J. Math. Soc. Japan
Vol. 71, No. 2 (2019) pp. 555–567
doi: 10.2969/jmsj/79217921

On bifurcations of cusps

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(Received Nov. 8, 2017)

Abstract. Let F_t, where $t \in \mathbb{R}$, be an analytic family of plane-to-plane mappings with F_0 having a critical point at the origin. The paper presents effective algebraic methods of computing the number of those cusp points of F_t, where $0 < |t| \ll 1$, emanating from the origin at which F_t has a positive/negative local topological degree.

1. Introduction.

Mappings between surfaces are a natural object of study in the theory of singularities. Whitney [**29**] proved that critical points of such a generic mapping are folds and cusps. There are several results concerning relations between the topology of surfaces and the topology of the critical locus of a mapping (see [**8**], [**18**], [**24**], [**28**], [**29**]). Singularities of map germs of the plane into the plane were studied in [**10**], [**11**], [**13**], [**21**], [**22**], [**25**].

Let F_t, where $t \in \mathbb{R}$, be an analytic family of plane-to-plane mappings with F_0 having a critical point at the origin. Under some natural assumptions there is a finite family of cusp points of F_t bifurcating from the origin. There are important results [**7**, Section 6.3], [**10**, Theorem 3.1], [**14**, Section 6], [**22**, Proposition 7.1] concerning the parity of the number of those points.

In this paper we show how to compute the number of cusps of F_t which are represented by germs having either positive or negative local topological degree (see Theorem 6.8).

The paper is organized as follows. In Sections 2 and 3, we collect some useful facts. The curve in $\mathbb{R} \times \mathbb{R}^2$ consisting of points (t, x), where x is a cusp point of F_t, is defined by three analytic equations, so that it is not a complete intersection. In Section 4 we show how to adopt in this case some more general techniques from [**23**] concerning curves in $\mathbb{R}^n$ defined by m equations, where $m \geq n$.

In Sections 5 and 6, we prove the main result. In Section 7 we present examples computed by a computer. We have implemented our algorithm with the help of Singular [**6**]. We have also used a computer program written by Łęcki [**19**].

2. Mappings between surfaces.

Let $(M, \partial M)$ and $(N, \partial N)$ be compact oriented connected surfaces, and let $f : M \to N$ be a smooth mapping such that $f^{-1}(\partial N) = \partial M$. Assume that

2010 *Mathematics Subject Classification.* Primary 14P15; Secondary 58K05.
Key Words and Phrases. singularities, bifurcations, cusps.

(i) every point in M is either a fold point, a cusp point or a regular point, and there are only a finite number of cusps which all belong to $M \setminus \partial M$,

(ii) the 1-dimensional manifold consisting of fold points is transverse to ∂M, so that $f|\partial M : \partial M \to \partial N$ is locally stable, i.e. its critical points are non-degenerate.

We shall write M^- for the closure in M of the set of regular points at which f does reverse the orientation.

If $p \in M \setminus \partial M$ is a cusp point, we define $\mu(p)$ to be the local topological degree of the germ $f : (M, p) \to (N, f(p))$. Put

$$\text{cusp deg}\,(f) = \sum \mu(p),$$

where p runs through the set of all cusp points of f.

Fukuda and Ishikawa [**10**] have generalized the results by Èliašberg [**8**] and Quine [**24**] concerning surfaces without boundary, proving

THEOREM 2.1. *Let M, N and f be as above and $\partial M \neq \emptyset$. Then*

$$\text{cusp deg}\,(f) = 2\chi(M^-) + (\deg f|\partial M)\chi(N) - \chi(M) - \#C(f|\partial M)/2,$$

where $C(f|\partial M)$ is the set of critical points of $f|\partial M$.

In fact, in [**10**] there is a stronger assumption that both $f : M \to N$ and $f|\partial M : \partial M \to \partial N$ are C^∞-stable mappings. However, if f satisfies (i), (ii), then there exists a C^∞-stable perturbation $\tilde{f}$, which is arbitrary close to f in C^∞-Whitney topology, such that all corresponding numbers associated to f and $\tilde{f}$ which appear in the above theorem stay the same.

Let $f = (f_1, f_2) : U \to \mathbb{R}^2$, where $U \subset \mathbb{R}^2$ is open, be a smooth mapping. Set $J = \partial(f_1, f_2)/\partial(x_1, x_2)$, $G_i = \partial(f_i, J)/\partial(x_1, x_2)$, $i = 1, 2$. Applying the same arguments as in the proof of [**17**, Proposition 2, p. 815] one gets

PROPOSITION 2.2. *The set of all common solutions in U of the system of equations $J = G_1 = G_2 = \partial(G_1, J)/\partial(x_1, x_2) = \partial(G_2, J)/\partial(x_1, x_2) = 0$ is empty if and only if the set of critical points of f consists of either fold or cusp points.*

If that is the case, then the set of cusp points is discrete and equals $\{J = G_1 = G_2 = 0\}$.

3. Families of germs.

In this section we recall some useful facts concerning 1-parameter families of real analytic germs.

For $r > 0$, let $D^n(r) = \{x \in \mathbb{R}^n \mid ||x|| \leq r\}$, and $S^{n-1}(r) = \partial D^n(r)$. We shall write $(t, x) = (t, x_1, \ldots, x_n) \in \mathbb{R} \times \mathbb{R}^n$. Assume $J(t, x) : \mathbb{R} \times \mathbb{R}^n, \mathbf{0} \to \mathbb{R}, 0$ is an analytic function defined in a neighbourhood of the origin having a critical point at $\mathbf{0}$. We shall write

$$L_0 = \{x \in S^{n-1}(r) \mid J(0, x) = 0\},$$

$$M_t^- = \{x \in D^n(r) \mid J(t,x) \leq 0\},$$

where $0 < |t| \ll r \ll 1$.

