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A new stability notion of closed hypersurfaces in the hyperbolic space

By Marco Antonio Lázaro Velásquez, Henrique Fernandes de Lima, Jonatan Floriano da Silva and Arlandson Matheus Silva Oliveira

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Abstract. In this article, we establish the notion of strong (r, k, a, b)-stability related to closed hypersurfaces immersed in the hyperbolic space $\mathbb{H}^{n+1}$, where r and k are nonnegative integers satisfying the inequality $0 \leq k < r \leq n-2$ and a and b are real numbers (at least one nonzero). In this setting, considering some appropriate restrictions on the constants a and b, we show that geodesic spheres are strongly (r, k, a, b)-stable. Afterwards, under a suitable restriction on the higher order mean curvatures H_{r+1} and H_{k+1}, we prove that if a closed hypersurface into the hyperbolic space $\mathbb{H}^{n+1}$ is strongly (r, k, a, b)-stable, then it must be a geodesic sphere, provided that the image of its Gauss mapping is contained in the chronological future (or past) of an equator of the de Sitter space.

1. Introduction.

The notion of stability concerning hypersurfaces of constant mean curvature in Riemannian ambient spaces was first studied by Barbosa and Do Carmo in [**6**], and Barbosa, Do Carmo and Eschenburg in [**7**], where they proved that spheres are the only stable critical points of the area functional for volume-preserving variations. In [**2**], Alencar, Do Carmo and Colares extended to hypersurfaces with constant scalar curvature the above stability result on constant mean curvature.

The natural generalization of mean and scalar curvatures for an n-dimensional hypersurface are the higher order mean curvatures H_r, $r \in \{1, \ldots, n\}$. In fact, H_1 is just the mean curvature H, and H_2 defines a geometric quantity which is directly related to the scalar curvature. In a space form, first Alencar, Do Carmo and Rosenberg in [**3**] and shortly after Barbosa and Colares [**5**] studied closed hypersurfaces with constant hight order mean curvature H_{r+1} and established the concept of r-stability. In this context, they showed that such hypersurfaces are r-stable if and only if they are geodesic spheres. Moreover, the first two authors of this paper together with De Sousa in [**15**] have studied the notion of (r, s)-stability concerning closed hypersurfaces with higher order mean curvatures linearly related in a space form, showing that, if such a hypersurface Σ^n is contained either in an open hemisphere of the Euclidean sphere $\mathbb{S}^{n+1}$ or in the hyperbolic

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space $\mathbb{H}^{n+1}$, then Σ^n is (r,s)-stable if and only if Σ^n is a geodesic sphere. In [**10**], the first two authors and Da Silva, through the development of a different technique, managed to complete this characterization of (r,s)-stable hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$.

More recently, the first two authors and De Sousa in [**11**] have established the notion of strong stability (that is, stability with respect to not necessarily volume-preserving variations) related to closed hypersurfaces satisfying $na_0H + n(n+1)a_1H_2 =$ constant, where a_0 and a_1 are nonnegative constants (with at least one nonzero), immersed in the hyperbolic space $\mathbb{H}^{n+1}$. These hypersurfaces are called *linear Weingarten hypersurfaces.* In this setting, initially we show that geodesic spheres are strongly stable. Afterwards, under a suitable restriction on the mean and scalar curvatures, they prove that if a closed linear Weingarten hypersurface into $\mathbb{H}^{n+1}$ is strongly stable and its image of its Gauss mapping is contained in the chronological future (or past) of an equator of the de Sitter space then it must be a geodesic sphere.

Motivated by these works, here we define the notion of strong (r,k,a,b)-stability (cf. Definition 1) concerning closed hypersurfaces immersed in the hyperbolic space $\mathbb{H}^{n+1}$, where r and k are nonnegative integers satisfying the inequality $0 \le k < r \le n-2$ and a and b are real numbers (at least one nonzero). Such concept arises considering the variational problem of minimizing the r-area functional for all variations, including those variations that preserve a linear combination of k-area functional and volume (cf. Section 3). A hypersurface Σ^n of $\mathbb{H}^{n+1}$ is a critical point of the variational problem described above when it has higher order mean curvatures H_{k+1} and H_{r+1} verifying $b_rH_{r+1}/(a\,b_kH_{k+1} - b) =$ constant, with $a\,b_kH_{k+1} - b \neq 0$ on Σ^n (cf. Proposition 1), where $b_j = (j+1)\binom{n}{j+1}$, $j \in \{k,r\}$. For such critical points, the second variation of the Jacobi functional associated to the corresponding variational problem is calculated (cf. Proposition 2), where appears naturally the differential operator $\widetilde{L}_{r,k,a,b}$, which is a certain linear combination of the linearized operators associated to the higher order mean curvatures H_r and H_k. In this context, considering some appropriate restrictions on the constants a and b related to such hypersurfaces, we show that geodesic spheres of $\mathbb{H}^{n+1}$ are strongly (r,k,a,b)-stable (cf. Proposition 3). Next, we consider an appropriated warped product model of the open subset $\mathbb{H}^{n+1}\setminus\{q\}$ of the hyperbolic space to calculate the $\widetilde{L}_{r,k,a,b}$ of a healthy function support (cf. Lemma 4) and, under a suitable restriction on the constants a and b and the higher order mean curvatures H_{k+1} and H_{r+1}, we show that if a closed hypersurface in $\mathbb{H}^{n+1}$ is strongly (r,k,a,b)-stable, then it must be a geodesic sphere, provided that the image of its Gauss mapping is contained in the chronological future (or past) of an equator of the de Sitter space (cf. Theorem 1). Finally, in Corollary 1 we rewrite our main result in the context of linear Weingarten hypersurfaces.

2. Preliminaries.

Let $\mathbb{L}^{n+2}$ denote the $(n+2)$-dimensional Lorentz–Minkowski space ($n \ge 2$), that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentz metric

$$\langle v, w\rangle = \sum_{i=1}^{n+1} v_iw_i - v_{n+2}w_{n+2},$$

for all $v, w \in \mathbb{R}^{n+2}$. We recall that the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$ can be regarded as the following hyperquadric

$$\mathbb{H}^{n+1} = \left\{ p \in \mathbb{L}^{n+2} : \langle p, p \rangle = -1 \quad \text{and} \quad p_{n+2} \geq 1 \right\},$$

which is a spacelike hypersurface of $\mathbb{L}^{n+2}$, that is, its induced metric via the inclusion $\iota : \mathbb{H}^{n+1} \hookrightarrow \mathbb{L}^{n+2}$ is a Riemannian metric on $\mathbb{H}^{n+1}$, indeed, this is its (complete) metric of constant sectional curvature -1. In this setting, we will denote by $C^\infty(\mathbb{H}^{n+1})$ the commutative ring of real functions of class C^∞ on $\mathbb{H}^{n+1}$, by $\mathfrak{X}(\mathbb{H}^{n+1})$ the $C^\infty(\mathbb{H}^{n+1})$-module of vector fields of class C^∞ on $\mathbb{H}^{n+1}$, by $d\mathbb{H}^{n+1}$ the volume element of $\mathbb{H}^{n+1}$ and by $\overline{\nabla}$ the Levi-Civita connection of $\mathbb{H}^{n+1}$.

