J. Math. Soc. Japan
Vol. 71, No. 1 (2019) pp. 117–145
doi: 10.2969/jmsj/78267826

Berkes' limit theorem

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(Received June 15, 2017)
(Revised July 6, 2017)

Abstract. In Berkes' striking paper of the early 1990s, he presented another limit theorem different from the central limit theorem for a lacunary trigonometric series not satisfying Erdős' lacunary condition. In this paper, we upgrade his result to the limit theorem having high versatility, which we would call Berkes' limit theorem. By this limit theorem, it is explained in a unified way that Fukuyama–Takahashi's counterexample and Takahashi's counterexample are all convergent to limiting distributions of the same type as Berkes.

Introduction.

When $\{n_i\}_{i=1}^{\infty}$ is a strictly increasing sequence of natural numbers, a sequence $\{\sqrt{2}\cos 2\pi n_i t\}_{i=1}^{\infty}$ of random variables on the Lebesgue probability space $([0,1),\mathcal{B}([0,1)),\mathbf{P})$ forms an orthonormal system of $L^2\big([0,1),\mathcal{B}([0,1)),\mathbf{P}\big)$, where $\mathcal{B}([0,1))$ is the Borel σ-algebra over $[0,1)$, and $\mathbf{P}$ is the Lebesgue measure restricted to $[0,1)$. So, for a real sequence $\{a_i\}_{i=1}^{\infty}$ with $\sum_{i=1}^{\infty} a_i^2 = \infty$, a sequence

$$\left\{\mathbf{P}\left(t \in [0,1); \frac{1}{A_N}\sum_{i=1}^{N} a_i \cos 2\pi n_i t \in \cdot\right)\right\}_N$$

of distributions on $\mathbb{R}$ is tight, where $A_N := \sqrt{(1/2)\sum_{i=1}^{N} a_i^2}$. Thus, as $N \to \infty$, some sort of limit theorems are expected for $(1/A_N)\sum_{i=1}^{N} a_i \cos 2\pi n_i t$. Among them, when $\{n_i\}$ and $\{a_i\}$ satisfy Erdős' lacunary condition (LC in this section only) or Takahashi's LC or Fukuyama–Takahashi's LC which is a generalization of the preceding two LCs, the central limit theorem (CLT)

$$\mathbf{P}\left(t \in [0,1); \frac{1}{A_N}\sum_{i=1}^{N} a_i \cos 2\pi n_i t \in \cdot\right) \to \text{the standard normal distribution} \quad \text{as } N \to \infty$$

holds ([**2**], [**5**], [**6**], [**3**]). Moreover, for this CLT, these LCs are best possible as well as sufficient in the sense that Erdős [**2**], Takahashi [**7**], Fukuyama–Takahashi [**3**] constructed a counterexample (CE in this section only) of $\{n_i\}$ and $\{a_i\}$ which does not satisfy their LC and for which CLT fails, respectively.

However, they only asserted that their CE is not convergent to the standard normal distribution, and mentioned nothing as to whether their CE is convergent or not. In

2010 *Mathematics Subject Classification.* Primary 60F05; Secondary 42A55.
Key Words and Phrases. Berkes' limit theorem, Lacunary condition.

addition, in the convergent case, they did not give any comment on another limit theorem. Some eight years ago from [**3**], Berkes tried to correct this defect. In [**1**], he constructed a CE which does not satisfy Erdős' LC and for which the limiting distribution is not the standard normal distribution, but another infinitely divisible distribution. Matsushita [**4**], imitating a manner of [**1**], found that Fukuyama–Takahashi's CE is convergent to a limiting distribution of the same type as Berkes. He also remarked that for Takahashi's CE, similar limit theorem to the above holds.

From these circumstantial evidences, we were convinced that there must be a general theory explaining three CEs above (i.e., Berkes', Fukuyama–Takahashi's and Takahashi's CE), and under this conviction, we investigated the argument and calculation of Berkes [**1**] and Matsushita [**4**], extracted common things from these, and arrived at some conditions for $\{n_i\}$ and $\{a_i\}$ having high versatility. In this paper, we show that Berkes type of limit theorem holds for a good many $\{n_i\}$ and $\{a_i\}$ satisfying these conditions. Since this limit theorem stems from pioneering paper [**1**] of Berkes, paying honor to him, let us call this Berkes' limit theorem.

In Section 1, Theorem and its corollary are presented. In Section 2, their proofs are given. In Section 3, it is verified that our results are applicable to Berkes', Fukuyama–Takahashi's and Takahashi's CE.

Remark 1. Erdős' CE is quite different from three other CEs. Unfortunately, this CE is not within the scope of our results. For the present, we know nothing about the limit theorem this CE satisfies.

The author would like to thank the referee for good advice which enabled him to make proofs clear and considerably short.

1. Presentation of theorem.

Let $j_0 \in \mathbb{N}$, $\{l(j)\}_{j=j_0}^{\infty}$ a sequence of natural numbers and $\{c_j\}_{j=j_0}^{\infty}$ a positive sequence such that

$$l(j) \to \infty \quad \text{as } j \to \infty, \tag{1}$$

$$B_m \nearrow \infty \quad \text{as } m \to \infty, \tag{2}$$

$$B_m \sim B_{m+1} \quad \text{as } m \to \infty, \tag{3}$$

$$\frac{B_m^2}{c_m^2} \to \frac{a}{4} \in (0, \infty) \quad \text{as } m \to \infty. \tag{4}$$

Here

$$B_m := \sqrt{\frac{1}{2} \sum_{j=j_0}^{m} \frac{c_j^2}{l(j)}} \quad (m \geq j_0). \tag{5}$$

In the following, we fix these $\{l(j)\}_{j=j_0}^{\infty}$ and $\{c_j\}_{j=j_0}^{\infty}$.

To state our theorem, we define the following:

Definition 1 (cf. Berkes [**1**]). (i) We denote by λ the Lebesgue measure on $\mathbb{R}$. Let λ_A be the restriction of λ to a Lebesgue measurable set $A \subset \mathbb{R}$. Note that $\mathbf{P} = \lambda_{[0,1)}$.
(ii) We define $F_+, F_- : (0,\infty) \to [0,\infty)$ by

$$F_+(t) := \lambda_{(0,\infty)}\Big(x; \frac{\sin x}{x} \geq t\Big), \quad F_-(t) := \lambda_{(0,\infty)}\Big(x; \frac{\sin x}{x} \leq -t\Big).$$

$F_\pm$ is continuous and non-increasing, $F_\pm(t) \leq 1/t$ ($^\forall t > 0$), $F_\pm = 0$ on $[1,\infty)$, and $F_\pm(0+) = \infty$.