Let $F : \mathbb{R} \times \mathbb{R}^n, \mathbf{0} \to \mathbb{R}^n, \mathbf{0}$ be an analytic mapping. Put $F_t(x) = F(t,x)$. Suppose that there exists a small $r > 0$ such that $F_0^{-1}(\mathbf{0}) \cap D^n(r) = \{\mathbf{0}\}$. For $0 < \delta \ll r$, put $\tilde{S}_t^{n-1}(\delta) = F_t^{-1}(S^{n-1}(\delta)) \cap D^n(r)$ and $\tilde{D}_t^n(\delta) = F_t^{-1}(D^n(\delta)) \cap D^n(r)$. We shall write

$$\tilde{L}_0 = \{x \in \tilde{S}_0^{n-1}(\delta) \mid J(0,x) = 0\},$$
$$\tilde{M}_t^- = \{x \in \tilde{D}_t^n(\delta) \mid J(t,x) \leq 0\},$$

where $0 < |t| \ll \delta \ll 1$.

Lemma 3.1. *We have* $\chi(\tilde{M}_t^-) = \chi(M_t^-)$ *and* $\chi(\tilde{L}_0) = \chi(L_0)$.

Proof. There exist small positive $\delta_1 < \delta_2$, $r_1 < r_2$ and t_0, such that for $0 < |t| < t_0$ we have

$$\{x \in \tilde{D}_t^n(\delta_1) \mid J(t,x) \leq 0\} \subset \{x \in D(r_1) \mid J(t,x) \leq 0\}$$
$$\subset \{x \in \tilde{D}_t^n(\delta_2) \mid J(t,x) \leq 0\} \subset \{x \in D(r_2) \mid J(t,x) \leq 0\},$$

and inclusions

$$\{x \in \tilde{D}_t^n(\delta_1) \mid J(t,x) \leq 0\} \subset \{x \in \tilde{D}_t^n(\delta_2) \mid J(t,x) \leq 0\},$$
$$\{x \in D(r_1) \mid J(t,x) \leq 0\} \subset \{x \in D(r_2) \mid J(t,x) \leq 0\}$$

induce isomorphisms of corresponding homology groups. Then

$$\chi(\tilde{M}_t^-) = \chi(\{x \in \tilde{D}_t^n(\delta_1) \mid J(t,x) \leq 0\})$$
$$= \chi(\{x \in D(r_2) \mid J(t,x) \leq 0\}) = \chi(M_t^-).$$

The proof of the second assertion is similar. □

Define a mapping $d_0 : \mathbb{R}^n, \mathbf{0} \to \mathbb{R}^n, \mathbf{0}$ by

$$d_0(x) = \left(\frac{\partial J}{\partial x_1}(0,x), \ldots, \frac{\partial J}{\partial x_n}(0,x)\right),$$

and mappings $d_1, d_2 : \mathbb{R} \times \mathbb{R}^n, \mathbf{0} \to \mathbb{R} \times \mathbb{R}^n, \mathbf{0}$, by

$$d_1(t,x) = \left(\frac{\partial J}{\partial t}(t,x), \frac{\partial J}{\partial x_1}(t,x), \ldots, \frac{\partial J}{\partial x_n}(t,x)\right),$$
$$d_2(t,x) = \left(J(t,x), \frac{\partial J}{\partial x_1}(t,x), \ldots, \frac{\partial J}{\partial x_n}(t,x)\right),$$

respectively. Applying directly results by Fukui [**12**] and Khimshiasvili [**15**], [**16**] we get

Theorem 3.2. *Suppose that the origin is isolated in* $d_0^{-1}(\mathbf{0})$, $d_1^{-1}(\mathbf{0})$ *and* $d_2^{-1}(\mathbf{0})$, *so that the local topological degrees* $\deg_{\mathbf{0}}(d_0)$, $\deg_{\mathbf{0}}(d_1)$ *and* $\deg_{\mathbf{0}}(d_2)$ *are defined.*

Then both $J(0,x)$ and $J(t,x)$ have an isolated critical point at the origin. If $0 \neq t$ is sufficiently close to zero, then we have

$$\chi(\tilde{M}_t^-) = \chi(M_t^-) = 1 - (\deg_{\mathbf{0}}(d_0) + \deg_{\mathbf{0}}(d_1) + \operatorname{sign}(t) \cdot \deg_{\mathbf{0}}(d_2))/2.$$

If n is even, then we have $\chi(\tilde{L}_0) = \chi(L_0) = 2 \cdot (1 - \deg_{\mathbf{0}}(d_0))$, and if n is odd, then $\chi(\tilde{L}_0) = 0$. In particular, if $n = 2$, then $\tilde{L}_0$ is finite and $\#\tilde{L}_0 = 2 \cdot (1 - \deg_{\mathbf{0}}(d_0))$.

It is proper to add that there exists an efficient computer program which can compute the local topological degree (see [**19**]).

4. Number of half-branches.

In this section we shall show how to adopt some techniques developed in [**23**], [**26**], [**27**] so as to compute the number of half-branches of an analytic set of dimension ≤ 1 emanating from a singular point.

Let $\mathcal{O}_{n+1} = \mathbb{R}\{t, x_1, \ldots, x_n\}$ denote the ring of germs at the origin of real analytic functions. If I is an ideal in $\mathcal{O}_{n+1}$, let $V(I) \subset \mathbb{R} \times \mathbb{R}^n$ denote the germ of zeros of I near the origin, and let $V_{\mathbb{C}}(I) \subset \mathbb{C} \times \mathbb{C}^n$ denote the germ of complex zeros of I.

Remark 4.1. If I is proper, then $\dim_{\mathbb{R}} \mathcal{O}_{n+1}/I < \infty$ if and only if $V_{\mathbb{C}}(I) = \{\mathbf{0}\}$.

Let $w_1, \ldots, w_m \in \mathcal{O}_{n+1}$, where $m \geq n$, be germs vanishing at the origin. We shall write $\langle w_1, \ldots, w_m \rangle$ for the ideal in $\mathcal{O}_{n+1}$ generated by $w_1, \ldots, w_m$.

Let $W \subset \mathcal{O}_{n+1}$ denote the ideal generated by $w_1, \ldots, w_m$ and all $n \times n$-minors of the Jacobian matrix of the mapping germ $(w_1, \ldots, w_m) : \mathbb{R} \times \mathbb{R}^n, \mathbf{0} \to \mathbb{R}^m, \mathbf{0}$. The ideal W is proper if and only if the rank of this matrix at the origin is $\leq n - 1$.

