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Finite formal model of toric singularities

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Abstract. We study the formal neighborhoods at rational non-degenerate arcs of the arc scheme associated with a toric variety. The first main result of this article shows that these formal neighborhoods are generically constant on each Nash component of the variety. Furthermore, using our previous work, we attach to every such formal neighborhood, and in fact to every toric valuation, a *minimal* formal model (in the class of stable isomorphisms) which can be interpreted as a measure of the singularities of the base-variety. As a second main statement, for a large class of toric valuations, we compute the dimension and the embedding dimension of such minimal formal models, and we relate the latter to the Mather discrepancy. The class includes the strongly essential valuations, that is to say those the center of which is a divisor in the exceptional locus of every resolution of singularities of the variety. We also obtain a similar result for monomial curves.

1. Introduction.

1.1. In [**23**], Nash introduced an intriguing connection between the arc scheme associated with an algebraic variety and the singularities of the variety (see Subsection 2.1, Remark 2.1 for details and references). Since this seminal work, the study of the geometry of arc scheme has become a current prominent topic in the broad field of singularity theory. One approach is to investigate the singular locus of the arc scheme itself by considering the analytic type of its singularities, in other word to study formal neighborhoods of arcs over the singular locus.

The first breakthrough in this way has been obtained by Reguera in [**25**]. The present work can be linked, in spirit, to the subsequent work [**22**].

1.2. In this article, we focus on the two following questions. The first one states the problem of the behaviour of the isomorphism class of the formal neighborhoods of the arc scheme under the variations (over the singular locus) of the considered arc.

Question 1.1. Let k be a field. Let V be a k-variety. Denote by $\mathscr{L}_\infty(V)$ the arc scheme associated with V, and, for every arc $\gamma \in \mathscr{L}_\infty(V)$, by $\mathscr{L}_\infty(V)_\gamma$ the formal neighborhood of γ in $\mathscr{L}_\infty(V)$. How does $\mathscr{L}_\infty(V)_\gamma$ vary when γ runs over the arcs whose center is in the singular locus of V? Can one relate this variation with the singularities of V?

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In general, one can observe on specific examples that the formal neighborhood varies with the arc γ over the singular locus (see [**1**]). The second question completes the first one.

Question 1.2. Let k be a field. Let V be a k-variety. Let γ be an arc over the singular locus. What part of the information of the singularities of V is carried on the formal neighborhood $\mathscr{L}_\infty(V)_\gamma$?

1.3. The aim of this article is to study these two questions when V is a toric variety and γ a rational non-degenerate arc, i.e., not contained in the arc scheme associated with the singular locus of V. The study of this specific class of arcs is motivated in particular by a theorem of Drinfeld and Grinberg-Kazhdan (see [**15**],[**12**] or [**1**]) which, for every such arc γ, constructs an affine pointed k-variety (S,s), with $s \in S(k)$, and an isomorphism of formal k-schemes:

$$\mathscr{L}_\infty(V)_\gamma \cong S_s \hat{\otimes}_k k[[(T_i)_{i\in\mathbf{N}}]]. \tag{1.1}$$

Every noetherian formal k-schemes of the shape S_s with (S,s) as before and which realizes isomorphism (1.1) is called a *finite dimensional formal model* of the formal neighborhood $\mathscr{L}_\infty(V)_\gamma$ (or of the arc γ). Such a formal k-scheme S_s is said to be *non-cancellable* if there does not exist an affine pointed k-variety (S',s') such that $S_s \cong S'_{s'}\hat{\otimes}_k k[[T]]$. If (S,s) and (S',s') are pointed k-varieties, the formal k-schemes S_s and $S'_{s'}$ are said to be *stably isomorphic* if there exist positive integers n and m such that $S_s\hat{\otimes}_k k[[T_1,\dots,T_n]] \cong S'_{s'}\hat{\otimes}_k k[[T_1,\dots,T_m]]$. As proved in [**1**], thanks to a cancellation lemma due to Gabber (see *loc. cit.*), there exists (up to isomorphism) a unique non-cancellable finite dimensional model of γ, such that every finite dimensional formal model of γ is stably isomorphic to it: we call this model the *minimal* (*finite dimensional*) *formal model* of γ.

In particular, for the class of non-degenerate arcs, the study of the formal neighborhood $\mathscr{L}_\infty(V)_\gamma$ is equivalent to that of its minimal formal model, which, as a noetherian object, is likely to be more tractable. Moreover, as justified in [**4**], the minimal formal model can be understood as a measure of the singularity of V at the origin of γ. More precisely, in *loc. cit*, the authors proved that the minimal formal model is trivial if and only if the branch at $\gamma(0)$ (which contains γ) is formally smooth.

1.4. Let us explain in more details the content of the present article (see Subsection 2.1 for details on the terminology and notation used here in the description of the Nash problem). In the direction of Question 1.1, we show the following statement (see Theorem 4.2 for a more precise statement).

Theorem 1.3. *Let k be a field. Let V be an affine toric k-variety. Let D be an exceptional divisor of a toric resolution $\pi\colon W \to V$. Let $\mathcal{N}_D \subset \mathscr{L}_\infty(V)$ be the associated Nash set. Then there exists a Zariski non-empty open subset U of $\mathcal{N}_D$ such that $\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}$ (hence also $\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}}$) is constant when the arc γ runs over $U(k)$.*

Let us stress that this theorem relates, in an original manner, the formal neighborhoods in the arc scheme of non-degenerate rational arcs to the Nash components, hence to the Nash problem. More generally, we may wonder whether this kind of statements may reflect some deep "equisingularity property" for the arc scheme (see also Remark 4.4). In the direction of Question 1.2 we prove in particular the following result.

THEOREM 1.4. *Let k be a field of characteristic zero. Let V be an affine toric k-variety. Let ν be a strongly essential toric valuation. Then the minimal formal model of a sufficiently generic rational non-degenerate arc lying in the Nash set associated with ν is of dimension zero, and its embedding dimension coincides with the Mather discrepancy of ν.*

Such a result can be compared to [**22**], where Reguera and Mourtada have obtained similar results in the computation of the embedding dimension of other types of formal neighborhoods in arc schemes.

1.5. Let us explain the organization of the paper. After a recollection on toric geometry in section 2, we establish a technical lemma dealing with toric valuations associated with arcs on toric varieties in section 3, making a crucial use of results of Ishii ([**16**]) and Ishii–Kollár ([**18**]). This lemma is used for proving Theorem 4.2 which answers Question 1.1 in the toric case for rational non-degenerate arcs. The proof of Theorem 6.3 is based of an alternative proof of the Drinfeld–Grinberg–Kazhdan theorem in the toric case, which exploits the binomial nature of ideals defining toric varieties and which we explain in section 5. Though both proofs share some tools such as the use of the Weierstrass division theorems, our approach produces a finite dimensional formal model which differs from the one computed by Drinfeld's and turns out to be much more suited to the computation of the minimal formal model and its embedding dimension. Theorem 6.3 identifies a large class of toric valuations for which one also obtains an explicit description of the minimal formal model, with a result for the embedding dimension. This allows us to show that the dimension of the minimal formal model of a rational non-degenerate arc may be arbitrarily large, even when restricting to 3-dimensional varieties. (See Subsection 8.1.) In the end, let us note that this approach also works more generally in the context of binomial varieties, in other words for non-normal toric varieties. For the sake of simplicity, we have not written up the full details, but as an illustration, we use similar arguments in section 7 to compute, for every monomial curve singularity, the minimal formal model at a primitive arc and its embedding dimension, generalizing in particular results of [**3**]. In the end, in section 8, we provide various examples and further problems in the direction of the present work.

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Convention, notation.

In this article, we fix a field k of arbitrary characteristic. A *k-variety* is a k-scheme of finite type. The non-smooth locus of the structural morphism of a k-variety V is the *singular locus* of V and its associated reduced k-variety is denoted by V_{sing}. To every k-variety V (or more generally to every scheme), one attaches its *arc scheme* $\mathscr{L}_\infty(V)$ (e.g., see [**26**] for a precise definition). A point in $\mathscr{L}_\infty(V)$ is an *arc*. An arc of V which does not belong to $\mathscr{L}_\infty(V_{\mathrm{sing}})$ is called a *non-degenerate* arc of V. The k-scheme $\mathscr{L}_\infty(V)$

is endowed with a canonical morphism of V-scheme $\pi_0\colon \mathscr{L}_\infty(V)\to V$ which sends every arc γ to its base point $\gamma(0)$. For every subset $Z\subset V$, we denote by $\mathscr{L}_\infty(V)^Z$ the subset of $\mathscr{L}_\infty(V)$ formed by the arcs of V whose base-point belongs to Z, i.e., $\mathscr{L}_\infty(V)^Z := \pi_0^{-1}(Z)$. A *test-ring* (over k) is a local k-algebra $(A,\mathfrak{m}_A)$, whose residue field is ($k-$)isomorphic to k and whose maximal ideal is nilpotent. Let us note that, for every arc $\gamma\in\mathscr{L}_\infty(V)(k)$, the formal neighborhood $\mathscr{L}_\infty(V)_\gamma$ is determined by the restriction of its functor of points to the category of test-rings. For every test-ring $(A,\mathfrak{m}_A)$, an *A-deformation* of γ is an element $\gamma_A\in\mathscr{L}_\infty(V)_\gamma(A)$. It will be useful to keep in mind that such an A-deformation corresponds to one of the following (equivalent) commutative diagram:

$$\begin{tikzcd} \mathcal{O}_{\mathscr{L}_\infty(V),\gamma} \arrow[r, "\gamma_A"] \arrow[d] & A \arrow[d, "\bmod\ \mathfrak{m}_A"] \\ k \arrow[r, equal] & k \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_{V,v} \arrow[r, "\gamma_A"] \arrow[d, "\gamma"'] & A[[T]] \arrow[d] \\ k[[T]] \arrow[r, equal] & k[[T]], \end{tikzcd} \qquad \begin{tikzcd} \widehat{\mathcal{O}_{V,v}} \arrow[r, "\gamma_A"] \arrow[d, "\gamma"'] & A[[T]] \arrow[d] \\ k[[T]] \arrow[r, equal] & k[[T]], \end{tikzcd} \tag{1.2}$$

(E.g., see [**1**] for details.)

2. Recollection on the Nash problem and toric geometry, terminology.

In this section, we recollect various useful material and fix the used terminology.

2.1. We recall important definitions and properties about the Nash problem and related concepts (see [**18**] for more details). Let k be a field and V be an algebraic k-variety which admits resolution of singularities. An *exceptional divisor over V* is a divisorial valuation ν_D defined by an irreducible exceptional divisor D of a resolution of singularities $\pi\colon W\to V$. The *Nash set* $\mathcal{N}_D$ attached to D is the closure in $\mathscr{L}_\infty(V)$ of the set $\mathscr{L}_\infty(\pi)(\mathscr{L}_\infty(W)^D)$; it depends only on ν_D and not on the choice of the pair (W,D). Since $\mathscr{L}_\infty(W)^D$ is irreducible, so is $\mathcal{N}_D$. An *essential divisor over V* (resp. a *strongly essential divisor over V*) is an exceptional divisor ν over V such that for every resolution of singularities $\pi\colon W\to V$ the center of ν on W is an irreducible component (resp. an irreducible component of codimension 1) of the exceptional locus $\pi^{-1}(V_{\mathrm{sing}})$. We shall also speak of (*strongly*) *essential valuations* on V. A *Nash component of V* is an irreducible component of $\mathscr{L}_\infty(V)^{V_{\mathrm{sing}}}\setminus\mathscr{L}_\infty(V_{\mathrm{sing}})$. For every Nash component C, one shows that there exists a unique essential divisor D over V such that $\mathcal{N}_D\setminus\mathscr{L}_\infty(V_{\mathrm{sing}}) = C$. The so-called *Nash problem* may be stated as the problem to determine whether $\mathcal{N}_D\setminus\mathscr{L}_\infty(V_{\mathrm{sing}})$ is a Nash component for every essential divisor D over V.

