)$ concentrates around 1 if the parameter p is sufficiently large.

THEOREM 3.10 (Theorem 1.1 in [11], spectral gap theorem in [13]). *Let $G \sim G(n, p)$ be the Erdős–Rényi graph. Let $\eta \geq 0$, $\delta > 0$, and $\varepsilon > 0$. Then, for sufficiently large n , if*

$$p \geq \frac{(1 + \eta + \delta) \log n}{n},$$

then $\mathbb{P}(\lambda_2[G] > 1 - \varepsilon) \geq 1 - \varepsilon n^{-\eta}$.

The following lemma concerns the structure of links in multi-parameter random complexes. Let $X \sim X(n, \mathbf{p})$ be a multi-parameter random complex with an n -point vertex set V that is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a simplex τ with $\mathbb{P}(\tau \in X) > 0$, we define a probability space $(\Omega_\tau, \mathcal{F}_\tau, \mathbb{P}_\tau)$ by

$$\Omega_\tau := \{\tau \in X\}, \mathcal{F}_\tau := \{B \in \mathcal{F} \mid B \subset \Omega_\tau\}, \text{ and } \mathbb{P}_\tau(\cdot) := \mathbb{P}(\cdot \mid \Omega_\tau).$$

Let $V_X(\tau)$ denote the vertex set of $\text{lk}_X(\tau)$ and $N_X(\tau)$ denote its cardinality; these are random variables on Ω_τ . The expectation with respect to \mathbb{P}_τ is denoted by \mathbb{E}_τ . Let $\text{Bin}(n, p)$ indicate the binomial distribution with parameters n and p .

LEMMA 3.11. *Let $0 \leq k \leq n - 2$ and consider $\tau \in \binom{V}{k}$. Provided that $\mathbb{P}(\tau \in X) > 0$, the distribution of $(\text{lk}_X(\tau))^{(1)}$ under \mathbb{P}_τ is $X(n - k, (r_{k-1}, r_k/r_{k-1}, 0, \dots, 0))$. In particular, the distribution of $N_X(\tau)$ under \mathbb{P}_τ is $\text{Bin}(n - k, r_{k-1})$.*

PROOF. A vertex $v \in V \setminus \tau$ belongs to $V_X(\tau)$ if and only if X contains every possible simplex that can be described as the union of $\{v\}$ and a subset of τ . For $0 \leq i \leq k$, there are $\binom{k}{i}$ such i -simplices. This implies

$$\mathbb{P}_\tau(v \in V_X(\tau)) = \prod_{i=0}^k p_i^{\binom{k}{i}} = r_{k-1}.$$

Moreover, events $\{v \in V_X(\tau)\}_{v \in V \setminus \tau}$ are independent under \mathbb{P}_τ since distinct events are described in terms of distinct simplices. Let $\hat{V} \subset V \setminus \tau$ with $\mathbb{P}_\tau(V_X(\tau) = \hat{V}) > 0$. An edge between vertices v and w in \hat{V} belongs to $\text{lk}_X(\tau)$ if and only if X contains every possible simplex described as the union of $\{v, w\}$ and a subset of τ . For $1 \leq i \leq k + 1$, there are $\binom{k}{i-1}$ such possible i -simplices. This implies

$$\mathbb{P}_\tau(\{v, w\} \in \text{lk}_X(\tau) \mid V_X(\tau) = \hat{V}) = \prod_{i=1}^{k+1} p_i^{\binom{k}{i-1}} = \frac{r_k}{r_{k-1}}.$$

Moreover, events $\{e \in \text{lk}_X(\tau)\}_{e \in \binom{\hat{V}}{2}}$ are independent under $\mathbb{P}_\tau(\cdot \mid V_X(\tau) = \hat{V})$ by the same reasoning as used above. Thus, the claim holds. □

PROOF OF PROPOSITION 3.4. From Theorem 2.5,

$$\begin{aligned} \mathbb{E}[\beta_k(X)] &\leq \sum_{\tau \in \binom{V}{k}} \mathbb{E}[N_X(\tau); \tau \in X_{k-1}, \lambda_2[\text{lk}_X(\tau)^{(1)}] \leq 1 - (k + 1)^{-1}] \\ &= \sum_{\tau \in \binom{V}{k}} \mathbb{P}(\tau \in X_{k-1}) \mathbb{E}_\tau[N_X(\tau); \lambda_2[\text{lk}_X(\tau)^{(1)}] \leq 1 - (k + 1)^{-1}] \\ &= \binom{n}{k} q_{k-1} \mathbb{E}[N; \lambda_2[Z] \leq 1 - (k + 1)^{-1}], \end{aligned}$$

where $Z \sim X(n - k, (r_{k-1}, r_k/r_{k-1}, 0, \dots, 0))$ and N is the number of vertices of Z , which follows $\text{Bin}(n - k, r_{k-1})$. The last identity follows from Lemma 3.11. Define $\mu := \mathbb{E}[N] = (n - k)r_{k-1}$ and recall the Chernoff bound

$$\mathbb{P}[|N - \mu| > \mu^{3/5}] \leq \exp\left(-\frac{\mu^{1/5}}{5}\right). \tag{3.6}$$

If $K_0 \geq 2k$ and (3.3) holds, then $\mu \geq nr_{k-1} - k \geq nr_{k-1}/2 \geq K_0/2$. Thus,

$$\frac{r_k}{r_{k-1}} \geq \frac{(\rho + \delta) \log(nr_{k-1})}{nr_{k-1}} \geq \sup_{m \geq \lfloor \mu - \mu^{3/5} \rfloor} \frac{(\rho + \delta/2) \log m}{m}$$

for K_0 sufficiently large. By combining these estimates with Theorem 3.10, for $\varepsilon = (k + 1)^{-1} \wedge 2^{-\rho}$, we have

$$\begin{aligned} & \mathbb{E}[N; \lambda_2[Z] \leq 1 - (k + 1)^{-1}] \\ & \leq \sum_{m=\lfloor \mu - \mu^{3/5} \rfloor}^{\lceil \mu + \mu^{3/5} \rceil} m \mathbb{P}(\lambda_2[Z] \leq 1 - (k + 1)^{-1} \mid N = m) \mathbb{P}(N = m) + \mathbb{E}[N; |N - \mu| > \mu^{3/5}] \\ & \leq \sum_{m=\lfloor \mu - \mu^{3/5} \rfloor}^{\lceil \mu + \mu^{3/5} \rceil} m \varepsilon m^{1-\rho} \mathbb{P}(N = m) + \mathbb{E}[N^2]^{1/2} \mathbb{P}(|N - \mu| > \mu^{3/5})^{1/2} \\ & \leq \varepsilon \mu (\lfloor \mu - \mu^{3/5} \rfloor)^{1-\rho} + (\mu + \mu^2)^{1/2} \exp\left(-\frac{\mu^{1/5}}{10}\right) \\ & \leq 2\varepsilon \mu^{2-\rho} \end{aligned}$$

when K_0 is sufficiently large. Here, note that the distribution of $\lambda_2[Z]$ under $\mathbb{P}(\cdot \mid N = m)$ is that of $\lambda_2[G]$ with $G \sim G(m, r_k/r_{k-1})$. Therefore,

$$\begin{aligned} \mathbb{E}[\beta_k(X)] & \leq 2\varepsilon \binom{n}{k} q_{k-1} \mu^{2-\rho} \\ & \leq 2\varepsilon n^k q_{k-1} nr_{k-1} \left(\frac{nr_{k-1}}{2}\right)^{1-\rho} \\ & \leq n^{k+1} q_k (nr_{k-1})^{1-\rho}. \end{aligned}$$

The estimate of $\mathbb{P}(\beta_k(X) \neq 0)$ is obtained in the same way. In this case, from Corollary 2.12, we have

$$\begin{aligned} \mathbb{P}(\beta_k(X) \neq 0) & \leq \mathbb{P}(\text{There exists } \tau \in X_{k-1} \text{ such that } \lambda_2[\text{lk}_X(\tau)^{(1)}] \leq 1 - (k + 1)^{-1}) \\ & \leq \sum_{\tau \in \binom{V}{k}} \mathbb{P}(\tau \in X_{k-1}, \lambda_2[\text{lk}_X(\tau)^{(1)}] \leq 1 - (k + 1)^{-1}) \\ & = \binom{n}{k} q_{k-1} \mathbb{P}(\lambda_2[Z] \leq 1 - (k + 1)^{-1}). \end{aligned}$$

Then, for $\varepsilon = (k + 1)^{-1} \wedge 2^{-\rho}$,

$$\begin{aligned} & \mathbb{P}(\lambda_2[Z] \leq 1 - (k + 1)^{-1}) \\ & \leq \sum_{m=\lfloor \mu - \mu^{3/5} \rfloor}^{\lceil \mu + \mu^{3/5} \rceil} \mathbb{P}(\lambda_2[Z] \leq 1 - (k + 1)^{-1} \mid N = m) \mathbb{P}(N = m) + \exp\left(-\frac{\mu^{1/5}}{5}\right) \\ & \leq \varepsilon(\lfloor \mu - \mu^{3/5} \rfloor)^{1-\rho} + \exp\left(-\frac{\mu^{1/5}}{5}\right) \leq 2\varepsilon\mu^{1-\rho} \end{aligned}$$

for K_0 sufficiently large. Therefore,

$$\begin{aligned} \mathbb{P}(\beta_k(X) \neq 0) & \leq 2\varepsilon \binom{n}{k} q_{k-1} \mu^{1-\rho} \\ & \leq 2\varepsilon n^k q_{k-1} \left(\frac{nr_{k-1}}{2}\right)^{1-\rho} \\ & \leq n^{k+1} q_k (nr_{k-1})^{-\rho}. \end{aligned} \quad \square$$

PROOF OF COROLLARY 3.5. Take K_0 in Proposition 3.4 in which we let ρ and δ be ν/θ and $\delta/(2\theta)$, respectively. From (3.4), for sufficiently large n , we have $r_{k-1} \geq r_k \geq K_0/n$ and

$$r_k \geq \frac{(\nu/\theta + \delta/(2\theta)) \log(nr_{k-1})}{n}.$$

Indeed, if we set $\bar{r} = (\nu + \delta)(\log n)/n$, then, for sufficiently large n ,

$$\begin{aligned} nr_k - \left(\frac{\nu}{\theta} + \frac{\delta}{2\theta}\right) \log(nr_{k-1}) & \geq nr_k - \left(\frac{\nu}{\theta} + \frac{\delta}{2\theta}\right) \log(nMr_k^{1-\theta}) \\ & = nr_k - \left(\frac{\nu}{\theta} + \frac{\delta}{2\theta}\right) (1 - \theta) \log(nr_k) - \left(\nu + \frac{\delta}{2}\right) \left(\log n - \frac{\log M}{\theta}\right) \\ & \geq n\bar{r} - \left(\frac{\nu}{\theta} + \frac{\delta}{2\theta}\right) (1 - \theta) \log(n\bar{r}) - \left(\nu + \frac{\delta}{2}\right) \left(\log n - \frac{\log M}{\theta}\right) \\ & = \frac{\delta}{2} \log n - \left(\frac{\nu}{\theta} + \frac{\delta}{2\theta}\right) \{(1 - \theta)(\log \log n + \log(\nu + \delta)) + \log M\} \\ & \geq 0. \end{aligned}$$

The conclusion follows from Proposition 3.4. □

3.3. Proof of Theorem 3.6.

Theorem 2.5 plays a key role in proving Theorem 3.6. We first discuss the eigenvalues of the averaging operator on the Erdős–Rényi graph.