If $V(W) = \{\mathbf{0}\}$, then by the implicit function theorem the germ $V(w_1, \ldots, w_m)$ is of dimension ≤ 1, so that this set is locally a union of a finite family of half-branches emanating from the origin. We shall say that $V(w_1, \ldots, w_m)$ is a curve having an algebraically isolated singularity at the origin if W is proper and $\dim_{\mathbb{R}} \mathcal{O}_{n+1}/W < \infty$.

From now on we shall assume that $m = 3$ and $n = 2$. Let $M(3,3)$ denote the space of all 3×3-matrices with coefficients in $\mathbb{R}$. By [**23**, Theorem 3.8] and comments in [**23**, p. 1012] we have

Theorem 4.2. *Assume that $V(w_1, w_2, w_3)$ is a curve having an algebraically isolated singularity at the origin. There exists a proper algebraic subset $\Sigma \subset M(3,3)$ such that for every non-singular matrix $[a_{sj}] \in M(3,3) \setminus \Sigma$ and $g_s = a_{s,1}w_1 + a_{s,2}w_2 + a_{s,3}w_3$, where $1 \leq s \leq 3$, the set $V(g_1, g_2)$ is a curve having an algebraically isolated singularity at the origin and $V(w_1, w_2, w_3) = V(g_1, g_2, g_3) \subset V(g_1, g_2)$.*

If that is the case and $J_p = \langle g_1, g_2, g_3^p \rangle$, where $p = 1, 2$, then we have $J_2 \subset J_1$ and $\dim_{\mathbb{R}}(J_1/J_2) < \infty$.

If $V(w_1, w_2)$ is a curve having an algebraically isolated singularity at the origin, then one can take $g_s = w_s$.

From now on we shall assume that

$$\dim_{\mathbb{R}} \mathcal{O}_3/\langle t, g_1, g_2\rangle < \infty. \tag{1}$$

As $\dim_{\mathbb{R}}(J_1/J_2) < \infty$ and $g_3(\mathbf{0}) = 0$, then by the Nakayama lemma $\xi = \min\{s \mid t^s \cdot g_3 \in J_2\}$ is finite. (In [**27**] there are presented effective methods for computing this number.) Let $k > \xi$ be an even positive integer.

Now we shall adopt to our case some arguments presented in [**27**, pp. 529–531]. There are germs $h_1, h_2, h_3 \in \mathcal{O}_3$ such that

$$t^\xi g_3 = h_1 g_1 + h_2 g_2 + h_3 g_3^2.$$

Let $Y_{\mathbb{C}} = V_{\mathbb{C}}(g_1, g_2) \setminus V_{\mathbb{C}}(g_3)$. By (1), the germ t^k does not vanish at points in $V_{\mathbb{C}}(g_1, g_2) \setminus \{\mathbf{0}\}$. If $(t, x_1, x_2) = (t, x) \in Y_{\mathbb{C}}$ lies sufficiently close to the origin, then $|h_3(t, x)| < M$ for some $M > 0$, $g_1(t, x) = g_2(t, x) = 0$ and $g_3(t, x) \neq 0$. Hence

$$|g_3(t, x)| \geq |t|^\xi / M > |t|^k.$$

Then the origin is isolated in both $V_{\mathbb{C}}(g_3 \pm t^k, g_1, g_2)$.

Take $(t, x) \in V(g_1, g_2) \setminus \{\mathbf{0}\}$ near the origin. By (1), $t \neq 0$. If $g_3(t, x) \neq 0$, then $g_3(t, x) \pm t^k$ has the same sign as $g_3(t, x)$. If $g_3(t, x) = 0$, then $g_3(t, x) + t^k > 0$ and $g_3(t, x) - t^k < 0$. Write b_+ (resp. b_-, b_0) for the number of half-branches of $V(g_1, g_2)$ on which g_3 is positive (resp. g_3 is negative, g_3 vanishes). Put

$$H_\pm = \left(\frac{\partial(g_3 \pm t^k, g_1, g_2)}{\partial(t, x_1, x_2)}, g_1, g_2\right) : \mathbb{R}^3, \mathbf{0} \to \mathbb{R}^3, \mathbf{0}.$$

By [**26**, Theorem 3.1] or [**27**, Theorem 2.3], the origin is isolated in both $H_\pm^{-1}(\mathbf{0})$ and

$$b_+ + b_0 - b_- = 2 \deg_{\mathbf{0}}(H_+),$$
$$b_+ - b_0 - b_- = 2 \deg_{\mathbf{0}}(H_-).$$

Theorem 4.3. *If $\dim_{\mathbb{R}} \mathcal{O}_3/\langle t, g_1, g_2\rangle < \infty$, then the number b_0 of half-branches of $V(w_1, w_2, w_3)$ emanating from the origin equals $\deg_{\mathbf{0}}(H_+) - \deg_{\mathbf{0}}(H_-)$.*

Proof. As the matrix $[a_{sj}]$ is non-singular, then $V(w_1, w_2, w_3) = V(g_1, g_2, g_3)$. Of course, b_0 equals the number of half-branches of $V(g_1, g_2, g_3)$. Moreover,

$$b_0 = \frac{1}{2}((b_+ + b_0 - b_1) - (b_+ - b_0 - b_1)) = \deg_{\mathbf{0}}(H_+) - \deg_{\mathbf{0}}(H_-). \qquad \square$$

Now we shall explain how to compute the number of half-branches of $V(w_1, w_2, w_3)$ in the region where $t > 0$.

Proposition 4.4. *Put $g_i'(t, x) = g_i(t^2, x)$. Then $\dim_{\mathbb{R}} \mathcal{O}_3/\langle t, g_1', g_2'\rangle < \infty$ and $V(g_1', g_2')$ has an isolated singularity at the origin.*

Proof. By (1), as $V_{\mathbb{C}}(t, g_1, g_2) = \{\mathbf{0}\}$ then $V_{\mathbb{C}}(t, g_1', g_2') = \{\mathbf{0}\}$. By Remark 4.1, $\dim_{\mathbb{R}} \mathcal{O}_3/\langle t, g_1', g_2'\rangle < \infty$. We have

$$\frac{\partial(g_i', g_j')}{\partial(t, x_p)}(t,x) = 2t\frac{\partial(g_i, g_j)}{\partial(t, x_p)}(t^2, x), \quad \frac{\partial(g_i', g_j')}{\partial(x_1, x_2)}(t,x) = \frac{\partial(g_i, g_j)}{\partial(x_1, x_2)}(t^2, x),$$

and then $V(g_1', g_2')$ is a curve having an algebraically isolated singularity at the origin. $\square$

REMARK 4.5. Let $J_p' = \langle g_1', g_2', (g_3')^p\rangle$. Put $\xi' = \min\{s \mid t^s \cdot g_3' \subset J_2'\}$. Of course, $\xi' \leq 2 \cdot \xi$.