REMARK 2.1. When the field k is assumed to be of characteristic zero, one knows that the answer to the above question fails to be true, in general, for varieties of dimension ≥ 3. (e.g., see [**20**] for details and references, and [**18**] for the first counterexample in dimension 4.) In positive characteristic, the question is open even in dimension two. By [**18**], we also know that this problem has a positive answer for the specific case of toric varieties.

REMARK 2.2. For the notion of essential divisors, we follow the terminology of [**18**]. Beware that in the works [**8**], [**7**], to be mentioned later, the slightly different terminology of Nash in [**23**] is followed. Thus what we call here an essential divisor is called there an *essential component*, whereas what we call here a strongly essential divisor is called there an essential divisor.

2.2. We now recall some standard facts about toric geometry, fixing along the way some notation used in the rest of the paper. (For more details on the suject, we refer to [**10**].) Let k be a field, $\mathcal{T}$ be a split algebraic k-torus of dimension d and $N := \mathrm{Hom}(\mathbf{G}_{m,k}, \mathcal{T})$ the group of its cocharacters. It defines a free abelian group of rank d. Let σ be a strictly convex N-rational polyhedral cone in $N \otimes_{\mathbf{Z}} \mathbf{R}$, in other words σ is the convex cone generated by finitely many elements of N, which moreover does not contain any line. Let $M := N^\vee = \mathrm{Hom}(N, \mathbf{Z})$ (which also is a free abelian group of dimension d) and $V(\sigma) := \mathrm{Spec}(k[\sigma^\vee \cap M])$ be the associated affine normal toric variety (often denoted by V if the cone σ is clear from the context). For every $m \in M$ we denote by χ^m the rational function on $V(\sigma)$ defined by m. Let $M_\sigma \subset \sigma^\vee \cap M$ be the minimal finite set generating the semigroup $\sigma^\vee \cap M$, which then forms a generating system of the $\mathbf{Q}$-vector space $M \otimes_{\mathbf{Z}} \mathbf{Q}$.

2.3. There is a natural bijection between the set of faces of σ and the set of $\mathcal{T}$-orbits in V. The orbit $\mathrm{orb}(\tau)$ associated with a face τ has codimension $\dim(\tau)$. In particular $\mathrm{orb}(\{0\})$ is the open orbit (which is isomorphic to $\mathcal{T}$) and $\mathrm{orb}(\sigma)$ is the only $\mathcal{T}$-invariant k-point on V. Moreover $x \in V$ is smooth if and only if it belongs to an orbit $\mathrm{orb}(\tau)$ with τ a N-regular face of σ (that is to say, τ may be generated by a part of a basis of N).

2.4. Recall that a N-primitive element of N is an element $n \in N$ such that the only elements of N which admit n as a multiple are n and $-n$. Each N-primitive element n of $\sigma \cap N$ determines a toric (i.e., torus-invariant) divisorial valuation on V, which maps the element of $k(V)$ corresponding to the character $m \in M$ to $\langle m\,,\, n\rangle$ (We stress that for us a divisorial valuation has multiplicity 1.). We shall frequently identify n and the associated toric valuation, thus speaking for example of (strongly) essential primitive elements of $\sigma \cap N$.

2.5. We now recall from [**18**] (resp. [**7**], [**8**]) the description of the essential divisors (resp. strongly essential divisors) of the affine toric variety $V = V(\sigma)$. First we introduce some terminology and notation, and make some remarks. Define a partial order on $\sigma \cap N$ by $n \prec_\sigma n'$ if and only if $n' \in n + \sigma$. Set

$$\mathrm{Sing}(\sigma) := \bigcup_{\substack{\tau \prec \sigma \\ \tau \text{ singular}}} \tau^\circ.$$

Example 2.3. If $d := \dim(X) = 2$ and V is singular, one has $\mathrm{Sing}(\sigma) = \sigma^\circ$.

Definition 2.4. Let $n \in \sigma \cap N$. We say that n is *indecomposable* if for every decomposition $n = n_1 + n_2$ with $n_1, n_2 \in \sigma \cap N$ one has either $n_1 = n$ or $n_2 = n$.

Remark 2.5. Let $n \in \sigma \cap N$ be a nonzero element. Then n is indecomposable if and only if n is a minimal element of $(\sigma \cap N) \setminus \{0\}$. In particular if n is an indecomposable element of $\mathrm{Sing}(\sigma)\cap N$ then n is a minimal element of $\mathrm{Sing}(\sigma)\cap N$. But in general there are minimal elements of $\mathrm{Sing}(\sigma)\cap N$ which are not indecomposable.

Example 2.6. Let N be a free $\mathbf{Z}$-module with basis $\{e_1, e_2, e_3\}$ and σ be the cone of $N\otimes_{\mathbf{Z}}\mathbf{R}$ generated by e_1, e_2, $f_1 := e_1+e_3$ and $f_2 = e_2+e_3$. Let $n = e_2+f_1 = e_1+f_2$. Then $\mathrm{Sing}(\sigma) = \sigma^\circ$ and n is a minimal element of $\mathrm{Sing}(\sigma)\cap N$, yet n is not indecomposable.

Theorem 2.7. *Let $n\in \mathrm{Sing}(\sigma)\cap N$ be a N-primitive element. Then n is essential if and only if n is a minimal element of* $\mathrm{Sing}(\sigma)\cap N$ *with respect to the order $\prec_\sigma$. Moreover n is strongly essential if and only if n is indecomposable.*

Proof. The first assertion follows from [**18**, Section 3] and the second assertion is a consequence of [**8**, Theorem 1.10] and [**7**, Theorem 1.2]. □

3. Toric valuations and stratifications on non-degenerate toric arcs.

In this section we recall some facts about toric valuations attached to arcs on toric varieties following [**16**], [**18**], and we prove a useful technical lemma (Lemma 3.6).

3.1. Let $V := V(\sigma)$ be an affine toric variety associated to the cone σ. We retain the previously introduced notation. We set $\mathscr{L}^\circ_\infty(V) := \mathscr{L}_\infty(V)\setminus\mathscr{L}_\infty(V\setminus\mathcal{T})$. Let us consider $\gamma\in\mathscr{L}^\circ_\infty(V)(k)$. As an element of $\mathscr{L}_\infty(V)(k)$, the rational arc γ corresponds to a k-algebra morphism $k[\sigma^\vee\cap M]\to k[[T]]$, or equivalently to a morphism of semigroups $\gamma_{sg}\colon \sigma^\vee\cap M\to k[[T]]$. Composing γ_{sg} with $T\mapsto 0$ gives the base-point $\gamma(0)$. Since γ does not lie in $\mathscr{L}_\infty(V\setminus\mathcal{T})$ and since the ideal of $V\setminus\mathcal{T}$ in $k[\sigma^\vee\cap M]$ is generated by the product of the $\{\chi^m\}_{m\in M_\sigma}$, no χ^m lies in $\mathrm{Ker}(\gamma)$. In other words, for every $m\in\sigma^\vee\cap M$, one has $\gamma_{sg}(m)\neq 0$. Thus one may compose γ with the usual map ord_T and obtain a morphism of semigroups

$$\mathrm{ord}(\gamma)\colon \sigma^\vee\cap M\to\mathbf{N}.$$

In this way, we observe that the functional $\mathrm{ord}(\gamma)$ in fact belongs to $\sigma\cap N$. More generally, for every arc $\gamma\in\mathscr{L}^\circ_\infty(V)$, one can define $\mathrm{ord}(\gamma)\in\sigma\cap N$ by considering the element of $\mathscr{L}^\circ_\infty(V)(\kappa(\gamma))$ induced by γ.

Remark 3.1. For every arc $\gamma\in\mathscr{L}^\circ_\infty(V)$, the functional $\mathrm{ord}(\gamma)$ determines in particular the toric stratum to which $\gamma(0)$ belongs. Indeed, by [**18**, Proposition 3.9(i)], for every face τ of σ, one has $\gamma(0)\in\mathrm{orb}(\tau)$ if and only if $\mathrm{ord}(\gamma)\in\tau^\circ$.

3.2. Since the torus $\mathcal{T}$ acts on V, the group scheme $\mathscr{L}_\infty(\mathcal{T})$ acts on $\mathscr{L}_\infty(V)$. This action has been studied in particular by S. Ishii, who proved the following result.

Lemma 3.2 ([**16**, Theorem 4.1(ii)]). *Let $\gamma_1,\gamma_2\in\mathscr{L}^\circ_\infty(V)(k)$. Then* $\mathrm{ord}(\gamma_1) = \mathrm{ord}(\gamma_2)$ *if and only if there exists $t\in\mathscr{L}_\infty(\mathcal{T})(k)$ such that $\gamma_1 = t\cdot\gamma_2$.*

In [**16**], the base field k is assumed to be algebraically closed; here k is arbitrary but due in particular to the fact that one works with split toric varieties, the arguments in

[**16**] are easily seen to be still valid.

Since for every $t \in \mathscr{L}_\infty(\mathcal{T})(k)$, $\gamma \mapsto t \cdot \gamma$ is an automorphism of $\mathscr{L}_\infty(V)$, we deduce the following consequence:

COROLLARY 3.3. *Let $\gamma \in \mathscr{L}^\circ_\infty(V)(k)$. Then the local ring $\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}$, depends only on* $\mathrm{ord}(\gamma)$. *In particular the completion $\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}}$ depends only on* $\mathrm{ord}(\gamma)$.

DEFINITION 3.4. For $n \in \sigma \cap N$, we denote by $\mathscr{S}_n$ the minimal formal model of any element γ of $\mathscr{L}^\circ_\infty(V)(k)$ such that $\mathrm{ord}(\gamma) = n$.

REMARK 3.5. In section 5, we shall give another proof of the result concerning $\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}}$ in Corollary 3.3. This proof in fact will provide an explicit description of a finite dimensional formal model at every arc γ such that $\mathrm{ord}(\gamma) = n$. Moreover, we will show that this description gives the minimal formal model $\mathscr{S}_n$ in case n is a strongly essential valuation, as well as in some other cases.

3.3. For every integer $\mu \in \mathbf{N}$ and every $m \in \sigma^\vee \cap M$, the set $\{n \in \sigma, \langle m\,, n\rangle \leq \mu\}$ will be simply denoted by $\{\langle m\,, \cdot\rangle \leq \mu\}$ (similar definition and notation with $\leq \mu$ replaced with $\geq \mu$). For every $n \in \sigma \cap N$, we set $\mathscr{L}^\circ_\infty(V)_n := \{\gamma \in \mathscr{L}^\circ_\infty(V), \mathrm{ord}(\gamma) = n\}$. More generally, for every $A \subset \sigma$, we set $\mathscr{L}^\circ_\infty(V)_A := \{\gamma \in \mathscr{L}^\circ_\infty(V), \mathrm{ord}(\gamma) \in A\}$. Recall that a *constructible subset* of a *separated and quasi-compact* scheme S is a finite union of subsets of the form $O_1 \cap (S \setminus O_2)$ for quasi-compact open subsets O_1, O_2 of S. A *proconstructible subset* of S is an intersection of constructible subsets. By construction, the arc scheme associated with S is separated and quasi-compact.

LEMMA 3.6. *For every integer $\mu \in \mathbf{N}$, $\mathscr{L}^\circ_\infty(V)_{\{\langle m\,,\cdot\rangle\leq\mu\}}$ is a constructible open subset of $\mathscr{L}^\circ_\infty(V)$ and $\mathscr{L}^\circ_\infty(V)_{\{\langle m\,,\cdot\rangle\geq\mu\}}$ is a constructible closed subset of $\mathscr{L}^\circ_\infty(V)$. For every $n \in \sigma \cap N$, the set $\mathscr{L}^\circ_\infty(V)_n$ is a dense constructible open subset of $\mathscr{L}^\circ_\infty(V)_{n+\sigma}$, and the set $\mathscr{L}^\circ_\infty(V)_{n+\sigma}$ is a proconstuctible closed subset in $\mathscr{L}^\circ_\infty(V)$.*

PROOF. Note that for $m \in \sigma^\vee \cap M$ and $\gamma \in \mathscr{L}^\circ_\infty(V)$, the integer $\langle m\,, \mathrm{ord}(\gamma)\rangle$ is nothing else but $\mathrm{ord}_T(\chi^m(\gamma))$. This shows the first assertion.