Definition 2. (i) By a Lévy measure, we mean a measure ν on $\mathbb{R}\setminus\{0\}$ satisfying $\int_{\mathbb{R}\setminus\{0\}}(x^2 \wedge 1)\nu(dx) < \infty$. We denote by $\mu_{m,v,\nu}$ the infinitely divisible distribution on $\mathbb{R}$ determined by triplet $(m, v, \nu) \in \mathbb{R}\times[0,\infty)\times\{\text{Lévy measure}\}$. That is, $\mu_{m,v,\nu}$ is the probability distribution on $\mathbb{R}$ whose characteristic function $\hat{\mu_{m,v,\nu}}(\xi) = \int_{\mathbb{R}} e^{\sqrt{-1}\xi t}\mu_{m,v,\nu}(dt)$ is given as

$$\exp\left\{\sqrt{-1}m\xi - \frac{v}{2}\xi^2 + \int_{\mathbb{R}\setminus\{0\}}\Big(e^{\sqrt{-1}\xi t} - 1 - \frac{\sqrt{-1}\xi t}{1+t^2}\Big)\nu(dt)\right\}.$$

(ii) For $A > 0$, we define $\nu_A \in \{\text{Lévy measure}\}$ and $m_A \in \mathbb{R}$ by

$$\nu_A(dt) := \frac{1}{A\pi}\left(\mathbf{1}_{(0,\infty)}(t)\frac{F_+(t/2\sqrt{A})}{t} + \mathbf{1}_{(-\infty,0)}(t)\frac{F_-(|t|/2\sqrt{A})}{|t|}\right)dt, \quad t \in \mathbb{R}\setminus\{0\},$$
$$m_A := -\int_{\mathbb{R}\setminus\{0\}} \frac{t^3}{1+t^2}\nu_A(dt).$$

Theorem. *Let $b \geq 2$ be an integer and $\{q(j)\}_{j=j_0}^\infty$ a sequence of natural numbers such that*

$$b^{q(j+1)-q(j)} > l(j) \quad (^\forall j \geq j_0), \tag{6}$$
$$\lim_{m\to\infty} \frac{1}{B_m}\sum_{j=j_0}^m c_j \frac{l(j)}{b^{q(j+1)-q(j)}} = 0. \tag{7}$$

We define $N_j \in 2\mathbb{N}$ $(j \geq j_0)$ by

$$N_j := \begin{cases} b^{q(j)} & \text{if } b \in 2\mathbb{N}, \\ 2b^{q(j)} & \text{if } b \in 2\mathbb{N}-1. \end{cases} \tag{8}$$

Then, for $^\forall S \in \mathcal{B}([0,1))$ with $\mathbf{P}(S) > 0$,

$$\mathbf{P}\Bigg(\frac{1}{B_m}\sum_{j=j_0}^m \frac{c_j}{l(j)}\sum_{l=1}^{l(j)} \cos 2\pi N_j l t \in \cdot \;\Bigg|\; S\Bigg) \to \mu_{m_{1/a},0,\nu_{1/a}} \quad \textit{as } m \to \infty.$$

In other words,

$$\lim_{m\to\infty}\mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)\sum_{j=j_0}^{m}(c_j/l(j))\sum_{l=1}^{l(j)}\cos 2\pi N_j lt}\ \Big|\ S\Big]$$
$$= \exp\biggl\{\int_{\mathbb{R}\setminus\{0\}}\bigl(e^{\sqrt{-1}\xi t}-1-\sqrt{-1}\xi t\bigr)\nu_{1/a}(dt)\biggr\}, \quad {}^\forall\xi\in\mathbb{R}.$$

Here $\mathbf{P}\bigl(\ \cdot\mid S\bigr)$ *and* $\mathbf{E}\bigl[\ \cdot\mid S\bigr]$ *denote a conditional probability given* S *and a conditional expectation given* S*, respectively. That is,*

$$\mathbf{P}\bigl(\ \cdot\mid S\bigr)=\frac{\mathbf{P}(\,\cdot\cap S)}{\mathbf{P}(S)},\quad \mathbf{E}\bigl[\ \cdot\mid S\bigr]=\frac{\mathbf{E}[\,\cdot\,\mathbf{1}_S]}{\mathbf{P}(S)}.$$

In this paper, $\mathbf{E}$ *stands for the expectation with respect to* $\mathbf{P}$*.*

As mentioned in Introduction, we restate Theorem in terms of $\{n_i\}$ and $\{a_i\}$:

Corollary 1. *By a partition of* $\mathbb{N}$ *as*

$$\mathbb{N}=\bigl[1,l(j_0)\bigr]+\sum_{j=j_0}^{\infty}\bigl(l(j_0)+\cdots+l(j),l(j_0)+\cdots+l(j)+l(j+1)\bigr],$$

we define a strictly increasing sequence $\{n_i\}_{i=1}^\infty$ *of natural numbers and a positive sequence* $\{a_i\}_{i=1}^\infty$ *by*

$$n_i:=\begin{cases} N_{j_0}i & \textit{if } 1\le i\le l(j_0),\\ N_{j+1}\Bigl(i-\bigl(l(j_0)+\cdots+l(j)\bigr)\Bigr) & \textit{if } l(j_0)+\cdots+l(j)<i\le l(j_0)+\cdots\\ & \qquad\cdots+l(j)+l(j+1)\quad (j\ge j_0),\end{cases}$$

$$a_i:=\begin{cases}\dfrac{c_{j_0}}{l(j_0)} & \textit{if } 1\le i\le l(j_0),\\ \dfrac{c_{j+1}}{l(j+1)} & \textit{if } l(j_0)+\cdots+l(j)<i\le l(j_0)+\cdots+l(j)+l(j+1)\quad (j\ge j_0).\end{cases}$$

Then, for ${}^\forall S\in\mathcal{B}([0,1))$ *with* $\mathbf{P}(S)>0$,

$$\mathbf{P}\biggl(\frac{1}{A_N}\sum_{i=1}^{N}a_i\cos 2\pi n_i t\in\cdot\ \bigg|\ S\biggr)\to\mu_{m_{1/a},0,\nu_{1/a}}\quad \textit{as } N\to\infty,$$

where

$$A_N=\sqrt{\frac{1}{2}\sum_{i=1}^{N}a_i^2}.$$

2. Proof of Theorem.

We begin with the following:

Claim 1. (i) $\displaystyle\lim_{m\to\infty}\frac{1}{B_m}\max_{j_0\le j\le m} c_j = \frac{2}{\sqrt{a}}$.

(ii) $\displaystyle\lim_{m\to\infty}\frac{1}{B_m^4}\sum_{j=j_0}^{m}\Big(\frac{c_j^2}{l(j)}\Big)^2 = 0.$

Proof. (i) For $j_0 \le m_0 \le m$,

$$\frac{1}{B_m}\max_{j_0\le j\le m} c_j \le \Big(\frac{1}{B_m}\max_{j_0\le j\le m_0} c_j\Big)\vee\Big(\max_{m_0\le j\le m}\frac{c_j}{B_j}\Big).$$

Letting $m\to\infty$ yields by (2) that

$$\limsup_{m\to\infty}\frac{1}{B_m}\max_{j_0\le j\le m} c_j \le \sup_{j\ge m_0}\frac{c_j}{B_j}.$$

And, letting $m_0\to\infty$ yields by (4) that

$$\limsup_{m\to\infty}\frac{1}{B_m}\max_{j_0\le j\le m} c_j \le \lim_{j\to\infty}\frac{c_j}{B_j} = \frac{2}{\sqrt{a}}.$$

On the other hand, since

$$\frac{1}{B_m}\max_{j_0\le j\le m} c_j \ge \frac{c_m}{B_m},$$

letting $m\to\infty$ yields that

$$\liminf_{m\to\infty}\frac{1}{B_m}\max_{j_0\le j\le m} c_j \ge \lim_{m\to\infty}\frac{c_m}{B_m} = \frac{2}{\sqrt{a}},$$

which, together with the preceding, implies the convergence of (i).
(ii) First, by (5),

$$\frac{1}{B_m^4}\sum_{j=j_0}^{m}\Big(\frac{c_j^2}{l(j)}\Big)^2 \le \frac{2}{B_m^2}\max_{j_0\le j\le m}\frac{c_j^2}{l(j)} \le \Big(\frac{2}{B_m^2}\max_{j_0\le j\le m_0}\frac{c_j^2}{l(j)}\Big)\vee\Big(2\max_{m_0\le j\le m}\frac{c_j^2}{B_j^2}\frac{1}{l(j)}\Big),$$

where $j_0\le m_0\le m$. Letting $m\to\infty$ yields that

$$\limsup_{m\to\infty}\frac{1}{B_m^4}\sum_{j=j_0}^{m}\Big(\frac{c_j^2}{l(j)}\Big)^2 \le 2\sup_{j\ge m_0}\frac{c_j^2}{B_j^2}\frac{1}{l(j)}.$$