Applying the same methods as above, one can compute the number b_0' of half-branches of $V(g_1', g_2', g_3')$. Obviously $b_0'/2$ equals the number of half-branches of $V(w_1, w_2, w_3)$ lying in the region where $t > 0$.

Other methods of computing the number of half-branches were presented in [**1**], [**2**] [**3**], [**4**], [**5**], [**9**], [**20**]. According to Khimshiashvili [**15**], [**16**], if a germ $f : \mathbb{R}^2, \mathbf{0} \to \mathbb{R}, 0$ has an isolated critical point at the origin, then the number of real half-branches in $f^{-1}(0)$ equals $2 \cdot (1 - \deg_{\mathbf{0}}(\nabla f))$, where $\nabla f : \mathbb{R}^2, \mathbf{0} \to \mathbb{R}^2, \mathbf{0}$ is the gradient of f.

5. Mappings between curves.

In this section we give sufficient conditions for a mapping between some smooth plane curves to have only non-degenerate critical points.

Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth mapping. Put $g = f_1^2 + f_2^2$. Assume that $\delta^2 > 0$ is a regular value of g and $P = g^{-1}(\delta^2)$ is non-empty, so that P is a smooth curve. Obviously, $P = f^{-1}(S^1(\delta))$ and $f|P : P \to S^1(\delta)$ is a smooth mapping between 1-dimensional manifolds.

At any $p \in P$ the gradient $\nabla g(p) = (\partial g/\partial x_1(p), \partial g/\partial x_2(p))$ is a non-zero vector perpendicular to P, and the vector $T(p) = (-\partial g/\partial x_2(p), \partial g/\partial x_1(p))$ obtained by rotating $\nabla g(p)$ counterclockwise by an angle of $\pi/2$ is tangent to P. This way $T : P \to \mathbb{R}^2$ is a non-vanishing tangent vector field along P.

Take $p \in P$. There exists a smooth mapping $x(t) = (x_1(t), x_2(t)) : \mathbb{R} \to P$ such that $x(0) = p$ and $x'(t) = T(x(t))$. Hence

$$\begin{aligned} x_1'(t) &= -2 \cdot \left(f_1 \frac{\partial f_1}{\partial x_2} + f_2 \frac{\partial f_2}{\partial x_2}\right)\bigg|_{(x(t))}, \\ x_2'(t) &= 2 \cdot \left(f_1 \frac{\partial f_1}{\partial x_1} + f_2 \frac{\partial f_2}{\partial x_1}\right)\bigg|_{(x(t))}. \end{aligned} \tag{2}$$

As $g(x(t)) = \delta^2$, then $f(x(t)) = (\delta \cos\theta(t), \delta \sin\theta(t))$ for some smooth function $\theta : \mathbb{R}, 0 \to \mathbb{R}$. Of course, $(\delta\cos\theta(0), \delta\sin\theta(0)) = f(x(0)) = f(p)$. Applying the complex numbers notation we can write

$$\delta \cdot e^{\mathbf{i}\theta} = f_1(x(t)) + \mathbf{i} f_2(x(t)), \text{ where } \mathbf{i} = \sqrt{-1}. \tag{3}$$

Put $J = \partial(f_1, f_2)/\partial(x_1, x_2)$ and $G_j = \partial(f_j, J)/\partial(x_1, x_2)$, where $j = 1, 2$.

LEMMA 5.1. *A point $p \in P$ is a critical point of $f|P : P \to S^1(\delta)$ if and only if $J(p) = 0$.*

Proof. By (2), the derivative of the equation (3) equals

$$\begin{aligned}\mathbf{i}\,\delta\,\theta'\cdot e^{\mathbf{i}\theta} &= \left(\frac{\partial f_1}{\partial x_1}\,x_1' + \frac{\partial f_1}{\partial x_2}\,x_2'\right) + \mathbf{i}\cdot\left(\frac{\partial f_2}{\partial x_1}\,x_1' + \frac{\partial f_2}{\partial x_2}\,x_2'\right)\\ &= 2\mathbf{i}(f_1+\mathbf{i}f_2)\cdot J = 2\mathbf{i}\,\delta\cdot e^{\mathbf{i}\theta}\cdot J.\end{aligned}$$

So $p\in P$ is a critical point of $f|P$ if and only if $\theta'(0)=0$, i.e. if $J(p)=0$. □

Lemma 5.2. *Suppose that $p\in P$ is a critical point of $f|P : P\to S^1(\delta)$. Then*

$$\operatorname{sign}(\theta''(0)) = \operatorname{sign}\,(f_1\cdot G_1 + f_2\cdot G_2)|_p\,.$$

In particular, a point $p\in P$ is a non-degenerate critical point of $f|P : P\to S^1(\delta)$ if and only if $J(p)=0$ and $(f_1\cdot G_1+f_2\cdot G_2)|_p\neq 0$.

Proof. Since $\theta'(0)=0$ and $J(p)=0$, after computing the second derivative of (3) the same way as above one gets

$$\begin{aligned}\mathbf{i}\,\delta\,\theta''\cdot e^{\mathbf{i}\theta}\big|_0 &= 2\mathbf{i}\,\delta\cdot e^{\mathbf{i}\theta}\cdot\left(\frac{\partial J}{\partial x_1}\,x_1' + \frac{\partial J}{\partial x_2}\,x_2'\right)\bigg|_0\\ &= 4\,\mathbf{i}\,\delta\cdot e^{\mathbf{i}\theta(0)}\cdot (f_1\cdot G_1+f_2\cdot G_2)|_p\,.\end{aligned}$$

□

Lemma 5.3. *Let $f=(f_1,f_2):\mathbb{R}^2,\mathbf{0}\to\mathbb{R}^2,\mathbf{0}$ be an analytic mapping such that $J(\mathbf{0})=0$, and the origin is isolated in both $f^{-1}(\mathbf{0})$ and $\nabla J^{-1}(\mathbf{0})$.*

If $0<\delta\ll r\ll 1$, then $\tilde{S}^1(\delta)=D(r)\cap f^{-1}(S^1(\delta))$ is diffeomorphic to a circle, $\tilde{D}^2(\delta)=D(r)\cap f^{-1}(D^2(\delta))$ is diffeomorphic to a disc, and $f:\tilde{S}^1(\delta)\to S^1(\delta)$ has only non-degenerate critical points. Moreover the one-dimensional set $J^{-1}(0)$ consisting of critical points of f is transverse to $\tilde{S}^1(\delta)$.