Let $n \in \sigma \cap N$. Since the set M_σ is finite (see Subsection 2.2), and since we have

$$\mathscr{L}^\circ_\infty(V)_n = \mathscr{L}^\circ_\infty(V)_{n+\sigma} \cap \left(\bigcap_{m\in M_\sigma} \mathscr{L}^\circ_\infty(V)_{\langle m\,,\cdot\rangle\leq\langle m\,,n\rangle}\right),$$

we deduce that $\mathscr{L}^\circ_\infty(V)_n$ is a constructible open subset of $\mathscr{L}^\circ_\infty(V)_{n+\sigma}$. By [**16**, proposition 4.8], the closure of $\mathscr{L}^\circ_\infty(V)_n$ is $\mathscr{L}^\circ_\infty(V)_{n+\sigma}$.

On the other hand, the equality

$$\mathscr{L}^\circ_\infty(V)_{n+\sigma} = \bigcap_{m\in M_\sigma} \mathscr{L}^\circ_\infty(V)_{\langle m\,,\cdot\rangle\geq\langle m\,,n\rangle}$$

shows that $\mathscr{L}^\circ_\infty(V)_{n+\sigma}$ is a proconstructible closed subset in $\mathscr{L}^\circ_\infty(V)$. □

3.4. For the convenience of the reader, we quickly provide a general definition of the notion of Mather discrepancy. Complements can be found, e.g., in [**17**], [**19**], [**13**], [**11**]. If V is a k-variety of dimension d, if $\pi\colon W \to V$ is a resolution of the singularities of V which factorizes through the Nash blow-up of V, then the image of the canonical morphism $\pi^*\Omega^d_{V/k} \to \Omega^d_{W/k}$ is an invertible sheaf, and thus may be written as $\Omega^d_{W/k}(-\hat{K}_{W/V})$, where $\hat{K}_{W/V}$ is a divisor called the *Mather discrepancy divisor.* This divisor is supported on the exceptional locus of π. For every exceptional divisor ν, one calls *the Mather discrepancy along* ν the integer $\hat{K}_\nu := \mathrm{ord}_D(\hat{K}_{W/V})$ where $\pi\colon W \to V$ is a resolution of the singularities of V which factorizes through the Nash blow-up and such that the center D of ν on W is an exceptional divisor.

Let us consider the particular case where $V = \mathscr{C}$ is an integral unibranch curve at the singular point x (with k assumed to be algebraically closed for simplicity). Let $\bar{x}$ be the preimage of x in the normalisation $\bar{\mathscr{C}}$ of $\mathscr{C}$. Then the so-called multiplicity $\mathrm{mult}(\mathscr{C}, x)$ of the germ $(\mathscr{C}, x)$ may be defined as the positive integer m such that T^m generates the image of the maximal ideal $\mathfrak{m}_{\mathscr{C},x}$ of $\mathcal{O}_{\mathscr{C},x}$ by the composition γ^* of the morphisms

$$\mathcal{O}_{\mathscr{C},x} \to \mathcal{O}_{\bar{\mathscr{C}},\bar{x}} \to \widehat{\mathcal{O}_{\bar{\mathscr{C}},\bar{x}}} \cong k[[T]].$$

In particular, one has $\gamma^*(d\mathfrak{m}_{\mathscr{C},x}) = T^{\mathrm{mult}(\mathscr{C},x)-1}k[[T]]dT$ which shows that $\hat{K}_{\bar{x}} = \mathrm{mult}(\mathscr{C}, x) - 1$.

In the toric case, one has an explicit formula for the Mather discrepancy in terms of the combinatorial data. We recall it here since it will be an ingredient in the proof of Theorem 6.3.

Lemma 3.7 ([**11**, Lemma 5.2]). *Let $V = V(\sigma)$ be an affine toric k-variety of dimension d. Let n be an N-primitive element of $\sigma \cap N$, as before identified with a toric divisorial valuation on V. Then, the Mather discrepancy $\hat{K}_n$ of n is given by the following formula*

$$\hat{K}_n = \min_{\substack{(m_1,\dots,m_d)\in M_\sigma^d \\ \{m_j\}\ \mathbf{Q}\text{-linearly independent}}} \left(-1 + \sum_{j=1}^{d} \langle m_j\,,\, n\rangle \right).$$

4. Generic behaviour of formal neighborhood of non-degenerate arcs on the Nash components of a toric variety.

In the direction of a comprehensive study of the formal neighborhoods of non-degenerate rational arcs, a challenging question is to understand how these formal neighborhoods change with the involved arcs. One of the motivation of this problem is the situation in the case of curves explained in Subsection 4.1. The main result of this section can be interpreted as a higher dimensional analog in toric geometry of the curves situation. In particular, Corollary 4.2 establishes the generic behaviour of the formal neighborhoods of arc schemes at rational non-degenerate arcs on the Nash components (see Subsection 2.1 for definitions.)

4.1. Let $\mathscr{C}$ be an integral curve defined over an algebraically closed field k of characteristic zero. Let $x \in \mathscr{C}(k)$ be a singular point. The set of the irreducible components of $\pi_0^{-1}(x)$ coincides with that of the sets $\mathcal{N}_y$ where y runs over the preimages of $\gamma(0)$ in the normalization of $\mathscr{C}$, i.e, the set of arcs on $\mathscr{C}$ which factorizes through the corresponding branch $\mathscr{B}_y$ (which is defined as an irreducible component of $\widehat{\mathcal{O}_{\mathscr{C},x}}$). For every non-constant arc $\gamma \in \mathscr{L}_\infty(\mathscr{C})$, with base-point $\gamma(0) = x$, the ideal generated by the image of the attached morphism of complete local k-algebras

$$\gamma^*\colon \widehat{\mathcal{O}_{\mathscr{C},x}} \to k[[T]]$$

is of the form $T^{m_\gamma}k[[T]]$. In this way, one constructs a map

$$\mathrm{mult}\colon \pi_0^{-1}(x) \setminus \{x\} \to \mathbf{N}$$

which sends γ to m_γ; in the above formula, in the expression $\{x\}$, one identifies x with its image by the zero section $\mathscr{C} \to \mathscr{L}_\infty(\mathscr{C})$. Let $\gamma_y \in \mathscr{L}_\infty(\mathscr{C})(k)$ be a primitive parametrization of $\mathscr{C}$ at x corresponding to a formal branch $\mathscr{B}_y$ of $\mathscr{C}$ at x (see, e.g., [**9**]). Let $m_x := \mathrm{mult}(\gamma_y)$. When $\mathscr{C}$ is unibranch at x, this integer is a possible definition for the so-called *multiplicity* of the germ $(\mathscr{C}, x)$ (see also Subsection 3.4). The constructible subset $\mathrm{mult}^{-1}(m_x)$ of $\mathscr{L}_\infty(\mathscr{C})$ induces a constructible *open* subset U of the irreducible component of $\pi_0^{-1}(x)$ corresponding to γ_y (or $\mathscr{B}_y$), i.e., $\mathcal{N}_y$, since on this component the integer m_x is the minimal possible value of the map mult. Every rational arc of U is a primitive parametrization of $\mathscr{B}_y$ which can be deduced from γ by a reparametrization of T induced by an automorphism of $k[[T]]$. As proved in [**1**], for every $\gamma' \in U(k)$, we have

$$\mathscr{L}_\infty(\mathscr{C})_{\gamma_y} \cong \mathscr{L}_\infty(\mathscr{C})_{\gamma'}.$$

4.2. Let us consider a toric resolution of singularities $\pi\colon W \to V$ of the affine toric variety $V = V(\sigma)$. Its exceptional divisors correspond to one-dimensional cones lying in σ. If D is such an exceptional divisor, we denote by n_D the primitive generator of the corresponding one-dimensional cone.

Lemma 4.1. *Let D be an exceptional divisor of a toric resolution $\pi\colon W \to V$, $\tau_D \subset \sigma$ be the corresponding one-dimensional cone and n_D be the primitive generator of τ_D. Then one has $\mathcal{N}_D \cap \mathscr{L}^\circ_\infty(V) = \mathscr{L}^\circ_\infty(V)_{n_D+\sigma}$ (see Section 3.3 for a definition). In particular $\mathscr{L}^\circ_\infty(V)_{n_D}$ is a dense open subset of $\mathcal{N}_D$.*

Proof. By [**18**, Proposition 3.9(ii)], one has $\mathscr{L}_\infty(\pi)(\mathscr{L}_\infty(W)^{\mathrm{orb}(\tau_D)} \setminus \mathscr{L}_\infty(W \setminus \mathcal{T})) = \mathscr{L}_\infty(V)^\circ_{\tau_D^\circ}$. Since W is assumed to be smooth and $\mathrm{orb}(\tau_D)$ is open in D (see 2.4), we conclude that $\mathscr{L}_\infty(W)^{\mathrm{orb}(\tau_D)} \setminus \mathscr{L}_\infty(W \setminus \mathcal{T})$ is a dense open subset of $\mathscr{L}_\infty(W)^D$. Thus, we deduce that $\mathscr{L}_\infty(V)_{\tau_D^\circ}$ is dense in $\mathscr{L}_\infty(\pi)(\mathscr{L}_\infty(W)^D)$, hence in $\mathcal{N}_D$ by the very definition of $\mathcal{N}_D$. But one has

$$\mathscr{L}^\circ_\infty(V)_{n_D} \subset \mathscr{L}^\circ_\infty(V)_{\tau_D^\circ} \subset \mathscr{L}^\circ_\infty(V)_{n_D+\sigma}$$

and by Lemma 3.6, $\mathscr{L}^\circ_\infty(V)_{n_D}$ is a dense open subset of $\mathscr{L}^\circ_\infty(V)_{n_D+\sigma}$. We infer that $\mathscr{L}^\circ_\infty(V)_{n_D+\sigma}$ is dense in $\mathcal{N}_D$, hence in $\mathcal{N}_D \cap \mathscr{L}^\circ_\infty(V)$. But again by Lemma 3.6,

$\mathscr{L}_\infty^\circ(V)_{n_D+\sigma}$ is closed in $\mathscr{L}_\infty^\circ(V)$, thus $\mathcal{N}_D \cap \mathscr{L}_\infty^\circ(V) = \mathscr{L}_\infty^\circ(V)_{n_D+\sigma}$ as claimed. Since $\mathscr{L}_\infty^\circ(V)_{n_D}$ is a dense open subset of $\mathscr{L}_\infty^\circ(V)_{n_D+\sigma}$ and $\mathcal{N}_D \cap \mathscr{L}_\infty^\circ(V)$ is open in $\mathcal{N}_D$, we conclude that $\mathscr{L}_\infty^\circ(V)_{n_D}$ is a dense open subset of $\mathcal{N}_D$. □

THEOREM 4.2. *Retain the previous notation. Let D be an exceptional divisor of a toric resolution $\pi\colon W \to V(\sigma)$. Then there exists a Zariski non-empty open subset U of $\mathcal{N}_D \cap \mathscr{L}_\infty^\circ(V(\sigma))$ such that $\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}$ (hence also $\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}}$) is constant for $\gamma \in U(k)$.*

PROOF. By Lemma 4.1 and Corollary 3.3, it suffices to take $U := \mathscr{L}_\infty^\circ(V(\sigma))_{n_D}$. □

4.3. Regarding Theorem 4.2, the case where $\mathcal{N}_D$ is a Nash components of V is thus of particular interest, since it allows us to understand "generically" the minimal formal model of arcs in $\mathscr{L}_\infty^\circ(V)^{V_{\text{sing}}}(k) \setminus \mathscr{L}_\infty(V_{\text{sing}})$. By [**18**, Theorem 3.16], if D is an exceptional divisor over V, we know that $\mathcal{N}_D$ is a Nash component of V if and only if D is an essential divisor of V (In other words, the Nash problem has an affirmative answer for toric varieties), and then D is necessarily a toric (i.e., torus-invariant) divisor.