Now, we consider hypersurfaces $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$, namely, isometric immersions from a connected, n-dimensional orientable Riemannian manifold Σ^n into $\mathbb{H}^{n+1}$. We also let $C^\infty(\Sigma^n)$, $\mathfrak{X}(\Sigma^n)$ and ∇ denote, respectively, the commutative ring of real functions of class C^∞ on Σ^n, the $C^\infty(\Sigma^n)$-module of vector fields of class C^∞ on Σ^n and the Levi-Civita connection of Σ^n.

Since Σ^n is orientable, one can choose a globally defined unit normal vector field N on Σ^n. Let A denote the shape operator with respect to N, so that, at each $p \in \Sigma^n$, A restricts to a self-adjoint linear map $A_p : T_p(\Sigma^n) \to T_p(\Sigma^n)$.

For $1 \leq r \leq n$, if we let $S_r(p)$ denote the *r-th elementary symmetric function* on the eigenvalues of A_p, we get n smooth functions $S_r : \Sigma^n \to \mathbb{R}$ such that

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. If $p \in \Sigma^n$ and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_p(\Sigma^n)$ formed by eigenvectors of A_p, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the *r-th elementary symmetric polynomial* on the indeterminates $X_1, \ldots, X_n$.

For $1 \leq r \leq n$, one defines the *r-th mean curvature* H_r of Σ^n by

$$\binom{n}{r} H_r = S_r = S_r(\lambda_1, \ldots, \lambda_n).$$

In particular, for $r = 1$, $H_1 = (1/n) \sum_{k=1}^{n} \lambda_k = H$ is the *mean curvature* of Σ^n, which is the main extrinsic curvature of the hypersurface. When $r = 2$, H_2 defines a geometric quantity which is related to the (intrinsic) *normalized scalar curvature* R of the hypersurface. More precisely, it follows from the Gauss equation of the hypersurface that

$$R = -1 + H_2. \tag{1}$$

We also define, for $0 \leq r \leq n$, the *r-th Newton transformation* P_r on Σ^n by setting $P_0 = I$ (the identity operator) and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = S_r I - AP_{r-1}.$$

A trivial induction shows that

$$P_r = (S_r I - S_{r-1}A + S_{r-2}A^2 - \cdots + (-1)^r r A^r),$$

so that Cayley–Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of T_pM diagonalizing A at $p \in \Sigma^n$ also diagonalize all of the P_r at p. Let $\{e_1, \ldots, e_n\}$ be such a basis. Denoting by A_i the restriction of A to $\langle e_i \rangle^{\perp} \subset T_p\Sigma$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1 \leq j_1 < \ldots < j_k \leq n \\ j_1, \ldots, j_k \neq i}} \lambda_{j_1} \cdots \lambda_{j_k}.$$

With the above notations, it is also immediate to check that

$$P_r e_i = S_r(A_i) e_i, \tag{2}$$

and hence (cf. Lemma 2.1 of [**5**])

$$\begin{cases} \operatorname{tr}(P_r) = (n-r)S_r = b_r H_r; \\ \operatorname{tr}(AP_r) = (r+1)S_{r+1} = b_r H_{r+1}; \\ \operatorname{tr}(A^2 P_r) = S_1 S_{r+1} - (r+2)S_{r+2} = n\dfrac{b_r}{r+1} H H_{r+1} - b_{r+1} H_{r+2}, \end{cases} \tag{3}$$

where $b_r = (r+1)\binom{n}{r+1} = (n-r)\binom{n}{r}$.

Associated to each Newton transformation P_r one has the *second order linear differential operator*

$$\begin{aligned} L_r : C^\infty(\Sigma^n) &\to C^\infty(\Sigma^n) \\ f \quad &\mapsto \; L_r(f) = \operatorname{tr}(P_r \operatorname{Hess} f). \end{aligned}$$

We remark that L_0 is the Laplacian operator Δ and L_1 is the Cheng–Yau's square operator $\square$ defined in [**9**].

3. The notion of strong (r, k, a, b)-stability.

For a closed hypersurface $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ as in the previous section, a *variation* of it is a smooth mapping $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{H}^{n+1}$ satisfying the following condition: for $t \in (-\epsilon, \epsilon)$, the map $X_t : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ given by $X_t(p) = X(t, p)$ is an immersion such that $X_0 = x$.

In all that follows, we let $d\Sigma_t$ denote the volume element of the metric induced on Σ^n by X_t and N_t the unit normal vector field along X_t.

The *variational field* associated to the variation $X : (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{H}^{n+1}$ is $(\partial X/\partial t)|_{t=0}$. Letting

$$f = \Big\langle \frac{\partial X}{\partial t}, N_t \Big\rangle, \tag{4}$$

we get $\partial X/\partial t = fN_t + (\partial X/\partial t)^\top$, where $(\,\cdot\,)^\top$ stands for tangential components.

The *balance of volume* of the variation $X : (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{H}^{n+1}$ is the functional

$$\begin{aligned} \mathcal{V} : (-\epsilon,\epsilon) &\to \ \mathbb{R} \\ t \quad &\mapsto \mathcal{V}(t) = \int_{\Sigma^n\times[0,t]} X^*(d\mathbb{H}^{n+1}), \end{aligned}$$

where $d\mathbb{H}^{n+1}$ denotes the volume element of $\mathbb{H}^{n+1}$. The following lemma is well known and can be found in [**5**].