Proof. If the origin is isolated in $J^{-1}(0)$, then $f|\mathbb{R}^2\setminus\{\mathbf{0}\}$ is a submersion near the origin, and so $f:\tilde{S}^1(\delta)\to S^1(\delta)$ has no critical points.

In the other case, $J^{-1}(0)\setminus\{\mathbf{0}\}$ is locally a finite union of analytic half-branches emanating from the origin. Let B be one of them. The gradient $\nabla J(p)$ is a non-zero vector perpendicular to T_pB at any $p\in B$.

The origin is isolated in $f^{-1}(\mathbf{0})$. By the curve selection lemma one can assume that $(f_1^2+f_2^2)|B$ has no critical points, so that ∇J and

$$\nabla(f_1^2+f_2^2) = \left(2f_1\frac{\partial f_1}{\partial x_1} + 2f_2\frac{\partial f_2}{\partial x_1},\, 2f_1\frac{\partial f_1}{\partial x_2} + 2f_2\frac{\partial f_2}{\partial x_2}\right)$$

are linearly independent along B. Then

$$0\neq \nabla J\times\nabla(f_1^2+f_2^2) = 2f_1\frac{\partial(J,f_1)}{\partial(x_1,x_2)} + 2f_2\frac{\partial(J,f_2)}{\partial(x_1,x_2)} = -2(f_1\cdot G_1+f_2\cdot G_2)$$

along B. By previous lemmas, $f:\tilde{S}^1(\delta)\to S^1(\delta)$ has only non-degenerate critical points. Other assertions are rather obvious. □

6. Families of self-maps of $\mathbb{R}^2$.

In this section we investigate 1-parameter families of plane-to-plane analytic mappings.

Let $F = (f_1, f_2) : \mathbb{R} \times \mathbb{R}^2, \mathbf{0} \to \mathbb{R}^2, \mathbf{0}$ be an analytic mapping defined in a neighbourhood of the origin. We shall write $F_t(x_1, x_2) = F(t, x_1, x_2)$ for t near zero. Define three germs $\mathbb{R} \times \mathbb{R}^2, \mathbf{0} \to \mathbb{R}$ by

$$J = \frac{\partial(f_1, f_2)}{\partial(x_1, x_2)},\ G_i = \frac{\partial(f_i, J)}{\partial(x_1, x_2)},\ i = 1, 2.$$

Put $J_t(x_1, x_2) = J(t, x_1, x_2)$.

From now on we shall also assume that

$$\begin{aligned} &\dim_{\mathbb{R}} \mathcal{O}_3/\langle t, f_1, f_2\rangle < \infty,\ \dim_{\mathbb{R}} \mathcal{O}_3/\langle t, G_1, G_2\rangle < \infty,\\ &J(\mathbf{0}) = 0,\ \dim_{\mathbb{R}} \mathcal{O}_3 \Big/ \left\langle t, \frac{\partial J}{\partial x_1}, \frac{\partial J}{\partial x_2} \right\rangle < \infty, \end{aligned} \tag{4}$$

i.e. the origin is isolated in both $(\{0\} \times \mathbb{C}^2) \cap V_{\mathbb{C}}(f_1, f_2)$ and $(\{0\} \times \mathbb{C}^2) \cap V_{\mathbb{C}}(G_1, G_2)$, and J_0 has an algebraically isolated critical point at the origin.

Lemma 6.1. *Let $Q = \mathcal{O}_3/\langle t, J, G_1, G_2\rangle$. Then $\dim_{\mathbb{R}} Q < \infty$, i.e. the origin is isolated in $(\{0\} \times \mathbb{C}^2) \cap V_{\mathbb{C}}(J, G_1, G_2)$.*

Proof. Of course $\langle t, G_1, G_2\rangle \subset \langle t, J, G_1, G_2\rangle$. Then $\dim_{\mathbb{R}} Q \leq \dim_{\mathbb{R}} \mathcal{O}_3/\langle t, G_1, G_2\rangle < \infty$. □

We shall write $g = f_1^2 + f_2^2$ and $g_t(x_1, x_2) = g(t, x_1, x_2)$. There exists a small $r_0 > 0$ such that $F_0^{-1}(\mathbf{0}) \cap D^2(r_0) = \{\mathbf{0}\}$. For $|t| \ll \delta \ll r_0$, put $\tilde{S}_t^1(\delta) = F_t^{-1}(S^1(\delta)) \cap D^2(r_0)$ and $\tilde{D}_t^2(\delta) = F_t^{-1}(D^2(\delta)) \cap D^2(r_0)$. If δ^2 is a regular value of $g_0|D^2(r_0)$, then it is also a regular value of $g_t|D^2(r_0)$. If that is the case, then $\tilde{S}_t^1(\delta)$ is diffeomorphic to $\tilde{S}_0^1(\delta) \simeq S^1(1)$. By the same argument, $\tilde{D}_t^2(\delta)$ is diffeomorphic to $\tilde{D}_0^2(\delta) \simeq D^2(1)$.

By Lemmas 5.2, 5.3 we get

Lemma 6.2. *Critical points of $F_0 : \tilde{S}_0^1(\delta) \to S^1(\delta)$ are non-degenerate, and $C(F_0|\tilde{S}_0^1(\delta)) = \tilde{S}_0^1(\delta) \cap \{J_0 = 0\}$.*

For t near zero, critical points of $F_t : \tilde{S}_t^1(\delta) \to S^1(\delta)$ are non-degenerate too, and the number of critical points $\#C(F_t|\tilde{S}_t^1(\delta))$ equals $\#(\tilde{S}_0^1(\delta) \cap \{J_0 = 0\})$. Moreover the set of critical points of F_t, i.e. $J_t^{-1}(0)$, is transverse to $\tilde{S}_t^1(\delta)$.