Let $G = V \sqcup E$ be a graph and let $h \in \mathbb{N}$. We call $w = (v_0, v_1, \dots, v_h) \in V^{h+1}$ a walk on G with length h if v_i is adjacent to v_{i+1} for all $i = 0, 1, \dots, h - 1$. A walk $w = (v_0, v_1, \dots, v_h) \in V^{h+1}$ where $v_0 = v_h$ is called a closed walk. We denote by $W_h(G)$ the set of all length- h closed walks on G . Given a graph G and a closed walk $w = (v_0, v_1, \dots, v_h) \in W_h(G)$, let $G(w) = V(w) \sqcup E(w)$ denote the subgraph of G induced by w , where

$$V(w) := \{v_0, v_1, \dots, v_{h-1}\} \text{ and } E(w) := \{\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{h-1}, v_h\}\}$$

are the vertex set and the edge set, respectively. The multiplicity $m_s(w)$ of $s \in V(w)$ is defined by

$$m_s(w) := \#\{j \in \{0, 1, \dots, h-1\} \mid v_j = s\}.$$

For $1 \leq v, e \leq h$, the set $W_h^{v,e}(G)$ is the set of all $w \in W_h(G)$ such that $\#V(w) = v$ and $\#E(w) = e$, and $w_h^{v,e}$ is the number of length- h closed walks on v unlabeled vertices that traverse exactly e edges (perhaps multiple times). That is,

$$w_h^{v,e} := \#W_h^{v,e}(K_v)/v!,$$

where K_v is the complete graph with v vertices.

LEMMA 3.12. *The following properties hold for $l \in \mathbb{N}$ and $1 \leq v, e \leq 2l$.*

- (1) *If $w_{2l}^{v,e} > 0$, then $e \geq v - 1$.*
- (2) *If $w_{2l}^{e+1,e} > 0$, then $e \leq l$.*

PROOF. Take $w \in W_{2l}^{v,e}(K_v)$. By applying the Euler–Poincaré formula to the graph $G(w)$, we have $v - e = 1 - \beta_1(G(w)) \leq 1$. This implies (1). For the proof of (2), take $w \in W_{2l}^{e+1,e}(K_{e+1})$. Since $\beta_1(G(w)) = 0$ (and $G(w)$ is connected), $G(w)$ is a tree. Then, since w is a closed walk, w passes through each edge of $G(w)$ at least twice. This implies $e \leq l$. □

LEMMA 3.13. *Let $G \sim G(n, p)$ be the Erdős–Rényi graph and let $\alpha > 0$. Let $\{\mu_i\}_{i=1}^n$ be all the (not necessarily distinct) eigenvalues of the averaging matrix $A[G]$. Then, for $l \in \mathbb{N}$ and $n \geq 2l$,*

$$\mathbb{E}[\#\{i \mid \mu_i \geq \alpha\}] \leq \frac{(2l)!}{\alpha^{2l}(n-2l+1)^{2l}p^{2l}} \sum_{\substack{1 \leq v, e \leq 2l; \\ e \geq v-1}} w_{2l}^{v,e} n^v p^e + \frac{n(1-p)^{n-1}}{\alpha^{2l}}.$$

PROOF. To proceed, we identify the vertex set of G with $\{1, 2, \dots, n\}$. Let $\{X_{ij}\}_{1 \leq i < j \leq n}$ be independent and identically distributed random variables that follow the Bernoulli distribution with parameter p . The Erdős–Rényi graph G can be generated by $\{X_{ij}\}_{1 \leq i < j \leq n}$: edge $\{i, j\}$ is supposed to belong to G if and only if $X_{ij} = 1$. In addition, we define $X_{ii} = 0$ for $1 \leq i \leq n$ and $X_{ji} = X_{ij}$ for $1 \leq i < j \leq n$. Then, a_{ij} , the (i, j) -component of $A[G]$, is given by

$$a_{ij} = \begin{cases} X_{ij} / \left(\sum_{s=1}^n X_{is} \right) & \text{if } \sum_{s=1}^n X_{is} \neq 0, \\ 1 & \text{if } \sum_{s=1}^n X_{is} = 0 \text{ and } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The obvious bound gives

$$\#\{i \mid \mu_i \geq \alpha\} \leq \frac{1}{\alpha^{2l}} \sum_{i=1}^n \mu_i^{2l} = \frac{1}{\alpha^{2l}} \text{tr}(A[G]^{2l}).$$

Let $I(G)$ denote the number of isolated vertices of G . With this,

$$\begin{aligned} \text{tr}(A[G]^{2l}) &= \sum_{1 \leq i_0, i_1, \dots, i_{2l-1} \leq n} a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{2l-1} i_0} \\ &= \sum_{\substack{1 \leq i_0, i_1, \dots, i_{2l-1} \leq n; \\ i_0 \neq i_1, i_1 \neq i_2, \dots, i_{2l-1} \neq i_0}} a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{2l-1} i_0} + I(G) \\ &= \sum_{w=(i_0, i_1, \dots, i_{2l}) \in W_{2l}(G)} a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{2l-1} i_{2l}} + I(G) \\ &= \sum_{w=(i_0, i_1, \dots, i_{2l}) \in W_{2l}(G)} \prod_{j=0}^{2l-1} \frac{1}{\sum_{s_j=1}^n X_{i_j s_j}} + I(G) \\ &\leq \sum_{w=(i_0, i_1, \dots, i_{2l}) \in W_{2l}(G)} \prod_{j=0}^{2l-1} \frac{1}{\sum_{s_j \notin V(w)} X_{i_j s_j} + 1} + I(G). \end{aligned} \tag{3.7}$$

Here, in the second line, we used the fact that, for each $i = 1, \dots, n$, $a_{ii} \neq 0$ if and only if $a_{is} = 0$ for every $s \neq i$. In the third line, recall that if vertex i is not adjacent to vertex $j \neq i$, then $a_{ij} = 0$. The last inequality follows from the fact that each i_j has at least one adjacent vertex in $V(w)$.

By using the independence of $\{X_{ij}\}_{1 \leq i < j \leq n}$, the expectation of the first term of the last line of (3.7) is equal to

$$\begin{aligned} &\sum_{w=(i_0, i_1, \dots, i_{2l}) \in W_{2l}(K_n)} \mathbb{E} \left[\prod_{j=0}^{2l-1} \frac{1}{\sum_{s_j \notin V(w)} X_{i_j s_j} + 1}; w \in W_{2l}(G) \right] \\ &= \sum_{w=(i_0, i_1, \dots, i_{2l}) \in W_{2l}(K_n)} \mathbb{P}(w \in W_{2l}(G)) \mathbb{E} \left[\prod_{j=0}^{2l-1} \frac{1}{\sum_{s_j \notin V(w)} X_{i_j s_j} + 1} \right] \\ &= \sum_{\substack{1 \leq v, e \leq 2l; \\ e \geq v-1}} \sum_{w=(i_0, i_1, \dots, i_{2l}) \in W_{2l}^{v, e}(K_n)} p^e \prod_{i \in V(w)} \mathbb{E}[(Z_v + 1)^{-m_i(w)}], \end{aligned} \tag{3.8}$$

where $Z_v \sim \text{Bin}(n - v, p)$. Here, Lemma 3.12(1) was used for the last identity. We also have, denoting $m_i(w)$ by m_i ,

$$\begin{aligned} &\mathbb{E}[(Z_v + 1)^{-m_i}] \\ &\leq m_i! \mathbb{E}[(Z_v + m_i)^{-1} (Z_v + m_i - 1)^{-1} \cdots (Z_v + 1)^{-1}] \\ &= m_i! \sum_{r=0}^{n-v} \frac{1}{(r + m_i)(r + m_i - 1) \cdots (r + 1)} \cdot \frac{(n - v)!}{r! (n - v - r)!} p^r (1 - p)^{n-v-r} \end{aligned}$$

$$\begin{aligned}
 &= \frac{m_i!}{(n-v+m_i)(n-v+m_i-1)\cdots(n-v+1)p^{m_i}} \\
 &\quad \times \sum_{r=0}^{n-v} \frac{(n-v+m_i)!}{(r+m_i)!(n-v-r)!} p^{r+m_i} (1-p)^{n-v-r} \\
 &\leq \frac{m_i!}{(n-v+1)^{m_i} p^{m_i}}.
 \end{aligned}$$

Since $\sum_{i \in V(w)} m_i(w) = 2l$, we have $\prod_{i \in V(w)} \{m_i(w)!\} \leq (2l)!$. By combining these estimates, (3.8) is dominated by

$$\begin{aligned}
 \sum_{\substack{1 \leq v, e \leq 2l; \\ e \geq v-1}} \sum_{w=(i_0, i_1, \dots, i_{2l}) \in W_{2l}^{v,e}(K_n)} \frac{p^e (2l)!}{(n-v+1)^{2l} p^{2l}} &\leq \sum_{\substack{1 \leq v, e \leq 2l; \\ e \geq v-1}} \frac{w_{2l}^{v,e} n^v p^e (2l)!}{(n-v+1)^{2l} p^{2l}} \\
 &\leq \frac{(2l)!}{(n-2l+1)^{2l} p^{2l}} \sum_{\substack{1 \leq v, e \leq 2l; \\ e \geq v-1}} w_{2l}^{v,e} n^v p^e.
 \end{aligned}$$

Since $\mathbb{E}[I(G)] = n(1-p)^{n-1}$, we reach the desired conclusion. □

We remark that $\gamma(G; 1-\alpha)$ is $\#\{i \mid \mu_i \geq \alpha\} - 1$ since $\mathcal{L}[G] = I_V - A[G]$. Now, we prove Theorem 3.6.

