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Arnold's problem on monotonicity of the Newton number for surface singularities

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Abstract. According to the Kouchnirenko Theorem, for a generic (meaning non-degenerate in the Kouchnirenko sense) isolated singularity f its Milnor number $\mu(f)$ is equal to the Newton number $\nu(\mathbf{\Gamma}_+(f))$ of a combinatorial object associated to f, the Newton polyhedron $\mathbf{\Gamma}_+(f)$. We give a simple condition characterizing, in terms of $\mathbf{\Gamma}_+(f)$ and $\mathbf{\Gamma}_+(g)$, the equality $\nu(\mathbf{\Gamma}_+(f)) = \nu(\mathbf{\Gamma}_+(g))$, for any surface singularities f and g satisfying $\mathbf{\Gamma}_+(f) \subset \mathbf{\Gamma}_+(g)$. This is a complete solution to an Arnold problem (No. 1982-16 in his list of problems) in this case.

1. Introduction.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a *holomorphic isolated singularity* (that is f possesses an isolated critical point at $0 \in \mathbb{C}^n$), in the sequel: a *singularity*, in short. The Milnor number $\mu(f)$ (see [**10**]) of a generic f can be expressed, as proved by Kouchnirenko [**8**], using a combinatorial object associated to f, the Newton polyhedron $\mathbf{\Gamma}_+(f) \subset \mathbb{R}^n_{\geq 0}$. More precisely, under an appropriate non-degeneracy condition imposed on f, it holds $\mu(f) = \nu(\mathbf{\Gamma}_+(f))$, where $\nu(\mathbf{\Gamma}_+(f))$ is the *Newton number of* $\mathbf{\Gamma}_+(f)$. For $\mathbf{\Gamma}_+(f)$ *convenient* (which means that the Newton polyhedron contains a point on each coordinate axis) the latter number is equal to

$$\nu(\mathbf{\Gamma}_+(f)) := n!\ V_n - (n-1)!\ V_{n-1} + \ldots + (-1)^{n-1}\ 1!\ V_1 + (-1)^n\ V_0,$$

where V_n is the n-dimensional volume of the (usually non-convex) polyhedron "under" $\mathbf{\Gamma}_+(f)$, V_{n-1} is the sum of $(n-1)$-dimensional volumes of the polyhedra "under" $\mathbf{\Gamma}_+(f)$ on all hyperplanes $\{x_i = 0\}$, V_{n-2} is the sum of $(n-2)$-dimensional volumes of the polyhedra "under" $\mathbf{\Gamma}_+(f)$ on all $\{x_i = x_j = 0\}$, $i \neq j$, and so on.

In his acclaimed list of problems, Arnold posed the following ([**1**, pp. 198–216]):

> 'Consider a Newton polyhedron Δ in $\mathbb{R}^n$ and the number $\mu(\Delta) = n!V - \Sigma(n-1)!V_i + \Sigma(n-2)!V_{ij} - \cdots$, where V is the volume under Δ, V_i is the volume under Δ on the hyperplane $x_i = 0$, V_{ij} is the volume under Δ on the hyperplane $x_i = x_j = 0$, and so on.
> Then $\mu(\Delta)$ grows (non strictly monotonically) as Δ grows (whenever Δ remains coconvex *and integer?*). *There is no elementary proof even for* $n = 2$.'

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The last sentence means that if $\Delta \subset \Delta'$ then $\mu(\Delta) \leq \mu(\Delta')$. Here, Arnold's terminology slightly differs from ours: Δ should be understood as $\mathbb{R}^n_{\geq 0} \setminus \mathbf{\Gamma}_+(f)$ for a singularity f, and then $\mu(\Delta) = \nu(\mathbf{\Gamma}_+(f))$.

In the comment to the problem, Lando [**1**, p. 417] wrote: 'A proof of a stronger statement, the semicontinuity of the spectrum of a singularity, exploiting mixed Hodge structures, was given by Varchenko in [**15**] and by Steenbrink in [**14**]. In the case $n = 2$ an elementary proof of the semicontinuity was given by the author of the present comment in 1981 (unpublished). I do not know whether an elementary proof in arbitrary dimension has ever been written.'

A proof of monotonicity (for $n = 2$) was eventually published by Lenarcik [**9**]. In the case of an arbitrary n, other proofs were offered by Furuya [**5**], Gwoździewicz [**7**] and Bivià-Ausina [**4**].

In the present paper we essentially complete the solution of the problem for surface singularities, i.e. for $n = 3$. More precisely, we prove not only the monotonicity but also we characterize when the equality holds. The characterization is a simple geometrical condition (Theorem 1), which we may describe in the following intuitive way: for any $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ such that $\mathbf{\Gamma}_+(f) \subset \mathbf{\Gamma}_+(g)$ one has $\nu(\mathbf{\Gamma}_+(f)) = \nu(\mathbf{\Gamma}_+(g))$ if, and only if, $\mathbf{\Gamma}_+(f)$ and $\mathbf{\Gamma}_+(g)$ differ by (possibly several) pyramids with bases in the coordinate planes and heights equal to 1. The proof we propose is purely geometrical and elementary. We believe that a similar result should be valid in the n-dimensional case (replacing in the assertion coordinate planes by $(n-1)$-dimensional coordinate hyperplanes).

Our result has interesting applications. For example, if f is a non-degenerate singularity then we may decide for which monomials $z^{\mathbb{i}}$ lying under $\mathbf{\Gamma}_+(f)$, elements of the deformation $f_t = f + t \cdot z^{\mathbb{i}}$ for t small, have the same Milnor numbers and in consequence (by the particular case of Lê-Ramanujam Theorem given by Parusiński [**13**]) are topologically equivalent.

We also expect that our result (and its potential multidimensional generalization) will find applications in effective singularity theory, e.g.: for computation of the Łojasiewicz exponent, jumps of the Milnor numbers in deformations of singularities, searching for tropisms of "partial" gradient ideals $(\partial f/\partial z_1, \ldots, \widehat{\partial f/\partial z_i}, \ldots, \partial f/\partial z_n)\mathcal{O}_n$ of an isolated singularity f, etc.

In a similar spirit, the problem of characterizing those f for which $\mu(f)$ is minimal (and equal to $\nu(\mathbf{\Gamma}_+(f))$) among all singularities with the same Newton polyhedron $\mathbf{\Gamma}_+(f)$ is given in the recent paper by Mondal [**11**].