Let I denote the ideal in the ring $\mathcal{O}_3$ generated by J, G_1, G_2, and let $V(I) \subset \mathbb{R} \times \mathbb{R}^2$ denote a representative of the germ of zeros of I near the origin. By Lemma 6.1, there exists $0 < \delta \ll 1$ such that $\{0\} \times \tilde{D}_0^2(\delta) \cap V(I) = \{\mathbf{0}\}$, and $\{t\} \times \tilde{S}_t^1(\delta) \cap V(I) = \emptyset$ for t sufficiently close to zero. Put $\Sigma_t = \{x \in \tilde{D}_t^2(\delta) \,|\, (t, x) \in V(I)\}$. Hence $\Sigma_0 = \{\mathbf{0}\}$ and Σ_t is contained in the interior of $\tilde{D}_t^2(\delta)$.

Let I' denote the ideal in $\mathcal{O}_3$ generated by germs J, G_1, G_2, $\partial(G_1, J)/\partial(x_1, x_2)$ and $\partial(G_2, J)/\partial(x_1, x_2)$. Suppose that $V(I') = \{\mathbf{0}\}$. Hence $\{t\} \times \tilde{D}^2(\delta) \cap V(I')$ is empty for $0 \neq t$ close to zero. By Proposition 2.2 one gets

LEMMA 6.3. *Suppose that $0 < \delta \ll 1$ and $0 \neq t$ is sufficiently close to zero. Then the set of critical points of $F_t : \tilde{D}^2_t(\delta) \to D^2(\delta)$ consists of fold points, and a finite family Σ_t of cusp points.*

REMARK 6.4. By [**10**, Theorem 3.1], if $0 \neq t$ is sufficiently close to zero, then $\#\Sigma_t \leq \dim_{\mathbb{R}} Q$ and $\#\Sigma_t \equiv \dim_{\mathbb{R}} Q \bmod 2$.

For $t \neq 0$ we shall write $\Sigma_t^{\pm} = \{x \in \Sigma_t \mid \mu_t(x) = \pm 1\}$, where $\mu_t(x)$ is the local topological degree of F_t at x. Put cusp $\deg(F_t) = \sum_{x \in \Sigma_t} \mu_t(x) = \#\Sigma_t^+ - \#\Sigma_t^-$. By Lemmas 5.3, 6.2, 6.3 and Theorem 2.1 we get

PROPOSITION 6.5. *Suppose that $0 < \delta \ll 1$, and $0 \neq t$ is sufficiently close to zero. Then*

(i) *the pair $(\tilde{D}^2_t(\delta), \tilde{S}^1_t(\delta))$ is diffeomorphic to $(D^2(1), S^1(1))$, and $F_t : \tilde{D}^2_t(\delta) \to D^2(\delta)$ is a mapping such that $F_t^{-1}(S^1(\delta)) = \tilde{S}^1_t(\delta)$,*

(ii) *every point in $\tilde{D}^2_t(\delta)$ is either a fold point, a cusp point or a regular point, and there is a finite family of cusps which all belong to $\tilde{D}^2_t(\delta) \setminus \tilde{S}^2_t(\delta)$,*

(iii) *$F_t|\tilde{S}^1_t : \tilde{S}^1_t(\delta) \to S^1_t(\delta)$ is locally stable, and the set of critical points of F_t, i.e. $J_t^{-1}(0)$, is transverse to $\tilde{S}^1_t(\delta)$,*

(iv) cusp $\deg(F_t) = 2\chi(\tilde{M}_t^-) + \deg(F_t|\tilde{S}^1_t(\delta)) - 1 - \#C(F_t|\tilde{S}^1_t(\delta))/2$

$$= 2\chi(\tilde{M}_t^-) + \deg_0(F_0) - \#C(F_0|\tilde{S}^1_0(\delta))/2 - 1,$$

where $\tilde{M}_t^- = \{x \in \tilde{D}^2_t(\delta) \mid J_t(x) \leq 0\}$.

Let $d_1, d_2 : \mathbb{R} \times \mathbb{R}^2, \mathbf{0} \to \mathbb{R} \times \mathbb{R}^2, \mathbf{0}$ be defined as in Section 3.

THEOREM 6.6. *Let $F = (f_1, f_2) : \mathbb{R} \times \mathbb{R}^2, \mathbf{0} \to \mathbb{R}^2, \mathbf{0}$ be an analytic mapping defined in a neighbourhood of the origin such that (4) holds. Suppose that the origin is isolated in $V(I')$, $d_1^{-1}(\mathbf{0})$ and $d_2^{-1}(\mathbf{0})$.*

Then there exists $r > 0$ such that the set of critical points of $F_t : D^2(r) \to \mathbb{R}^2$, where $0 \neq t$ is sufficiently close to zero, consists of fold points, and a finite family Σ_t of cusp points. Moreover, the origin is isolated in $F_0^{-1}(\mathbf{0})$ and

$$\text{cusp}\deg(F_t) = \deg_{\mathbf{0}}(F_0) - \deg_{\mathbf{0}}(d_1) - \operatorname{sign}(t) \cdot \deg_{\mathbf{0}}(d_2).$$

PROOF. For any small $\delta > 0$ there is $r > 0$ such that $D^2(r) \subset \tilde{D}^2_0(\delta) \setminus \tilde{S}^1_0(\delta)$, so that also $D^2(r) \subset \tilde{D}^2_t(\delta) \setminus \tilde{S}^1_t(\delta)$ if $|t|$ is small.

By Lemma 6.3, the set of critical points of $F_t|\tilde{D}^2_t(\delta)$ consists of fold points, and a finite family Σ_t of cusp points. Because $\Sigma_0 = \{\mathbf{0}\}$ then Σ_t is the set of cusp points of $F_t|D^2(r)$.

By (4), the germ $d_0 = \nabla J_0 : \mathbb{R}^2, \mathbf{0} \to \mathbb{R}^2, \mathbf{0}$ has an isolated zero at the origin. By Theorem 3.2 and Lemma 6.2, we have

$$\#C(F_t|\tilde{S}^1_t(\delta)) = \#(\tilde{S}^1_0(\delta)\cap\{J_0=0\}) = 2\cdot(1-\deg_{\mathbf{0}}(d_0)),$$

for $0\neq t$ sufficiently close to zero. Our assertion is then a consequence of Proposition 6.5 and Theorem 3.2. $\square$

Put $J' = J(t^2,x_1,x_2)$, $G_i' = G_i(t^2,x_1,x_2)$, $i=1,2$.