REMARK 4.3. It seems to us an interesting problem to understand whether Theorem 4.2 extend to arbitrary varieties.

REMARK 4.4. In [**6**], [**5**], [**24**], various statements for a global version of the Drinfeld–Grinberg–Kazhdan have been established. It seems to us interesting to understand, at least in the toric framework, the precise link between these statements and Theorem 4.2. We strongly believe that this will be connected to equisingularity properties of $\mathscr{L}_\infty(V)$.

5. A computation of finite dimensional formal models for binomial varieties.

In this section, we provide an alternative proof of the Drinfeld–Grinberg–Kazdhan theorem for varieties defined by binomial ideals. It produces a presentation of a finite dimensional formal model different from the one given by Drinfeld's general approach. The presentation that we obtain turns out to be much more suited to the determination of the minimal formal model and its embedding dimension. For the sake of simplicity, we focus on the case of toric varieties. In section 7, we shall explain how this approach allows to study the minimal formal model of monomial curve singularities.

5.1. Let us fix first some definitions and notation. If $d \leq e$ are integers, we denote by $[\![d, e]\!]$ the set $\{n \in \mathbf{Z}, d \leq n \leq e\}$. Let I and J be *finite* sets, $(d_j) \in \mathbf{N}^J$ be a family of non-negative integers and $(\Pi_i)_{i\in I}$ be a family of elements of $k[(X_j)_{j\in J}]$ such that, for every element $i \in I$, one has $\Pi_i((T^{d_j})_{j\in J}) = 0$. We define the affine k-scheme $W((d_j)_{j\in J}, (\Pi_i)_{i\in I})$ as the closed subscheme of the affine space $\mathbf{A} := \operatorname{Spec}(k[(p_{j,a})_{\substack{j\in J\\ a\in[\![1,d_j]\!]}}])$ whose ideal is generated by all the coefficients with respect to the variable T in the polynomials

$$\Pi_i\left(\left(T^{d_j}+\sum_{a\in[\![1,d_j]\!]}p_{j,a}T^{d_j-a}\right)_{j\in J}\right)\tag{5.1}$$

for $i\in I$. Note that this ideal is contained in the ideal $\langle(p_{j,a})_{\substack{j\in J\\ a\in[\![1,d_j]\!]}}\rangle$ which is the origin $\mathfrak{o}$ of the scheme $\mathbf{A}$. We denote by $\mathscr{W}((d_j),(\Pi_i))$ the formal completion of $W((d_j),(\Pi_i))$ along this origin $\mathfrak{o}$. If $j_0\in J$, we define $W^{(j_0)}((d_j),(\Pi_i))$ to be the closed subscheme of $W((d_j),(\Pi_i))$ whose ideal is generated by the image of $p_{j_0,1}$ in $\mathcal{O}(W((d_j),(\Pi_i)))$. Its formal completion of $W^{(j_0)}((d_j),(\Pi_i))$ along the origin $\mathfrak{o}$ is denoted by $\mathscr{W}^{(j_0)}((d_j),(\Pi_i))$.

Lemma 5.1. *Assume* $\mathrm{char}(k)$ *and* d_{j_0} *are coprime. Let* $j_0\in J$ *and* $(Q_{j,a})_{\substack{j\in J\\ a\in[\![1,d_j]\!]}}$ *be the family of elements of* $k[(p_{j_0,a})_{\substack{j\in J\\ a\in[\![1,d_j]\!]}}]$ *defined by*

$$T^{d_j}+\sum_{a\in[\![1,d_j]\!]}p_{j,a}T^{d_j-a}=\left(T-\frac{p_{j_0,1}}{d_{j_0}}\right)^{d_j}+\sum_{a\in[\![1,d_j]\!]}Q_{j,a}\cdot\left(T-\frac{p_{j_0,1}}{d_{j_0}}\right)^{d_j-a}$$

(in particular $Q_{j_0,1}=0$*).*

Then the automorphism of k*-algebras of* $k[(p_{j,a})_{\substack{j\in J\\ a\in[\![1,d_j]\!]}}]$ *mapping* $p_{j,a}$ *to* $p_{j,a}$ *for* $(j,a)=(j_0,1)$ *and* $p_{j,a}$ *to* $Q_{j,a}$ *for any other value of the pair* (j,a) *induces isomorphisms:*

$$W((d_j),(\Pi_i))\cong W^{(j_0)}((d_j),(\Pi_i))\times_k\mathbf{A}^1_k$$

and

$$\mathscr{W}((d_j),(\Pi_i))\cong\mathscr{W}^{(j_0)}((d_j),(\Pi_i))\hat{\times}_k\mathbf{D}_k.$$

Proof. This comes from the fact that, setting $U=T-(p_{j_0,1})/(d_{j_0})$, expression (5.1) may be rewritten

$$\Pi_i\left(U^{d_{j_0}}+\sum_{a\in[\![2,d_{j_0}]\!]}Q_{j_0,a}T^{d_{j_0}-a},\left(U^{d_j}+\sum_{a\in[\![1,d_j]\!]}Q_{j,a}U^{d_j-a}\right)_{j\in J\setminus\{j_0\}}\right).\tag{5.2}$$

$\square$

5.2. We keep the notation introduced in the previous sections. In particular, $V(\sigma)$ designates an affine toric variety of arbitrary dimension d. Let I be a finite set and $(\alpha_i)\in(\mathbf{Z}^{M_\sigma})^I$ be a finite collection of nontrivial integral linear relations between the elements of M_σ that is to say, for every $i\in I$,

$$\sum_{m\in M_\sigma}\alpha_{i,m}m=0,\tag{5.3}$$

such that moreover

$$\left\{ \prod_{\substack{m\in M_\sigma \\ \alpha_{i,m}>0}} X_m^{\alpha_{i,m}} - \prod_{\substack{m\in M_\sigma \\ \alpha_{i,m}<0}} X_m^{-\alpha_{i,m}} \right\}_{i\in I}$$

is a set of binomial generators of the kernel of the morphism $k[(X_m)_{m\in M_\sigma}] \to k[\sigma^\vee \cap M]$ sending X_m to χ^m.

5.3. Let $(\nu_m) \in \mathbf{N}^{M_\sigma}$ be a family of non-negative integers such that, for every element $i \in I$,

$$\sum_{m\in M_\sigma} \alpha_{i,m}\nu_m = 0.$$

Then there exists a unique $n \in \sigma \cap N$ such that, for every $m \in M_\sigma$, we have

$$\langle m\,,\, n\rangle = \nu_m.$$

This may be deduced from the fact that M_σ necessarily contains a $\mathbf{Z}$-basis of M (consider a M-regular subdivision of $\sigma^\vee$, which exists by [**14**, Section 2.6]).

5.4. Let $\widetilde{M_\sigma} \subset M_\sigma$ be a subset of cardinality d whose elements are $\mathbf{Q}$-linearly independent. In particular $\widetilde{M_\sigma}$ is a $\mathbf{Q}$-basis of $M \otimes_{\mathbf{Z}} \mathbf{Q}$. (See Subsection 2.2.) Thus, every $m \in M_\sigma \setminus \widetilde{M_\sigma}$ is a $\mathbf{Q}$-linear combination of elements of $\widetilde{M_\sigma}$. Then, for every $m \in M_\sigma \setminus \widetilde{M_\sigma}$, there exists a positive integer α such that αm is a $\mathbf{Z}$-linear combination of elements of $\widetilde{M_\sigma}$. We enlarge I by this kind of relations. Thus from now on the set I is such that relations (5.3) verify that, for every $m \in M_\sigma \setminus \widetilde{M_\sigma}$, there exists $i \in I$ such that

$$\{m' \in M_\sigma, \alpha_{i,m'} \neq 0\} \setminus \widetilde{M_\sigma} = \{m\}.$$

5.5. Let $n \in \sigma \cap N$. We set

$$\mathscr{W}_n := \mathscr{W}\left((\langle m\,,\, n\rangle)_{m\in M_\sigma}, \left(\prod_{\substack{m\in M_\sigma \\ \alpha_{i,m}>0}} X_m^{\alpha_{i,m}} - \prod_{\substack{m\in M_\sigma \\ \alpha_{i,m}<0}} X_m^{-\alpha_{i,m}} \right)_{i\in I} \right). \tag{5.4}$$

and, for every $m_0 \in M_\sigma$ such that $\langle m_0\,,\, n\rangle \geq 1$,

$$\mathscr{W}_n^{(m_0)} := \mathscr{W}^{(m_0)}\left((\langle m\,,\, n\rangle)_{m\in M_\sigma}, \left(\prod_{\substack{m\in M_\sigma \\ \alpha_{i,m}>0}} X_m^{\alpha_{i,m}} - \prod_{\substack{m\in M_\sigma \\ \alpha_{i,m}<0}} X_m^{-\alpha_{i,m}} \right)_{i\in I} \right). \tag{5.5}$$

Let us state and prove the main theorem of this section.

THEOREM 5.2. *Let $n \in \sigma \cap N$ and $\gamma \in \mathscr{L}_\infty^\circ(X)_n(k)$. Then, the formal k-scheme $\mathscr{W}_n$ (defined by formula (5.4)) is a finite dimensional formal model of $\mathscr{L}_\infty(V)_\gamma$.*

The proof given below provides in particular an alternative proof of the Drinfeld–Grinberg–Kazdhan theorem for $\gamma \in \mathscr{L}_\infty^\circ(X)_n(k)$. Note also that Theorem 5.2 allows us to recover the fact (obtained in Corollary 3.3) that the completion $\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}}$ only depends on $\mathrm{ord}(\gamma)$. Furthermore, by Lemma 5.1, we directly obtain the following consequence.

COROLLARY 5.3. *Let $n \in \sigma \cap N$ and $\gamma \in \mathscr{L}_\infty^\circ(X)_n(k)$. Let $m_0 \in M_\sigma$ such that $\langle m_0\,,\, n\rangle \geq 1$. Assume that $\langle m_0\,,\, n\rangle$ and* $\mathrm{char}(k)$ *are coprime. Then, the formal k-scheme $\mathscr{W}_n^{(m_0)}$ (defined by formula* (5.5)*) is a finite dimensional formal model of $\mathscr{L}_\infty(V)_\gamma$.*

PROOF. (of Theorem 5.2) Recall that γ corresponds to the datum of a morphism of semigroups

$$\gamma_{sg} \colon \sigma^\vee \cap M \to k[[T]]$$

such that $\gamma_{sg}^{-1}(\{0\}) = \emptyset$ and $\mathrm{ord}_T(\gamma_{sg}(m)) = \langle m\,,\, n\rangle$ for every $m \in \sigma^\vee \cap M$. Composing γ_{sg} with $T \mapsto 0$ gives the base-point $\gamma(0)$. Let $\tau \prec \sigma$ be the unique face such that $n \in \tau^\circ$.