2. Polyhedra.

According to the standard definitions (see e.g. Berger [**3**]), a *convex n-polyhedron in* $\mathbb{R}^n$ is an intersection of a finite family of closed half-spaces of $\mathbb{R}^n$, having non-empty interior. An *n-polyhedron in* $\mathbb{R}^n$ is a union of finitely many convex n-polyhedra in $\mathbb{R}^n$. Let $k \leq n$; a *k-polyhedron in* $\mathbb{R}^n$ is a finite union of k-polyhedra in k-dimensional affine subspaces of $\mathbb{R}^n$. A compact connected k-polyhedron in $\mathbb{R}^n$ is called a *k-polytope in* $\mathbb{R}^n$.

For convenience, we introduce the following notations. Let $\mathbf{P}, \mathbf{Q} \subset \mathbb{R}^n$ be two k-polyhedra. The *polyhedral difference* (*p-difference*) of $\mathbf{P}$ and $\mathbf{Q}$ is the closure of their set-theoretical difference, in symbols

$$\mathbf{P} - \mathbf{Q} := \overline{\mathbf{P} \setminus \mathbf{Q}}.$$

One can check that $\mathbf{P} - \mathbf{Q}$ is also a k-polyhedron in $\mathbb{R}^n$, or an empty set.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the set of non-negative integers. We define the Newton polyhedra in an abstract way, without any relation to singularities. A subset $\mathbf{\Gamma}_+ \subset \mathbb{R}^n_{\geq 0}$ is said to be a *Newton polyhedron* when there exists a subset $A \subset \mathbb{N}_0^n$ such that

$$\mathbf{\Gamma}_+ = \operatorname{conv}\left(\bigcup_{\mathbb{i} \in A} (\mathbb{i} + \mathbb{R}^n_{\geq 0}) \right),$$

where $\operatorname{conv}(X)$ for $X \subset \mathbb{R}^n$ means the convex hull of X. For such an A we will write $\mathbf{\Gamma}_+ = \mathbf{\Gamma}_+(A)$. In the sequel we will assume that there are no superfluous points in A, implying A is precisely the set of all the vertices of $\mathbf{\Gamma}_+$.

Remark 1. In the context of singularity theory, we take $A = \operatorname{supp} f$, where $f = \sum_{\mathbb{i} \in \mathbb{N}_0^n} a_{\mathbb{i}} z^{\mathbb{i}}$ around 0 and $\operatorname{supp} f := \{\mathbb{i} \in \mathbb{N}_0^n : a_{\mathbb{i}} \neq 0\}$.

A Newton polyhedron $\mathbf{\Gamma}_+$ is called *convenient* if $\mathbf{\Gamma}_+$ intersects all coordinate axes of $\mathbb{R}^n$. Since $\mathbb{N}_0^n$ is a lattice in $\mathbb{R}^n_{\geq 0}$, the boundary of a convenient polyhedron $\mathbf{\Gamma}_+$ is a finite union of convex $(n-1)$-polytopes (*compact facets*) and a finite union of convex unbounded $(n-1)$-polyhedra (*unbounded facets*) lying in coordinate hyperplanes. By $\mathbf{\Gamma}$ we will denote the set of these compact facets, and sometimes – depending on the context – also their set-theoretic union. The closure of the complement of $\mathbf{\Gamma}_+$ in $\mathbb{R}^n_{\geq 0}$ will be denoted by $\mathbf{\Gamma}_-$, i.e.

$$\mathbf{\Gamma}_- := \mathbb{R}^n_{\geq 0} - \mathbf{\Gamma}_+.$$

It is an n-polytope in $\mathbb{R}^n$ provided $\mathbf{\Gamma}_- \neq \emptyset$. Hence, $\mathbf{\Gamma}_-$ has finite n-dimensional volume (in short n-volume). Similarly, for any $\emptyset \neq I \subset \{1, \ldots, n\}$, $\mathbf{\Gamma}_-$ restricted to the coordinate hyperplane $\mathbb{R}^I_{\geq 0} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} : x_i = 0 \text{ for } i \notin I\}$, that is $\mathbf{\Gamma}^I_- := \mathbf{\Gamma}_- \cap \mathbb{R}^I_{\geq 0}$, has finite $(\#I)$-volume. Consequently, we may define the *Newton number* $\nu(\mathbf{\Gamma}_+)$ *of convenient* $\mathbf{\Gamma}_+$ by the formula

$$\nu(\mathbf{\Gamma}_+) := n!\, V_n - (n-1)!\, V_{n-1} + \ldots + (-1)^{n-1}\, 1!\, V_1 + (-1)^n\, V_0,$$

where V_i denotes the sum of i-volumes of $\mathbf{\Gamma}^I_-$, for all $I \subset \{1, \ldots, n\}$ satisfying $\#I = i$. Note that $V_0 = 1$ if $\mathbf{\Gamma}_- \neq \emptyset$ and $V_0 = 0$ if $\mathbf{\Gamma}_- = \emptyset$. Hence $\nu(\mathbb{R}^n_{\geq 0}) = 0$. Clearly, we may also extend the domain of this definition to any n-polytope $\mathbf{P}$ in $\mathbb{R}^n$; thus $\nu(\mathbf{P})$ makes sense. Then for any Newton polyhedron $\mathbf{\Gamma}_+$ we have $\nu(\mathbf{\Gamma}_+) = \nu(\mathbf{\Gamma}_-)$. We will use both notations interchangeably.

The following notions will be useful in our proof. Let B be a compact $(n-1)$-polyhedron in an $(n-1)$-dimensional hyperplane $H \subset \mathbb{R}^n$ and $Q \in \mathbb{R}^n \setminus H$. A *pyramid* $\mathbf{Pyr}(B, Q)$ *with apex* Q *and base* B is by definition the (compact) cone with vertex Q and base B. By [**3**, 12.2.2, p. 13], the n-volume of $\mathbf{Pyr}(B, Q)$ can be computed using the elementary formula

$$\mathrm{vol}_n\,\mathbf{Pyr}(B,Q)=\frac{\mathrm{vol}_{n-1}(B)\ \mathrm{dist}(Q,H)}{n},\tag{1}$$

where $\mathrm{vol}_n(X)$ means n-volume of X.