And, letting $m_0\to\infty$ yields by (1) and (4) that

$$\limsup_{m\to\infty}\frac{1}{B_m^4}\sum_{j=j_0}^{m}\Big(\frac{c_j^2}{l(j)}\Big)^2 \le 2\lim_{j\to\infty}\frac{c_j^2}{B_j^2}\frac{1}{l(j)} = 0. \qquad\square$$

To prove Theorem, we define the following:

Definition 3. Fix $b\in\mathbb{Z}_{\ge 2}$ and a sequence $\{q(j)\}_{j=j_0}^{\infty}$ of natural numbers satisfying (6) and (7). We define sequences $\{X_j\}_{j=j_0}^{\infty}$ and $\{\widetilde{X}_j\}_{j=j_0}^{\infty}$ of random variables on the Lebesgue probability space $\big([0,1),\mathcal{B}([0,1)),\mathbf{P}\big)$ by

$$X_j(t) := \sum_{l=1}^{l(j)} \cos 2\pi N_j l t, \tag{9}$$

$$\widetilde{X}_j(t) := \sum_{l=1}^{l(j)} \cos 2\pi N_j l \bigg(\sum_{q(j)<k\le q(j+1)} \frac{d^{(k)}(t)}{b^k} \bigg). \tag{10}$$

Here

$$d^{(k)}(t) := \lfloor b^k t \rfloor - b\lfloor b^{k-1} t \rfloor, \quad k \in \mathbb{N}. \tag{11}$$

Note that $|X_j(t)| \le l(j)$.

Since, by (6), $\{1\} \cup \big\{\sqrt{2}\cos 2\pi N_j l t; 1 \le l \le l(j), j \ge j_0\big\}$ form an orthonormal system of $L^2\big([0,1),\mathcal{B}([0,1)),\mathbf{P}\big)$,

$$\mathbf{E}\big[X_j\big] = 0, \tag{12}$$

$$\mathbf{E}\big[X_j^2\big] = \frac{l(j)}{2}, \tag{13}$$

$$\mathbf{E}\bigg[\Big(\sum_{j=j_0}^{m} \frac{c_j}{l(j)} X_j\Big)^2\bigg] = \frac{1}{2}\sum_{j=j_0}^{m} \frac{c_j^2}{l(j)} = B_m^2. \tag{14}$$

Also, by (10),

$$\widetilde{X}_j \text{ is } \sigma\big(d^{(k)}; q(j) < k \le q(j+1)\big)\text{-measurable.} \tag{15}$$

And the following holds:

Lemma 1. $\big|X_j(t) - \widetilde{X}_j(t)\big| \le 4\pi \dfrac{l(j)^2}{b^{q(j+1)-q(j)}}$.

Proof. First note that

$$t = \sum_{k=1}^{\infty} \frac{d^{(k)}(t)}{b^k} \quad \big(t \in [0,1)\big) \quad \big(\text{cf. (11)}\big).$$

By substituting this into (9),

$$\begin{aligned} X_j(t) &= \sum_{l=1}^{l(j)} \cos 2\pi N_j l \bigg(\sum_{1\le k\le q(j)} \frac{d^{(k)}(t)}{b^k} + \sum_{k>q(j)} \frac{d^{(k)}(t)}{b^k} \bigg) \\ &= \sum_{l=1}^{l(j)} \cos 2\pi N_j l \sum_{k>q(j)} \frac{d^{(k)}(t)}{b^k} \end{aligned}$$

because $N_j/b^k \in \mathbb{N}$ $(1 \le k \le q(j))$ by (8). By the inequality $|\cos u - \cos v| \le |u-v|$ $(u, v \in \mathbb{R})$,

$$\begin{aligned}\bigl|X_j(t) - \widetilde{X}_j(t)\bigr| &\le \sum_{l=1}^{l(j)} 2\pi N_j l \sum_{k>q(j+1)} \frac{d^{(k)}(t)}{b^k} \\ &\le 2\pi l(j)^2 N_j \frac{1}{b^{q(j+1)}} \le 4\pi \frac{l(j)^2}{b^{q(j+1)-q(j)}}. \qquad \square\end{aligned}$$

DEFINITION 4. We define a sequence $\{\mathcal{B}_n\}_{n=1}^{\infty}$ of sub σ-algebras of $\mathcal{B}([0,1))$ by

$$\mathcal{B}_n := \sigma\bigl(d^{(k)}; 1 \le k \le n\bigr). \tag{16}$$

Note that $\mathcal{B}_n \nearrow$ (i.e., $\mathcal{B}_n \subset \mathcal{B}_{n+1}$),

$$\mathcal{B}_n = \text{the } \sigma\text{-algebra generated by } \Bigl[\frac{k-1}{b^n}, \frac{k}{b^n}\Bigr), \ k = 1, \dots, b^n, \tag{17}$$

$$\sigma\Bigl(\bigcup_{n=1}^{\infty} \mathcal{B}_n\Bigr) = \mathcal{B}([0,1)). \tag{18}$$

CLAIM 2. *For* $^\forall S \in \bigcup_{n=1}^{\infty} \mathcal{B}_n$ *and* $^\forall \xi \in \mathbb{R}$,

$$\lim_{m\to\infty} \biggl| \mathbf{E}\Bigl[e^{\sqrt{-1}\xi(1/B_m)\sum_{j=j_0}^{m}(c_j/l(j))X_j}; S\Bigr] - \Bigl(\prod_{j=j_0}^{m} \mathbf{E}\bigl[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j}\bigr]\Bigr)\mathbf{P}(S)\biggr| = 0.$$

PROOF. Fix $^\forall \xi \in \mathbb{R}$. First note that

$$\{d^{(k)}\}_{k=1}^{\infty} \text{ is, under } \mathbf{P}, \text{ a sequence of i.i.d. random variables.} \tag{19}$$

Let $S \in \mathcal{B}_n$. Clearly $q(j_0 + n - 1) \ge n$ by $q(j+1) - q(j) \ge 1$ and $q(j_0) \ge 1$. Since, by this, (15) and (19), $\mathbf{1}_S$, $\widetilde{X}_{j_0+n-1}$, $\widetilde{X}_{j_0+n}$, $\dots$, $\widetilde{X}_m$ are independent, it follows that

$$\mathbf{E}\Bigl[e^{\sqrt{-1}\xi(1/B_m)\sum_{j=j_0+n-1}^{m}(c_j/l(j))\widetilde{X}_j}; S\Bigr] = \Bigl(\prod_{j=j_0+n-1}^{m} \mathbf{E}\bigl[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))\widetilde{X}_j}\bigr]\Bigr)\mathbf{P}(S).$$