Lemma 1. *If $X : (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{H}^{n+1}$ is a variation of a closed hypersurface $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$, then*

$$\mathcal{V}'(t) = \int_{\Sigma^n} f d\Sigma_t,$$

where f is the function defined in (4)*.*

Following the ideas of [**5**], we define the *r-area functional* associated to the variation $X : (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{H}^{n+1}$ by

$$\begin{aligned} \mathcal{A}_r : (-\epsilon,\epsilon) &\to \quad \mathbb{R} \\ t \quad &\mapsto A_r(t) = \int_{\Sigma^n} F_r(S_1,S_2,\ldots,S_r)d\Sigma_t, \end{aligned}$$

where $S_r = S_r(t)$ and F_r is recursively defined by setting $F_0 = 1$, $F_1 = S_1$ and, for $2 \le r \le n-1$,

$$F_r = S_r - \frac{(n-r+1)}{r-1}F_{r-2}.$$

We notice that if $r = 0$, the functional $\mathcal{A}_0$ is the classical area functional. The following result follows from Proposition 4.1 of [**5**].

Lemma 2. *If $X : (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{H}^{n+1}$ is a variation of $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$, then*

$$\frac{\partial H_{r+1}}{\partial t} = \frac{r+1}{b_r}\left\{L_r f + \big(-\mathrm{tr}(P_r) + \mathrm{tr}(A^2P_r)\big) f\right\} + \Big\langle \Big(\frac{\partial X}{\partial t}\Big)^\top, \nabla H_{r+1}\Big\rangle, \tag{5}$$

where $b_r = (r+1)\binom{n}{r+1}$ and f is the function defined in (4)*.*

The previous lemma allows us to obtain the first variation of the r-area functional (see, for example, Lemma 3.4 of [**15**]).

Lemma 3. *If $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{H}^{n+1}$ is a variation of a closed hypersurface $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$, then*

$$\mathcal{A}_r'(t) = -b_r \int_{\Sigma^n} H_{r+1} f d\Sigma_t,$$

where $b_r = (r+1)\binom{n}{r+1}$ and f is the function defined in (4).

Let r and k be nonnegative integers, satisfying the inequality $0 \leq k < r \leq n-2$, and consider real numbers a and b (with at least one nonzero). In order to characterize hypersurfaces whose elementary functions satisfy a certain constant quotient, we define

$$\begin{aligned} \mathcal{C}_{k,a,b} : (-\epsilon, \epsilon) \to & \quad \mathbb{R} \\ t \quad & \mapsto \mathcal{C}_{k,a,b}(t) = a\, \mathcal{A}_k(t) + b\, \mathcal{V}(t), \end{aligned}$$

and we say that the variation $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{H}^{n+1}$ of $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ preserves the linear combination $\mathcal{C}_{k,a,b}$ if $\mathcal{C}_{k,a,b}(t) = \mathcal{C}_{k,a,b}(0)$ for all $t \in (-\epsilon, \epsilon)$.

Now, we consider the variational problem of minimizing the r-area functional $\mathcal{A}_r$ for all variations that preserve the functional $\mathcal{C}_{k,a,b}$. The *Jacobi functional* associated to the problem is given by

$$\begin{aligned} \mathcal{J}_{r,k,a,b} : (-\epsilon, \epsilon) \to & \quad \mathbb{R} \\ t \quad & \mapsto \mathcal{J}_{r,k,a,b}(t) = \mathcal{A}_r(t) + \varrho\, \mathcal{C}_{k,a,b}(t), \end{aligned}$$

where ϱ is a constant to be determined. As an immediate consequence of Lemmas 3 and 1 we get

$$\mathcal{J}_{r,k,a,b}'(t) = -\int_{\Sigma^n} \{b_r H_{r+1} + \varrho\,(a\, b_k H_{k+1} - b)\}\, f d\Sigma_t,$$

where f is the function defined in (4). To choose ϱ, let

$$\overline{\mathcal{H}} = \frac{1}{\mathcal{A}_0(0)} \int_{\Sigma} \left\{ \frac{b_r H_{r+1}(0)}{a\, b_k H_{k+1}(0) - b} \right\} d\Sigma$$

be the mean of the function $b_r H_{r+1}(0)/\{a\, b_k H_{k+1}(0) - b\}$ along Σ^n, where $H_j(0)$ stands for the j-th mean curvature of the immersion $X_0 = x$. We call the attention to the fact that, when $b_r H_{r+1}(0)/\{a\, b_k H_{k+1}(0) - b\}$ is constant, one has

$$\overline{\mathcal{H}} = \frac{b_r H_{r+1}(0)}{a\, b_k H_{k+1}(0) - b} = \frac{b_r H_{r+1}}{a\, b_k H_{k+1} - b}, \tag{6}$$

and this notation will be used in what follows without further comments. Therefore, if we choose $\varrho = -\overline{\mathcal{H}}$, we arrive at

$$\mathcal{J}_{r,k,a,b}'(t) = -\int_{\Sigma^n} \left\{ b_r H_{r+1} - \overline{\mathcal{H}}\,(a\, b_k H_{k+1} - b) \right\} f d\Sigma_t, \tag{7}$$

where f is the function defined in (4).

Now, following the same ideas of Proposition 2.7 in [**6**] we can establish, from (7), the following result (see [**15**, Proposition 3.6]).

Proposition 1. *Let r and k be nonnegative integers satisfying the inequality $0 \leq k < r \leq n-2$, and let $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ be a closed hypersurface. Are equivalent*

(a) *$x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ have higher order mean curvatures H_{k+1} and H_{r+1} verifying*

$$\frac{b_r H_{r+1}}{a\, b_k H_{k+1} - b} = \text{constant},$$

with $a\, b_k H_{k+1} - b \neq 0$ on Σ^n, where a and b are real numbers (with at least one nonzero) and $b_j = (j+1)\binom{n}{j+1}$ for $j \in \{k, r\}$;

(b) *for all variations $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{H}^{n+1}$ of x that preserve the functional $\mathcal{C}_{k,a,b}$, we have $\mathcal{A}'_r(0) = 0$;*

(c) *for all variations $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{H}^{n+1}$ of x, we have $\mathcal{J}'_{r,k,a,b}(0) = 0$.*

At this point, motivated by the ideas established in [**11**], we exchanged our problem and now we want to detect closed hypersurfaces $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ that minimize the Jacobi functional $\mathcal{J}_{r,k,a,b}$ for all variations $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{H}^{n+1}$ of x. Next, Proposition 1 shows that the critical points of $\mathcal{J}_{r,k,a,b}$ are hypersurfaces $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ such that its higher order mean curvatures H_{k+1} and H_{r+1} verify

$$\frac{b_r H_{r+1}}{a\, b_k H_{k+1} - b} = \text{constant},$$

with $a\, b_k H_{k+1} - b \neq 0$ on Σ^n. So, for such a hypersurface, we aim to compute the second variation of $\mathcal{J}_{r,k,a,b}$. This will motivate us to establish the following notion of stability.