3. The Main Theorem.

Let $\mathbf{\Gamma}_+$, $\widetilde{\mathbf{\Gamma}}_+$ be two convenient Newton polyhedra such that $\mathbf{\Gamma}_+ \subsetneq \widetilde{\mathbf{\Gamma}}_+$. Then

$$\widetilde{\mathbf{\Gamma}}_+=\mathrm{conv}(\mathbf{\Gamma}_+\cup\{P_1,\dots,P_k\}),$$

for some points $P_1,\dots,P_k$ lying under $\mathbf{\Gamma}_+$, i.e. $P_i\in\mathbb{N}_0^n\setminus\mathbf{\Gamma}_+$. In such situation $\widetilde{\mathbf{\Gamma}}_+$ will also be denoted by $\mathbf{\Gamma}_++\{P_1,\dots,P_k\}$ or $\mathbf{\Gamma}_+^{P_1,\dots,P_k}$. Clearly, $\mathbf{\Gamma}_++\{P_1,\dots,P_k\}=(\mathbf{\Gamma}_++\{P_1,\dots,P_{k-1}\})+P_k$ and hence $\nu(\mathbf{\Gamma}_++\{P_1,\dots,P_k\})=\nu((\mathbf{\Gamma}_++\{P_1,\dots,P_{k-1}\})+P_k)$. Since moreover

$$\nu(\mathbf{\Gamma}_+)-\nu(\mathbf{\Gamma}_++\{P_1,\dots,P_k\})=\sum_{1\le i\le k}(\nu(\mathbf{\Gamma}_++\{P_1,\dots,P_{i-1}\})-\nu(\mathbf{\Gamma}_++\{P_1,\dots,P_i\})),$$

it suffices to study the monotonicity of the Newton number for polyhedra defined by sets which differ in one point only, i.e. for Newton polyhedra $\mathbf{\Gamma}_+$ and $\mathbf{\Gamma}_+^P$, for some $P\in\mathbb{N}_0^n\setminus\mathbf{\Gamma}_+$.

We now give a nice formula for the difference $\nu(\mathbf{\Gamma}_+)-\nu(\mathbf{\Gamma}_+^P)$ in terms of simplexes of a triangulation of $\mathbf{\Gamma}_+^P\setminus\mathbf{\Gamma}_+$. It has been suggested to us by the referee of the first version of this paper. The formula has considerably improved and shortened our original proof of the main theorem and is likely the starting point for a proof in $n\ge 4$ dimensions.

Let $\mathbf{\Gamma}_+$ be a convenient Newton polyhedron and $P\in\mathbb{N}_0^n\setminus\mathbf{\Gamma}_+$. Let $\mathbb{R}^{I(P)}$ be the "smallest" coordinate hyperplane to which P belongs, i.e. $I(P)\subset\{1,\dots,n\}$ and $P\in(\mathbb{R}^*)^{I(P)}$. Consider $\mathbf{\Gamma}_+^P\setminus\mathbf{\Gamma}_+$ as a disjoint union of open simplexes "emanating" from the point P

$$\mathbf{\Gamma}_+^P\setminus\mathbf{\Gamma}_+=\bigsqcup_{\mathbf{S}\in\mathcal{S}}\mathbf{S},$$

where each $\mathbf{S}\in\mathcal{S}$ is of the form $\mathbf{S}=\{P+\sum_{i=1}^k\lambda_i\,\mathbb{a}_i:\lambda_i>0,\sum_{i=1}^k\lambda_i<1\}$, for some $\mathbb{a}_i\in\mathbb{R}^n$ such that $P+\mathbb{a}_i$ is a vertex of $\mathbf{\Gamma}_+$ and some $k\in\{0,\dots,n\}$. Notice that one of simplexes of $\mathcal{S}$ is the point P itself, $\mathbf{S}=\{P\}\in\mathcal{S}$. To each $\mathbf{S}\in\mathcal{S}$ we associate the "half-open" parallelepiped

$$\mathbf{P}(\mathbf{S}):=\left\{P+\sum_{i=1}^k\lambda_i\,\mathbb{a}_i:0\le\lambda_i<1\right\}.$$

We put $\mathcal{P}:=\{\mathbf{P}(\mathbf{S}):\mathbf{S}\in\mathcal{S}\}$. For any $I\subset\{1,\dots,n\}$ we let

$$\mathcal{S}(I):=\{\mathbf{S}\in\mathcal{S}:\mathbf{S}\subset\mathbb{R}^I,\dim\mathbf{S}=\#I\},$$
$$\mathcal{P}(I):=\{\mathbf{P}(\mathbf{S}):\mathbf{S}\in\mathcal{S}(I)\}.$$

By classical formulas (see e.g. [**2**, Lemma 10.3]), given any $\mathbf{S} \in \mathcal{S}(I)$ and putting $l := \#I$, we may write

$$l! \operatorname{vol}_l(\mathbf{S}) = \operatorname{vol}_l \mathbf{P}(\mathbf{S}) = \#(\mathbf{P}(\mathbf{S}) \cap \mathbb{N}_0^I) = \#(\mathbf{P}(\mathbf{S}) \cap \mathbb{N}_0^n). \tag{2}$$

Notice that $\mathcal{S}(I) \neq \emptyset$ implies $I(P) \subset I$. In fact, if $\mathbf{S} \in \mathcal{S}(I)$ then $P \in \overline{\mathbf{S}} \subset \overline{\mathbb{R}^I} = \mathbb{R}^I$. Hence $I(P) \subset I$.