By virtue of this expression,

$$\begin{aligned}&\biggl| \mathbf{E}\Bigl[e^{\sqrt{-1}\xi(1/B_m)\sum_{j=j_0}^{m}(c_j/l(j))X_j}; S\Bigr] - \Bigl(\prod_{j=j_0}^{m} \mathbf{E}\bigl[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j}\bigr]\Bigr)\mathbf{P}(S)\biggr| \\ &\quad \le I_1(m) + I_2(m) + I_3(m) + I_4(m),\end{aligned}$$

where

$$\begin{aligned}I_1(m) &= \mathbf{E}\Bigl[\bigl|e^{\sqrt{-1}\xi(1/B_m)\sum_{j=j_0}^{m}(c_j/l(j))X_j} - e^{\sqrt{-1}\xi(1/B_m)\sum_{j=j_0+n-1}^{m}(c_j/l(j))X_j}\bigr|\Bigr], \\ I_2(m) &= \mathbf{E}\Bigl[\bigl|e^{\sqrt{-1}\xi(1/B_m)\sum_{j=j_0+n-1}^{m}(c_j/l(j))X_j} - e^{\sqrt{-1}\xi(1/B_m)\sum_{j=j_0+n-1}^{m}(c_j/l(j))\widetilde{X}_j}\bigr|\Bigr],\end{aligned}$$

$$I_3(m)=\Big|\prod_{j=j_0+n-1}^{m}\mathbf{E}\big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))\widetilde{X}_j}\big]-\prod_{j=j_0+n-1}^{m}\mathbf{E}\big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j}\big]\Big|,$$
$$I_4(m)=\Big|\prod_{j=j_0+n-1}^{m}\mathbf{E}\big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j}\big]-\prod_{j=j_0}^{m}\mathbf{E}\big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j}\big]\Big|.$$

Since, by the inequalities $|e^{\sqrt{-1}u}-e^{\sqrt{-1}v}|\le|u-v|$ $(u,v\in\mathbb{R})$ and $\big|\prod_{i=1}^k z_i-\prod_{i=1}^k w_i\big|\le\sum_{i=1}^k|z_i-w_i|$ $(z_1,\dots,z_k,w_1,\dots,w_k\in\mathbb{C}$ with $|z_i|,|w_i|\le 1)$,

$$I_1(m),\ I_4(m)\le\sum_{j_0\le j<j_0+n-1}\mathbf{E}\Big[|\xi|\frac{1}{B_m}\frac{c_j}{l(j)}|X_j|\Big]\le|\xi|\frac{1}{B_m}\sum_{j_0\le j<j_0+n-1}c_j,$$
$$I_2(m),\ I_3(m)\le\sum_{j=j_0+n-1}^{m}|\xi|\frac{1}{B_m}\frac{c_j}{l(j)}\mathbf{E}\big[|X_j-\widetilde{X}_j|\big]\le 4\pi|\xi|\frac{1}{B_m}\sum_{j=j_0}^{m}c_j\frac{l(j)}{b^{q(j+1)-q(j)}},$$

the assertion of the claim follows immediately from (2) and (7). □

Claim 3. *For* $^\forall\xi\in\mathbb{R}$,

$$\lim_{m\to\infty}\prod_{j=j_0}^{m}\mathbf{E}\big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j}\big]=\exp\Big\{\int_{\mathbb{R}\setminus\{0\}}\big(e^{\sqrt{-1}\xi t}-1-\sqrt{-1}\xi t\big)\nu_{1/a}(dt)\Big\}.$$

We simplify this claim:

Lemma 2. *For* $^\forall\xi\in\mathbb{R}$, $\displaystyle\lim_{m\to\infty}\sum_{j=j_0}^{m}\Big|\mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j}-1\Big]\Big|^2=0.$

Proof. Fix $\xi\in\mathbb{R}$. By (12), (13) and the inequality $\big|e^{\sqrt{-1}u}-1-\sqrt{-1}u\big|\le u^2/2$ $(u\in\mathbb{R})$,

$$\begin{aligned}\Big|\mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j}-1\Big]\Big|&=\Big|\mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j}-1-\sqrt{-1}\xi\frac{1}{B_m}\frac{c_j}{l(j)}X_j\Big]\Big|\\&\le\mathbf{E}\Big[\frac{1}{2}\xi^2\frac{1}{B_m^2}\Big(\frac{c_j}{l(j)}\Big)^2X_j^2\Big]=\frac{\xi^2}{4}\frac{1}{B_m^2}\frac{c_j^2}{l(j)}.\end{aligned}$$

Squaring this, and then summing over $j_0\le j\le m$, we have

$$\sum_{j=j_0}^{m}\Big|\mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j}-1\Big]\Big|^2\le\Big(\frac{\xi^2}{4}\Big)^2\frac{1}{B_m^4}\sum_{j=j_0}^{m}\Big(\frac{c_j^2}{l(j)}\Big)^2,$$

which tends to 0 as $m\to\infty$ from Claim 1(ii). □

Since, by this lemma,

$$\max_{j_0\le j\le m}\Big|\mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j}-1\Big]\Big|$$

$$\leq \left(\sum_{j=j_0}^{m} \left| \mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j} - 1 \Big] \right|^2 \right)^{1/2} \to 0 \quad \text{as } m \to \infty,$$

we can take $m(\xi) \geq j_0$ such that

$$\left| \mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j} - 1 \Big] \right| \leq \frac{1}{2}, \quad j_0 \leq {}^\forall j \leq m,\ {}^\forall m \geq m(\xi).$$

Note that for $z \in \mathbb{C} \setminus (-\infty, -1]$,

$$1 + z = e^z e^{R(z)}, \tag{20}$$

$$|R(z)| \leq |z|^2 \quad \left(|z| \leq \frac{1}{2} \right), \tag{21}$$

where $R(z) = -z^2 \int_0^1 s/(1+zs)ds$. Then, for each $m \geq m(\xi)$,

$$\begin{aligned} &\prod_{j=j_0}^{m} \mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j} \Big] \\ &= \exp\left\{ \sum_{j=j_0}^{m} \mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j} - 1 \Big] \right\} \exp\left\{ \sum_{j=j_0}^{m} R\Big(\mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j} - 1 \Big] \Big) \right\}. \end{aligned}$$

Since, by Lemma 2 and (21),

$$\sum_{j=j_0}^{m} R\Big(\mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j} - 1 \Big] \Big) \to 0 \quad \text{as } m \to \infty,$$

Claim 3 reduces to the following claim:

CLAIM 4.
$$\begin{aligned} &\lim_{m\to\infty} \sum_{j=j_0}^{m} \mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j} - 1 \Big] \\ &\qquad = \int_{\mathbb{R}\setminus\{0\}} \big(e^{\sqrt{-1}\xi t} - 1 - \sqrt{-1}\xi t \big) \nu_{1/a}(dt), \quad \xi \in \mathbb{R}. \end{aligned}$$

In what follows, we show this claim.