PROOF OF THEOREM 3.6. Since $\mathbb{E}[f_k(X)] = \binom{n}{k+1} q_k$, the second inequality of (3.5) implies that

$$\mathbb{E}[\beta_k(X)] \leq n^{k+1} q_k. \tag{3.9}$$

Thus, it suffices to prove

$$\mathbb{E}[\beta_k(X)] \leq C n^{k+1} q_k (nr_k)^{-l}$$

for some constant C that depends on only k and l . Take K_0 in Proposition 3.4 with $\rho = l + 1$ and $\delta = 1$. Take $K_1 \geq K_0 \vee 1$ such that $x^{1/l} \geq (l + 2) \log x$ for all $x \geq K_1$. Suppose that

$$r_k \geq \frac{K_1}{n} \vee \frac{(nr_{k-1})^{1/l}}{n}. \tag{3.10}$$

Then, $r_{k-1} \geq r_k \geq K_1/n$ and $r_k \geq (l + 2)\{\log(nr_{k-1})\}/n$ hold. Thus, from Proposition 3.4 with $\rho = l + 1$ and $\delta = 1$,

$$\mathbb{E}[\beta_k(X)] \leq n^{k+1} q_k (nr_{k-1})^{-l} \leq n^{k+1} q_k (nr_k)^{-l}.$$

Next, consider the constraint

$$\frac{K_2}{n} \leq r_k \leq \frac{(nr_{k-1})^{1/l}}{n} \tag{3.11}$$

for some constant $K_2 \geq K_1$ that will be specified later. By applying Theorem 2.5 to X with $D = k + 1$, we have

$$\begin{aligned} \mathbb{E}[\beta_k(X)] &\leq \mathbb{E}\left[\sum_{\tau \in X_{k-1}} \gamma(\text{lk}_X(\tau)^{(1)}; 1 - (k + 1)^{-1})\right] \\ &= \sum_{\tau \in \binom{V}{k}} \mathbb{P}(\tau \in X_{k-1}) \mathbb{E}_\tau[\gamma(\text{lk}_X(\tau)^{(1)}; 1 - (k + 1)^{-1})] \\ &= \binom{n}{k} q_{k-1} \mathbb{E}[\gamma(Z; 1 - (k + 1)^{-1})], \end{aligned}$$

where $Z \sim X(n - k, (r_{k-1}, r_k/r_{k-1}, 0, \dots, 0))$. The last identity follows from Lemma 3.11. Denote by N the number of vertices of Z , which follows $\text{Bin}(n - k, r_{k-1})$. Define $\mu := \mathbb{E}[N] = (n - k)r_{k-1}$. Take $K_2 \geq K_1 \vee 2k$ so that $\lfloor x - x^{3/5} \rfloor \geq 2l$ for all $x \geq K_2/2$ and suppose that (3.11) holds. Consequently, we have $\mu \geq nr_{k-1}/2 \geq K_2/2$ and $\lfloor \mu - \mu^{3/5} \rfloor \geq 2l$. Then, Lemma 3.13 implies that

$$\begin{aligned} &\mathbb{E}[\gamma(Z; 1 - (k + 1)^{-1})] \\ &\leq \sum_{m=\lfloor \mu - \mu^{3/5} \rfloor}^{\lceil \mu + \mu^{3/5} \rceil} \mathbb{E}[\gamma(Z; 1 - (k + 1)^{-1}) + 1 \mid N = m] \mathbb{P}(N = m) \\ &\quad + \mathbb{E}[\gamma(Z; 1 - (k + 1)^{-1}) + 1; |N - \mu| > \mu^{3/5}] - 1 \\ &\leq \sum_{m=\lfloor \mu - \mu^{3/5} \rfloor}^{\lceil \mu + \mu^{3/5} \rceil} \left\{ \frac{(2l)!(k + 1)^{2l}}{(m - 2l + 1)^{2l}(r_k/r_{k-1})^{2l}} \sum_{\substack{1 \leq v, e \leq 2l; \\ e \geq v-1}} w_{2l}^{v,e} m^v \left(\frac{r_k}{r_{k-1}}\right)^e \right. \\ &\quad \left. + (k + 1)^{2l} m \left(1 - \frac{r_k}{r_{k-1}}\right)^{m-1} \right\} \mathbb{P}(N = m) + \mathbb{E}[N; |N - \mu| > \mu^{3/5}] - 1 \\ &\leq (2l)!(k + 1)^{2l} nr_{k-1} \sum_{\substack{1 \leq v, e \leq 2l; \\ e \geq v-1}} \frac{w_{2l}^{v,e} A_v}{(nr_{k-1})^{2l-v+1}(r_k/r_{k-1})^{2l-e}} \\ &\quad + (k + 1)^{2l} \mathbb{E}\left[N \left(1 - \frac{r_k}{r_{k-1}}\right)^{N-1}\right] + \mathbb{E}[N^2]^{1/2} \mathbb{P}(|N - \mu| > \mu^{3/5})^{1/2} - 1, \quad (3.12) \end{aligned}$$

where

$$A_v = \sum_{m=\lfloor \mu - \mu^{3/5} \rfloor}^{\lceil \mu + \mu^{3/5} \rceil} \left(\frac{nr_{k-1}}{m - 2l + 1}\right)^{2l} \left(\frac{m}{nr_{k-1}}\right)^v \mathbb{P}(N = m).$$

If K_2 (which depends on only k and l) is chosen to be larger in advance, each A_v becomes less than 2. In what follows, C is a positive constant depending on only k and l ; it may vary from line to line. Concerning the first term of the last line of (3.12), we have

$$\begin{aligned}
 & \sum_{\substack{1 \leq v, e \leq 2l; \\ e \geq v-1}} \frac{w_{2l}^{v,e}}{(nr_{k-1})^{2l-v+1}(r_k/r_{k-1})^{2l-e}} \\
 &= \sum_{m=0}^{2l} \sum_{\substack{1 \leq v, e \leq 2l; \\ v-e=1-m}} \frac{w_{2l}^{v,e}}{(nr_{k-1})^m(nr_k)^{2l-e}} \\
 &= \sum_{e=1}^l \frac{w_{2l}^{e+1,e}}{(nr_k)^{2l-e}} + \sum_{m=1}^{2l} \sum_{\substack{1 \leq v, e \leq 2l; \\ v-e=1-m}} \frac{w_{2l}^{v,e}}{(nr_{k-1})^m(nr_k)^{2l-e}} \quad (\text{from Lemma 3.12(2)}) \\
 &\leq (nr_k)^{-l} \sum_{e=1}^l w_{2l}^{e+1,e} + (nr_{k-1})^{-1} \sum_{m=1}^{2l} \sum_{\substack{1 \leq v, e \leq 2l; \\ v-e=1-m}} w_{2l}^{v,e} \\
 &\leq C(nr_k)^{-l}.
 \end{aligned}$$

Here, the first inequality follows from the relations $nr_{k-1} \geq nr_k \geq K_2 \geq 1$, and the last one follows from the inequality $(nr_{k-1})^{-1} \leq (nr_k)^{-l}$ in (3.11).

Considering the second term of (3.12),

$$\begin{aligned}
 \mathbb{E} \left[N \left(1 - \frac{r_k}{r_{k-1}} \right)^{N-1} \right] &= \mathbb{E}[IZ] \\
 &= \sum_{v \in \{1, 2, \dots, n-k\}} \mathbb{P}(v \text{ is an isolated vertex in } Z) \\
 &= (n-k)r_{k-1}(1-r_k)^{n-k-1} \\
 &\leq Cnr_{k-1}(nr_k)^{-l}.
 \end{aligned}$$

The last inequality follows from the inequalities $(1-x)^m \leq e^{-mx} \leq C(mx)^{-l}$ for $0 \leq x \leq 1$ and $m \geq 0$, and $\exp(r_k(k+1)) \leq e^{k+1}$.

The third term of (3.12) is dominated by $(\mu^2 + \mu)^{1/2} \exp(-\mu^{1/5}/10)$ from the Chernoff bound (3.6), which is less than 1 for K_2 greater than some absolute constant.

Combining these estimates, we obtain

$$\mathbb{E}[\beta_k(X)] \leq Cn^k q_{k-1} nr_{k-1} (nr_k)^{-l} = Cn^{k+1} q_k (nr_k)^{-l}.$$

Lastly, if

$$r_k \leq \frac{K_2}{n}, \tag{3.13}$$

then, from (3.9),

$$\mathbb{E}[\beta_k(X)] \leq n^{k+1} q_k \leq K_2^l n^{k+1} q_k (nr_k)^{-l}.$$

Since (3.10), (3.11), and (3.13) cover all cases, the proof is completed. □

4. Estimates of lifetime sums of random simplicial complex processes.

4.1. Persistent homology.

Let X be a simplicial complex. A family $\mathcal{X} = \{X(t)\}_{t \geq 0}$ of subcomplexes of X is a right-continuous filtration of X if $X(s) \subset X(t)$ for $0 \leq s \leq t$ and $X(t) = \bigcap_{t' > t} X(t')$ for $t \geq 0$. Since X is a finite simplicial complex, $X(t)$ differs from $\bigcup_{t' < t} X(t')$ only finitely many times. Here, $X(t)$ can be the empty set, which is considered to be a (-1) -dimensional simplicial complex.