Definition 1. Let r and s be nonnegative integers satisfying the inequality $0 \leq k < r \leq n-2$, and let $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ be a closed hypersurface whose higher order mean curvatures H_{k+1} and H_{r+1} satisfy

$$\frac{b_r H_{r+1}}{a\, b_k H_{k+1} - b} = \text{constant},$$

with $a\, b_k H_{k+1} - b \neq 0$ on Σ^n, where a and b are real numbers (with at least one nonzero) and $b_j = (j+1)\binom{n}{j+1}$ for $j \in \{k, r\}$. We say that $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ is *strongly (r,k,a,b)-stable* if $\mathcal{J}''_{r,k,a,b}(0)(f) \geq 0$ for all $f \in C^\infty(\Sigma^n)$.

The sought formula for the second variation of $\mathcal{J}_{r,k,a,b}$ is a straightforward consequence of Lemmas 2 and 3.

Proposition 2. *Let r and k be nonnegative integers satisfying the inequality $0 \leq k < r \leq n-2$, and let $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ be a closed hypersurface whose higher order mean curvatures H_{k+1} and H_{r+1} satisfy*

$$\frac{b_r H_{r+1}}{a\, b_k H_{k+1} - b} = \text{constant},$$

with $a\,b_k H_{k+1} - b \neq 0$ *on* Σ^n, *where* a *and* b *are real numbers* (*with at least one nonzero*) *and* $b_j = (j+1)\binom{n}{j+1}$ *for* $j \in \{k, r\}$. *If* $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{H}^{n+1}$ *is a variation of* x, *then* $\mathcal{J}''_{r,k,a,b}(0)$ *is given by*

$$\mathcal{J}''_{r,k,a,b}(0)(f) = -(r+1)\int_{\Sigma^n} \Big\{\widetilde{L}_{r,k,a,b}(f) + \Big(-\operatorname{tr}(P_r) + \operatorname{tr}(A^2P_r) \\ - \Lambda_{r,k,a,b}\big(-\operatorname{tr}(P_k) + \operatorname{tr}(A^2P_k)\big)\Big)f\Big\} f d\Sigma, \tag{8}$$

for any $f \in C^\infty(\Sigma^n)$, *where* $\widetilde{L}_{r,k,a,b}$ *is the differential operator*

$$\begin{aligned} \widetilde{L}_{r,k,a,b} : C^\infty(\Sigma^n) &\to C^\infty(\Sigma^n) \\ f &\mapsto \widetilde{L}_{r,k,a,b}(f) = L_r f - \Lambda_{r,k,a,b}\, L_k f, \end{aligned} \tag{9}$$

and

$$\Lambda_{r,k,a,b} = \frac{a(k+1)b_r H_{r+1}}{a(r+1)b_k H_{k+1} - (r+1)b}. \tag{10}$$

PROOF. From (5), (6) and (7), we obtain

$$\begin{aligned} \mathcal{J}''_{r,k,a,b}(0) &= \frac{\partial}{\partial t}\left(-\int_\Sigma \left\{b_r H_{r+1} - \overline{\mathcal{H}}\,(a\,b_k H_{k+1} - b)\right\} f d\Sigma_t\right)\Big|_{t=0} \\ &= -\int_{\Sigma^n}\left(b_r \frac{\partial H_{r+1}}{\partial t}\Big|_{t=0} - \overline{\mathcal{H}} a\, b_k \frac{\partial H_{k+1}}{\partial t}\Big|_{t=0}\right) f dM \\ &\quad - \int_{\Sigma^n}\left(\underbrace{b_r H_{r+1} - \overline{\mathcal{H}}\,(a\,b_k H_{k+1} - b)}_{0}\right)\frac{\partial}{\partial t}\,(f dM_t)\Big|_{t=0} \\ &= -(r+1)\int_{\Sigma^n}\Big\{(L_r - \Lambda_{r,k,a,b}L_k)(f) + \big(-\operatorname{tr}(P_r) + \operatorname{tr}(A^2P_r) \\ &\qquad -\Lambda_{r,k,a,b}\left(-\operatorname{tr}(P_k) + \operatorname{tr}(A^2P_k)\right)\big) f\Big\} f d\Sigma \\ &\quad - \int_{\Sigma^n}\Big\langle \left(\frac{\partial X}{\partial t}\right)^\top, \underbrace{\nabla\left(b_r H_{r+1} - \overline{\mathcal{H}}\,(a\,b_k H_{k+1} - b)\right)}_{0}\Big\rangle f d\Sigma. \end{aligned}$$

To finish the proof, we observe that the above expression depends only on the hypersurface $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ and on the function $f \in C^\infty(\Sigma^n)$. □

4. Strongly (r, k, a, b)-stable hypersurfaces in $\mathbb{H}^{n+1}$.

We establish a similar result to the statements found in Proposition 5.1 of [**5**] and Proposition 4.1 of [**15**].

PROPOSITION 3. *If* $\Lambda_{r,k,a,b}$ *is nonpositive, then the geodesic spheres of* $\mathbb{H}^{n+1}$ *are strongly* (r, k, a, b)*-stable.*

PROOF. Let $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ be a geodesic sphere of $\mathbb{H}^{n+1}$. Since Σ^n is totally umbilical then its principal curvatures are all equal to a certain constant λ. By choosing a suitably normal vector we may assume that $\lambda > 0$. Thus we have

$$S_j = \binom{n}{j}\lambda^j, \qquad H_j = \lambda^j, \qquad S_j(A_i) = \binom{n-1}{j}\lambda^j$$

and, if $e_1, \dots, e_n$ are the principal directions of Σ^n,

$$L_j(f) = \sum_{i=1}^{n} \big\langle \mathrm{Hess}(f)(e_i), P_j(e_i)\big\rangle = \binom{n-1}{j}\lambda^j \Delta f.$$

for any $j \in \{0, \dots, n\}$ and all $f \in C^\infty(\Sigma^n)$, where we use (2) in the last step. Next, for nonnegative integers r and k, satisfying the inequality $0 \le k < r \le n-2$, and real numbers a and b (with at least one nonzero) such that $a(k+1)\binom{n}{k+1}\lambda^{k+1} \neq b$, we have $b_r H_{r+1}/\{a\, b_k H_{k+1} - b\} = b_r \lambda^{r+1}/\{a\, b_k \lambda^{k+1} - b\} = \text{constant}$, where $b_j = (j+1)\binom{n}{j+1}$ for $j \in \{k, r\}$. Then, from (3) and (8) we obtain