Fix $\xi \in \mathbb{R}$. First note that

$$\begin{aligned} e^{\sqrt{-1}\xi\eta} - 1 = {}& \sqrt{-1}\xi \int_0^\infty \big(e^{\sqrt{-1}\xi c} - 1 - \sqrt{-1}\xi c \big) \mathbf{1}_{\eta \geq c} dc \\ & - \sqrt{-1}\xi \int_0^\infty \big(e^{-\sqrt{-1}\xi c} - 1 - (-\sqrt{-1}\xi c) \big) \mathbf{1}_{\eta \leq -c} dc \\ & + \sqrt{-1}\xi\eta - \frac{\xi^2}{2}\eta^2, \quad \eta \in \mathbb{R}. \end{aligned}$$

By letting $\eta = (1/B_m)(c_j/l(j))X_j$,

$$
\begin{aligned}
e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j}-1 = {} & \sqrt{-1}\xi\int_0^\infty \bigl(e^{\sqrt{-1}\xi c}-1-\sqrt{-1}\xi c\bigr)\mathbf{1}_{(1/B_m)(c_j/l(j))X_j\ge c}dc \\
& -\sqrt{-1}\xi\int_0^\infty \bigl(e^{-\sqrt{-1}\xi c}-1-(-\sqrt{-1}\xi c)\bigr)\mathbf{1}_{(1/B_m)(c_j/l(j))X_j\le -c}dc \\
& +\sqrt{-1}\xi\frac{1}{B_m}\frac{c_j}{l(j)}X_j-\frac{\xi^2}{2}\frac{1}{B_m^2}\Bigl(\frac{c_j}{l(j)}\Bigr)^2X_j^2.
\end{aligned}
$$

Here, by noting that

$$
\Bigl|\frac{1}{B_m}\frac{c_j}{l(j)}X_j\Bigr| = \frac{1}{B_m}\frac{c_j}{l(j)}\bigl|X_j\bigr| \le \frac{c_j}{B_m} \le \frac{1}{B_m}\max_{j_0\le j\le m} c_j,
$$

the expression above is turned into

$$
\begin{aligned}
& e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j}-1 \\
& = \sqrt{-1}\xi\int_0^{(1/B_m)\max_{j_0\le j\le m}c_j} \bigl(e^{\sqrt{-1}\xi c}-1-\sqrt{-1}\xi c\bigr)\mathbf{1}_{(1/B_m)(c_j/l(j))X_j\ge c}dc \\
& \quad -\sqrt{-1}\xi\int_0^{(1/B_m)\max_{j_0\le j\le m}c_j} \bigl(e^{-\sqrt{-1}\xi c}-1-(-\sqrt{-1}\xi c)\bigr)\mathbf{1}_{(1/B_m)(c_j/l(j))X_j\le -c}dc \\
& \quad +\sqrt{-1}\xi\frac{1}{B_m}\frac{c_j}{l(j)}X_j-\frac{\xi^2}{2}\frac{1}{B_m^2}\Bigl(\frac{c_j}{l(j)}\Bigr)^2X_j^2.
\end{aligned}
$$

Taking expectation, and then summing over $j_0\le j\le m$ yield by (12) and (13) that

$$
\sum_{j=j_0}^m \mathbf{E}\Bigl[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j}-1\Bigr] = I_5^+(m)+I_5^-(m)+I_6^+(m)+I_6^-(m)-\frac{\xi^2}{2}, \tag{22}
$$

where

$$
\begin{aligned}
& I_5^\pm(m) \\
& = \pm\sqrt{-1}\xi\int_{2/\sqrt{a}}^{(1/B_m)\max_{j_0\le j\le m}c_j}\bigl(e^{\pm\sqrt{-1}\xi c}-1-(\pm\sqrt{-1}\xi c)\bigr)\sum_{j=j_0}^m\mathbf{P}\Bigl(\pm X_j\ge cB_m\frac{l(j)}{c_j}\Bigr)dc, \\
& I_6^\pm(m) \\
& = \pm\sqrt{-1}\xi\int_0^{2/\sqrt{a}}\bigl(e^{\pm\sqrt{-1}\xi c}-1-(\pm\sqrt{-1}\xi c)\bigr)\sum_{j=j_0}^m\mathbf{P}\Bigl(\pm X_j\ge cB_m\frac{l(j)}{c_j}\Bigr)dc.
\end{aligned}
$$

Lemma 3. $\displaystyle \Bigl|\bigl(e^{\pm\sqrt{-1}\xi c}-1-(\pm\sqrt{-1}\xi c)\bigr)\sum_{j=j_0}^m\mathbf{P}\Bigl(\pm X_j\ge cB_m\frac{l(j)}{c_j}\Bigr)\Bigr| \le \frac{\xi^2}{2}, \quad {}^\forall c>0.$

Proof. By Chebyshev's inequality,

$$
\text{the left-hand side} \le \frac{1}{2}\xi^2c^2\sum_{j=j_0}^m\mathbf{P}\Bigl(\bigl|X_j\bigr|\ge cB_m\frac{l(j)}{c_j}\Bigr)
$$

$$\leq \frac{1}{2}\xi^2 c^2 \sum_{j=j_0}^{m} \mathbf{E}\Big[\Big(\frac{|X_j|}{cB_m l(j)/c_j}\Big)^2\Big] = \frac{\xi^2}{2}. \qquad \square$$

CLAIM 5. *For* $0 < {}^\forall c < 2/\sqrt{a}$,

$$\lim_{m\to\infty} \sum_{j=j_0}^{m} \mathbf{P}\Big(\pm X_j \geq cB_m \frac{l(j)}{c_j}\Big) = \frac{a}{\pi}\int_{c\sqrt{a}/2}^{1} \frac{F_\pm(s)}{s} ds.$$

Recognizing this claim as correct, we here give the proof of Claim 4:

PROOF OF CLAIM 4. First we check the convergence of each term of the right-hand side of (22).

Since, by Claim 1(i) and Lemma 3,

$$\begin{aligned}
&\Big|\int_{2/\sqrt{a}}^{(1/B_m)\max_{j_0\leq j\leq m} c_j} \big(e^{\pm\sqrt{-1}\xi c} - 1 - (\pm\sqrt{-1}\xi c)\big) \sum_{j=j_0}^{m} \mathbf{P}\Big(\pm X_j \geq cB_m \frac{l(j)}{c_j}\Big) dc\Big| \\
&\leq \int_{(2/\sqrt{a})\wedge((1/B_m)\max_{j_0\leq j\leq m} c_j)}^{(2/\sqrt{a})\vee((1/B_m)\max_{j_0\leq j\leq m} c_j)} \Big|\big(e^{\pm\sqrt{-1}\xi c} - 1 - (\pm\sqrt{-1}\xi c)\big) \sum_{j=j_0}^{m} \mathbf{P}\Big(\pm X_j \geq cB_m \frac{l(j)}{c_j}\Big)\Big| dc \\
&\leq \Big|\frac{1}{B_m}\max_{j_0\leq j\leq m} c_j - \frac{2}{\sqrt{a}}\Big| \frac{\xi^2}{2} \to 0 \quad \text{as } m\to\infty,
\end{aligned}$$

it follows that $I_5^\pm(m) \to 0$. Since, by Claim 5, Lemma 3 and the Lebesgue convergence theorem,

$$\begin{aligned}
&\int_0^{2/\sqrt{a}} \big(e^{\pm\sqrt{-1}\xi c} - 1 - (\pm\sqrt{-1}\xi c)\big) \sum_{j=j_0}^{m} \mathbf{P}\Big(\pm X_j \geq cB_m \frac{l(j)}{c_j}\Big) dc \\
&\quad \to \int_0^{2/\sqrt{a}} \big(e^{\pm\sqrt{-1}\xi c} - 1 - (\pm\sqrt{-1}\xi c)\big) dc \frac{a}{\pi}\int_{c\sqrt{a}/2}^{1} \frac{F_\pm(s)}{s} ds \quad \text{as } m\to\infty,
\end{aligned}$$

it follows that

$$I_6^\pm(m) \to \pm\sqrt{-1}\xi \int_0^{2/\sqrt{a}} \big(e^{\pm\sqrt{-1}\xi c} - 1 - (\pm\sqrt{-1}\xi c)\big) dc \frac{a}{\pi}\int_{c\sqrt{a}/2}^{1} \frac{F_\pm(s)}{s} ds.$$