Let $\mathbb{R}[\mathbb{R}_{\geq 0}]$ be an \mathbb{R} -vector space of formal linear combinations of finite elements in $\mathbb{R}_{\geq 0}$. Each element of $\mathbb{R}[\mathbb{R}_{\geq 0}]$ is expressed as a linear combination of monomials z^t ($t \in \mathbb{R}_{\geq 0}$), where z is an indeterminate. The product of two elements is given by the linear extension of $az^s \cdot bz^t := abz^{s+t}$ ($a, b \in \mathbb{R}, s, t \in \mathbb{R}_{\geq 0}$). This operation equips $\mathbb{R}[\mathbb{R}_{\geq 0}]$ with a graded ring structure. For $k \geq 0$, the k -th persistent homology $\text{PH}_k(\mathcal{X})$ of $\mathcal{X} = \{X(t)\}_{t \geq 0}$ is defined as the result of taking a direct sum in the k -th homology vector space:

$$\text{PH}_k(\mathcal{X}) := \bigoplus_{t \geq 0} H_k(X(t)).$$

It follows that $\text{PH}_k(\mathcal{X})$ has a graded module structure over the graded ring $\mathbb{R}[\mathbb{R}_{\geq 0}]$ with isomorphisms from $H_k(X(s))$ to $H_k(X(t))$ ($0 \leq s \leq t$) induced by the inclusion from $X(s)$ to $X(t)$. The following theorem is called the structure theorem of the persistent homology.

THEOREM 4.1 ([17], [10]). *For each $k \geq 0$, there exist unique indices $p, q \in \mathbb{Z}_{\geq 0}$ and $\{b_i\}_{i=1}^{p+q}, \{d_i\}_{i=1}^p \subset \mathbb{R}_{\geq 0}$ such that $b_i < d_i$ for all $i = 1, \dots, p$. Furthermore, the following graded module isomorphism holds.*

$$\text{PH}_k(\mathcal{X}) \simeq \bigoplus_{i=1}^p ((z^{b_i})/(z^{d_i})) \oplus \bigoplus_{i=p+1}^{p+q} (z^{b_i}), \tag{4.1}$$

where (z^a) expresses an ideal in $\mathbb{R}[\mathbb{R}_{\geq 0}]$ generated by the monomial z^a .

Here, b_i and d_i are called the k -th birth and death times, respectively, and they indicate the times of the appearance and disappearance (again, respectively) of k -dimensional holes in the filtration $\mathcal{X} = \{X(t)\}_{t \geq 0}$. The lifetime l_i is defined by $l_i := d_i - b_i$. We set $d_i = \infty$ for $p + 1 \leq i \leq p + q$. We define the lifetime sum $L_k(\mathcal{X})$ by

$$L_k(\mathcal{X}) = \sum_{i=1}^{p+q} (d_i - b_i).$$

The following formula is a generalization of (1.1) to filtrations.

THEOREM 4.2 (Lifetime formula [10, Proposition 2.2]). *Let $\mathcal{X} = \{X(t)\}_{t \geq 0}$ be a right-continuous filtration of a simplicial complex X . Then, for each $k \geq 0$,*

$$L_k(\mathcal{X}) = \int_0^\infty \beta_k(X(t)) dt.$$

Let $k \geq 0$ and $T \geq 0$. We also define the k -th lifetime sum until time T by

$$(L_k(\mathcal{X}))_T = \sum_{i=1}^{p+q} ((d_i \wedge T) - (b_i \wedge T)).$$

As an analogue of Theorem 4.2, the equality

$$(L_k(\mathcal{X}))_T = \int_0^T \beta_k(X(t)) dt$$

holds.

4.2. Lifetime sums of random simplicial complex processes.

We consider a class of random simplicial complex processes associated with an n -point vertex set V and moving multi-parameters $\mathbf{p} = (p_0, p_1, \dots, p_{n-1})$. Let $n \in \mathbb{N}$ and denote the complete $(n - 1)$ -dimensional simplicial complex by $K(n)$, that is, the family of all nonempty subsets of V . To each simplex $\sigma \in K(n)$ we assign an independent non-negative random variable u_σ with distribution function F_σ . We assume that these distribution functions F_σ are identical for equal-dimensional simplices and, for each $\sigma \in K(n)$ with $\dim \sigma = i$, we denote F_σ by $p_i(\cdot)$. We define a random simplicial complex process $\mathcal{X}_n = \{X_n(t)\}_{t \geq 0}$ by

$$X_n(t) := \{\sigma \in K(n) \mid u_\tau \leq t \text{ for every simplex } \emptyset \neq \tau \subset \sigma\}.$$

We call this process the multi-parameter random complex process with n vertices and multi-parameter function $\mathbf{p}(\cdot) = (p_0(\cdot), p_1(\cdot), \dots, p_{n-1}(\cdot))$. Note that $X_n(t) \sim X(n, \mathbf{p}(t))$ for fixed t and that \mathcal{X}_n is a right-continuous filtration of $K(n)$. Note also that $X_n(t)$ can be expressed as

$$X_n(t) = \{\sigma \in K(n) \mid w_\sigma \leq t\},$$

where $w_\sigma := \max\{u_\tau \mid \emptyset \neq \tau \subset \sigma\}$.

In what follows, each $p_i(t)$ is assumed to be independent of n . We write $q_k(t)$ and $r_k(t)$, respectively, for the q_k and r_k defined in (3.1) that are associated with $X_n(t)$. The generalized inverse function \check{r}_k of r_k is defined by

$$\check{r}_k(u) = \inf\{t \geq 0 \mid r_k(t) > u\} \quad \text{for } u < 1,$$

and $\check{r}_k(1) = \infty$. We additionally define

$$Q_k(t) = \int_0^t q_k(s) ds \quad \text{for } t \geq 0,$$

$$\Phi_k(u) = Q_k(\check{r}_k(u)) \text{ and } \Psi_k(u) = Q_k(\check{r}_{k-1}(u)) \quad \text{for } u \in [0, 1].$$

We note that $\Phi_k \geq \Psi_k$ since \check{r}_k is nondecreasing with respect to k . Moreover, \check{r}_k ,

Φ_k , and Ψ_k are nondecreasing right-continuous functions. The following relations are fundamental. For $t \geq 0$, $u \in [0, 1)$, and $\varepsilon > 0$,

- $\check{r}_k(r_k(t) - \varepsilon) \leq t \leq \check{r}_k(r_k(t))$; and
- $r_k(\check{r}_k(u) - \varepsilon) \leq u \leq r_k(\check{r}_k(u))$ if $\check{r}_k(u) \geq \varepsilon$.

We provide results for the asymptotic behavior of the lifetime sum of \mathcal{X}_n in the following theorems.

THEOREM 4.3. (1) *If there exist $A \in (0, 1]$ and $u_0 \in (0, 1)$ such that*

$$\Phi_k(u_0) > 0 \quad \text{and} \quad \Phi_k\left(\frac{u}{2}\right) \geq A\Phi_k(u) \quad \text{for } 0 \leq u \leq u_0, \tag{4.2}$$

then, for any $m \in \mathbb{N}$, there exists a constant $C \geq 0$ such that for sufficiently large n ,

$$\mathbb{E}[(L_k(\mathcal{X}_n))_T] \leq Cn^{k+1}\Phi_k\left(\frac{1}{n}\right)(1 + Tn^{-m}) \quad \text{for } T > 0.$$

Moreover, if $\int_0^\infty t^{1+\delta} dq_{k+1}(t) < \infty$ for some $\delta > 0$, then

$$\mathbb{E}[L_k(\mathcal{X}_n)] = O\left(n^{k+1}\Phi_k\left(\frac{1}{n}\right)\right).$$

(2) *If $\Phi_k(u) = O(u^m)$ as $u \rightarrow 0$ for all $m \geq 0$, then, for all $m \in \mathbb{N}$, there exists a constant $C \geq 0$ such that for sufficiently large n ,*

$$\mathbb{E}[(L_k(\mathcal{X}_n))_T] \leq C(1 + T)n^{-m} \quad \text{for } T > 0.$$

Moreover, if $\int_0^\infty t^{1+\delta} dq_{k+1}(t) < \infty$ for some $\delta > 0$, then $\mathbb{E}[L_k(\mathcal{X}_n)] = O(n^{-m})$ for all $m \geq 0$.

THEOREM 4.4. *Suppose that there exist $u_0 \in (0, 1)$, $B > 1$, and $D \in [0, (k + 2)/(4(k + 1)))$ such that*

$$\Phi_k(Du) \geq B\Psi_k(u) \quad \text{for } 0 \leq u \leq u_0.$$

Then, there exists some $c > 0$ such that for $T > \check{r}_k(0)$,

$$\mathbb{E}[(L_k(\mathcal{X}_n))_T] \geq cn^{k+1}\Phi_k\left(\frac{1}{n}\right) \quad \text{for sufficiently large } n.$$

In particular, $\mathbb{E}[L_k(\mathcal{X}_n)] = \Omega(n^{k+1}\Phi_k(1/n))$.

Before proceeding to the proof, we consider typical situations in which

$$\Phi_k(u) = \Theta(u^\alpha) \text{ as } u \rightarrow 0 \tag{4.3}$$

for some constant $0 \leq a \leq \infty$. Here, we write $f(u) = \Theta(g(u))$ as $u \rightarrow 0$ to indicate that there exist $c_1 > 0$, $c_2 > 0$, and $u_0 \in (0, 1)$ such that $c_1g(u) \leq f(u) \leq c_2g(u)$ for all $u \in (0, u_0]$. Note also that $u^\infty = 0$ and $0^0 = 1$ by convention. Then, we have the following.

- Theorem 4.3(1) holds when $a < \infty$;
- Theorem 4.3(2) holds when $a = \infty$;
- Theorem 4.4 holds when $\Psi_k(u) = o(\Phi_k(u))$ as $u \rightarrow 0$.

In particular, we have the following result.