Thus, the arc γ corresponds to the datum of a family $(\gamma_m(T))_{m\in M_\sigma}$ of elements of $k[[T]] \setminus \{0\}$ such that, for every $m \in M_\sigma$, one has $\mathrm{ord}_T(\gamma_m) = \langle m\,,\, n\rangle$ and, for every $i \in I$,

$$\prod_{\substack{m\in M_\sigma\\ \alpha_{i,m}>0}} \gamma_m(T)^{\alpha_{i,m}} = \prod_{\substack{m\in M_\sigma\\ \alpha_{i,m}<0}} \gamma_m(T)^{-\alpha_{i,m}}.$$

Moreover, for every $m \in \tau^\perp$, one has $\gamma_m(0) = k^\times$. Thus, for every test-ring A, an A-deformation of γ is a collection $(\gamma_{m,A}(T))_{m\in M_\sigma}$ of elements of $A[[T]]$ such that, for every $m \in M_\sigma$,

$$\gamma_{m,A}(T) = \gamma_m(T) \pmod{\mathfrak{m}_A[[T]]} \tag{5.6}$$

and, for every $i \in I$,

$$\prod_{\substack{m\in M_\sigma\\ \alpha_{i,m}>0}} \gamma_{m,A}(T)^{\alpha_{i,m}} = \prod_{\substack{m\in M_\sigma\\ \alpha_{i,m}<0}} \gamma_{m,A}(T)^{-\alpha_{i,m}}. \tag{5.7}$$

Let $(\gamma_{m,A}(T))_{m\in M_\sigma}$ be a collection of elements of $A[[T]]$ such that formula (5.6) holds. For $m \in M_\sigma$, let

$$\gamma_{m,A}(T) = p_{m,A}(T)u_{m,A}(T)$$

be the Weierstrass decomposition of $\gamma_{m,A}(T)$ (see [**21**, IV, Theorem 9.2]). In particular, the polynomial $p_{m,A}(T)$ is a Weierstrass polynomial (see *loc. cit.*) with degree $\langle m\,,\, n\rangle$ and $u_{m,A}(T) \in A[[T]]^\times$. In case $m \in \tau^\perp$, one has $p_{m,A} = 1$ and $\gamma_{m,A}(T) = u_{m,A}(T)$.

By the uniqueness in the Weierstrass decomposition, the deformation $(\gamma_{m,A}(T))_{m\in M_\sigma}$ satisfies relation (5.7) if and only if the following relations hold true:

$$\forall i \in I \qquad \prod_{\substack{m\in M_\sigma\\ \alpha_{i,m}>0}} p_{m,A}(T)^{\alpha_{i,m}} = \prod_{\substack{m\in M_\sigma\\ \alpha_{i,m}<0}} p_{m,A}(T)^{-\alpha_{i,m}} \tag{5.8}$$

and

$$\forall i \in I \quad \prod_{\substack{m\in M_\sigma \\ \alpha_{i,m}>0}} u_{m,A}(T)^{\alpha_{i,m}} = \prod_{\substack{m\in M_\sigma \\ \alpha_{i,m}<0}} u_{m,A}(T)^{-\alpha_{i,m}}. \tag{5.9}$$

Now, the functor on the category of test-rings defined by identifying T^i-coefficients in relations (5.9) is isomorphic to $A \mapsto \mathfrak{m}_A^{\mathbf{N}}$. Indeed, relations (5.9) describe the A-deformations of the arc $\widetilde{\gamma} := (\gamma_m(T)/T^{\langle m\,,\,n\rangle})$, whose origin lies in $\mathcal{T}$ and is therefore a non-singular point of V, which implies $\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\widetilde{\gamma}}} \cong \mathrm{Spf}(k[[(T_i)_{i\in\mathbf{N}}]])$. Relations (5.8) define $\mathscr{W}_n(A)$ by formula (5.4). Thus the functor $A \mapsto \mathrm{Def}_\gamma(A)$ is isomorphic to the functor $A \mapsto \mathscr{W}_n(A) \times \mathfrak{m}_A^{\mathbf{N}}$. That concludes the proof. $\square$

EXAMPLE 5.4. We consider the toric surface

$$\mathscr{S} = \mathrm{Spec}(k[X_0,X_1,X_2,X_3]/\langle X_0X_2 - X_1^2, X_1X_3 - X_2^3\rangle$$

and the arc $\gamma(T) = (T,T,T,T^2)$. Using Corollary 5.3 we obtain a finite formal model of γ whose presentation in $\mathrm{Spf}(k[[p_{0,0},p_{1,0},p_{3,1},p_{3,0}]])$ is given by the T-coefficients of the polynomials

$$T(T+p_{0,0}) - (T+p_{1,0})^2 \quad \text{and} \quad (T+p_{1,0})(T^2+p_{3,1}T+p_{3,0}) - T^3.$$

After identification and elimination, we find that this finite model is isomorphic to $\mathrm{Spf}(k[[p_{1,0}]]/\langle p_{1,0}^2\rangle)$, hence is minimal.

6. Minimal formal model of toric singularities.

Theorem 5.2 and Corollary 5.3 explain how to compute finite dimensional formal models for toric singularities. In this section, we apply this result to provide a first element of answer in the direction of Question 1.2 in toric geometry. (See Theorem 6.3.)

6.1. Let us begin by a useful technical lemma.

LEMMA 6.1. *Let I_1, I_2 be two non-empty disjoint finite sets. For every $j \in \{1,2\}$, let $(d_\ell)_{\ell\in I_j}$ (resp. $(\alpha_\ell)_{\ell\in I_j}$) be two families of non-negative (resp. positive) integers such that*

$$\sum_{r\in I_1} d_r\alpha_r = \sum_{s\in I_2} d_s\alpha_s =: N.$$

Let $A := \mathbf{Z}[(p_{i,a})_{i\in I_j,\, j\in\{1,2\},\, a\in[\![1,d_i]\!]}]$. Consider the polynomial of $A[T]$ given by

$$\prod_{r\in I_1}\left(T^{d_r} + \sum_{a\in[\![1,d_r]\!]} p_{r,a}T^{d_r-a}\right)^{\alpha_r} - \prod_{s\in I_2}\left(T^{d_s} + \sum_{a\in[\![1,d_s]\!]} p_{s,a}T^{d_s-a}\right)^{\alpha_s}. \tag{6.1}$$

Then, for every element $i \in I_1$ and every $b \in [\![1,d_i]\!]$, there exists a polynomial $Q_{i,b} \in A$, whose expression does not contain the variable $p_{i,b}$, such that the T^{N-b}-coefficient of polynomial (6.1) reads

$$\alpha_i p_{i,b} + Q_{i,b}.$$

PROOF. Let C_{N-b} be the T^{N-b}-coefficient of polynomial (6.1). By a direct computation, we observe that the monomial $\alpha_i p_{i,b}$ appears in C_{N-b}. Let us set $Q_{i,b} := C_{N-b} - \alpha_i p_{i,b}$. We have to prove that the variable $p_{i,b}$ does not appear in the expression of C_{N-b}. If we attribute the weight a to $p_{i,a}$ (in particular each variable has a positive weight), polynomial (6.1) (as a polynomial in $A[T]$) is isobaric of weight N. So, every monomial appearing in C_{N-b} has weight $b = \mathrm{weight}(p_{i,b})$. This shows the result. □

6.2. For every element $n \in \sigma \cap N$, by a *decomposition of* n we mean a decomposition of n into a sum of a finite number of elements of the semigroup $\sigma \cap N$. We shall sometimes identify a decomposition of n with a finite family $(n_i)_{i\in I}$ of elements of $\sigma \cap N$ such that $\sum_{i\in I} n_i = n$. We denote by $\ell(n)$ the maximal number of terms in a decomposition of n. For example, we observe that n is indecomposable if and only if $\ell(n) = 1$. We introduce the following partial order $\prec$ on the set of decomposition of n: if $\mathcal{D}_1 \colon n = \sum_{i=1}^{r} n_i$ and $\mathcal{D}_2 \colon n = \sum_{j=1}^{s} n_j$ are two decompositions of n, we say that $\mathcal{D}_2 \prec \mathcal{D}_1$ if there exists a partition of $[\![1, s]\!]$ into r non empty sets $J_1, \ldots, J_r$ such that for every $i \in [\![1, r]\!]$ one has $n_i = \sum_{j\in J_i} n_j$. In particular, with respect to this order, the minimal decompositions of n are the decompositions of n into a sum of indecomposable elements. We say that *property* $\mathcal{P}_n$ *is satisfied* if the supremum of the set of the minimal decompositions of n is the trivial decomposition $n = n$.

EXAMPLE 6.2. If n is indecomposable, property $\mathcal{P}_n$ is satisfied. The decomposable element n described in Example 2.6 also satisfies property $\mathcal{P}_n$.

6.3. We state and prove the main result of this section.

THEOREM 6.3. *Assume that the base-field k is of characteristic zero. Let $n \in \sigma \cap N$ be a N-primitive element. Let $M_{\sigma,n} \subset M_\sigma$ be a set of $\mathbf{Q}$-linearly independent elements with cardinality d and such that $\hat{K}_n = \langle \sum_{m\in M_{\sigma,n}} m, n\rangle - 1$ (see Subsection 3.4). Let $m_0 \in M_{\sigma,n}$ such that $\langle m_0 \,,\, n\rangle \geq 1$. Let*

$$Y_n := W^{(m_0)}\left((\langle m \,,\, n\rangle)_{m\in M_\sigma}, \left(\prod_{\substack{m\in M_\sigma \\ \alpha_{i,m}>0}} X_m^{\alpha_{i,m}} - \prod_{\substack{m\in M_\sigma \\ \alpha_{i,m}<0}} X_m^{-\alpha_{i,m}} \right)_{i\in I} \right)$$

(resp. $\mathscr{Y}_n := \mathscr{W}^{(m_0)}(\ldots)$, resp. $\tilde{Y}_n := W(\ldots)$, resp. $\tilde{\mathscr{Y}}_n := \mathscr{W}(\ldots)$).

1. *One has* $\mathrm{embdim}(\mathscr{Y}_n) = \hat{K}_n$.
2. *Assume that n is indecomposable. Then, the formal k-scheme $\mathscr{Y}_n$ has dimension 0; in particular it is non-cancellable.*
3. *Assume that n satisfies property $\mathcal{P}_n$. Then, the formal k-scheme $\mathscr{Y}_n$ has dimension $\ell(n) - 1$, and it is non-cancellable.*

By Theorem 5.2, Corollary 5.3, the formal k-scheme $\mathscr{Y}_n$ is a finite formal model of $\mathscr{L}_\infty(V)_\gamma$ for every arc $\gamma \in \mathscr{L}^\circ_\infty(X)_n$. Then, Theorem 6.3 implies the following corollary.

COROLLARY 6.4. *Let $n \in \mathrm{Sing}(\sigma) \cap N$ be a N-primitive element satisfying property $\mathcal{P}_n$. Then the minimal finite formal model $\mathscr{S}_n$ is of dimension $\ell(n) - 1$ and embedding dimension $\hat{K}_n$.*

By Theorem 2.7 and Example 6.2, the conclusion holds in particular if n is strongly essential.

REMARK 6.5. Let us note that Theorem 6.3 presents analogies with the main result of [**22**].

PROOF. (of Theorem 6.3) Recall that $\tilde{Y}_n$ (resp. Y_n) is the closed subscheme of

$$\mathrm{Spec}(k[(p_{m,a})_{\substack{m \in M_\sigma \\ a \in [\![1, \langle m\,,\,n\rangle]\!]}}])$$

whose ideal $I_{\tilde{Y}_n}$ (resp. I_{Y_n}) is generated by (resp. $p_{m_0,1}$ and) all the coefficients with respect to the variable T in the polynomials

$$\prod_{\substack{m \in M_\sigma \\ \alpha_{i,m} > 0}} p_m(T)^{\alpha_{i,m}} - \prod_{\substack{m \in M_\sigma \\ \alpha_{i,m} < 0}} p_m(T)^{-\alpha_{i,m}}, \quad i \in I,$$

where

$$p_m(T) := T^{\langle m\,,\,n\rangle} + \sum_{a \in [\![1, \langle m\,,\,n\rangle]\!]} p_{m,a} T^{\langle m\,,\,n\rangle - a}.$$

Moreover $\tilde{\mathscr{Y}_n} = \mathrm{Spf}(k[[(p_{m,a})]]/I_{\tilde{Y}_n}) =: \mathrm{Spf}(\tilde{\mathcal{B}})$ (resp. $\mathscr{Y}_n = \mathrm{Spf}(k[[(p_{m,a})]]/I_{Y_n}) =: \mathrm{Spf}(\mathcal{B})$) is the completion of $\tilde{Y}_n$ (resp. Y_n) along the origin, and by Lemma 5.1, one has $\tilde{\mathscr{Y}_n} \cong \mathscr{Y}_n \hat{\times}_k \mathbf{D}_k$.