Thus, combining them, we obtain that

$$\begin{aligned}
&\lim_{m\to\infty} \sum_{j=j_0}^{m} \mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)(c_j/l(j))X_j} - 1\Big] \\
&= -\frac{\xi^2}{2} + \sum_{h\in\{+,-\}} h\sqrt{-1}\xi \int_0^{2/\sqrt{a}} \big(e^{h\sqrt{-1}\xi c} - 1 - h\sqrt{-1}\xi c\big) dc \frac{a}{\pi}\int_{c\sqrt{a}/2}^{1} \frac{F_h(s)}{s} ds. \quad (23)
\end{aligned}$$

Next, we compute the right-hand side above. To this end, note that

$$\int_0^1 tdt \int_t^1 \frac{F_\pm(s)}{s} ds \le \int_0^1 tdt \int_t^1 \frac{1}{s^2} ds = \frac{1}{2} < \infty,$$

$$\begin{aligned}\int_0^1 \big(sF_+(s) + sF_-(s)\big) ds &= \int_0^1 sds \Big(\int_0^\infty \mathbf{1}_{(\sin x)/x \ge s} dx + \int_0^\infty \mathbf{1}_{-(\sin x)/x \ge s} dx \Big) \\ &= \int_0^1 sds \int_0^\infty \mathbf{1}_{|(\sin x)/x| \ge s} dx \\ &= \int_0^\infty dx \int_0^{1 \wedge |(\sin x)/x|} sds = \frac{1}{2} \int_0^\infty \Big(\frac{\sin x}{x} \Big)^2 dx = \frac{\pi}{4}.\end{aligned}$$

Then, by the change of variables $t = (\sqrt{a}/2)c$,

$$\begin{aligned}&\text{the right-hand side of (23)} \\ &\quad = -\frac{\xi^2}{2} + \sum_{h \in \{+,-\}} \frac{a}{\pi} \int_0^1 h\sqrt{-1}\zeta \big(e^{h\sqrt{-1}\zeta t} - 1 - h\sqrt{-1}\zeta t \big) dt \int_t^1 \frac{F_h(s)}{s} ds,\end{aligned}$$

where $\xi 2/\sqrt{a} =: \zeta$ for simplicity. And, by Fubini's theorem,

$$\begin{aligned}&\text{the last expression} \\ &= -\frac{\xi^2}{2} + \sum_{h \in \{+,-\}} \frac{a}{\pi} \int_0^1 \frac{F_h(s)}{s} \Big(e^{h\sqrt{-1}\zeta s} - 1 - h\sqrt{-1}\zeta s + \frac{\zeta^2}{2} s^2 \Big) ds \\ &= -\frac{\xi^2}{2} + \frac{\zeta^2}{2} \frac{a}{\pi} \int_0^1 \big(sF_+(s) + sF_-(s) \big) ds \\ &\quad + \sum_{h \in \{+,-\}} \frac{a}{\pi} \int_0^1 \big(e^{h\sqrt{-1}\zeta s} - 1 - h\sqrt{-1}\zeta s \big) \frac{F_h(s)}{s} ds \\ &= \int_{\mathbb{R}\setminus\{0\}} \Big(e^{\sqrt{-1}\xi(2/\sqrt{a})s} - 1 - \sqrt{-1}\xi \frac{2}{\sqrt{a}} s \Big) \frac{a}{\pi} \Big(\mathbf{1}_{(0,1)}(s) \frac{F_+(s)}{s} + \mathbf{1}_{(-1,0)}(s) \frac{F_-(|s|)}{|s|} \Big) ds\end{aligned}$$

because $\zeta = \xi 2/\sqrt{a}$. Finally, by the change of variables $(2/\sqrt{a})s = t$,

$$\text{the last expression} = \int_{\mathbb{R}\setminus\{0\}} \big(e^{\sqrt{-1}\xi t} - 1 - \sqrt{-1}\xi t \big) \nu_{1/a}(dt),$$

which shows the assertion of Claim 4. □

PROOF OF CLAIM 5. It is divided into 3 steps.

$\underline{1^\circ}$ Fix $0 < f < 1$. Set $j_m(f) := \min\big\{ j \in \{j_0, \dots, m\}; B_m/B_j \le 1/f \big\}$. For $m \gg 1$, $j_m(f) > j_0$, $B_m/B_{j_m(f)} \le 1/f < B_m/B_{j_m(f)-1}$, and thus $j_m(f) \to \infty$ as $m \to \infty$. Since $B_m/B_j > 1/f$ $(j_0 \le j < j_m(f))$, so $\mathbf{P}\big(\pm X_j \ge B_m/B_j l(j) f \big) = \mathbf{P}(\emptyset) = 0$, it is seen that

$$\sum_{j=j_0}^m \mathbf{P}\Big(\pm X_j \ge \frac{B_m}{B_j} l(j) f \Big) = \sum_{j_m(f) \le j \le m} \mathbf{P}\Big(\pm X_j \ge \frac{B_m}{B_j} l(j) f \Big). \tag{24}$$

Set

$$t_{m,j,\pm} := 2\frac{B_m}{B_j}l(j)f \pm 1 \quad (j_0 \le j \le m). \tag{25}$$

Note that $t_{m,j,\pm} > 1 \ \ (j_m(f) \le j \le m)$ for $m \gg 1$ because $t_{m,j,\pm} \ge 2l(j)f \pm 1 \to \infty$ as $j \to \infty$. Since, by (9), $X_j(t) = \sum_{l=1}^{l(j)} \cos 2\pi l N_j t = g_j(N_j t)$, where $g_j(\cdot)$ is periodic with period 1, and since, by (8), $N_j/2 \in \mathbb{N}$ and the transformation $[0,1) \ni t \mapsto \{nt\} \in [0,1)$ is measure preserving for each $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbf{P}\Big(\pm X_j \ge \frac{B_m}{B_j}l(j)f\Big) &= \mathbf{P}\Big(\pm g_j(N_j t) \ge \frac{B_m}{B_j}l(j)f\Big)\\
&= \mathbf{P}\Big(\pm g_j(2y) \ge \frac{B_m}{B_j}l(j)f\Big)\\
&= \frac{1}{2\pi}\lambda_{(0,2\pi)}\Big(\pm g_j\Big(\frac{x}{\pi}\Big) \ge \frac{B_m}{B_j}l(j)f\Big)\\
&= \frac{1}{2\pi}\lambda_{(0,2\pi)}\Big(\pm\Big(\frac{\sin(2l(j)+1)x}{2\sin x} - \frac{1}{2}\Big) \ge \frac{B_m}{B_j}l(j)f\Big)\\
&= \frac{1}{2\pi}\lambda_{(0,2\pi)}\Big(\pm\frac{\sin(2l(j)+1)x}{\sin x} \ge t_{m,j,\pm}\Big).
\end{aligned}$$

By Lemma 1 in Berkes [**1**], there exists $\Theta_{m,j,\pm} \in [0,1]$ such that

$$\text{the last expression} = \frac{2}{\pi}\frac{1}{2l(j)+1}F_\pm\Big(\frac{t_{m,j,\pm}}{2l(j)+1}\Big(1 - \frac{\Theta_{m,j,\pm}}{t_{m,j,\pm}^2}\Big)\Big),$$

so that

$$\begin{aligned}
&\sum_{j_m(f)\le j\le m} \mathbf{P}\Big(\pm X_j \ge \frac{B_m}{B_j}l(j)f\Big)\\
&\quad = \frac{2}{\pi}\sum_{j=j_m(f)}^{m} \frac{1}{2l(j)+1}F_\pm\Big(\frac{t_{m,j,\pm}}{2l(j)+1}\Big(1 - \frac{\Theta_{m,j,\pm}}{t_{m,j,\pm}^2}\Big)\Big).
\end{aligned} \tag{26}$$