The last fact implies that in order to study the difference $\mathbf{\Gamma}_+^P \setminus \mathbf{\Gamma}_+$ we may limit ourselves to investigating only these simplexes which belong to $\mathcal{S}(I)$ for some $I \supset I(P)$. Thus, by definition of the Newton number and (2), we get

$$\nu(\mathbf{\Gamma}_+) - \nu(\mathbf{\Gamma}_+^P) = \sum_{\substack{I \subset \{1,\ldots,n\} \\ I(P) \subset I}} \sum_{\mathbf{P} \in \mathcal{P}(I)} (-1)^{n-\#I} \#(\mathbf{P} \cap \mathbb{N}_0^n). \tag{3}$$

Since $\mathbf{P} \cap \mathbb{N}_0^n$ is the union of sets of points in the relative interior $\mathbf{P}^{\mathrm{o}}$ of $\mathbf{P}$ and relative interiors of its facets "emanating" from P, formula (3) gets the form

$$\nu(\mathbf{\Gamma}_+) - \nu(\mathbf{\Gamma}_+^P) = \sum_{\mathbf{P} \in \mathcal{P}} \#(\mathbf{P}^{\mathrm{o}} \cap \mathbb{N}_0^n)\, \chi(\mathbf{P}), \tag{4}$$

where $\chi(\mathbf{P}) := \sum\limits_{\substack{\mathbf{Q} \in \mathfrak{P} \\ \mathbf{P} \subset \mathbf{Q}}} (-1)^{n-\dim \mathbf{Q}}$ and $\mathfrak{P} := \bigsqcup\limits_{\substack{I \subset \{1,\ldots,n\} \\ I(P) \subset I}} \mathcal{P}(I)$.

As there is a one-to-one correspondence between parallelepipeds in $\mathcal{P}$ and simplexes in $\mathcal{S}$, formula (4) can be equivalently stated in the language of simplexes as follows

$$\nu(\mathbf{\Gamma}_+) - \nu(\mathbf{\Gamma}_+^P) = \sum_{\mathbf{S} \in \mathcal{S}} \#(\mathbf{P}(\mathbf{S})^{\mathrm{o}} \cap \mathbb{N}_0^n)\, \chi(\mathbf{S}), \tag{5}$$

where $\chi(\mathbf{S}) := \sum\limits_{\substack{\mathbf{T} \in \mathfrak{S} \\ \mathbf{S} \subset \overline{\mathbf{T}}}} (-1)^{n-\dim \mathbf{T}}$ and $\mathfrak{S} := \bigsqcup\limits_{\substack{I \subset \{1,\ldots,n\} \\ I(P) \subset I}} \mathcal{S}(I)$. Notice $\chi(\mathbf{S}) = \chi(\mathbf{P}(\mathbf{S}))$.

In the sequel we shall use both of the above formulas (4) and (5) interchangeably, whichever seems more convenient for the purpose at hand. Now, we state the main theorem of the paper.

THEOREM 1. *Let $\mathbf{\Gamma}_+$ be a convenient Newton polyhedron in $\mathbb{R}^3_{\geq 0}$ and let a lattice point P lie under $\mathbf{\Gamma}_+$ i.e. $P \in \mathbb{N}_0^3 \setminus \mathbf{\Gamma}_+$. Then*

1. $\nu(\mathbf{\Gamma}_+^P) \leq \nu(\mathbf{\Gamma}_+)$,
2. $\nu(\mathbf{\Gamma}_+^P) = \nu(\mathbf{\Gamma}_+)$ *if, and only if, there exists a coordinate plane H such that $P \in H$ and $\mathbf{\Gamma}_+^P - \mathbf{\Gamma}_+$ is a pyramid with base $(\mathbf{\Gamma}_+^P - \mathbf{\Gamma}_+) \cap H$ and of height equal to* 1.

REMARK 2. We believe that the same theorem is true, *mutatis mutandis*, in the n-dimensional case. In the simpler case $n = 2$ the theorem is well-known ([**7**], [**9**] or [**6**]).

REMARK 3. In the particular case when $\mathbf{\Gamma}_+^P - \mathbf{\Gamma}_+$ is a 3-dimensional simplex item 2 follows from Lemma 2.2 in [**12**].

Example 1. Let us illustrate the second item of the theorem with some figures. Let P lying under $\mathbf{\Gamma}_+$ be such that $\nu(\mathbf{\Gamma}_+^P) = \nu(\mathbf{\Gamma}_+)$. Up to permutation of the variables, we have the following, essentially different, possible locations for P:

1. P lies in the plane $\{z = 0\}$ and not on axes (Figure 1 (a)),
2. P lies in the plane $\{z = 0\}$ and on the axis $Ox := \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$ (Figure 1 (b)).

Remark 4. Item 2 of Theorem 1 can be equivalently stated as follows:

2'. $\nu(\mathbf{\Gamma}_+^P) < \nu(\mathbf{\Gamma}_+)$ if, and only if, one of the following two conditions is satisfied:

(a) P lies in the interior of $\mathbf{\Gamma}_-$ i.e. $P \in \mathrm{Int}(\mathbf{\Gamma}_-)$,

(b) for each coordinate plane H such that $P \in H$ the p-difference $\mathbf{\Gamma}_+^P - \mathbf{\Gamma}_+$ is either a pyramid with base $(\mathbf{\Gamma}_+^P - \mathbf{\Gamma}_+) \cap H$ and of height greater than or equal to 2, or a 3-polytope with at least two vertices outside of H.

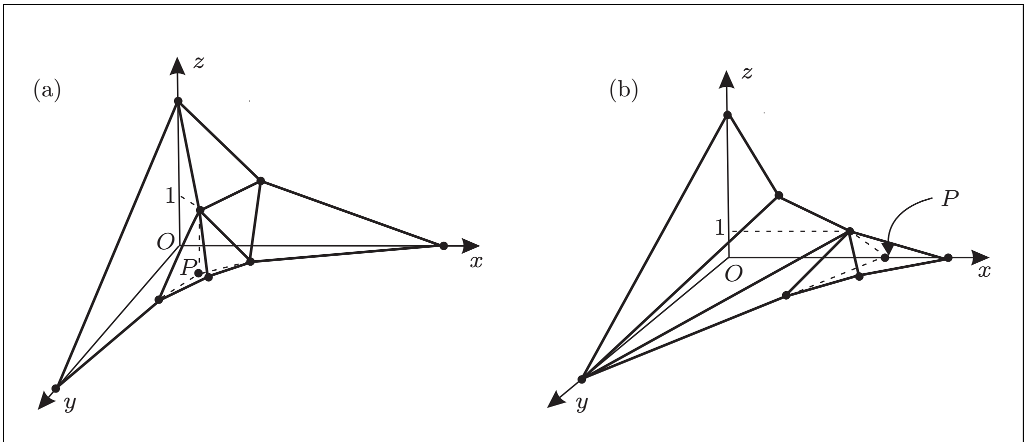

Figure 1. (a) P lies in the plane and not on axes. (b) P lies in the plane and on an axis.