COROLLARY 4.5. *Suppose that Φ_k satisfies (4.3) with $0 \leq a < \infty$ and $\Psi_k(u) = o(\Phi_k(u))$ as $u \rightarrow 0$. Then, for each $T > 0$,*

$$\mathbb{E}[(L_k(\mathcal{X}_n))_T] \asymp n^{k+1-a}.$$

Moreover, if $\int_0^\infty t^{1+\delta} dq_{k+1}(t) < \infty$ for some $\delta > 0$, then $\mathbb{E}[L_k(\mathcal{X}_n)] \asymp n^{k+1-a}$.

The case $\Psi_k(u) = \Theta(\Phi_k(u))$ as $u \rightarrow 0$ is sensitive, and we cannot provide a simple answer for it because more detailed relations between Φ_k and Ψ_k affect the asymptotic behaviors of $\mathbb{E}[(L_k(\mathcal{X}_n))_T]$ and $\mathbb{E}[L_k(\mathcal{X}_n)]$, and Theorem 4.3 does not always give proper upper estimates. Here, we just consider the case where $\Psi_k(u)$ is identically $\Phi_k(u)$ for all $u \in [0, 1)$.

THEOREM 4.6. *If $\Phi_k(u) = \Psi_k(u)$ for all $u \in [0, 1)$, then $L_k(\mathcal{X}_n) = 0$ almost surely for all $n \in \mathbb{N}$.*

Theorems 1.2 and 1.5 are special cases of Corollary 4.5, as will be shown below. We can modify the range of the parameter t to $[0, \infty)$ by setting $p_i(t) = 1$ for $t \geq 1$ and all i : in other words, $X_n(t) = K(n)$ for $t \geq 1$. This modification does not affect the lifetimes because the one-point set $\{1\}$ is a Lebesgue null set and all dimensional homologies of $K(n)$ vanish. When we are interested in a filtration with parameters in a finite interval, we will make such a modification, if necessary, without explicitly mentioning it.

EXAMPLE 4.7. Let $n > d \geq 1$ be fixed and define $\mathbf{p}(t) = (p_0(t), \dots, p_{n-1}(t))$ by

$$p_i(t) := \begin{cases} 1 & (0 \leq i \leq d-1), \\ t & (i = d), \\ 0 & (d+1 \leq i \leq n-1) \end{cases} \quad \text{for } 0 \leq t \leq 1.$$

The corresponding process $\mathcal{K}_n^{(d)} = \{K_n^{(d)}(t)\}_{0 \leq t \leq 1}$ is called the d -Linial–Meshulam complex process. This is a higher-dimensional analogue of the Erdős–Rényi graph process $\mathcal{K}_n = \{K_n(t)\}_{0 \leq t \leq 1}$, which is identified with the 1-Linial–Meshulam complex process. We can easily confirm that

$$(\Phi_k(u), \Psi_k(u)) = \begin{cases} (0, 0) & (k < d - 1), \\ (u, 0) = (u^1, u^\infty) & (k = d - 1), \\ (1/2, u^2/2) = (\Theta(u^0), \Theta(u^2)) & (k = d), \\ (0, 0) & (k > d). \end{cases}$$

From Corollary 4.5 and Theorem 4.6, we have

$$\mathbb{E}[L_k(\mathcal{K}_n^{(d)})] \asymp \begin{cases} 0 & (k \neq d - 1, d), \\ n^{d-1} & (k = d - 1), \\ n^{d+1} & (k = d). \end{cases}$$

The case $k = d - 1$ corresponds to Theorem 1.2.

EXAMPLE 4.8. Let $n > d \geq 1$ be fixed and define $\mathbf{p}(t) = (p_0(t), \dots, p_{n-1}(t))$ by

$$p_i(t) := \begin{cases} 1 & (0 \leq i \leq d - 1), \\ t & (i = d), \\ 1 & (d + 1 \leq i \leq n - 1) \end{cases} \quad \text{for } 0 \leq t \leq 1.$$

We call the corresponding process $\mathcal{C}_n^{(d)} = \{C_n^{(d)}(t)\}_{0 \leq t \leq 1}$ the d -flag complex process. Note that the case $d = 1$ corresponds to the random clique complex process $\mathcal{C}_n = \{C_n(t)\}_{0 \leq t \leq 1}$.

By straightforward computation, we have

$$(\Phi_k(u), \Psi_k(u)) = \begin{cases} (0, 0) & (k < d - 1), \\ (u, 0) = (u^1, u^\infty) & (k = d - 1), \\ \left(\Theta\left(u^{(k+1-d)/(d+1) + \binom{k+1}{d}^{-1}}\right), \Theta\left(u^{(k+1)/(d+1) + \binom{k}{d}^{-1}}\right) \right) & (k \geq d). \end{cases}$$

From Corollary 4.5 and Theorem 4.6, we have

$$\mathbb{E}[L_k(\mathcal{C}_n^{(d)})] \asymp \begin{cases} 0 & (k < d - 1), \\ n^{(k+2)d/(d+1) - \binom{k+1}{d}^{-1}} & (k \geq d - 1). \end{cases}$$

The case $d = 1$ corresponds to Theorem 1.5.

PROOF OF THEOREM 4.3. (1) From (4.2), for any $j \in \mathbb{N}$,

$$\Phi_k\left(\frac{u}{2^j}\right) \geq A^j \Phi_k(u) \quad \text{for } u \in [0, u_0].$$

Thus, the following estimate holds for $0 \leq K \leq 1$.

$$\Phi_k(Ku) \geq AK^\gamma \Phi_k(u) \quad \text{for } u \in [0, u_0], \tag{4.4}$$

where $\gamma = -\log_2 A \geq 0$. In particular, we have

$$\Phi_k(v) \geq A\Phi_k(u_0)v^\gamma \quad \text{for } v \in [0, u_0] \tag{4.5}$$

by letting $u = u_0$ and $K = v/u_0$ in (4.4). Let $m \in \mathbb{N}$. Take $l \in \mathbb{N}$ such that $l \geq \gamma + m$. Theorem 3.6 implies that there exists some $C \geq 0$ such that for all n ,

$$\mathbb{E}[\beta_k(X_n(t))] \leq n^{k+1}q_k(t)\{1 \wedge C(nr_k(t))^{-l}\}.$$

Let $T > 0$. In what follows, $n \in \mathbb{N}$ is taken to be sufficiently large and independent of T . Define $S_n = \check{r}_k(1/n)$. Then,

$$\int_0^{S_n \wedge T} \mathbb{E}[\beta_k(X_n(t))] dt \leq n^{k+1}\Phi_k\left(\frac{1}{n}\right).$$

Next, suppose $t > S_n$. Then, $r_k(t) \geq r_k(S_n) \geq 1/n$. Let $\Xi(t) = -(nr_k(t))^{-l}$, which is a right-continuous nondecreasing function. Then, denoting $\check{r}_k(u_0)$ by t_0 ,

$$\begin{aligned} & \int_{S_n \wedge T}^T \mathbb{E}[\beta_k(X_n(t))] dt \\ &= \left(\int_{(S_n \wedge T, t_0 \wedge T]} + \int_{(t_0 \wedge T, T]} \right) \mathbb{E}[\beta_k(X_n(t))] dt \\ &\leq -Cn^{k+1} \int_{(S_n, t_0]} Q'_k(t)\Xi(t) dt + TCn^{k+1}q_k(T)(nr_k(t_0))^{-l} \\ &= -Cn^{k+1} \left([Q_k(t)\Xi(t)]_{t=S_n}^{t=t_0} - \int_{(S_n, t_0]} Q_k(t) d\Xi(t) \right) + TCn^{k+1}q_k(T)(nr_k(t_0))^{-l} \\ &\leq Cn^{k+1} \left(\Phi_k(u_0)(nu_0)^{-l} + \lim_{\varepsilon \downarrow 0} \int_{(S_n + \varepsilon, t_0]} Q_k(t - \varepsilon) d\Xi(t) + T(nu_0)^{-l} \right). \end{aligned}$$

By noting that $\Phi_k(u_0) > 0$, from (4.5), we have

$$(\Phi_k(u_0) + T)(nu_0)^{-l} \leq \frac{\Phi_k(u_0) + T}{Au_0^l \Phi_k(u_0)} n^{-m} \Phi_k\left(\frac{1}{n}\right) \leq C'(1 + Tn^{-m})\Phi_k\left(\frac{1}{n}\right),$$

where $C' > 0$ is a constant independent of n and T . Writing $\check{\Xi}(u) = \inf\{t \geq 0 \mid \Xi(t) > u\}$ for $u \in \mathbb{R}$ and taking a small $\varepsilon > 0$, we have

$$\begin{aligned} \int_{(S_n + \varepsilon, t_0]} Q_k(t - \varepsilon) d\Xi(t) &\leq \int_{(S_n + \varepsilon, t_0]} \Phi_k(r_k(t - \varepsilon)) d\Xi(t) \\ &\leq \int_{(S_n + \varepsilon, t_0]} \frac{1}{A} (nr_k(t - \varepsilon))^\gamma \Phi_k\left(\frac{1}{n}\right) d\Xi(t) \\ &= \frac{1}{A} \Phi_k\left(\frac{1}{n}\right) \int_{(S_n + \varepsilon, t_0]} (-\Xi(t - \varepsilon))^{-\gamma/l} d\Xi(t) \\ &= \frac{1}{A} \Phi_k\left(\frac{1}{n}\right) \int_{(\Xi(S_n + \varepsilon), \Xi(t_0))} (-\Xi(\check{\Xi}(u) - \varepsilon))^{-\gamma/l} du \\ &\leq \frac{1}{A} \Phi_k\left(\frac{1}{n}\right) \int_{\Xi(S_n + \varepsilon)}^{\Xi(t_0)} (-u)^{-\gamma/l} du \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{A} \Phi_k \left(\frac{1}{n} \right) \cdot \frac{-1}{1 - \gamma/l} \{(-\Xi(t_0))^{1-\gamma/l} - (-\Xi(S_n + \varepsilon))^{1-\gamma/l}\} \\
 &\leq \frac{1}{A(1 - \gamma/l)} \Phi_k \left(\frac{1}{n} \right) (nr_k(S_n))^{\gamma-l} \\
 &\leq \frac{1}{A(1 - \gamma/l)} \Phi_k \left(\frac{1}{n} \right).
 \end{aligned}$$

Here, we used (4.4) with $K = 1/(nr_k(t - \varepsilon))$ and $u = r_k(t - \varepsilon)$ in the second line, noting that $K \leq 1$ and $u \leq u_0$ for $t \in (S_n + \varepsilon, t_0]$. In the fourth line, we used the change of variable formula with $t = \tilde{\Xi}(u)$. Thus,

$$\int_{S_n \wedge T}^T \mathbb{E}[\beta_k(X_n(t))] dt \leq C \left(C' + \frac{1}{A(1 - \gamma/l)} \right) n^{k+1} \Phi_k \left(\frac{1}{n} \right) (1 + Tn^{-m})$$

and the first result given in Theorem 4.3 (1) follows.