Here noting that for $j_m(f) \le j \le m$,

$$\begin{aligned}
&f \le \frac{B_m}{B_j}f \le 1,\\
&\Big|\frac{B_m}{B_j}f - \frac{t_{m,j,\pm}}{2l(j)+1}\Big| = \frac{1}{2l(j)+1}\Big|\frac{B_m}{B_j}f \mp 1\Big| \le \frac{2}{2l(j)+1},\\
&\Big|\frac{B_m}{B_j}f - \frac{t_{m,j,\pm}}{2l(j)+1}\Big(1 - \frac{\Theta_{m,j,\pm}}{t_{m,j,\pm}^2}\Big)\Big| \le \Big|\frac{B_m}{B_j}f - \frac{t_{m,j,\pm}}{2l(j)+1}\Big| + \frac{1}{2l(j)+1} \le \frac{3}{2l(j)+1},
\end{aligned}$$

we obtain that for $m \gg 1$,

$$\Big|\sum_{j=j_m(f)}^{m} \frac{1}{2l(j)+1}F_\pm\Big(\frac{t_{m,j,\pm}}{2l(j)+1}\Big(1 - \frac{\Theta_{m,j,\pm}}{t_{m,j,\pm}^2}\Big)\Big) - \sum_{j=j_m(f)}^{m} \frac{1}{2l(j)+1}F_\pm\Big(\frac{B_m}{B_j}f\Big)\Big|$$

$$
\begin{aligned}
&\le \sum_{j=j_m(f)}^{m} \frac{1}{2l(j)+1}\left|F_\pm\Big(\frac{t_{m,j,\pm}}{2l(j)+1}\Big(1-\frac{\Theta_{m,j,\pm}}{t_{m,j,\pm}^2}\Big)\Big) - F_\pm\Big(\frac{B_m}{B_j}f\Big)\right| \\
&\le \Big(\sum_{j=j_m(f)}^{m} \frac{1}{2l(j)+1}\Big) \sup\left\{\big|F_\pm(u)-F_\pm(v)\big|;\ \begin{matrix} f\le v\le 1, \\ |u-v| \le \sup\limits_{j\ge j_m(f)} 3/(2l(j)+1)\end{matrix}\right\}.
\end{aligned}
\tag{27}
$$

Letting $m\to\infty$ yields by the continuity of $F_\pm$ and (1) that

$$
\lim_{m\to\infty} \sup\left\{\big|F_\pm(u)-F_\pm(v)\big|;\ \begin{matrix} f\le v\le 1, \\ |u-v| \le \sup\limits_{j\ge j_m(f)} 3/(2l(j)+1)\end{matrix}\right\} = 0. \tag{28}
$$

Also

$$
\limsup_{m\to\infty} \sum_{j=j_m(f)}^{m} \frac{1}{2l(j)+1} < \infty. \tag{29}
$$

For, if, for simplicity, we set

$$
x_j^{(m)} := \frac{B_m^2}{B_j^2} \quad (j_0 \le j \le m), \tag{30}
$$

then

$$
\begin{aligned}
&1 = x_m^{(m)} < x_{m-1}^{(m)} < \cdots < x_{j_m(f)}^{(m)} \le \frac{1}{f^2} < x_{j_m(f)-1}^{(m)}, \\
&x_{j-1}^{(m)} - x_j^{(m)} \ge \frac{x_{j-1}^{(m)} - x_j^{(m)}}{x_{j-1}^{(m)}} = \frac{c_j^2}{B_j^2}\frac{1}{2l(j)} > \frac{c_j^2}{B_j^2}\frac{1}{2l(j)+1} \quad (j_0 < j \le m),
\end{aligned}
$$

so that

$$
\begin{aligned}
\sum_{j=j_m(f)}^{m} \frac{1}{2l(j)+1} &< \sum_{j=j_m(f)}^{m} \frac{B_j^2}{c_j^2}\big(x_{j-1}^{(m)} - x_j^{(m)}\big) \\
&\le \Big(\sup_{j\ge j_0} \frac{B_j^2}{c_j^2}\Big)\big(x_{j_m(f)-1}^{(m)} - x_m^{(m)}\big) \\
&\le \Big(\sup_{j\ge j_0} \frac{B_j^2}{c_j^2}\Big)\Big(x_{j_m(f)-1}^{(m)} - x_{j_m(f)}^{(m)} + \frac{1}{f^2} - 1\Big).
\end{aligned}
$$

Since, by (3),

$$
\begin{aligned}
\max_{j_m(f)\le j\le m} \big(x_{j-1}^{(m)} - x_j^{(m)}\big) &= \max_{j_m(f)\le j\le m} \Big(\frac{B_j^2}{B_{j-1}^2} - 1\Big)x_j^{(m)} \\
&\le \frac{1}{f^2} \sup_{j\ge j_m(f)} \Big(\frac{B_j^2}{B_{j-1}^2} - 1\Big) \to 0 \quad \text{as } m\to\infty,
\end{aligned}
\tag{31}
$$

it follows that

$$\limsup_{m\to\infty} \sum_{j=j_m(f)}^{m} \frac{1}{2l(j)+1} \le \Big(\sup_{j\ge j_0} \frac{B_j^2}{c_j^2}\Big)\Big(\frac{1}{f^2}-1\Big) < \infty.$$

Therefore, by combining (28), (29) and (27),

$$\lim_{m\to\infty}\Big| \sum_{j=j_m(f)}^{m} \frac{1}{2l(j)+1} F_{\pm}\Big(\frac{t_{m,j,\pm}}{2l(j)+1}\Big(1-\frac{\Theta_{m,j,\pm}}{t^2_{m,j,\pm}}\Big)\Big) - \sum_{j=j_m(f)}^{m} \frac{1}{2l(j)+1} F_{\pm}\Big(\frac{B_m}{B_j} f\Big)\Big| = 0,$$

which, together with (26), implies that

$$\lim_{m\to\infty}\Big| \sum_{j=j_m(f)}^{m} \mathbf{P}\Big(\pm X_j \ge \frac{B_m}{B_j} l(j) f\Big) - \frac{2}{\pi} \sum_{j=j_m(f)}^{m} \frac{1}{2l(j)+1} F_{\pm}\Big(\frac{B_m}{B_j} f\Big)\Big| = 0.$$

<u>2°</u> For simplicity, set $\varphi_{\pm}(x) := F_{\pm}(\sqrt{x} f)$ $(x>0)$. $\varphi_{\pm} : (0,\infty) \to [0,\infty)$ is continuous. First

$$\begin{aligned}
&\sum_{j=j_m(f)}^{m} \frac{1}{2l(j)+1} F_{\pm}\Big(\frac{B_m}{B_j} f\Big) \\
&= \sum_{j=j_m(f)}^{m} \frac{1}{1+(1/2l(j))} \frac{B_j^2}{c_j^2} \frac{B_{j-1}^2}{B_j^2} \frac{\varphi_{\pm}(x_j^{(m)})}{x_j^{(m)}} \big(x_{j-1}^{(m)} - x_j^{(m)}\big)
\end{aligned}$$

$$\begin{cases}
\le \Big(\sup\limits_{j\ge j_m(f)} \dfrac{B_j^2}{c_j^2}\Big) \sum\limits_{j=j_m(f)}^{m} \dfrac{\varphi_{\pm}(x_j^{(m)})}{x_j^{(m)}} \big(x_{j-1}^{(m)} - x_j^{(m)}\big), \\[2ex]
\ge \Big(1+\dfrac{1}{2\inf\limits_{j\ge j_m(f)} l(j)}\Big)^{-1} \Big(\inf\limits_{j\ge j_m(f)} \dfrac{B_j^2}{c_j^2}\Big)\Big(\inf\limits_{j\ge j_m(f)} \Big(\dfrac{B_{j-1}}{B_j}\Big)^2\Big) \\[2ex]
\qquad \times \sum\limits_{j=j_m(f)}^{m} \dfrac{\varphi_{\pm}(x_j^{(m)})}{x_j^{(m)}} \big(x_{j-1}^{(m)} - x_j^{(m)}\big).
\end{cases}$$