$$\begin{aligned}\mathcal{J}''_{r,k,a,b}(0)(f) = -(r+1)\int_{\Sigma^n} \Big\{\Gamma^{\,n-1}_{r,k,a,b}\,\Delta f + \Big(-(n-r)S_r + S_1 S_{r+1} - (r+2)S_{r+2}\Big) f \\ - \Lambda_{r,k,a,b}\Big(-(n-k)S_k + S_1 S_{k+1} - (k+2)S_{k+2}\Big) f\Big\} f\, d\Sigma,\end{aligned}$$

where

$$\Gamma^{\,n-1}_{r,k,a,b} = \binom{n-1}{r}\lambda^r - \Lambda_{r,k,a,b}\binom{n-1}{k}\lambda^k \tag{11}$$

and $\Lambda_{r,k,a,b}$ is defined in (10). Thus,

$$\begin{aligned}\mathcal{J}''_{r,k,a,b}(0)(f) &= -(r+1)\int_{\Sigma^n} \Big\{\Gamma^{\,n-1}_{r,k,a,b}\,\Delta f + \Big(-(n-r)\binom{n}{r}\lambda^r \\ &\quad + n\binom{n}{r+1}\lambda^{r+2} - (r+2)\binom{n}{r+2}\lambda^{r+2}\Big) f \\ &\quad - \Lambda_{r,k,a,b}\Big(-(n-k)\binom{n}{k}\lambda^k + n\binom{n}{k+1}\lambda^{k+2} \\ &\quad - (k+2)\binom{n}{k+2}\lambda^{k+2}\Big) f\Big\} f\, d\Sigma \\ &= -(r+1)\int_{\Sigma^n} \Big\{\Gamma^{\,n-1}_{r,k,a,b}\,\Delta f \\ &\quad - \Big((n-r)\binom{n}{r}\lambda^r - \Lambda_{r,k,a,b}(n-k)\binom{n}{k}\lambda^k\Big) f \\ &\quad + \lambda^{r+2}\Big(n\binom{n}{r+1} - (r+2)\binom{n}{r+2}\Big) f \\ &\quad - \lambda^{k+2}\Lambda_{r,k,a,b}\Big(n\binom{n}{k+1} - (k+2)\binom{n}{k+2}\Big) f\Big\} f\, d\Sigma\end{aligned}$$

$$= -(r+1)\int_{\Sigma^n}\Big\{\Gamma^{n-1}_{r,k,a,b}\,\Delta f - n\,\Gamma^{n-1}_{r,k,a,b}\,f + n\Gamma^{n-1}_{r,k,a,b}\,\lambda^2 f\Big\} f d\Sigma$$
$$= (r+1)\Gamma^{n-1}_{r,k,a,b}\int_{\Sigma^n}\left\{-f\Delta f - n(-1+\lambda^2)f^2\right\} d\Sigma, \tag{12}$$

for any $f \in C^\infty(\Sigma^n)$. Hence, if η_1 denote the first eigenvalue of the Laplacian of Σ^n and considering the assumption of function $\Lambda_{r,k,a,b}$, from (11) and (12) we get

$$\mathcal{J}''_{r,k,a,b}(0)(f) \;\geq\; (r+1)\Gamma^{n-1}_{r,k,a,b}\int_{\Sigma^n}\left\{\eta_1 - n(-1+\lambda^2)\right\} f^2 d\Sigma \;=\; 0,$$

for any $f \in C^\infty(\Sigma^n)$, where the last equality was obtained by observing that Σ^n is isometric to an n-dimensional Euclidean sphere with constant sectional curvature equal to $\lambda^2 - 1$; hence $\eta_1 = n(\lambda^2 - 1)$. Therefore, we conclude that $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ is strongly (r, k, a, b)-stable. $\square$

In order to prove our main result, we note that, according to Example 4.3 in [**13**], the hyperbolic space $\mathbb{H}^{n+1}$ (minus a certain point q) can be regarded of as the following warped product

$$\mathbb{H}^{n+1}\setminus\{q\} \;\simeq\; (0,+\infty)\times_{\sinh\tau}\mathbb{S}^n, \quad \tau\in(0,+\infty), \tag{13}$$

($\simeq$ means *isometric to*) where $\mathbb{S}^n$ stands for the Euclidean unit sphere. More precisely, if $d\tau^2$ and $d\sigma^2$ denote the metrics of $(0,+\infty)$ and $\mathbb{S}^n$, respectively, then

$$\langle\,,\rangle = (\pi_1)^*\left(d\tau^2\right) + (\sinh\tau)^2(\pi_{\mathbb{S}^n})^*\left(d\sigma^2\right),$$

is the tensor metric of $(0,+\infty)\times_{\sinh\tau}\mathbb{S}^n$, where π_1 and $\pi_{\mathbb{S}^n}$ denote the projections onto the $(0,+\infty)$ and $\mathbb{S}^n$ factors, respectively. We note that, in this warped product model, the *slices*

$$\Sigma^n_{\tau_0} = \{\tau_0\}\times\mathbb{S}^n, \;\; \tau_0\in(0,+\infty),$$

are, exactly, the geodesic spheres of $\mathbb{H}^{n+1}$. Moreover, if we orient such slices by the unit normal vector field $-\partial/\partial\tau$, then the j-th mean curvature of $\Sigma^n_{\tau_0}$ is constant equal to

$$H_j = (\coth\tau_0)^j, \quad j\in\{1,\ldots,n\}. \tag{14}$$

Finally, we observe that

$$W = (\sinh\tau)\frac{\partial}{\partial\tau} \tag{15}$$

is a conformal, and closed vector field (in the sense that its dual 1-form is closed), namely, $\overline{\nabla}_Y W = (\cosh\tau)\,Y$ for any tangent vector field Y defined in $\mathbb{H}^{n+1}\setminus\{q\}$.

We need the following result, whose proof is a consequence of a suitable formula due to Barros and Sousa [**8**].