Example 2. The (weaker) requirement that the p-difference $\mathbf{\Gamma}_+^P - \mathbf{\Gamma}_+$ should lie in "a wall" of thickness 1 around a coordinate plane is not sufficient for the equality $\nu(\mathbf{\Gamma}_+^P) = \nu(\mathbf{\Gamma}_+)$. In fact, if $\mathbf{\Gamma}_+$ is the Newton polyhedron of the surface singularity $f(x, y, z) := x^6 + 2y^6 + z(x^2 + y^2) + z^4$ and $P = (3, 2, 0)$, then:

1. $\nu(\mathbf{\Gamma}_+) = 15$, $\nu(\mathbf{\Gamma}_+^P) = 13$,
2. $\mathbf{\Gamma}_+^P - \mathbf{\Gamma}_+$ is a 3-polytope with "base" $(\mathbf{\Gamma}_+^P - \mathbf{\Gamma}_+) \cap Oxy$ and of height 1, but it is not a pyramid with base in a coordinate plane; it has two vertices above Oxy, where $Oxy = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$.

4. Proof of Theorem 1.

First we prove a lemma which by formula (4) gives us the monotonicity of the Newton number, i.e. item 1 in Theorem 1.

LEMMA 1. *Let $\mathbf{\Gamma}_+$ be a convenient Newton polyhedron in $\mathbb{R}^3_{\geq 0}$ and $P \in \mathbb{N}^3_0 \setminus \mathbf{\Gamma}_+$. Then, in notation of the previous section, for each $\mathbf{P} \in \mathcal{P}$, $\chi(\mathbf{P}) \geq 0$.*

PROOF. Consider cases:

1. $\dim(\mathbf{P}) = 3$. Then the only $\mathbf{Q} \in \mathfrak{P}$ for which $\mathbf{P} \subset \mathbf{Q}$ is $\mathbf{P}$ itself. Hence $\chi(\mathbf{P}) = 1$.

2. $\dim(\mathbf{P}) = 2$. Consider subcases:

2 a). $\mathbf{P}$ does not lie in any coordinate plane. Then the only $\mathbf{Q} \in \mathfrak{P}$ for which $\mathbf{P} \subset \mathbf{Q}$ are 3-dimensional parallelepipeds in $\mathcal{P}$ with $\mathbf{P}$ in the boundary. Hence $\chi(\mathbf{P}) = k > 0$, where k is the number of such parallelepipeds.

2 b). $\mathbf{P}$ lies in a coordinate plane. Then there are but two $\mathbf{Q} \in \mathfrak{P}$ for which $\mathbf{P} \subset \mathbf{Q}$: $\mathbf{P}$ itself and one 3-dimensional parallelepiped. Hence $\chi(\mathbf{P}) = -1 + 1 = 0$.

3. $\dim(\mathbf{P}) = 1$. Consider subcases:

3 a). $\mathbf{P}$ does not lie in any coordinate plane. Repeat the reasoning from case 2 a).

3 b). $\mathbf{P}$ lies in a coordinate plane but not in axes. Let $\mathbf{P} = \mathbf{P}(\mathbf{S})$, $\mathbf{S} \in \mathcal{S}$. Then $\mathbf{S}$ is in the boundary of one or two triangles in $\mathcal{S}$ lying in this coordinate plane. Since every such triangle generates a 3-simplex in $\mathcal{S}$, we get $\chi(\mathbf{P}) = \chi(\mathbf{S}) = -1 + 1 + k \geq 0$ in the first case or $\chi(\mathbf{P}) = \chi(\mathbf{S}) = -2 + 2 + k \geq 0$ in the second one, where k is the number of extra 3-simplexes in $\mathcal{S}$ with $\mathbf{S}$ in the boundary.

3 c). $\mathbf{P}$ lies in a coordinate axis. Let $\mathbf{P} = \mathbf{P}(\mathbf{S})$, $\mathbf{S} \in \mathcal{S}$. Then $\{\mathbf{S}\} \in \mathfrak{S}$ and $\mathbf{S}$ is in the boundary of exactly two triangles in $\mathfrak{S}$ lying in two different coordinate planes. Hence $\chi(\mathbf{P}) = \chi(\mathbf{S}) = 1 - 2 + k \geq 0$, where $k > 0$ is the number of 3-dimensional simplexes in $\mathcal{S}$ with $\mathbf{S}$ in the boundary.

4. $\dim(\mathbf{P}) = 0$. We have only the case $\mathbf{P} = \{P\}$. Consider subcases:

4 a). $\mathbf{P}$ does not lie in any coordinate plane. The reasoning is similar to case 3 a).

4 b). $\mathbf{P}$ lies in a coordinate plane but not in axes. The reasoning is similar to case 3 b).

4 c). $\mathbf{P}$ lies in a coordinate axis and $P \neq 0$. The reasoning is similar to case 3 c).

4 d). $\mathbf{P} = \{0\}$, i.e. $P = 0$. Let $\mathbf{T}^{xy}_1, \dots, \mathbf{T}^{xy}_k$ be triangles in $\mathcal{S}$ lying in Oxy and similarly $\mathbf{T}^{xz}_1, \dots, \mathbf{T}^{xz}_l$ and $\mathbf{T}^{yz}_1, \dots, \mathbf{T}^{yz}_m$ for planes Oxz and Oyz, respectively. Each of these triangles uniquely generates one 3-simplex in $\mathcal{S}$. Denote them by $\mathbf{S}^{xy}_1, \dots, \mathbf{S}^{xy}_k$ and respectively $\mathbf{S}^{xz}_1, \dots, \mathbf{S}^{xz}_l$ and $\mathbf{S}^{yz}_1, \dots, \mathbf{S}^{yz}_m$. However, some of them may be identical, but only one of $\mathbf{S}^{xy}_1, \dots, \mathbf{S}^{xy}_k$ may be equal to one of $\mathbf{S}^{xz}_1, \dots, \mathbf{S}^{xz}_l$ (when they have the same edge in Ox) and similarly for remaining pairs of coordinate planes (Oxy, Oyz) and (Oxz, Oyz). In the worst case only three pairs in the sequence of all simplexes $\mathbf{S}^{xy}_1, \dots, \mathbf{S}^{yz}_m$ may be identical.