(1) We show the statement on the embedding dimension. If $\mathcal{A}$ is a complete local k-algebra, one has $\mathrm{embdim}(\mathcal{A}[[T]]) = \mathrm{embdim}(\mathcal{A}) + 1$. Thus it suffices to show that $\mathrm{embdim}(\tilde{\mathscr{Y}_n}) = \hat{K}_n + 1$. We claim that the maximal ideal $\mathfrak{m}_{\tilde{\mathcal{B}}}$ is generated by the classes of the elements of the set

$$P := \{p_{m,a}\}_{\substack{m \in M_{\sigma,n} \\ a \in [\![1, \langle m\,,\,n\rangle]\!]}}.$$

Indeed, for every $m \in M_\sigma \setminus M_{\sigma,n}$, Lemma 6.1 and Section 5.4 allow to eliminate the variables $\{p_{m,a}\}_{a \in [\![1, \langle m\,,\,n\rangle]\!]}$.

Let us now show that the classes of the elements of P constitute a basis of $\mathfrak{m}_{\tilde{\mathcal{B}}}/\mathfrak{m}^2_{\tilde{\mathcal{B}}}$, which will prove the claimed statement about the embedding dimension by the very definition of $M_{\sigma,n}$. Let A be the test ring $k[S]/\langle S^2\rangle$ and s be the image of S in A. It suffices to show that the following property, called condition $\mathcal{C}$, is satisfied: for every $\mu \in M_{\sigma,n}$ and every $b \in [\![1, \langle \mu\,,\,n\rangle]\!]$ there exists a morphism from $\tilde{\mathcal{B}} = k[[(p_{m,a})]]/I_{\tilde{Y}_n}$ to A satisfying the following property, called condition $\mathcal{C}_{\mu,b}$: it maps $p_{\mu,b}$ to s and $p \in P \setminus \{p_{\mu,b}\}$

to 0.

For $a \in [\![0, \max_{m\in M_\sigma} \langle m\,,\,n\rangle - 1]\!]$, set

$$M^+_{\sigma,n,a} := \{m \in M_{\sigma,n}, \quad a \leq \langle m\,,\,n\rangle - 1\} \quad \text{and} \quad M^-_{\sigma,n,a} := M_{\sigma,n} \setminus M^+_{\sigma,n,a}.$$

Let $(x_{m,a})_{\substack{m\in M_\sigma\\ a\in[\![1,\langle m\,,\,n\rangle]\!]}}$ be a family of elements of k and consider the morphism $\varphi\colon k[[(p_{m,a})]] \to A$ defined by $\varphi(p_{m,a}) = x_{m,a}s$. Let $\mu \in M_{\sigma,n}$ and $b \in [\![1, \langle \mu\,,\,n\rangle]\!]$ (such that $b \neq 1$ in case $\mu = m_0$). The morphism φ factorizes through $\tilde{\mathcal{B}}$ and induces a morphisme $\tilde{\mathcal{B}} \to A$ satisfying condition $\mathcal{C}_{\mu,b}$ if and only if, for every $i \in I$, the following identities holds in $A[T]$:

$$\prod_{\substack{m\in M_\sigma\\ \alpha_{i,m}>0}} (T^{\langle m\,,\,n\rangle} + s \sum_{a\in[\![1,\langle m\,,\,n\rangle]\!]} x_{m,a}T^{\langle m\,,\,n\rangle - a})^{\alpha_{i,m}}$$

$$= \prod_{\substack{m\in M_\sigma\\ \alpha_{i,m}<0}} (T^{\langle m\,,\,n\rangle} + s \sum_{a\in[\![1,\langle m\,,\,n\rangle]\!]} x_{m,a}T^{\langle m\,,\,n\rangle - a})^{-\alpha_{i,m}}$$

and moreover the following conditions hold:

$$\begin{cases} x_{m,a} &= 0 \ \forall m \in M_{\sigma,n} \ \setminus \{\mu\} \ \forall a \in [\![1, \langle m\,,\,n\rangle]\!] \\ x_{\mu,a} &= 0 \ \forall a \in [\![1, \langle \mu\,,\,n\rangle]\!] \setminus \{b\} \\ x_{\mu,b} &= 1. \end{cases}$$

Now, expanding the above polynomial relations in A, it is easy to see that condition $\mathcal{C}$ is equivalent to the following property: for every $a \in [\![0, \max_{m\in M_\sigma} \langle m\,,\,n\rangle - 1]\!]$, for every $\mu \in M^+_{\sigma,n,a}$, there exists a solution $(x_{m,\langle m\,,\,n\rangle - a})_{\substack{m\in M_\sigma\\ a\leq\langle m\,,\,n\rangle-1}}$ with values in $\mathbf{Q}$ to the linear system with constraints

$$\sum_{\substack{m\in M_\sigma\\ a\leq\langle m\,,\,n\rangle-1}} \alpha_{i,m}x_{m,\langle m\,,\,n\rangle-a} = 0, \quad i \in I \quad \text{and} \quad \begin{cases} x_{m,\langle m\,,\,n\rangle-a} = 0 \text{ if } m \in M^+_{\sigma,n,a} \setminus \{\mu\} \\ x_{\mu,\langle \mu\,,\,n\rangle-a} = 1 \end{cases}. \tag{6.2}$$

Since $M_{\sigma,n}$ is a $\mathbf{Q}$-basis of $M \otimes_{\mathbf{Z}} \mathbf{Q}$, there certainly exists $\tilde{n} \in N \otimes_{\mathbf{Z}} \mathbf{Q}$ such that $\begin{cases} \langle m\,,\,\tilde{n}\rangle = 0 \text{ if } m \in M^+_{\sigma,n,a} \setminus \{\mu\} \\ \langle \mu\,,\,\tilde{n}\rangle = 1 \end{cases}$. Let $m' \in M_\sigma$ such that $\langle m'\,,\,n\rangle < a+1$. Let us show that $\langle m'\,,\,\tilde{n}\rangle = 0$. Assume that $\langle m'\,,\,\tilde{n}\rangle \neq 0$. Then

$$\{m\}_{m\in M_{\sigma,n}\setminus\{\mu\}} \cup \{m'\}$$

is a family of elements of $\sigma \cap M$ which constitute a $\mathbf{Q}$-basis of $M \otimes_{\mathbf{Z}} \mathbf{Q}$. But since $\langle m'\,,\,n\rangle < a+1 \leq \langle \mu\,,\,n\rangle$, one has

$$\left\langle m' + \sum_{m \in M_{\sigma,n} \setminus \{\mu\}} m \,,\, n \right\rangle < \left\langle \sum_{m \in M_{\sigma,n}} m \,,\, n \right\rangle$$

which contradicts the definition of $M_{\sigma,n}$. Thus for every $m \in M_\sigma$ such that $\langle m \,,\, n\rangle < a+1$ one has $\langle m \,,\, \tilde{n}\rangle = 0$. In particular, for every $i \in I$, one has

$$0 = \sum_{m \in M_\sigma} \alpha_{i,m} \langle m \,,\, \tilde{n}\rangle = \sum_{\substack{m \in M_\sigma \\ a \leq \langle m \,,\, n\rangle - 1}} \alpha_{i,m} \langle m \,,\, \tilde{n}\rangle$$

hence $(x_{m,\langle m \,,\, n\rangle - a}) := (\langle m \,,\, \tilde{n}\rangle)$ provides an adequate solution to the system with constraints (6.2).

(2) Let us now assume that n is indecomposable and show that $\dim(\mathscr{Y}_n) = 0$. It suffices to show that for every k-extension K, the set $Y_n(K)$ is a singleton. So let $\{p_{m,a}\}_{\substack{m \in M_\sigma \\ a \in [\![1, \langle m \,,\, n\rangle]\!]}}$ be a family of elements of K by an element of $Y_n(K)$. In particular, we have $p_{m_0,1} = 0$. For every $m \in M_\sigma$, we set

$$\pi_{m,n}(T) := T^{\langle m \,,\, n\rangle} + \sum_{a \in [\![1, \langle m \,,\, n\rangle]\!]} p_{m,a} T^{\langle m \,,\, n\rangle - a} \in K[T].$$

Let x be a root of $\pi_{m_0,n}$ in an extension of K. For every $m \in M_\sigma$, let $\mu_m(x)$ be the multiplicity of x as a zero of $\pi_{m,n}(T)$. In particular one has, for every $m \in M_\sigma$, $\mu_m(x) \leq \langle m \,,\, n\rangle$, and, for every $i \in I$,

$$\sum_{\substack{m \in M_\sigma \\ \alpha_{i,m} > 0}} \alpha_{i,m} \mu_m(x) = \sum_{\substack{m \in M_\sigma \\ \alpha_{i,m} < 0}} \alpha_{i,m} \mu_m(x).$$

By Subsection 5.3, one may find $n_1(x), n_2(x) \in \sigma \cap N$ such that, for every $m \in M_\sigma$,

$$\langle m \,,\, n_1(x)\rangle = \mu_m(x), \quad \langle m \,,\, n_2(x)\rangle = \langle m \,,\, n\rangle - \mu_m(x).$$

In particular, we deduce that $n = n_1(x) + n_2(x)$. Since n is indecomposable and $\langle m_0 \,,\, n_1(x)\rangle = \mu_{m_0}(x) > 0$, one necessarily has $n_1(x) = n$; hence, $\mu_m(x) = \langle m \,,\, n\rangle = \deg(\pi_{m,n}(T))$ for every $m \in M_\sigma$. In other words, the element x is a zero of maximal multiplicity of every polynomial $\pi_{m,n}(T)$. So, we conclude that $\pi_{m,n}(T) = (T - x)^{\langle m \,,\, n\rangle}$ for every $m \in M_\sigma$. Since $p_{m_0,1} = 0$, one necessarily has $x = 0$; hence, for every $m \in M_\sigma$, $\pi_{m,n}(T) = T^{\langle m \,,\, n\rangle}$. It implies that all the $p_{m,a}$'s are zero. This shows the second assertion of the theorem.

(3) Next we study the dimension of the irreducible components of $\mathscr{Y}_n$. It suffices to study the dimension of the minimal prime ideals of $\mathcal{A} := k[(p_{m,a})_{\substack{m \in M_\sigma \\ a \in [\![1, \langle m \,,\, n\rangle]\!]}}]/I_{Y_n}$ contained in $\mathfrak{m}_{\mathcal{A}} := \langle p_{m,a}\rangle$.

Let B be an integral k-algebra. A morphism $\varphi\colon \mathcal{A} \to B$ corresponds to the datum of a family of polynomials

$$p_{m,n,\varphi}(T) := T^{\langle m\,,\,n\rangle} + \sum_{a\in[\![1,\langle m\,,\,n\rangle]\!]} \varphi(p_{m,a})T^{\langle m\,,\,n\rangle-a} \in B[T].$$

such that $\varphi(p_{m_0,1}) = 0$ and

$$\prod_{\substack{m\in M_\sigma\\ \alpha_{i,m}>0}} p_{m,n,\varphi}(T)^{\alpha_{i,m}} = \prod_{\substack{m\in M_\sigma\\ \alpha_{i,m}<0}} p_{m,n,\varphi}(T)^{-\alpha_{i,m}}, \quad i\in I.$$

Let K be an extension of $\mathrm{Frac}(B)$ splitting all the polynomials $\{p_{m,n,\varphi}(T)\}$. Then there exists a finite subset $K^{(0)}$ of K and a unique decomposition $\mathrm{Dec}(\varphi) := (n_x)_{x\in K^{(0)}}$ of n such that, for every $x\in K^{(0)}$ and every $m\in M_\sigma$, the multiplicity of x in $p_{m,n,\varphi}(T)$ is $\langle m\,,\,n_x\rangle$ and, for every $x\in K\setminus K^{(0)}$ and every $m\in M_\sigma$, the element x is not a root of $p_{m,n,\varphi}(T)$. Note that the set $\mathrm{Dec}(\varphi)$ does not depend on the choice of the extension K.