LEMMA 4. *Let r and k be nonnegative integers satisfying the inequality $0 \leq k < r \leq n-2$, and let $x : \Sigma^n \hookrightarrow (0,+\infty) \times_{\sinh\tau} \mathbb{S}^n$ be a hypersurface whose higher order mean curvatures H_{k+1} and H_{r+1} satisfy*

$$\frac{b_r H_{r+1}}{a\, b_k H_{k+1} - b} = \text{constant},$$

with $a\, b_k H_{k+1} - b \neq 0$ on Σ^n, where a and b are real numbers (with at least one non-zero) and $b_j = (j+1)\binom{n}{j+1}$ for $j \in \{k, r\}$. If N is the Gauss map on Σ^n and $\eta = \langle(\sinh\tau)\partial/\partial\tau, N\rangle$ then

$$\begin{aligned}\widetilde{L}_{r,k,a,b}(\eta) = &-\left\{\left(-\mathrm{tr}(P_r) + \mathrm{tr}(A^2P_r)\right) - \Lambda_{r,k,a,b}\left(-\mathrm{tr}(P_k) + \mathrm{tr}(A^2P_k)\right)\right\}\eta \\ &-\{b_rH_r - \Lambda_{r,k,a,b}\, b_kH_k\}\left\langle\frac{\partial}{\partial\tau}, N\right\rangle \sinh\tau \\ &-\{b_rH_{r+1} - \Lambda_{r,k,a,b}\, b_kH_{k+1}\}\cosh\tau, \end{aligned} \tag{16}$$

where $\widetilde{L}_{r,k,a,b}$ is the differential operator defined in (9) *and $\Lambda_{r,k,a,b}$ is defined in* (10).

PROOF. From Theorem 2 of [**8**],

$$\begin{aligned}L_j(\eta) = &-\left\{\mathrm{tr}(A^2P_j) - \mathrm{tr}(P_j)\right\}\eta - b_jH_jN(\cosh\tau) - b_jH_{j+1}\cosh\tau \\ &-\frac{b_j}{j+1}\langle(\sinh\tau)\frac{\partial}{\partial\tau}, \nabla H_{j+1}\rangle\end{aligned}$$

for $j \in \{k, r\}$. Thus,

$$\begin{aligned}\widetilde{L}_{r,k,a,b}(\eta) = L_r(\eta) - \Lambda_{r,k,a,b}L_k(\eta) \;=\; &-\left\{-\mathrm{tr}(P_r) + \mathrm{tr}(A^2P_r)\right\}\eta \\ &-b_rH_rN(\cosh\tau) - b_rH_{r+1}\cosh\tau - \frac{b_r}{r+1}\langle(\sinh\tau)\frac{\partial}{\partial\tau}, \nabla H_{r+1}\rangle \\ &-\Lambda_{r,k,a,b}\Bigg(-\left\{-\mathrm{tr}(P_k) + \mathrm{tr}(A^2P_k)\right\}\eta - b_kH_kN(\cosh\tau) \\ &-b_kH_{k+1}\cosh\tau - \frac{b_k}{k+1}\langle(\sinh\tau)\frac{\partial}{\partial\tau}, \nabla H_{k+1}\rangle\Bigg) \\ = &-\left\{\left(-\mathrm{tr}(P_r) + \mathrm{tr}(A^2P_r)\right) - \Lambda_{r,k,a,b}\left(-\mathrm{tr}(P_k) + \mathrm{tr}(A^2P_k)\right)\right\}\eta \\ &-\{b_rH_r - \Lambda_{r,k,a,b}\, b_kH_k\}N(\cosh\tau) \\ &-\{b_rH_{r+1} - \Lambda_{r,k,a,b}\, b_kH_{k+1}\}\cosh\tau \\ &-\Big\langle(\sinh\tau)\frac{\partial}{\partial\tau}, \underbrace{\nabla\left(-\frac{b_r}{r+1}H_{r+1} + \Lambda_{r,k,a,b}\frac{b_k}{k+1}H_{k+1}\right)}_{0}\Big\rangle. \end{aligned} \tag{17}$$

Now, observing that

$$\overline{\nabla}\cosh\tau = \left\langle\overline{\nabla}\cosh\tau, \frac{\partial}{\partial\tau}\right\rangle\frac{\partial}{\partial\tau} = (\cosh\tau)'\frac{\partial}{\partial\tau} = (\sinh\tau)\frac{\partial}{\partial\tau},$$

we have that

$$N(\cosh\tau) = \langle \overline{\nabla} \cosh\tau, N\rangle = \left\langle \frac{\partial}{\partial\tau}, N \right\rangle \sinh\tau. \tag{18}$$

Finally, substituting (18) into (17) we obtain (16). $\square$

Returning to the hyperquadric model of $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$, let $\boldsymbol{a} \in \mathbb{L}^{n+2}$ be an unit timelike vector (that is, $\langle \boldsymbol{a}, \boldsymbol{a}\rangle = -1$). Then, we easily verify that

$$V(p) = \boldsymbol{a} + \langle p, \boldsymbol{a}\rangle p\,, \quad p \in \mathbb{H}^{n+1}, \tag{19}$$

is a closed conformal vector field globally defined in $\mathbb{H}^{n+1}$. Consequently, from Proposition 1 in [**13**], we have that such a vector field V foliates $\mathbb{H}^{n+1}$ by means of totally umbilical spheres, which can be characterized as the following level sets

$$L_\delta = \{\ p \in \mathbb{H}^{n+1}\ :\ \langle p, \boldsymbol{a}\rangle = \delta\ \}, \quad \delta^2 > 1.$$

At this point, we recall that the $(n+1)$-dimensional de Sitter space $\mathbb{S}_1^{n+1}$ is defined as being the following hyperquadric of $\mathbb{L}^{n+2}$

$$\mathbb{S}_1^{n+1} = \left\{ p \in \mathbb{L}^{n+2}\ :\ \langle p, p\rangle = 1 \right\}.$$

The induced metric from $\langle\,,\rangle$ makes $\mathbb{S}_1^{n+1}$ into a Lorentz manifold with constant sectional curvature one.