Assume first there are less such identical pairs (i.e. ≤ 2). Then there are the following simplexes $\mathbf{Q} \in \mathfrak{P}$ for which $\mathbf{P} \subset \mathbf{Q}$: one 0-dimensional ($\mathbf{P}$ itself), three 1-dimensional (lying in axes), $(k+l+m)$ 2-dimensional (lying in planes Oxy, Oxz, Oyz) and at least $(k+l+m-2)$ 3-dimensional. Hence

$$\chi(\mathbf{P}) \geq -1+3-(k+l+m)+(k+l+m-2)=0.$$

If there are 3 pairs of identical simplexes in the sequence $\mathbf{S}_1^{xy}, \ldots, \mathbf{S}_k^{xy}$, $\mathbf{S}_1^{xz}, \ldots, \mathbf{S}_l^{xz}$, $\mathbf{S}_1^{yz}, \ldots, \mathbf{S}_m^{yz}$, then a simple geometric reasoning gives us that

• either $k=l=m=1$ and $\mathbf{S}_1^{xy}=\mathbf{S}_1^{xz}=\mathbf{S}_1^{yz}$ and then

$$\chi(\mathbf{P})=-1+3-3+1=0,$$

•• or there are extra 3-simplexes in $\mathcal{S}$ (besides $\mathbf{S}_1^{xy}, \ldots, \mathbf{S}_m^{yz}$) and then

$$\chi(\mathbf{P})=-1+3-(k+l+m)+(k+l+m-3)+s \geq 0,$$

where $s>0$ is the number of these extra 3-simplexes in $\mathcal{S}$. □

From the above lemma and formula (4) we get item 1 in Theorem 1. Let us pass to the proof of item 2.

"$\Leftarrow$" First, we prove that the combinatorial condition in item 2 implies the equality $\nu(\mathbf{\Gamma}_+)=\nu(\mathbf{\Gamma}_+^P)$. Without loss of generality we may assume that, having fixed coordinates (x,y,z) in $\mathbb{R}^3$, we have: $H=\{z=0\}$, $P \in H$ and $\mathbf{\Gamma}_+^P-\mathbf{\Gamma}_+$ is a pyramid with base $(\mathbf{\Gamma}_+^P-\mathbf{\Gamma}_+)\cap H$ and of height equal to 1. We must show that $\nu(\mathbf{\Gamma}_+^P)=\nu(\mathbf{\Gamma}_+)$. Since $\mathbf{\Gamma}_+^P-\mathbf{\Gamma}_+=\mathbf{\Gamma}_--\mathbf{\Gamma}_-^P$, but the latter is a p-difference of two polytopes, we prefer to reason in terms of $\mathbf{\Gamma}_-$ and $\mathbf{\Gamma}_-^P$ instead.

We have three possibilities:

1. P does not lie on any axis, that is $P \notin Ox \cup Oy$. Then the polytopes $\mathbf{\Gamma}_-$ and $\mathbf{\Gamma}_-^P$ are identical on Ox, Oy, Oz, Oxz and Oyz. Denoting by $\mathbf{W}$ the p-difference polygon of $\mathbf{\Gamma}_-$ and $\mathbf{\Gamma}_-^P$ in Oxy, we have by definition of the Newton number and (1)

$$\nu(\mathbf{\Gamma}_+)-\nu(\mathbf{\Gamma}_+^P)=\frac{3!\ \mathrm{vol}_2(\mathbf{W})\cdot 1}{3}-2!\ \mathrm{vol}_2(\mathbf{W})=0.$$

2. P lies on Ox or Oy and $P \neq 0$. Up to renaming of the variables, we may assume that $P \in Ox$. Hence and by the assumption that $\mathbf{\Gamma}_+$ is convenient, the apex of the pyramid must lie in the plane Oxz. Setting $\mathbf{W}:=(\mathbf{\Gamma}_--\mathbf{\Gamma}_-^P)\cap Oxy$ and $\mathbf{L}:=(\mathbf{\Gamma}_--\mathbf{\Gamma}_-^P)\cap Ox$, we have

$$\nu(\mathbf{\Gamma}_+)-\nu(\mathbf{\Gamma}_+^P)=\frac{3!\ \mathrm{vol}_2(\mathbf{W})\cdot 1}{3}-2!\ \mathrm{vol}_2(\mathbf{W})-\frac{2!\ \mathrm{vol}_1(\mathbf{L})\cdot 1}{2}+\mathrm{vol}_1(\mathbf{L})=0.$$

3. $P=0$. Then $\mathbf{\Gamma}_-^P=\emptyset$. By the assumption that $\mathbf{\Gamma}_+$ is convenient, the apex of the pyramid must be $(0,0,1)$. Hence, if we denote by $\mathbf{L}^x$, $\mathbf{L}^y$, $\mathbf{L}^z$, $\mathbf{W}^{xy}$, $\mathbf{W}^{xz}$, $\mathbf{W}^{yz}$ the intersections of $\mathbf{\Gamma}_-$ with coordinate axes and planes, respectively, then

$$\begin{aligned}\nu(\mathbf{\Gamma}_+)-\nu(\mathbf{\Gamma}_+^P)&=\nu(\mathbf{\Gamma}_+)\\&=\frac{3!\ \mathrm{vol}_2(\mathbf{W}^{xy})\cdot 1}{3}-2!\ \mathrm{vol}_2(\mathbf{W}^{xy})-2!\ \mathrm{vol}_2(\mathbf{W}^{xz})-2!\ \mathrm{vol}_2(\mathbf{W}^{yz})\end{aligned}$$

$$+\operatorname{vol}_1(\mathbf{L}^x) + \operatorname{vol}_1(\mathbf{L}^y) + \operatorname{vol}_1(\mathbf{L}^z) - 1$$
$$= 0.$$

"$\Rightarrow$" We now show the inverse implication in item 2 of Theorem 1. To the contrary, assume that the combinatorial condition in item 2 does not hold. Then we have to prove that $\nu(\mathbf{\Gamma}_+) - \nu(\mathbf{\Gamma}_+^P) > 0$. In some cases we shall use formula (5). According to this formula and Lemma 1, it suffices to find just one $\mathbf{S} \in \mathcal{S}$ for which $\mathbf{P}(\mathbf{S})^o \cap \mathbb{N}_0^n \neq \emptyset$ and $\chi(\mathbf{S}) > 0$. Consider possible cases:

1. P does not lie in any coordinate plane i.e. $I(P) = \{1, 2, 3\}$. Then the p-difference $\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^P$ is a 3-polytope, disjoint from all the coordinate planes. Hence, by definition of the Newton number

$$\nu(\mathbf{\Gamma}_+) - \nu(\mathbf{\Gamma}_+^P) = 3!\ \operatorname{vol}_3(\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^P) > 0.$$

2. P lies in a coordinate plane, but not on any axis. Without loss of generality, we may assume that $P \in Oxy \backslash (Ox \cup Oy)$. It means $I(P) = \{1, 2\}$. Then the p-difference 3-polytope $\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^P$ is disjoint from the planes Oxz and Oyz, but $\mathbf{W} := (\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^P) \cap Oxy \neq \emptyset$ (Figure 2 (a)). According to Remark 4 (b), we should examine the following possibilities:

a). $\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^P$ is a pyramid with base $\mathbf{W}$ and of height $h \geq 2$. We have

$$\nu(\mathbf{\Gamma}_+) - \nu(\mathbf{\Gamma}_+^P) = \frac{3!\ \operatorname{vol}_2(\mathbf{W})\, h}{3} - 2!\ \operatorname{vol}_2(\mathbf{W}) = 2\ \operatorname{vol}_2(\mathbf{W})\,(h-1) > 0.$$

b). There are at least two vertices of $\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^P$ lying above Oxy. Then there exists a 3-simplex $\mathbf{T} \in \mathcal{S}$ with two vertices above Oxy. Hence $\mathbf{T}$ has no side in Oxy. If we denote by $\mathbf{W}_1, \ldots, \mathbf{W}_k$ the triangles of $\mathbf{W}$ (see Figure 2 (b)), then each $\mathbf{W}_i$ is a side of a 3-simplex $\mathbf{T}_i$ and the $\mathbf{T}_i$ are different from $\mathbf{T}$. Hence, taking $\mathbf{S} = \{P\}$ in formula (5), we get $\mathbf{P}(\mathbf{S}) = \mathbf{S}$, $(\mathbf{S}^o \cap \mathbb{N}_0^n) = \{P\} \neq \emptyset$.

Moreover, there are only the following simplexes $\mathbf{Q} \in \mathfrak{P}$ for which $P \in \mathbf{Q}$: k 2-simplexes lying in Oxy and at least $(k+1)$ 3-simplexes. Hence $\chi(\mathbf{S}) \geq -k + k + 1 > 0$. We obtain $\nu(\mathbf{\Gamma}_+) - \nu(\mathbf{\Gamma}_+^P) > 0$.

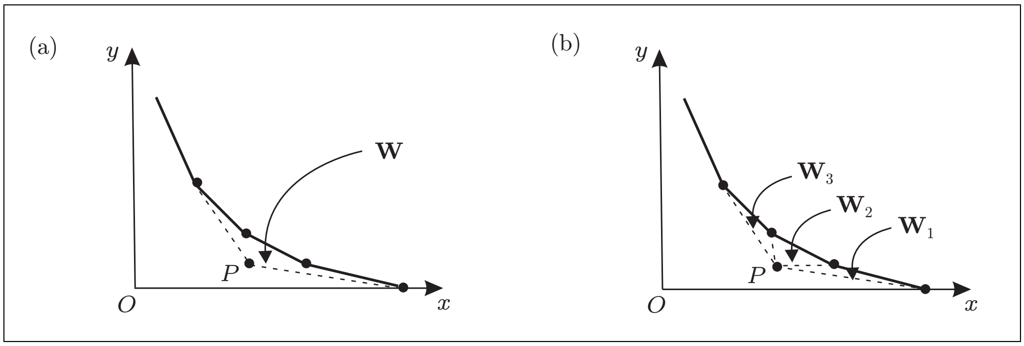

Figure 2. (a) p-difference $\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^P$ in Oxy.
(b) Triangulation of $\mathbf{W}$.

3. P lies on an axis and $P \neq 0$. Without loss of generality we may assume $P \in Ox$ i.e. $I(P) = \{1\}$. Then the p-difference 3-polytope $\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^P$ is disjoint from the plane Oyz and from the axes Oy and Oz. The polygons $\mathbf{W}^{xy} := (\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^P) \cap Oxy$ and $\mathbf{W}^{xz} := (\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^P) \cap Oxz$ are unions of triangles $\mathbf{W}_1^{xy}, \dots, \mathbf{W}_k^{xy}, k \geq 1$ and $\mathbf{W}_1^{xz}, \dots, \mathbf{W}_l^{xz}, l \geq 1$, respectively, all of them having P as a vertex (see Figure 3).

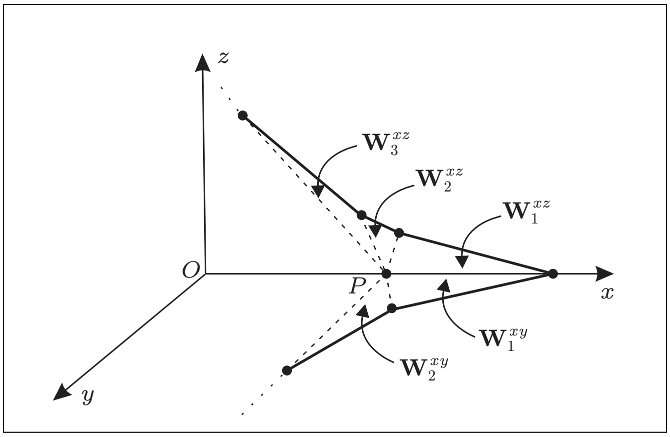

Figure 3. Triangulation of $\mathbf{W}^{xy}$ and $\mathbf{W}^{xz}$.