Now, suppose that $M := \int_0^\infty t^{1+\delta} dq_{k+1}(t) < \infty$ for some $\delta > 0$. Let $w_\sigma = \max\{u_\tau \mid \emptyset \neq \tau \subset \sigma\}$ for $\sigma \in K(n)$ and define $U_n = \max\{w_\sigma \mid \sigma \in \binom{V}{k+2}\}$. The distribution function of w_σ for $\sigma \in \binom{V}{k+2}$ is equal to $q_{k+1}(\cdot)$. Take $m \in \mathbb{N}$ such that $m \geq (\gamma + k + 2)/\delta$. From the first conclusion with this m , there exists a constant $C \geq 0$ such that, for sufficiently large n ,

$$\mathbb{E}[(L_k(\mathcal{X}_n))_{n^m}] \leq Cn^{k+1} \Phi_k \left(\frac{1}{n} \right)$$

by letting $T = n^m$. Then,

$$\begin{aligned}
 \mathbb{E}[L_k(\mathcal{X}_n)] &= \mathbb{E} \left[\int_0^{U_n} \beta_k(X_n(t)) dt \right] \\
 &= \mathbb{E} \left[\int_0^{U_n} \beta_k(X_n(t)) dt ; U_n \leq n^m \right] + \mathbb{E} \left[\int_0^{U_n} \beta_k(X_n(t)) dt ; U_n > n^m \right] \\
 &\leq \mathbb{E}[(L_k(\mathcal{X}_n))_{n^m}] + n^{k+1} \mathbb{E}[U_n ; U_n > n^m] \\
 &\leq Cn^{k+1} \Phi_k \left(\frac{1}{n} \right) + n^{k+1} n^{-m\delta} \mathbb{E}[U_n^{1+\delta}].
 \end{aligned}$$

For the second term, we have

$$\begin{aligned}
 n^{-m\delta} \mathbb{E}[U_n^{1+\delta}] &\leq n^{-m\delta} \mathbb{E} \left[\sum_{\sigma \in \binom{V}{k+2}} w_\sigma^{1+\delta} \right] \\
 &\leq n^{-(\gamma+k+2)} \binom{n}{k+2} \int_0^\infty t^{1+\delta} dq_{k+1}(t) \\
 &\leq Mn^{-\gamma} \\
 &\leq \frac{M}{A\Phi_k(u_0)} \Phi_k \left(\frac{1}{n} \right) \quad (\text{from (4.5)}).
 \end{aligned}$$

The final conclusion in part (1) of the theorem follows by combining these estimates.

(2) Let $l \in \mathbb{N}$ and $\varepsilon_n = n^{-1/2}$. From the assumption, there exist $c > 0$ and $u_0 \in (0, 1)$ such that $\Phi_k(u) \leq cu^l$ for all $u \in (0, u_0]$. Moreover, $r_k(t) \geq \varepsilon_n$ for $t \geq \check{r}_k(\varepsilon_n)$. Thus, from Theorem 3.6, for sufficiently large n ,

$$\begin{aligned} \mathbb{E}[(L_k(\mathcal{X}_n))_T] &= \int_0^{\check{r}_k(\varepsilon_n) \wedge T} \mathbb{E}[\beta_k(X_n(t))] dt + \int_{\check{r}_k(\varepsilon_n) \wedge T}^T \mathbb{E}[\beta_k(X_n(t))] dt \\ &\leq \binom{n}{k+1} \Phi_k(\varepsilon_n) + Cn^{k+1} \int_{\check{r}_k(\varepsilon_n) \wedge T}^T q_k(t)(nr_k(t))^{-l} dt \\ &\leq cn^{k+1-l/2} + Cn^{k+1-l/2} \int_0^T q_k(t) dt \\ &= (c + CT)n^{k+1-l/2} \end{aligned}$$

for $T > 0$.

The second conclusion of part (2) of the theorem follows in the same manner as that of part (1). □

PROOF OF THEOREM 4.4. Let $T > \check{r}_k(0)$ and $\alpha = k/2 + 1$. Define

$$\tilde{S}_n = \check{r}_{k-1} \left(\frac{\alpha}{Dn} \right) \text{ and } \tilde{T}_n = \check{r}_k \left(\frac{\alpha}{n} \right)$$

for large enough n such that $\alpha/(Dn) \leq u_0$ and $\tilde{T}_n \leq T$.

Suppose $\tilde{S}_n \geq \tilde{T}_n$ for some n . Then, $\Psi_k(\alpha/(Dn)) = Q_k(\tilde{S}_n) \geq Q_k(\tilde{T}_n) = \Phi_k(\alpha/n)$, while $\Phi_k(\alpha/n) \geq B\Psi_k(\alpha/(Dn))$ by assumption. Therefore, $\Phi_k(\alpha/n) = 0$, which implies $\Phi_k(u) = 0$ for $u \in [0, \alpha/n]$. In this case, the conclusion is trivially true.

Thus, we may assume that $\tilde{S}_n < \tilde{T}_n$ for every n . If $t \in [\tilde{S}_n, \tilde{T}_n)$, then

$$r_{k-1}(t) \geq r_{k-1}(\tilde{S}_n) \geq \frac{\alpha}{Dn}$$

and

$$r_k(t) \leq \lim_{\varepsilon \downarrow 0} r_k(\tilde{T}_n - \varepsilon) \leq \frac{\alpha}{n}.$$

Since

$$\frac{(k+1)D}{\alpha} + \frac{\alpha}{k+2} = \frac{2(k+1)D}{k+2} + \frac{1}{2} < 1,$$

we can apply Proposition 3.3 to obtain the existence of $n_0 \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that, if $n \geq n_0$, then

$$\mathbb{E}[\beta_k(X_n(t))] \geq \varepsilon_0 n^{k+1} q_k(t) \text{ for } t \in [\tilde{S}_n, \tilde{T}_n).$$

Thus, for $n \geq n_0$,

$$\begin{aligned}
 \mathbb{E}[(L_k(\mathcal{X}_n))_T] &\geq \int_{\tilde{S}_n}^{\tilde{T}_n} \mathbb{E}[\beta_k(X_n(t))] dt \\
 &\geq \varepsilon_0 n^{k+1} \int_{\tilde{S}_n}^{\tilde{T}_n} q_k(t) dt \\
 &= \varepsilon_0 n^{k+1} \left(\Phi_k\left(\frac{\alpha}{n}\right) - \Psi_k\left(\frac{\alpha}{Dn}\right) \right) \\
 &\geq \varepsilon_0 n^{k+1} \left(\Phi_k\left(\frac{\alpha}{n}\right) - B^{-1} \Phi_k\left(\frac{\alpha}{n}\right) \right) \\
 &\geq \varepsilon_0 (1 - B^{-1}) n^{k+1} \Phi_k\left(\frac{1}{n}\right). \quad \square
 \end{aligned}$$

PROOF OF THEOREM 4.6. Note that for any $0 \leq k \leq n - 1$ and $t \geq 0$,

$$q_k(t) \geq \prod_{i=0}^k (p_i(t))^{(i+1)\binom{k+1}{i+1}} = \prod_{i=0}^k (p_i(t))^{(k+1)\binom{k}{i}} = (r_{k-1}(t))^{k+1}.$$

Then, for $u \in (0, 1)$ and $t \geq \check{r}_{k-1}(u)$, the inequality $q_k(t) \geq (r_{k-1}(t))^{k+1} \geq u^{k+1}$ holds. Thus,

$$0 = \Phi_k(u) - \Psi_k(u) = \int_{\check{r}_{k-1}(u)}^{\check{r}_k(u)} q_k(t) dt \geq u^{k+1} (\check{r}_k(u) - \check{r}_{k-1}(u)).$$

Therefore, $\check{r}_{k-1}(u) = \check{r}_k(u)$ for $u \in (0, 1)$, which implies that $r_{k-1}(t) = r_k(t)$ for $t \geq 0$ from the (weak) monotonicity and right-continuity of r_k . Therefore, for $t \geq 0$,

$$0 = r_{k-1}(t) - r_k(t) = r_{k-1}(t) \left(1 - \prod_{i=1}^{k+1} p_i(t)^{\binom{k}{i-1}} \right) \geq q_k(t) \left(1 - \prod_{i=1}^{k+1} p_i(t)^{\binom{k}{i-1}} \right). \quad (4.6)$$

Here, in the inequality above, we used the relation $r_{k-1}(t) \geq r_{k-1}(t)q_{k-1}(t) = q_k(t)$. Let $t \geq 0$. (4.6) implies that $q_k(t) = 0$ or $p_i(t) = 1$ for all $i = 1, 2, \dots, k + 1$. If $q_k(t) = 0$, then $\beta_k(X_n(t)) = 0$ almost surely. Suppose that $p_i(t) = 1$ for all $i = 1, 2, \dots, k + 1$. Then, $\beta_k(X_n(t)) = 0$ almost surely since $X_n(t)$ includes the complete $(k + 1)$ -dimensional skeleton. Thus, in all cases, $\beta_k(X_n(t)) = 0$ almost surely. This implies $L_k(\mathcal{X}_n) = 0$ a.s. □

4.3. Limiting constants.

As a refinement of Theorems 4.3 and 4.4, it is natural to consider the behavior of the normalized k -th lifetime sum

$$\bar{L}_k(\mathcal{X}_n) := \frac{L_k(\mathcal{X}_n)}{n^{k+1} \Phi_k(1/n)}$$

of a filtration \mathcal{X}_n . It is not yet known what general conditions are needed for $\bar{L}_k(\mathcal{X}_n)$ to converge in a certain sense as $n \rightarrow \infty$. In the case where \mathcal{X}_n is the d -Linial–Meshulam complex process $\mathcal{K}_n^{(d)} = \{K_n^{(d)}(t)\}_{0 \leq t \leq 1}$, Hiraoka and Shirai [10, Section 7.1] made a formal argument and conjectured that the expectation of $\bar{L}_{d-1}(\mathcal{K}_n^{(d)})$ converges to some

positive constant I_{d-1} . Their argument was based on recent work by Linial and Peled [15] on the convergence of $K_n^{(d)}(c/n)$ for fixed $c \geq 0$. We justify their argument and prove Theorem 1.4 in a more general form by using the upper estimate in Example 3.7.