In a dual manner of that of the hyperbolic space, taking again a unit timelike vector $\boldsymbol{a} \in \mathbb{L}^{n+2}$, we have the vector field

$$K(p) = \boldsymbol{a} - \langle p, \boldsymbol{a}\rangle p\,, \quad p \in \mathbb{S}_1^{n+1},$$

is a conformal and closed timelike vector field globally defined in $\mathbb{S}_1^{n+1}$. From Proposition 1 in [**14**], we see that such a vector field K foliates $\mathbb{S}_1^{n+1}$ by means of totally umbilical round spheres, which are described as the following level sets

$$\mathcal{L}_\varepsilon = \{\ p \in \mathbb{S}_1^{n+1} : \langle p, \boldsymbol{a}\rangle = \varepsilon\ \}, \quad \varepsilon \in \mathbb{R}.$$

In particular, the level set $\{\ p \in \mathbb{S}_1^{n+1}\ :\ \langle p, \boldsymbol{a}\rangle = 0\ \}$ defines a round sphere of radius one which is a totally geodesic hypersurface in $\mathbb{S}_1^{n+1}$. According to [**1**], we will refer to that sphere as the *equator* of $\mathbb{S}_1^{n+1}$ determined by $\boldsymbol{a}$. This equator divides $\mathbb{S}_1^{n+1}$ into two connected components, the *chronological future* which is given by

$$\{\ p \in \mathbb{S}_1^{n+1}\ :\ \langle \boldsymbol{a}, p\rangle < 0\ \}, \tag{20}$$

and the *chronological past*, given by

$$\{\ p \in \mathbb{S}_1^{n+1}\ :\ \langle \boldsymbol{a}, p\rangle > 0\ \}.$$

On the other hand, we observe that the unit normal vector field N of $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ can be regarded as a map $N : \Sigma^n \hookrightarrow \mathbb{S}_1^{n+1}$, called *Gauss mapping* of x. In this setting, the image $N(\Sigma^n)$ will be called the *Gauss image* of x.

REMARK 1. By fixing a unit timelike vector $\boldsymbol{a} \in \mathbb{L}^{n+2}$ and considering in $\mathbb{H}^{n+1}$ as well as in $\mathbb{S}_1^{n+1}$ the foliations previously described, we can follow the same ideas of Section 3 in [**4**] in order to verify that the Gauss mapping of a geodesic sphere L_δ of $\mathbb{H}^{n+1}$ is given by

$$N(p) = -\frac{1}{\sqrt{\delta^2 - 1}}(\boldsymbol{a} + \delta p)\,, \quad p \in L_\delta.$$

Consequently, we have that $N(L_\delta) \subset \mathcal{L}_\varepsilon$ for $\varepsilon = -\sqrt{\delta^2 - 1} < 0$. Therefore, we conclude that the Gauss image of a geodesic sphere of $\mathbb{H}^{n+1}$ is contained in the chronological future (or past) of the equator of $\mathbb{S}_1^{n+1}$ determined by $\boldsymbol{a}$.

Now, we are in position to state and prove our main result.

THEOREM 1. *Let r and k be nonnegative integers satisfying the inequality $0 \le k < r \le n-2$, let a and b be real numbers, with $b \ne 0$, and let $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ be a strongly (r,k,a,b)-stable closed hypersurface. Suppose that $\Lambda_{r,k,a,b}$ is nonpositive and the higher order mean curvatures H_{k+1} and H_{r+1} of $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ satisfy*

$$H_{j+1} \;\ge\; H_j, \quad j \in \{k, r\}. \tag{21}$$

If the Gauss image of x is contained in the chronological future (or past) of an equator of $\mathbb{S}_1^{n+1}$ then $x(\Sigma^n)$ is a geodesic sphere of $\mathbb{H}^{n+1}$.

PROOF. Initially, we affirm that $H_j > 0$ everywhere on Σ^n, for all $j \in \{0, \dots, r+1\}$. In fact, as Σ^n is closed in $\mathbb{H}^{n+1}$, we may assume that the orientation N of Σ^n is considered such that its principal curvatures are positive at a point $p_0 \in \Sigma^n$. Moreover, from the strong (r,k,a,b)-stability of $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ we obtain that the quotient $b_r H_{r+1}/\{a\, b_k H_{k+1} - b\}$ is constant, with $a\, b_k H_{k+1} - b \ne 0$ on Σ^n. Let

$$\beta := \frac{(r+1)S_{r+1}}{a\, b_k H_{k+1} - b}(p_0) \equiv \frac{b_r H_{r+1}}{a\, b_k H_{k+1} - b}. \tag{22}$$

If $a\, b_k H_{k+1} - b < 0$ then $\beta < 0$, so, by (22), $H_{r+1} = b_r^{-1}\beta\{a\, b_k H_{k+1} - b\} > 0$ on Σ^n. On the other hand, if $a\, b_k H_{k+1} - b > 0$, we have $\beta > 0$, so, again by (22), $H_{r+1} = b_r^{-1}\beta\{a\, b_k H_{k+1} - b\} > 0$ on Σ^n. Anyway $H_{r+1} > 0$ on Σ^n. Finally, our assertion follows directly from the classical inequalities of Gårding [**12**].

Now, let us suppose that, without loss of generality, the Gauss image $N(\Sigma^n)$ of the hypersurface $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ is contained in the chronological future of the equator of $\mathbb{S}_1^{n+1}$ determined by a unit timelike vector $\boldsymbol{a} \in \mathbb{L}^{n+2}$. For such vector $\boldsymbol{a}$, let us also consider the warped product given in (13), which models the hyperbolic space $\mathbb{H}^{n+1}$ (minus a point) as being $(0, +\infty) \times_{\sinh \tau} \mathbb{S}^n$.

In this setting, we consider the *normal angle* θ of $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$, which is the smooth function $\theta : \Sigma^n \to [0, \pi]$ given by

$$\cos\theta(p) = -\left\langle \Phi_* N, \frac{\partial}{\partial \tau} \right\rangle_{(\Phi \circ x)(p)}, \tag{23}$$

where Φ stands for an isometry between the hyperquadric and warped product models of $\mathbb{H}^{n+1}$. From (23) and (15), for any $p \in \Sigma^n$ we have that

$$\begin{aligned}\cos\theta(p) &= -\Big\langle \Phi_* N((\Phi\circ x)(p)), \frac{W((\Phi\circ x)(p))}{|W((\Phi\circ x)(p))|}\Big\rangle \\ &= -\frac{1}{|\Phi_*^{-1}W(x(p))|}\Big\langle N(x(p)), \Phi_*^{-1}W(x(p))\Big\rangle. \end{aligned} \tag{24}$$

Since $\Phi_*^{-1}W = V$, where V is the closed conformal vector field given by (19), from (24) we get

$$\begin{aligned}\cos\theta(p) &= -\frac{1}{|V(x(p))|}\Big\langle N(x(p)), \boldsymbol{a} + \langle \boldsymbol{a}, x(p)\rangle x(p)\Big\rangle \\ &= -\frac{1}{|V(x(p))|}\Big\langle N(x(p)), \boldsymbol{a}\Big\rangle. \end{aligned} \tag{25}$$