LEMMA 6.7. *Suppose that* $V(I') = \{\mathbf{0}\}$. *Then* $\dim V(J,G_1,G_2)\leq 1$ *and* $\dim V(J',G_1',G_2')\leq 1$.

Moreover, if $\dim_{\mathbb{R}}\mathcal{O}_3/I' < \infty$, *then* $V(J',G_1',G_2')$, *as well as* $V(J,G_1,G_2)$, *is a curve having an algebraically isolated singularity at the origin.*

PROOF. We have

$$\{\mathbf{0}\} = V(I') = V(J,G_1,G_2)\cap V\left(\frac{\partial(G_1,J)}{\partial(x_1,x_2)},\frac{\partial(G_2,J)}{\partial(x_1,x_2)}\right),$$

so by the implicit function theorem $\dim V(J,G_1,G_2)\leq 1$. Of course, $(t,x_1,x_2)\in V(J',G_1',G_2')$ if and only if $(t^2,x_1,x_2)\in V(J,G_1,G_2)$. Hence $\dim V(J',G_1',G_2')\leq 1$ too.

The ideal

$$K = \left\langle J',G_1',G_2',\frac{\partial(G_1',J')}{\partial(x_1,x_2)},\frac{\partial(G_2',J')}{\partial(x_1,x_2)}\right\rangle\subset\mathcal{O}_3$$

is contained in the ideal L generated by J',G_1',G_2' and all 2×2-minors of the derivative matrix of (J',G_1',G_2').

As $\dim_{\mathbb{R}}\mathcal{O}_3/I'<\infty$, by the local Nullstellensatz, the origin is isolated in the set of complex zeros of I'. Since

$$\frac{\partial(G_i',J')}{\partial(x_1,x_2)}(t,x_1,x_2) = \frac{\partial(G_i,J)}{\partial(x_1,x_2)}(t^2,x_1,x_2),$$

the origin is isolated in the set of complex zeros of K. Hence $\dim_{\mathbb{R}}\mathcal{O}_3/L\leq\dim_{\mathbb{R}}\mathcal{O}_3/K<\infty$, and then $V(J',G_1',G_2')$ is a curve having an algebraically isolated singularity at the origin. The proof of the last assertion is similar. $\square$

Suppose that the origin is isolated in $V(I')$. Let b_0 (resp. b_0') be the number of half branches in $V(J,G_1,G_2)$ (resp. $V(J',G_1',G_2')$) emanating from the origin.

By Lemma 6.1, no half-branch is contained in $\{0\}\times\mathbb{R}^2$. Then by the curve selection lemma the family of half-branches is a finite union of graphs of continuous functions $t\mapsto x^i(t)\in\mathbb{R}^2$, where t belongs either to $(-\epsilon,0]$ or to $[0,\epsilon)$, $0<\epsilon\ll 1$, $x^i(0)=\mathbf{0}$, $1\leq i\leq b_0$ (resp. $1\leq i\leq b_0'$), and those graphs meet only at the origin.

Hence, if $0<t\ll 1$, then we have

$$b_0 = \#\Sigma_t + \#\Sigma_{-t} = \#\Sigma_t^+ + \#\Sigma_t^- + \#\Sigma_{-t}^+ + \Sigma_{-t}^-,$$
$$b_0'/2 = \#\Sigma_t = \#\Sigma_t^+ + \#\Sigma_t^-.$$

By Theorem 6.6, we have

$$\deg_{\mathbf{0}}(F_0) - \deg_{\mathbf{0}}(d_1) - \deg_{\mathbf{0}}(d_2) = \#\Sigma_t^+ - \#\Sigma_t^-,$$
$$\deg_{\mathbf{0}}(F_0) - \deg_{\mathbf{0}}(d_1) + \deg_{\mathbf{0}}(d_2) = \#\Sigma_{-t}^+ - \#\Sigma_{-t}^-.$$

Then we have

THEOREM 6.8. *Suppose that assumptions of Theorem* 6.6 *hold. Then numbers* $\#\Sigma_{\pm t}^{\pm}$*, where* $t > 0$ *is small, are determined by* $b_0, b_0', \deg_{\mathbf{0}}(F_0), \deg_{\mathbf{0}}(d_1)$ *and* $\deg_{\mathbf{0}}(d_2)$.

Moreover, if $\dim \mathcal{O}_3/I' < \infty$*, then* $V(J, G_1, G_2)$ *and* $V(J', G_1', G_2')$ *are curves having an algebraically isolated singularity at the origin. In that case one can apply Theorem* 4.3 *so as to compute* b_0 *and* b_0'*. In particular, if* $\dim_{\mathbb{R}} \mathcal{O}_3/I'' < \infty$*, where*

$$I'' = \left\langle G_1, G_2, \frac{\partial(G_1, G_2)}{\partial(t, x_1)}, \frac{\partial(G_1, G_2)}{\partial(t, x_2)}, \frac{\partial(G_1, G_2)}{\partial(x_1, x_2)} \right\rangle,$$

then $V(G_1, G_2)$ *is a curve having an algebraically isolated singularity at the origin. In that case one can take* $g_1 = G_1$*,* $g_2 = G_2$*,* $g_3 = J$.

7. Examples.

Examples presented in this section were calculated with the help of SINGULAR [**6**] and the computer program written by Andrzej Łęcki [**19**].