One of the special features of $\text{PH}_{d-1}(\mathcal{K}_n^{(d)})$ is that all the birth times b_i in (4.1) are zero because $K_n^{(d)}(0)$ is the complete $(d - 1)$ -dimensional simplicial complex. Given this, we can obtain a formula for the generalized sums of lifetimes as follows. Let d_i ($i = 1, \dots, p+q$) be the death times in (4.1) for the k -th persistent homology of a general filtration $\mathcal{X} = \{X(t)\}_{t \geq 0}$.

PROPOSITION 4.9. *Let φ be a right-continuous nondecreasing function on $[0, \infty)$ with $\varphi(0) = 0$. Suppose that $b_i = 0$ for $i = 1, \dots, p+q$. Then,*

$$\sum_{i=1}^{p+q} \varphi(d_i-) = \int_{[0, \infty)} \beta_k(X(t)) d\varphi(t),$$

where $\varphi(t-) = \lim_{\varepsilon \downarrow 0} \varphi(t - \varepsilon)$.

PROOF. This is proved by simple calculation:

$$\begin{aligned} \sum_{i=1}^{p+q} \varphi(d_i-) &= \sum_{i=1}^{p+q} \int_0^\infty 1_{[0, d_i)}(t) d\varphi(t) \\ &= \int_0^\infty \left(\sum_{i=1}^{p+q} 1_{[0, d_i)}(t) \right) d\varphi(t) \\ &= \int_0^\infty \beta_k(X(t)) d\varphi(t). \end{aligned} \quad \square$$

Let $\alpha > 0$ and $d \in \mathbb{N}$. We consider the d -Linial–Meshulam complex process $\mathcal{K}_n^{(d)}$ and define

$$L_{d-1}^{(\alpha)}(\mathcal{K}_n^{(d)}) = \sum_{i=1}^{p+q} d_i^\alpha,$$

which is the sum of the α -th powers of the $(d - 1)$ -th lifetimes of $\mathcal{K}_n^{(d)}$. Clearly, $L_{d-1}^{(1)}(\mathcal{K}_n^{(d)}) = L_{d-1}(\mathcal{K}_n^{(d)})$. From Proposition 4.9,

$$L_{d-1}^{(\alpha)}(\mathcal{K}_n^{(d)}) = \alpha \int_0^1 \beta_{d-1}(K_n^{(d)}(t)) t^{\alpha-1} dt.$$

Below, we study the precise asymptotic behavior of $L_{d-1}^{(\alpha)}(\mathcal{K}_n^{(d)})$ as $n \rightarrow \infty$.

We recall some results in [15]. For $d \geq 2$, let t_d^* be the unique root in $(0, 1)$ of the equation

$$(d + 1)(1 - t) + (1 + dt) \log t = 0,$$

and define the constant $c_d^* = \psi_d(t_d^*) > 0$, where

$$\psi_d(t) = \frac{-\log t}{(1-t)^d}, \quad t \in (0, 1).$$

For $d = 1$, define $t_1^* = c_1^* = 1$. For $c \geq c_d^*$, let t_c denote the smallest positive root of the equation $\psi_d(t) = c$. Note that $t_c \leq t_d^*$.

THEOREM 4.10. For $c \geq 0$,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[\beta_d(K_n^{(d)}(c/n))]}{\binom{n}{d}} = g_d(c), \tag{4.7}$$

where

$$g_d(c) = \begin{cases} 0 & (c < c_d^*), \\ ct_c(1-t_c)^d + \frac{c}{d+1}(1-t_c)^{d+1} - (1-t_c) & (c \geq c_d^*). \end{cases}$$

Moreover, for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\left| \frac{\beta_d(K_n^{(d)}(c/n))}{\binom{n}{d}} - g_d(c) \right| > \varepsilon \right] = 0. \tag{4.8}$$

PROOF. The claim for $c \neq c_d^*$ follows from the results in [15]. When $c = c_d^*$, the assertion follows from the monotonicity of $\beta_d(K_n^{(d)}(\cdot))$ and the continuity of g_d with $g_d(c_d^*) = 0$. Indeed, for $\varepsilon > 0$, take $c' > c_d^*$ such that $g_d(c') \leq \varepsilon/2$. Then,

$$\begin{aligned} \mathbb{P} \left[\frac{\beta_d(K_n^{(d)}(c_d^*/n))}{\binom{n}{d}} > \varepsilon \right] &\leq \mathbb{P} \left[\frac{\beta_d(K_n^{(d)}(c_d^*/n))}{\binom{n}{d}} > \frac{\varepsilon}{2} + g_d(c') \right] \\ &\leq \mathbb{P} \left[\frac{\beta_d(K_n^{(d)}(c'/n))}{\binom{n}{d}} - g_d(c') > \frac{\varepsilon}{2} \right]. \end{aligned}$$

Since the last term converges to 0 as $n \rightarrow \infty$, we obtain (4.8) for $c = c_d^*$. The proof of (4.7) with $c = c_d^*$ is similar. \square

Since

$$\beta_d(K_n^{(d)}(t)) - \beta_{d-1}(K_n^{(d)}(t)) = f_d(K_n^{(d)}(t)) - \binom{n-1}{d}$$

by the Euler–Poincaré formula, we have

$$\frac{\beta_{d-1}(K_n^{(d)}(c/n))}{\binom{n}{d}} = \frac{1}{\binom{n}{d}} \left(\beta_d \left(K_n^{(d)} \left(\frac{c}{n} \right) \right) + \binom{n-1}{d} - f_d \left(K_n^{(d)} \left(\frac{c}{n} \right) \right) \right).$$

Write

$$Z_n := \frac{f_d(K_n^{(d)}(c/n))}{\binom{n}{d}}.$$

Then, $\mathbb{E}[Z_n] = (c/(d + 1))(1 - d/n)$. Note that $f_d(K_n^{(d)}(c/n)) \sim \text{Bin}(\binom{n}{d+1}, c/n)$ for the d -Linial–Meshulam complex. Thus, by direct computation,

$$\mathbb{E}[(Z_n - \mathbb{E}[Z_n])^2] = \frac{c}{d + 1} \left(1 - \frac{d}{n}\right) \left(1 - \frac{c}{n}\right) \binom{n}{d}^{-1} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, we obtain $\lim_{n \rightarrow \infty} \mathbb{E}[(Z_n - c/(d + 1))^2] = 0$.

From these estimates, for each $c \geq 0$,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[\beta_{d-1}(K_n^{(d)}(c/n))]}{\binom{n}{d}} = h_d(c)$$

and, for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\left| \frac{\beta_{d-1}(K_n^{(d)}(c/n))}{\binom{n}{d}} - h_d(c) \right| > \varepsilon \right] = 0, \tag{4.9}$$

where

$$h_d(c) := 1 - \frac{c}{d + 1} + g_d(c) \geq 0. \tag{4.10}$$

For $\alpha > 0$, define

$$I_{d-1}^{(\alpha)} = \frac{\alpha}{d!} \int_0^\infty h_d(s) s^{\alpha-1} ds. \tag{4.11}$$

The constant in Theorem 1.4 is then defined by $I_{d-1} := I_{d-1}^{(1)}$. The following theorem is the main result of this subsection. Theorem 1.4 is a particular case of Theorem 4.11 with $\alpha = 1$.

THEOREM 4.11. *Let $d \geq 1$ and $\alpha > 0$. Then $I_{d-1}^{(\alpha)}$ is finite, and for any $r \in [1, \infty)$,*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left| \frac{L_{d-1}^{(\alpha)}(\mathcal{K}_n^{(d)})}{n^{d-\alpha}} - I_{d-1}^{(\alpha)} \right|^r \right] = 0.$$

In particular, $\mathbb{E}[L_{d-1}^{(\alpha)}(\mathcal{K}_n^{(d)})]/n^{d-\alpha}$ converges to $I_{d-1}^{(\alpha)}$ as $n \rightarrow \infty$.