Let $\mathcal{D}$ be a decomposition of n and

$$B_{\mathcal{D}} := k[(X_\nu)_{\nu\in\mathcal{D}}]/\langle \sum_{\nu\in\mathcal{D}} \langle m_0\,,\,\nu\rangle\, X_\nu\rangle.$$

Let us denote by $\mathfrak{p}_{\mathcal{D}}$ the kernel of the unique morphism of k-algebras $\varphi_{\mathcal{D}}\colon k[(p_{m,a})]\to B_{\mathcal{D}}$ mapping $p_{m,a}$ to the $T^{\langle m\,,\,n\rangle-a}$-coefficient of $\prod_{\nu\in\mathcal{D}}(T-X_\nu)^{\langle m\,,\,\nu\rangle}$. By the very definition, the ideal $\mathfrak{p}_{\mathcal{D}}$ is a prime ideal containing I_{Y_n} and contained in $\mathfrak{m}_{\mathcal{A}}$. In the sequel, we will identify $\mathfrak{p}_{\mathcal{D}}$ with its image in $\mathcal{A}$. Moreover, if $\nu\in\mathcal{D}$ and $m\in M_\sigma$ is such that $\langle m\,,\,\nu\rangle\neq 0$, the image of X_ν in $B_{\mathcal{D}}$ is a root of the monic T-polynomial $\prod_{\nu\in\mathcal{D}}(T-X_\nu)^{\langle m\,,\,\nu\rangle}$, whose coefficients are the $\varphi_{\mathcal{D}}(p_{m,a})$. Thus $B_{\mathcal{D}}$ is an integral extension of $\varphi_{\mathcal{D}}(\mathcal{A})$. In particular, $\dim(\mathcal{A}/\mathfrak{p}_{\mathcal{D}}) = \dim(B_{\mathcal{D}}) = \mathrm{card}(\mathcal{D})-1$. Moreover the polynomial $p_{m,n,\varphi_{\mathcal{D}}}$ splits over $\mathrm{Frac}(B_{\mathcal{D}})$ and one has $\mathrm{Dec}(\varphi_{\mathcal{D}}) = \mathcal{D}$.

We claim that, for every integral k-algebra B and every morphism $\varphi\colon\mathcal{A}\to B$, the ideal $\mathrm{Ker}(\varphi)$ contains $\mathfrak{p}_{\mathcal{D}}$ if and only if one has $\mathcal{D}\prec\mathrm{Dec}(\varphi)$. First assume that $\mathrm{Ker}(\varphi)$ contains $\mathfrak{p}_{\mathcal{D}}$. Pick a prime ideal $\mathfrak{q}$ of the B-algebra $B_{\mathcal{D}}\otimes_{\varphi_{\mathcal{D}}(\mathcal{A})} B$ (which is integral over B) whose trace on B is $\langle 0\rangle$, and note that the polynomials $\{p_{m,n,\varphi}\}$ split in the $\mathrm{Frac}(B)$-extension $K := \mathrm{Frac}((B_{\mathcal{D}}\otimes_{\varphi_{\mathcal{D}}(\mathcal{A})} B)/\mathfrak{q})$. Moreover, denoting by x_ν the image of X_ν in K for $\nu\in\mathcal{D}$, one has in $K[T]$

$$p_{m,n,\varphi}(T) = \prod_{\nu\in\mathcal{D}}(T-x_\nu)^{\langle m\,,\,\nu\rangle}.$$

By the definition of $\mathrm{Dec}(\varphi)$, this shows that $\mathcal{D}\prec\mathrm{Dec}(\varphi)$.

Now assume that $\mathcal{D}\prec\mathrm{Dec}(\varphi)$. In particular, there exist an extension K of $\mathrm{Frac}(B)$ and a family $\{x_\nu\}_{\nu\in\mathcal{D}}$ of elements of K satisfying

$$p_{m,n,\varphi}(T) = \prod_{\nu\in\mathcal{D}}(T-x_\nu)^{\langle m\,,\,\nu\rangle}$$

for every $m\in M_\sigma$ and $\sum_{\nu\in\mathcal{D}}\langle m_0\,,\,\nu\rangle\, x_\nu = 0$. Now consider the morphism $\mathcal{A}\to K$ obtained by composing $\varphi_{\mathcal{D}}$ with the morphism $B_{\mathcal{D}}\to K$ mapping X_ν to x_ν: its kernel contains $\mathfrak{p}_{\mathcal{D}}$, its image is contained in B and the induced morphism $\mathcal{A}\to B$ is φ.

Let $\mathcal{D}$, $\mathcal{D}'$ be two decompositions of n. Let us now show that $\mathcal{D}\prec\mathcal{D}'$ if and only if

$\mathfrak{p}_{\mathcal{D}} \subset \mathfrak{p}_{\mathcal{D}'}$. First assume $\mathcal{D} \prec \mathcal{D}'$. If $\varphi_{\mathcal{D},\mathcal{D}'}$ is the morphism $B_{\mathcal{D}} \to B_{\mathcal{D}'}$ mapping X_ν to $X_{\nu'}$ where ν' is the unique element of $\mathcal{D}'$ containing ν, it is clear that $\varphi_{\mathcal{D}'} = \varphi_{\mathcal{D},\mathcal{D}'}\varphi_{\mathcal{D}}$ hence $\mathfrak{p}_{\mathcal{D}} \subset \mathfrak{p}_{\mathcal{D}'}$. Now assume $\mathfrak{p}_{\mathcal{D}} \subset \mathfrak{p}_{\mathcal{D}'}$. Then $\mathcal{D} \prec \mathrm{Dec}(\varphi_{\mathcal{D}'}) = \mathcal{D}'$.

In particular for every decomposition $\mathcal{D}$ of n one has $\mathfrak{p}_{\mathcal{D}} \subset \mathfrak{p}_{\{n\}} = \mathfrak{m}_{\mathcal{A}}$ and the minimal prime ideals of $\mathcal{A}$ contained in $\mathfrak{m}_{\mathcal{A}}$ are exactly the $\mathfrak{p}_{\mathcal{D}}$ where $\mathcal{D}$ is a minimal decomposition of n. This shows that the irreducible components of $\mathrm{Spf}(\widehat{\mathcal{A}_{\mathfrak{m}_{\mathcal{A}}}})$ are in one-to-one correspondence with the minimal decompositions of n; the irreducible component corresponding to the minimal decomposition $\mathcal{D}$ has dimension $\mathrm{card}(\mathcal{D}) - 1$.

Let us assume that propery $\mathcal{P}_n$ holds, in other words that the supremum of the minimal decompositions of n is $\{n\}$. This property implies that

$$\sum_{\mathcal{D}\text{ minimal}} \mathfrak{p}_{\mathcal{D}} = \mathfrak{p}_{\{n\}} = \mathfrak{m}_{\mathcal{A}}.$$

On the other hand if one had an isomorphism $\widehat{\mathcal{A}_{\mathfrak{m}_{\mathcal{A}}}} \cong C[[T]]$, the variable T would not contained in the sum of the minimal prime ideals of $\widehat{\mathcal{A}_{\mathfrak{m}_{\mathcal{A}}}}$, in other words the intersection of all the irreducible components would be of dimension ≥ 1. Hence $\mathrm{Spf}(\widehat{\mathcal{A}_{\mathfrak{m}_{\mathcal{A}}}})$ is non-cancellable. □

7. Minimal formal model of monomial curves singularities.

7.1. Let $N \geq 2$ be an integer, $(d_j)_{j\in[\![1,N]\!]}$ be an increasing sequence of coprime positive integers, with $d_1 \geq 2$, $\mathcal{I}$ be the kernel of the k-algebra morphism $k[(X_j)]_{j\in[\![1,N]\!]} \to k[T]$ sending X_j to T^{d_j}. We set $\mathscr{C} := \mathrm{Spec}(k[(X_j)_{j\in[\![1,N]\!]}]/\mathcal{I})$. We then observe that the origin $\mathfrak{o}$ of $\mathbf{A}^N_k$ is a singular point of $\mathscr{C}$, and that $k[(X_j)_{j\in[\![1,N]\!]}]/\mathcal{I} \to k[T]$ is the normalization morphism. In particular, one has $\mathrm{mult}(\mathscr{C}, \mathfrak{o}) = d_1$ (see Subsection 3.4) and $(T^{d_j})_{j\in[\![1,N]\!]}$ is a primitive arc of the germ $(\mathscr{C}, \mathfrak{o})$. The germ $(\mathscr{C}, \mathfrak{o})$ is called a germ of monomial curve (see, e.g., [**27**]).

Let I be a finite set and $(\alpha_i) \in (\mathbf{Z}^N)^I$ be a finite collection of nontrivial integral linear relations between the d_j's, that is to say, for every $i \in I$,

$$\sum_{j\in[\![1,N]\!]} \alpha_{i,j} d_j = 0, \tag{7.1}$$

such that moreover

$$\left\{ \prod_{\substack{j\in[\![1,N]\!]\\ \alpha_{i,j}>0}} X_j^{\alpha_{i,j}} - \prod_{\substack{j\in[\![1,N]\!]\\ \alpha_{i,j}<0}} X_j^{-\alpha_{i,j}} \right\}_{i\in I} \tag{7.2}$$

is a set of binomial generators of $\mathcal{I}$. Upon enlarging I, one may and shall assume that, for every integer $j \in [\![2, N]\!]$, this set of generators contains $X_1^{d_j} - X_j^{d_1}$.

REMARK 7.1. Let $(\nu_j)_{j\in[\![1,N]\!]}$ be a family of integers such that, for every $i \in I$, we have

$$\sum_{j\in[\![1,N]\!]} \alpha_{i,j}\nu_j = 0.$$

Then, by our assumption, we conclude that, for every integer $j \geq 2$ one has $d_j\nu_1 = d_1\nu_j$. Let p be a prime divisor of ν_1 and r be the p-adic valuation of ν_1. Since the (d_j) are coprime, there exists $j \geq 2$ such that p does not divide ν_j. Hence p^r must divide d_1. We conclude that ν_1 divides d_1, and finally that d_j divides ν_j for every $j \in [\![1,N]\!]$.

7.2. Let us state and prove the main theorem of the section.

THEOREM 7.2. *Keep the notation of Subsection* 7.1. *Assume that the base-field k has characteristic zero. Let γ be a primitive k-parametrization of $\mathscr{C}$ at $\mathfrak{o}$. Let*

$$Y := W^{(1)}\left((d_j)_{j\in[\![1,N]\!]}, \left(\prod_{\substack{j\in[\![1,N]\!]\\ \alpha_{i,j}>0}} X_j^{\alpha_{i,j}} - \prod_{\substack{j\in[\![1,N]\!]\\ \alpha_{i,j}<0}} X_j^{-\alpha_{i,j}}\right)_{i\in I}\right)$$

and $\mathscr{Y}$ be the completion of Y along the origin $\mathfrak{o}$. Then, the formal k-scheme $\mathscr{Y}$ is the minimal formal model of γ. It is of dimension zero and of embedding dimension $\mathrm{mult}(\mathscr{C},\mathfrak{o}) - 1$.

PROOF. The first assertion is proved using the same kind of argument than in the proof of Theorem 5.2.

∘ Let us show the assertion on the embedding dimension. Recall that the k-scheme Y is the closed subscheme of

$$\mathrm{Spec}(k[(p_{j,a})_{\substack{j\in[\![1,N]\!]\\ a\in[\![1,d_j]\!]}}])$$

whose ideal I_Y is generated by $p_{1,1}$ and all the coefficients with respect to the variable T in the polynomials

$$\prod_{j,\alpha_{i,j}>0} p_j(T)^{\alpha_{i,j}} - \prod_{j,\alpha_{i,j}<0} p_j(T)^{-\alpha_{i,j}} \tag{7.3}$$

for every $i \in I$, where

$$p_j(T) = T^{d_j} + \sum_{a\in[\![1,d_j]\!]} p_{j,a}T^{d_j-a}.$$

Moreover $\mathscr{Y} = \mathrm{Spf}(k[[(p_{j,a})]]/I_Y) =: \mathrm{Spf}(\mathcal{B})$ is the completion of Y along the origin.