Hence, since we are supposing that the Gauss image $N(\Sigma^n)$ is contained in the chronological future of $\mathbb{S}_1^{n+1}$ determined by $\boldsymbol{a}$, from (20) and (25) we conclude that

$$0 < \cos\theta \leq 1. \tag{26}$$

On the other hand, since $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ is strongly (r,k,a,b)-stable, from Definition 1 and equation (8) we obtain

$$\begin{aligned} 0 \;\leq\; \mathcal{J}''_{r,k,a,b}(0)(f) = -(r+1)\int_{\Sigma^n} \Big\{ &\widetilde{L}_{r,k,a,b}(f) + \big(-\mathrm{tr}(P_r) + \mathrm{tr}(A^2P_r) \\ &-\Lambda_{r,k,a,b}\left(-\mathrm{tr}(P_k) + \mathrm{tr}(A^2P_k)\right)\big) f\Big\} f d\Sigma, \end{aligned}$$

for all $f \in C^\infty(\Sigma^n)$, where $\widetilde{L}_{r,k,a,b}$ is the differential operator defined in (9) and $\Lambda_{r,k,a,b}$ is defined in (10). In particular, taking

$$f = \eta\circ\Phi^{-1} = \Big\langle (\sinh\tau)\frac{\partial}{\partial\tau}, \Phi_*N\Big\rangle = -\sinh\tau\cos\theta$$

and (for simplicity of notation) considering $N = \Phi_*N$ and $H_j = H_j\circ\Phi^{-1}$ for $j \in \{k, k+1, r, r+1\}$, from Lemma 4 we obtain

$$\begin{aligned} 0 &\leq (r+1)\int_{\Phi(\Sigma^n)} \{(b_rH_r - \Lambda_{r,k,a,b}\, b_kH_k)\sinh\tau\cos\theta \\ &\qquad - (b_rH_{r+1} - \Lambda_{r,k,a,b}\, b_kH_{k+1})\cosh\tau\}\sinh\tau\cos\theta\, d\Phi(\Sigma) \\ &\leq (r+1)\int_{\Phi(\Sigma^n)} \{(b_rH_r - \Lambda_{r,k,a,b}\, b_kH_k)\cos\theta \\ &\qquad - (b_rH_{r+1} - \Lambda_{r,k,a,b}\, b_kH_{k+1})\}\cosh\tau\sinh\tau\cos\theta\, d\Phi(\Sigma), \end{aligned} \tag{27}$$

where $b_j = (j+1)\binom{n}{j+1} = (n-j)\binom{n}{j}$, with $j \in \{k, r\}$. But, since $\Lambda_{r,k,a,b} \leq 0$, from (21) we also have that

$$b_k\left(H_{k+1} - H_k\right)\Lambda_{r,k,a,b} - b_r\left(H_{r+1} - H_r\right) \leq 0.$$

Equivalently,

$$b_r H_{r+1} - \Lambda_{r,k,a,b}\, b_k H_{k+1} \geq b_r H_r - \Lambda_{r,k,a,b}\, b_k H_k. \tag{28}$$

Substituting (28) into (27) we have

$$0 \;\leq\; (r+1)\int_{\Phi(\Sigma^n)} (b_r H_r - \Lambda_{r,k,a,b}\, b_k H_k)(\cos\theta - 1)\cosh\tau \sinh\tau \cos\theta \, d\Phi(\Sigma) \;\leq\; 0,$$

where we have used (26), that $\Lambda_{r,k,a,b} \leq 0$, $H_r > 0$, $H_k > 0$ and that $\tau > 0$. Hence, $\cos\theta = 1$ and, consequently, there exists $\tau_0 \in (0, +\infty)$ such that

$$(\Phi \circ x)(\Sigma^n) = \{\tau_0\} \times \mathbb{S}^n. \qquad \square$$

REMARK 2. We would like to point out that, taking into account that the higher order mean curvatures of each slice $\Sigma^n_{\tau_0} = \{\tau_0\} \times \mathbb{S}^n$ verify $H_{r+1} > H_r \geq H_{k+1} > H_k > 1$ (as can be observed from (14)) for any entire numbers r and k satisfying the inequality $0 \leq k < r \leq n-2$, our restriction on the values of the higher order mean curvatures H_{k+1} and H_{r+1} in Theorem 1 constitutes a mild hypothesis in the sense that, in the light of Proposition 3 and Remark 1, it is natural to detect geodesic spheres of $\mathbb{H}^{n+1}$.

We recall that a hypersurface $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ is *linear Weingarten* when the mean curvature H and normalized scalar curvature R satisfy

$$\delta_0 H + \delta_1 R = \delta_2,$$

for some constants $\delta_0, \delta_1, \delta_2 \in \mathbb{R}$. Then, making $r = 1$ and $k = 0$ in Theorem 1, from (1) we get the following result.

COROLLARY 1. *Let a and b be real numbers, with $b \neq 0$, and let $x : \Sigma^n \hookrightarrow \mathbb{H}^{n+1}$ be a strongly $(1, 0, a, b)$-stable closed linear Weingarten hypersurface such that*

$$n(n-1)(R+1) - naH\delta = -b\delta,$$

where $\delta \in \mathbb{R} \setminus \{0\}$. Suppose that $\Lambda_{1,0,a,b}$ is nonpositive and $1 \leq H \leq R+1$. If the Gauss image of x is contained in the chronological future (or past) of an equator of $\mathbb{S}^{n+1}_1$ then Σ^n is a sphere and x is its inclusion as a geodesic sphere of $\mathbb{H}^{n+1}$.

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Marco Antonio Lázaro Velásquez
Departamento de Matemática
Universidade Federal de Campina Grande
58109-970 Campina Grande
Paraíba, Brazil
E-mail: marco.velasquez@mat.ufcg.edu.br

Henrique Fernandes de Lima
Departamento de Matemática
Universidade Federal de Campina Grande
58109-970 Campina Grande
Paraíba, Brazil
E-mail: henrique@mat.ufcg.edu.br

Jonatan Floriano da Silva
Departamento de Matemática
Universidade Federal do Ceará
60.455-760 Fortaleza
Ceará, Brazil
E-mail: jonatanfloriano@gmail.com

Arlandson Matheus Silva Oliveira
Departamento de Matemática
Universidade Federal de Campina Grande
58109-970 Campina Grande
Paraíba, Brazil
E-mail: arlandsonm@gmail.com