EXAMPLE 7.1. Let $F = (f_1, f_2) = (x_1^3 + x_2^2 + tx_1, x_1x_2)$. Since $\dim_{\mathbb{R}} \mathcal{O}_3/\langle t, f_1, f_2\rangle = 5$, $\dim_{\mathbb{R}} \mathcal{O}_3/\langle t, G_1, G_2\rangle = 7$ and $\dim_{\mathbb{R}} \mathcal{O}_3/\langle t, \partial J/\partial x_1, \partial J/\partial x_2\rangle = 2$, (4) holds. Moreover, we have $\dim_{\mathbb{R}} \mathcal{O}_3/I' = 8$, $\dim_{\mathbb{R}} \mathcal{O}_3/\langle \partial J/\partial t, \partial J/\partial x_1, \partial J/\partial x_2\rangle = 1$, and $\dim_{\mathbb{R}} \mathcal{O}_3/\langle J, \partial J/\partial x_1, \partial J/\partial x_2\rangle = 3$. Then the origin is isolated in $V(I')$, $d_1^{-1}(\mathbf{0})$ and $d_2^{-1}(\mathbf{0})$. Using the computer program by Łęcki one can compute $\deg_{\mathbf{0}}(F_0) = -1$, $\deg_{\mathbf{0}}(d_1) = +1$ and $\deg_{\mathbf{0}}(d_2) = -1$. By Theorem 6.6, $\operatorname{cusp}\deg(F_t) = \operatorname{sign}(t) - 2$ for $0 \neq t$ sufficiently close to zero.

By Lemma 6.7, the set $V(J, G_1, G_2)$, as well as $V(J', G_1', G_2')$, is a curve having an algebraically isolated singularity at the origin. Hence we can apply techniques presented in Section 4 so as to compute the number of half-branches of those curves.

One can verify that $\dim_{\mathbb{R}} \mathcal{O}_3/I'' = 8$, so that $V(G_1, G_2)$ is a curve with an algebraically isolated singularity at the origin.

Put $J_p = \langle G_1, G_2, J^p\rangle$, where $p = 1, 2$. In that case $\xi = 2$, and so $k = 4$. As $\dim_{\mathbb{R}} \mathcal{O}_3/\langle t, G_1, G_2\rangle < \infty$, then (1) holds. Set

$$H_{\pm} = \left(\frac{\partial(J \pm t^4, G_1, G_2)}{\partial(t, x_1, x_2)}, G_1, G_2 \right) : \mathbb{R}^3, \mathbf{0} \to \mathbb{R}^3, \mathbf{0}.$$

One can compute $\deg_{\mathbf{0}}(H_+) = +2$, $\deg_{\mathbf{0}}(H_-) = -2$. By Theorem 4.3, $V(J, G_1, G_2)$ is a union of four half-branches emanating from the origin, i.e. $b_0 = 4$.

Now we shall apply the same techniques so as to compute the number of half-branches of $V(J', G_1', G_2')$. By Proposition 4.4, $V(G_1', G_2')$ is a curve with an algebraically isolated singularity at the origin. Put $J_p' = \langle G_1', G_2', (J')^p\rangle$, where $p = 1, 2$. By Remark 4.5, $\xi' \leq 4$ and so one can take $k = 6$. Let

$$H'_{\pm} = \left(\frac{\partial(J' \pm t^6, G'_1, G'_2)}{\partial(t, x_1, x_2)}, G'_1, G'_2 \right) : \mathbb{R}^3, \mathbf{0} \to \mathbb{R}^3, \mathbf{0}.$$

One can compute $\deg_{\mathbf{0}}(H'_+) = +1$, $\deg_{\mathbf{0}}(H'_-) = -1$. Then $V(J', G'_1, G'_2)$ is a union of two half-branches emanating from the origin, i.e. $b'_0/2 = 1$. Hence, if $0 < t \ll 1$, then $\#\Sigma_t^+ = 0$, $\#\Sigma_t^- = 1$, $\#\Sigma_{-t}^+ = 0$ and $\#\Sigma_{-t}^- = 3$.

Example 7.2. Let $F = (f_1, f_2) = (x_1^4 + x_2^4 + x_1^2 x_2^2 + tx_1, x_1 x_2 + tx_2)$. In that case $\dim_{\mathbb{R}} \mathcal{O}_3/\langle t, f_1, f_2\rangle = 8$, $\dim_{\mathbb{R}} \mathcal{O}_3/\langle t, G_1, G_2\rangle = 24$, $\dim_{\mathbb{R}} \mathcal{O}_3/\langle t, \partial J/\partial x_1, \partial J/\partial x_2\rangle = 9$, $\dim_{\mathbb{R}} \mathcal{O}_3/I' = 33$, $\dim_{\mathbb{R}} \mathcal{O}_3/\langle \partial J/\partial t, \partial J/\partial x_1, \partial J/\partial x_2\rangle = 3$, and $\dim_{\mathbb{R}} \mathcal{O}_3/\langle J, \partial J/\partial x_1, \partial J/\partial x_2\rangle = 12$. Then the origin is isolated in $V(I')$, $d_1^{-1}(\mathbf{0})$ and $d_2^{-1}(\mathbf{0})$. One can compute $\deg_{\mathbf{0}}(F_0) = 0$, $\deg_{\mathbf{0}}(d_1) = +1$ and $\deg_{\mathbf{0}}(d_2) = 0$. By Theorem 6.6, $\operatorname{cusp}\deg(F_t) = -1$ for $0 \neq t$ sufficiently close to zero, i.e. $\#\Sigma_t^+ - \#\Sigma_t^- = -1$.

As $\dim_{\mathbb{R}} \mathcal{O}_3/I'' = 45$ then $V(G_1, G_2)$ is a curve having an isolated singularity at the origin. Let J_p be defined the same way as in the previous example. One can verify that $\xi = 2$, and so $k = 4$. Put

$$H_{\pm} = \left(\frac{\partial(J \pm t^4, G_1, G_2)}{\partial(t, x_1, x_2)}, G_1, G_2 \right) : \mathbb{R}^3, \mathbf{0} \to \mathbb{R}^3, \mathbf{0}.$$

One can compute $\deg_{\mathbf{0}}(H_+) = 0$, $\deg_{\mathbf{0}}(H_-) = -2$. Then $V(J, G_1, G_2)$ is an union of two half-branches emanating from the origin, i.e. $b_0 = 2$.

Because $F_t(x_1, x_2) = F_{-t}(-x_1, -x_2)$, then $b'_0/2 = 1$ and $\#\Sigma_t^+ = \#\Sigma_{-t}^+$, $\#\Sigma_t^- = \#\Sigma_{-t}^-$. So in this case there is no need to compute $\deg_{\mathbf{0}}(H'_{\pm})$. Hence, if $t > 0$, then $\#\Sigma_t^+ = \#\Sigma_{-t}^+ = 0$ and $\#\Sigma_t^- = \#\Sigma_{-t}^- = 1$.

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