We claim that the maximal ideal $\mathfrak{m}_{\mathcal{B}}$ is generated by the classes of the elements of the set $P := \{p_{1,a}\}_{a\in[\![2,d_1]\!]}$. Indeed, for every $j \in [\![2,N]\!]$, Lemma 6.1 and the fact that $X_1^{d_j} - X_j^{d_1}$ is in the set (7.2) allow to eliminate the variables $\{p_{j,a}\}_{a\in[\![1,d_j]\!]}$.

Let us now show that the classes of the elements of P constitute a basis of $\mathfrak{m}_{\mathcal{B}}/\mathfrak{m}_{\mathcal{B}}^2$, which will prove the claimed statement about the embedding dimension, since $d_1 = \mathrm{mult}(\mathscr{C},\mathfrak{o})$. Arguing similarly as in the toric case, we see that it boils down to show the following: for every integer $a_0 \in [\![2, d_1]\!]$, there exists a solution $(x_{j,a})_{\substack{j\in[\![1,N]\!]\\ a\in[\![1,d_j]\!]}}$ with values in $\mathbf{Q}$ to the linear system

$$\sum_{\substack{j\in[\![1,N]\!]\\ d_j\geq a}} \alpha_{i,j} x_{j,a} = 0, \quad i \in I, \quad a \in [\![d_1, \max(d_j)]\!] \tag{7.4}$$

with constraints

$$\begin{cases} x_{1,a_0} = 1 \\ \ x_{1,a} = 0 \quad \text{if } a \in [\![1, d_1]\!] \setminus \{a_0\} \end{cases}. \tag{7.5}$$

Thanks to (7.1), such a solution is given as follows: for $j \in [\![1, N]\!]$, set $x_{j,a_0} = (d_j)/(d_1)$ and take all the other variables $x_{j,a}$ equal to zero. Here we use that $d_1 = \min(d_j)$, thus in particular for every j one has $a_0 \leq d_j$; if it were not the case, in case $d_1 \geq a_0 > d_j$ for at least one j, the non-trivially solvable part of the above system with constraints would read

$$\sum_{\substack{j\in[\![1,N]\!]\\ d_j\geq a_0}} \alpha_{i,j} x_{j,a_0} = 0, \quad i \in I \tag{7.6}$$

and the above solution does not work due to the restriction $d_j \geq a_0$.

∘ Let us show that $\dim(\mathscr{Y}) = 0$. Arguing similarly as in the toric case, we see that it suffices to show that for every family $(\pi_j(T))_{j\in[\![1,N]\!]}$ of monic polynomials with coefficients in a field satisfying $\deg(\pi_j) = d_j$ and

$$\prod_{j,\alpha_{i,j}>0} \pi_j(T)^{\alpha_{i,j}} = \prod_{j,\alpha_{i,j}<0} \pi_j(T)^{-\alpha_{i,j}},$$

and for every root x of the polynomial π_1, then for every $j \in [\![1, N]\!]$ the multiplicity $\mu_j(x)$ of x in $\pi_j(T)$ is d_j. But under the previous assumptions one has for every integer $j \in [\![1, N]\!]$, the inequality $\mu_j(x) \leq d_j$, and, for every $i \in I$, the formula

$$\sum_{j\in[\![1,N]\!]} \alpha_{i,j}\mu_j(x) = 0.$$

Now, since $\mu_1(x) > 0$, we deduce by Remark 7.1 that $\mu_j(x) = d_j = \deg(\pi_j(T))$ for every integer j. □

Example 7.3. Let k be a field of characteristic zero, Let us consider the affine plane cusp $\mathscr{C} = \operatorname{Spec}(k[X_0, X_1]/\langle X_0^3 - X_1^2\rangle$ and $\gamma(T) = (T^2, T^3)$. Using Theorem 7.2, one may check that the minimal formal model is $\operatorname{Spf}(k[[p_{0,0}]]/\langle p_{0,0}^2\rangle)$.

Example 7.4. Let k be a field of characteristic zero. Let $\mathscr{C}$ be the k-curve defined by the datum of the polynomials $X_1^2 - X_0\,X_2$, $X_1\,X_2 - X_0^3$ and $X_2^2 - X_0^2\,X_1$ in $k[X_0, X_1, X_2]$, $\mathfrak{o}$ be the origin of $\mathbf{A}_k^3$ and γ be the primitive arc (T^3, T^4, T^5). Note that the monomial curve singularity $(\mathscr{C}, \mathfrak{o})$ does not satisfy the Gorenstein relation, thus it is not plane. Using Theorem 7.2, we deduce after performing suitable eliminations that the minimal formal model is isomorphic to $\mathrm{Spf}(k[[p_{0,0},p_{0,1}]]/\langle p_{0,0}^2,p_{0,1}^3,p_{0,0}p_{0,1}\rangle)$.

8. Further comments, examples and problems.

We conclude the article by various comments, examples and problems.

8.1. If one puts no particular restriction on the dimension of the varieties under consideration, a variant of a construction of Drinfeld in [**12**] shows that there exist rational non-degenerate arcs the minimal formal model of which has arbitrarily large dimension (see [**2**]). Using Corollary 6.4, one can obtain the following new result: there exist rational non-degenerate arcs the minimal formal model of which has arbitrarily large dimension, even when restricting to arcs on 3-dimensional (toric) varieties. Indeed, let us consider $N=\mathbf{Z}^3$, D be a positive integer and σ be the cone generated by $(1,0,1)$, $(1,2,1)$, $(0,0,1)$ and $(D,1,0)$. One checks that $(1,0,1)$, $(1,1,1)$, $(0,0,1)$ and $(D,1,0)$ are indecomposable elements of the semigroup $\sigma\cap N$. Let n be the primitive element $(D,1,D)$. Since

$$n=(D,1,0)+D(0,0,1)=(1,1,1)+(D-1)(1,0,1),$$

property $\mathcal{P}_n$ holds and $\ell(n)\geq D+1$. Thus by Corollary 6.4 the minimal formal model $\mathscr{S}_n$ is of dimension $\ell(n)-1\geq D$.

8.2. If $n\in\sigma\cap N$ is a primitive element such that $\mathcal{P}_n$ holds, one saw that the minimal formal model $\mathscr{S}_n$ is of dimension $\ell(n)-1$ and may be obtained from the formal model $\mathscr{W}_n$ (defined by formula (5.4)) by the cancellation described by Lemma 5.1. Note that in case one has $\ell(n)\geq 2$, the arguments show in particular that the minimal formal model $\mathscr{S}_n$ is not irreducible. More generally, it may happen that $\mathcal{P}_n$ does not hold (even if n is a minimal element of $\mathrm{Sing}(\sigma)\cap N$) but one still obtains $\mathscr{S}_n$ from $\mathscr{W}_n$ by the cancellation described by Lemma 5.1 (and in particular the embedding dimension of $\mathscr{S}_n$ will still equal the Mather discrepancy $\hat{K}_n$); on the other hand, $\mathscr{S}_n$ may be irreducible even if it has positive dimension. Let us give a specific example in $N=\mathbf{Z}^3$. We consider the cone σ generated by $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(-1,1,2)$. One checks that these four elements generate the rays of σ and form a Hilbert basis of the semigroup $\sigma\cap N$. One also checks that $\mathrm{Sing}(\sigma)=\sigma^\circ$, that $n:=(0,1,1)$ is the unique minimal element of $\mathrm{Sing}(\sigma)\cap N$ and that $n=(0,1,0)+(0,0,1)$ is the only decomposition of n. In particular property $\mathcal{P}_n$ does not hold. On the other hand, a presentation of the toric variety $V(\sigma)$ is $\mathrm{Spec}(k[X_0,X_1,X_2,X_3,X_4]/\langle X_4^2-X_0X_3,X_0X_2-X_1X_4,X_1X_3-X_2X_4\rangle)$ and $\gamma(T):=(T,T,T,T,T)$ satisfies $\mathrm{ord}(\gamma)=n$. Thus, after the cancellation of $\mathscr{W}_n$ corresponding to $T\mapsto T+p_4$, we obtain a finite formal model of γ whose presentation in $\mathrm{Spf}(k[[p_0,p_1,p_2,p_3]])$ is given by the T-coefficients of the polynomials

$$T^2-(T+p_0)(T+p_3),\quad (T+p_0)(T+p_2)-(T+p_1)T\quad\text{and}\quad (T+p_1)(T+p_3)-(T+p_2)T.$$

One easily eliminates p_2 and p_3 and obtains that the latter model is isomorphic to $\mathscr{Y}:=\mathrm{Spf}(k[[p_0,p_1]]/\langle p_0^2,p_0p_1\rangle$. It is thus irreducible of dimension 1. Let us show that it is non cancellable, in other words that $\mathscr{Y}=\mathscr{S}_n$. If one has an isomorphism

$\theta\colon k[[p_0, p_1]]/\langle p_0^2, p_0p_1\rangle \cong A[[u]]$, one may assume after a k-linear transformation on u that $\theta(p_0) = u$ or $\theta(p_1) = u$. But this contradicts the fact that u is not a zero divisor in $A[[u]]$.

8.3. As a matter of fact, for a general primitive element n of $\sigma \cap N$, the minimal formal model $\mathscr{S}_n$ may not always be obtained from the formal model $\mathscr{W}_n$ (defined by formula (5.4)) by the cancellation described by Lemma 5.1. One obvious obstruction is the existence of a non-trivial toric splitting, that is to say, denoting by τ the minimal face of σ containing n, the existences of non-trivial lattices N_1, N_2 and cones τ_1 in N_1, and τ_2 in N_2 such that $N \cap \mathrm{Vect}(\tau) \cong N_1 \times N_2$ and $\tau \cap N \cong \tau_1 \cap N_1 \times \tau_2 \cap N_2$. In this case, one reduces to a computation on each factor thus one may perform at least two cancellations described by lemme 5.1. Note however that such a situation can not occur when n is a minimal element of $\mathrm{Sing}(\sigma) \cap N$. It would be interesting to know whether $\mathscr{S}_n$ may always be obtained from the formal model $\mathscr{W}_n$ by a single cancellation in case n is a minimal element of $\mathrm{Sing}(\sigma) \cap N$ (which would imply in particular that in this case the embedding dimension coincides with the Mather discrepancy $\hat{K}_n$).

8.4. On the other hand, let us give an example showing that even when there is no non-trivial toric splitting, it may happen that at least two cancellations (or more) are necessary to obtain $\mathscr{S}_n$ from $\mathscr{W}_n$; in particular the embedding dimension of $\mathscr{S}_n$ no longer coincides with the Mather discrepancy $\hat{K}_n$. Let us consider $N = \mathbf{Z}^2$, q a positive integer, σ be the cone generated by $(1, 0)$ and $(1, 2)$ and $n := (q, 1)$. One has $V(\sigma) \cong \mathrm{Spec}(k[X_0, X_1, X_2]/\langle X_0X_2 - X_1^2\rangle)$ and $\gamma(T) = (T, T^q, T^{2q-1})$ satisfies $\mathrm{ord}(\gamma) = n$. Thus, after the cancellation of $\mathscr{W}_n$ corresponding to $T \mapsto T + p_{0,0}$ we obtain a finite formal model of γ whose presentation in $\mathrm{Spf}(k[[p_{1,0}, \ldots, p_{1,q-1}, p_{2,0}, \ldots, p_{2,2q-1}]])$ is given by the T-coefficients of the polynomial

$$T\left(T^q + \sum_{i=0}^{2q-1} p_{2,i}T^i\right) - \left(T^q + \sum_{i=0}^{q-1} p_{1,i}T^i\right)^2.$$

One easily eliminates the $p_{2,i}$ and obtains a finite formal model isomorphic to $\mathrm{Spf}(k[[p_{1,0}, \ldots, p_{1,q-1}]]/\langle p_{1,0}^2\rangle)$ which *is* cancellable; the minimal formal model is $\mathrm{Spf}(k[[p_{1,0}]]/\langle p_{1,0}^2\rangle)$.

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