Each of them is a side of a 3-simplex in $\mathcal{S}$. Denote these simplexes by $\mathbf{S}_1^{xy}, \dots, \mathbf{S}_k^{xy}$ and $\mathbf{S}_1^{xz}, \dots, \mathbf{S}_l^{xz}$, respectively. Clearly, they are all different except possibly $\mathbf{S}_1^{xy} = \mathbf{S}_1^{xz}$. If there are other 3-simplexes in $\mathcal{S}$ besides $\mathbf{S}_1^{xy}, \dots, \mathbf{S}_k^{xy}, \mathbf{S}_1^{xz}, \dots, \mathbf{S}_l^{xz}$ or $\mathbf{S}_1^{xy} \neq \mathbf{S}_1^{xz}$, then there are at least $(k+l)$ 3-simplexes in $\mathcal{S}$. We take $\mathbf{S} = \{P\}$ in formula (5). Then $\mathbf{P}(\mathbf{S}) = \mathbf{S}$, $(\mathbf{S}^{\text{o}} \cap \mathbb{N}_0^n) = \{P\} \neq \emptyset$. Moreover, there are only the following simplexes $\mathbf{Q} \in \mathfrak{P}$ for which $P \in \mathbf{Q}$: one 1-simplex lying in Ox, $(k+l)$ 2-simplexes lying in Oxy and Oxz, and at least $(k+l)$ 3-simplexes. Hence

$$\chi(\mathbf{S}) \geq +1 - (k+l) + (k+l) > 0.$$

So, we may assume that the only 3-simplexes in $\mathcal{S}$ are $\mathbf{S}_1^{xy} = \mathbf{S}_1^{xz}, \mathbf{S}_2^{xy}, \dots, \mathbf{S}_k^{xy}, \mathbf{S}_2^{xz}, \dots, \mathbf{S}_l^{xz}$. We claim that they have no vertices outside Oxy and Oxz. Indeed, if, to the contrary, $\mathbf{S}_i^{xy}$ had a vertex outside Oxz, say R, then $\mathbf{S}_{i-1}^{xy}$ would also have the same vertex R, and so on. At the end we would conclude that $\mathbf{S}_1^{xy}$ also has R as its vertex, which is impossible ($\mathbf{S}_1^{xy} = \mathbf{S}_1^{xz}$). Consider cases:

i). $k = l = 1$. Since $\mathbf{\Gamma}_+^P - \mathbf{\Gamma}_+$ is not a pyramid of height 1, the triangles $\mathbf{W}_1^{xy}$ and $\mathbf{W}_1^{xz}$ with bases in Ox have heights ≥ 2. Then, by definition of the Newton number, we easily get $\nu(\mathbf{\Gamma}_+) - \nu(\mathbf{\Gamma}_+^P) > 0$.

ii). $l = 1$ and $k \geq 2$. Since $\mathbf{S}_1^{xy}, \dots, \mathbf{S}_k^{xy}$ have vertices in Oxz and $l = 1$, all of them share the same vertex, which is also the vertex of $\mathbf{W}_1^{xz}$. As $\mathbf{\Gamma}_+^P - \mathbf{\Gamma}_+$ is not a pyramid of height 1, the triangle $\mathbf{W}_1^{xz}$ with base in Ox has its height ≥ 2. Hence, again by definition of the Newton number, we can check that $\nu(\mathbf{\Gamma}_+) - \nu(\mathbf{\Gamma}_+^P) > 0$.

iii). $k = 1$ and $l \geq 2$. Analogously as in ii).

iv). $k \geq 2$ and $l \geq 2$. In this case we can easily prove that among $\mathbf{S}_2^{xy}, \ldots, \mathbf{S}_k^{xy}, \mathbf{S}_2^{xz}, \ldots, \mathbf{S}_l^{xz}$ there are ones with height ≥ 2. Then, again by definition of the Newton number, we find that $\nu(\mathbf{\Gamma}_+) - \nu(\mathbf{\Gamma}_+^P) > 0$.

4. $P = 0$. Then $\mathbf{\Gamma}_-^P = \emptyset$ and consequently $\nu(\mathbf{\Gamma}_+) - \nu(\mathbf{\Gamma}_+^P) = \nu(\mathbf{\Gamma}_-) - \nu(\mathbf{\Gamma}_-^P) = \nu(\mathbf{\Gamma}_-)$. By assumption $\mathbf{\Gamma}_-$ is not a pyramid with base in a coordinate plane and height 1. Hence, $\mathbf{\Gamma}_+$ intersects all axes at points with coordinates greater than or equal to 2. Take auxiliary point $\widetilde{P} = (1, 0, 0)$ on axis Ox, lying under $\mathbf{\Gamma}_+$. Then $\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^{\widetilde{P}}$ is not a pyramid of height 1, either. Hence, by 3., $\nu(\mathbf{\Gamma}_+) - \nu(\mathbf{\Gamma}_+^{\widetilde{P}}) > 0$. Since $\mathbf{\Gamma}_-^{\widetilde{P}}$ is a pyramid of height 1, $\nu(\mathbf{\Gamma}_-^{\widetilde{P}}) = 0$. But $\mathbf{\Gamma}_- = (\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^{\widetilde{P}}) \cup \mathbf{\Gamma}_-^{\widetilde{P}}$ and, of course, $\nu(\mathbf{\Gamma}_-) = \nu((\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^{\widetilde{P}}) \cup \mathbf{\Gamma}_-^{\widetilde{P}}) = \nu(\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^{\widetilde{P}}) + \nu(\mathbf{\Gamma}_-^{\widetilde{P}})$. We get $\nu(\mathbf{\Gamma}_-) = \nu(\mathbf{\Gamma}_- - \mathbf{\Gamma}_-^{\widetilde{P}}) = \nu(\mathbf{\Gamma}_-) - \nu(\mathbf{\Gamma}_-^{\widetilde{P}}) = \nu(\mathbf{\Gamma}_+) - \nu(\mathbf{\Gamma}_+^{\widetilde{P}}) > 0$. This ends the proof of Theorem 1.

Corollary 1. *Let $\mathbf{\Gamma}_+$ be a convenient Newton polyhedron in $\mathbb{R}^3_{\geq 0}$. Then $\nu(\mathbf{\Gamma}_+) \geq 0$. Moreover, $\nu(\mathbf{\Gamma}_+) = 0$ if, and only if, either $\mathbf{\Gamma}_+ = \mathbb{R}^3_{\geq 0}$ or $\mathbf{\Gamma}_+$ intersects one of the axes at the point with coordinate equal to* 1.

Remark 5. The last corollary was proved by Furuya [**5**, Corollary 2.4], in n-dimensional case.

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