PROOF. We may assume without loss of generality that all random variables are defined in a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote the L^r -norm on $(\Omega, \mathcal{F}, \mathbb{P})$ by $\|\cdot\|_{L^r}$. From the second inequality of (3.5) and (4.9), for $s \geq 0$, we have

$$0 \leq \frac{\beta_{d-1}(K_n^{(d)}(s/n))}{n^d} 1_{[0,n]}(s) \leq \frac{1}{d!} \text{ for } n \in \mathbb{N} \tag{4.12}$$

and

$$\frac{\beta_{d-1}(K_n^{(d)}(s/n))}{n^d} 1_{[0,n]}(s) \rightarrow \frac{1}{d!} h_d(s) \text{ in probability as } n \rightarrow \infty. \tag{4.13}$$

Take $l \in \mathbb{N}$ such that $l > 1 \vee r\alpha$. From Example 3.7 (1), there exists $C \geq 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[\frac{\beta_{d-1}(K_n^{(d)}(s/n))}{n^d} 1_{[0,n]}(s) \right] \leq 1 \wedge Cs^{-l} \quad \text{for } s \geq 0. \tag{4.14}$$

Applying Fatou’s lemma to an appropriate subsequence, we obtain $(1/d!)h_d(s) \leq 1 \wedge Cs^{-l}$. From this estimate, $I_{d-1}^{(\alpha)}$ must be finite. From Proposition 4.9 and Minkowski’s inequality,

$$\begin{aligned} & \left\| L_{d-1}^{(\alpha)}(\mathcal{K}_n^{(d)})/n^{d-\alpha} - I_{d-1}^{(\alpha)} \right\|_{L^r} \\ &= \left\| \alpha \int_0^\infty \left(\frac{\beta_{d-1}(K_n^{(d)}(s/n))}{n^d} 1_{[0,n]}(s) - \frac{1}{d!} h_d(s) \right) s^{\alpha-1} ds \right\|_{L^r} \\ &\leq \alpha \int_0^\infty U_n(s) s^{\alpha-1} ds, \end{aligned}$$

where

$$U_n(s) = \left\| \frac{\beta_{d-1}(K_n^{(d)}(s/n))}{n^d} 1_{[0,n]}(s) - \frac{1}{d!} h_d(s) \right\|_{L^r}.$$

Combining (4.12) and (4.13), we obtain $\lim_{n \rightarrow \infty} U_n(s) = 0$ for each $s \geq 0$. Moreover, from (4.12) and (4.14),

$$\begin{aligned} \sup_{n \in \mathbb{N}} U_n(s) &\leq \sup_{n \geq s} \mathbb{E} \left[\left(\frac{1}{d!} \right)^{r-1} \frac{\beta_{d-1}(K_n^{(d)}(s/n))}{n^d} \right]^{1/r} + \frac{1}{d!} h_d(s) \\ &\leq \left(\frac{1}{d!} \right)^{(r-1)/r} (1 \wedge C^{1/r} s^{-l/r}) + \frac{1}{d!} h_d(s). \end{aligned}$$

Thus, $\sup_{n \in \mathbb{N}} U_n(s) s^{\alpha-1}$ is Lebesgue integrable over $[0, \infty)$. The dominated convergence theorem implies that $\int_0^\infty U_n(s) s^{\alpha-1} ds$ converges to 0 as $n \rightarrow \infty$. This completes the proof. \square

4.4. The expression of the constant $I_{d-1}^{(\alpha)}$.

We now provide more concrete expressions for $I_{d-1}^{(\alpha)}$. By an argument similar to that in [10, Section 7.1], $I_{d-1}^{(\alpha)}$ can be expressed as

$$I_{d-1}^{(\alpha)} = \frac{1}{d! (\alpha + 1)} \left(\int_0^{t_d^*} \frac{(-\log s)^{\alpha+1}}{(1-s)^{d\alpha}} ds + (c_d^*)^\alpha \int_{t_d^*}^1 (-\log s) ds \right).$$

For any $d \in \mathbb{N}$ and $\alpha > 0$,

$$\int_0^{t_d^*} \frac{(-\log s)^{\alpha+1}}{(1-s)^{d\alpha}} ds = \int_{-\log t_d^*}^\infty \frac{t^{\alpha+1}}{(1-e^{-t})^{d\alpha}} e^{-t} dt$$

$$\begin{aligned}
 &= \int_{-\log t_d^*}^{\infty} t^{\alpha+1} \sum_{k=0}^{\infty} \binom{d\alpha - 1 + k}{k} e^{-(k+1)t} dt \\
 &= \sum_{k=0}^{\infty} \binom{d\alpha - 1 + k}{k} \int_{-\log t_d^*}^{\infty} t^{\alpha+1} e^{-(k+1)t} dt \\
 &= \sum_{k=0}^{\infty} \binom{d\alpha - 1 + k}{k} \frac{1}{(k+1)^{\alpha+2}} \int_{-(k+1)\log t_d^*}^{\infty} u^{\alpha+1} e^{-u} du.
 \end{aligned}$$

Here, we used the change of variable formulae with $t = -\log s$ in the first line and with $u = (k + 1)t$ in the fourth line. For $d \in \mathbb{N}$ and $x > 0$, define

$$J_{d,k}(x) = \int_{-(k+1)\log t_d^*}^{\infty} u^{x-1} e^{-u} du.$$

Then, using integration by parts,

$$J_{d,k}(x + 1) = xJ_{d,k}(x) + (t_d^*)^{k+1}(k + 1)^x(-\log t_d^*)^x.$$

As $J_{d,k}(1) = (t_d^*)^{k+1}$, we have

$$J_{d,k}(x + 1) = x!(t_d^*)^{k+1} \sum_{j=0}^x \frac{(k + 1)^j (-\log t_d^*)^j}{j!}$$

for $x \in \mathbb{N} \cup \{0\}$. Then, for any $d, \alpha \in \mathbb{N}$,

$$\begin{aligned}
 \int_0^{t_d^*} \frac{(-\log s)^{\alpha+1}}{(1-s)^{d\alpha}} ds &= \sum_{k=0}^{\infty} \binom{d\alpha - 1 + k}{k} \frac{J_{d,k}(\alpha + 2)}{(k + 1)^{\alpha+2}} \\
 &= (\alpha + 1)! \sum_{k=0}^{\infty} \binom{d\alpha - 1 + k}{k} \frac{(t_d^*)^{k+1}}{(k + 1)^{\alpha+2}} \sum_{j=0}^{\alpha+1} \frac{(k + 1)^j (-\log t_d^*)^j}{j!} \\
 &= (\alpha + 1)! \sum_{j=0}^{\alpha+1} \frac{(-\log t_d^*)^j}{j!} \sum_{k=0}^{\infty} \binom{d\alpha - 1 + k}{k} \frac{(t_d^*)^{k+1}}{(k + 1)^{\alpha+2-j}} \\
 &= \frac{(\alpha + 1)!}{(d\alpha - 1)!} \sum_{j=0}^{\alpha+1} \frac{(-\log t_d^*)^j}{j!} \sum_{i=0}^{d\alpha-1} \begin{bmatrix} d\alpha - 1 \\ i \end{bmatrix} \sum_{k=0}^{\infty} \frac{(t_d^*)^{k+1}}{(k + 1)^{\alpha+2-i-j}} \\
 &= \frac{(\alpha + 1)!}{(d\alpha - 1)!} \sum_{i=0}^{d\alpha-1} \begin{bmatrix} d\alpha - 1 \\ i \end{bmatrix} \sum_{j=0}^{\alpha+1} \frac{(-\log t_d^*)^j}{j!} \text{Li}_{\alpha+2-i-j}(t_d^*).
 \end{aligned}$$

Here, $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes Stirling numbers of the first kind, that is, the coefficients of the identity

$$x(x + 1) \cdots (x + n - 1) = \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix} x^i,$$

where $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$ by convention, and $\text{Li}_s(x)$ is the polylogarithm

$$\text{Li}_s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^s} \quad (s \in \mathbb{Z}, 0 \leq x \leq 1).$$

Thus, for any $d, \alpha \in \mathbb{N}$,

$$I_{d-1}^{(\alpha)} = \frac{1}{d!} \left\{ \frac{\alpha!}{(d\alpha - 1)!} \sum_{i=0}^{d\alpha-1} \binom{d\alpha - 1}{i} \sum_{j=0}^{\alpha+1} \frac{(-\log t_d^*)^j}{j!} \text{Li}_{\alpha+2-i-j}(t_d^*) + \frac{(c_d^*)^\alpha \{-\log t_d^* - (1 - t_d^*)\}}{d(\alpha + 1)} \right\}, \tag{4.15}$$

where we used the identity $(d + 1)(1 - t_d^*) + (1 + dt_d^*) \log t_d^* = 0$ for the last term. In particular, noting that $t_1^* = 1$, for any $\alpha \in \mathbb{N}$,

$$I_0^{(\alpha)} = \alpha \sum_{i=0}^{\alpha-1} \binom{\alpha - 1}{i} \text{Li}_{\alpha+2-i}(1) = \alpha \sum_{i=0}^{\alpha-1} \binom{\alpha - 1}{i} \zeta(\alpha + 2 - i).$$

In particular, we have the specific values

$$\begin{aligned} I_0 &= I_0^{(1)} = \zeta(3), \\ I_0^{(2)} &= 2\zeta(3), \\ I_0^{(3)} &= 3(\zeta(3) + \zeta(4)), \\ I_0^{(4)} &= 4(\zeta(3) + 3\zeta(4) + 2\zeta(5)), \\ I_0^{(5)} &= 5(\zeta(3) + 6\zeta(4) + 11\zeta(5) + 6\zeta(6)). \end{aligned}$$

Also, by letting $\alpha = 1$ and $d \geq 2$ in (4.15), we obtain

$$I_{d-1} = \frac{1}{d!} \left[\frac{1}{(d-1)!} \sum_{i=0}^{d-1} \binom{d-1}{i} \left\{ \text{Li}_{3-i}(t_d^*) + (-\log t_d^*) \text{Li}_{2-i}(t_d^*) + \frac{(-\log t_d^*)^2}{2} \text{Li}_{1-i}(t_d^*) \right\} + \frac{(-\log t_d^*) \{-\log t_d^* - (1 - t_d^*)\}}{2d(1 - t_d^*)^d} \right]. \tag{4.16}$$

In particular,

$$\begin{aligned} I_1 &= \frac{1}{2} \left[\text{Li}_2(t_2^*) + (\log t_2^*) \log(1 - t_2^*) + \frac{t_2^* (\log t_2^*)^2}{2(1 - t_2^*)} + \frac{(\log t_2^*) \{\log t_2^* + (1 - t_2^*)\}}{4(1 - t_2^*)^2} \right], \\ I_2 &= \frac{1}{12} \left[\text{Li}_2(t_3^*) + (\log t_3^* - 1) \log(1 - t_3^*) + \frac{t_3^* (\log t_3^*) (\log t_3^* - 2)}{2(1 - t_3^*)} + \frac{t_3^* (\log t_3^*)^2}{2(1 - t_3^*)^2} + \frac{(\log t_3^*) \{\log t_3^* + (1 - t_3^*)\}}{3(1 - t_3^*)^3} \right]. \end{aligned}$$

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