Next

$$\begin{aligned}
&\Big| \sum_{j=j_m(f)}^{m} \frac{\varphi_{\pm}(x_j^{(m)})}{x_j^{(m)}} \big(x_{j-1}^{(m)} - x_j^{(m)}\big) - \int_1^{1/f^2} \frac{\varphi_{\pm}(x)}{x} dx \Big| \\
&= \Big| \sum_{j=j_m(f)+1}^{m} \int_{x_j^{(m)}}^{x_{j-1}^{(m)}} \Big(\frac{\varphi_{\pm}(x_j^{(m)})}{x_j^{(m)}} - \frac{\varphi_{\pm}(x)}{x}\Big) dx
\end{aligned}$$

$$+ \int_{x_{j_m(f)}^{(m)}}^{1/f^2} \Big(\frac{\varphi_\pm(x_{j_m(f)}^{(m)})}{x_{j_m(f)}^{(m)}} - \frac{\varphi_\pm(x)}{x} \Big) dx + \int_{1/f^2}^{x_{j_m(f)-1}^{(m)}} \frac{\varphi_\pm(x_{j_m(f)}^{(m)})}{x_{j_m(f)}^{(m)}} dx \Big|$$

$$\le \Big(\frac{1}{f^2} - 1 \Big) \sup \left\{ \Big| \frac{\varphi_\pm(u)}{u} - \frac{\varphi_\pm(v)}{v} \Big| ; \begin{array}{l} 1 \le u, v \le 1/f^2, \\ |u - v| \le \max\limits_{j_m(f) \le j \le m} \big(x_{j-1}^{(m)} - x_j^{(m)} \big) \end{array} \right\}$$

$$+ \big(x_{j_m(f)-1}^{(m)} - x_{j_m(f)}^{(m)} \big) \sup \Big\{ \frac{\varphi_\pm(u)}{u} ; 1 \le u \le \frac{1}{f^2} \Big\}.$$

Letting $m \to \infty$ yields by (31) and the continuity of $\varphi_\pm(\cdot)$ that

$$\lim_{m \to \infty} \sum_{j=j_m(f)}^{m} \frac{\varphi_\pm(x_j^{(m)})}{x_j^{(m)}} \big(x_{j-1}^{(m)} - x_j^{(m)} \big) = \int_1^{1/f^2} \frac{\varphi_\pm(x)}{x} dx.$$

This, together with (1), (3) and (4), implies that

$$\lim_{m \to \infty} \sum_{j=j_m(f)}^{m} \frac{1}{2l(j)+1} F_\pm \Big(\frac{B_m}{B_j} f \Big) = \frac{a}{4} \int_1^{1/f^2} \frac{\varphi_\pm(x)}{x} dx = \frac{a}{2} \int_f^1 \frac{F_\pm(s)}{s} ds.$$

<u>3°</u> First, by 1° and 2°,

$$\lim_{m \to \infty} \sum_{j=j_0}^{m} \mathbf{P} \Big(\pm X_j \ge \frac{B_m}{B_j} l(j) f \Big) = \frac{a}{\pi} \int_f^1 \frac{F_\pm(s)}{s} ds, \quad 0 < f < 1. \tag{32}$$

Fix $0 < {}^\forall c < 2/\sqrt{a}$. By (4), $\lim\limits_{j \to \infty} cB_j/c_j = c\sqrt{a}/2 \in (0,1)$, so that for $0 < {}^\forall \varepsilon < \big(c\sqrt{a}/2 \big)^{-1} - 1$,

$${}^\exists j_1 = j_1(c, \varepsilon) \ge j_0 \ \text{ s.t. } \ \frac{1}{1+\varepsilon} c \frac{\sqrt{a}}{2} < c \frac{B_j}{c_j} < (1+\varepsilon) c \frac{\sqrt{a}}{2} \quad ({}^\forall j \ge j_1).$$

Then it is easily seen that

$$cB_m \frac{l(j)}{c_j} = \frac{B_m}{B_j} l(j) c \frac{B_j}{c_j} \begin{cases} > \dfrac{1}{1+\varepsilon} c \dfrac{\sqrt{a}}{2} \dfrac{B_m}{B_j} l(j), \\[2ex] < (1+\varepsilon) c \dfrac{\sqrt{a}}{2} \dfrac{B_m}{B_j} l(j) \end{cases} \quad ({}^\forall j \ge j_1).$$

From this, it follows that

$$\sum_{j=j_0}^{m} \mathbf{P} \Big(\pm X_j \ge cB_m \frac{l(j)}{c_j} \Big)$$
$$= \sum_{j_0 \le j < j_1} \mathbf{P} \Big(\pm X_j \ge cB_m \frac{l(j)}{c_j} \Big) + \sum_{j=j_1}^{m} \mathbf{P} \Big(\pm X_j \ge cB_m \frac{l(j)}{c_j} \Big)$$

$$\begin{cases} \leq \displaystyle\sum_{j_0 \leq j < j_1} \mathbf{P}\Big(\pm X_j \geq cB_m \frac{l(j)}{c_j}\Big) + \sum_{j=j_0}^{m} \mathbf{P}\Big(\pm X_j \geq \frac{1}{1+\varepsilon} c \frac{\sqrt{a}}{2} \frac{B_m}{B_j} l(j)\Big), \\ \geq - \displaystyle\sum_{j_0 \leq j < j_1} \mathbf{P}\Big(\pm X_j \geq (1+\varepsilon) c \frac{\sqrt{a}}{2} \frac{B_m}{B_j} l(j)\Big) + \sum_{j=j_0}^{m} \mathbf{P}\Big(\pm X_j \geq (1+\varepsilon) c \frac{\sqrt{a}}{2} \frac{B_m}{B_j} l(j)\Big). \end{cases}$$

Here, since $B_m \to \infty$ as $m \to \infty$,

$$\lim_{m \to \infty} \sum_{j_0 \leq j < j_1} \mathbf{P}\Big(\pm X_j \geq cB_m \frac{l(j)}{c_j}\Big) = 0,$$
$$\lim_{m \to \infty} \sum_{j_0 \leq j < j_1} \mathbf{P}\Big(\pm X_j \geq (1+\varepsilon) c \frac{\sqrt{a}}{2} \frac{B_m}{B_j} l(j)\Big) = 0,$$

which, together with (32), implies that

$$\limsup_{m \to \infty} \sum_{j=j_0}^{m} \mathbf{P}\Big(\pm X_j \geq cB_m \frac{l(j)}{c_j}\Big) \leq \frac{a}{\pi} \int_{(1/(1+\varepsilon))c\sqrt{a}/2}^{1} \frac{F_\pm(s)}{s} ds,$$
$$\liminf_{m \to \infty} \sum_{j=j_0}^{m} \mathbf{P}\Big(\pm X_j \geq cB_m \frac{l(j)}{c_j}\Big) \geq \frac{a}{\pi} \int_{(1+\varepsilon)c\sqrt{a}/2}^{1} \frac{F_\pm(s)}{s} ds.$$

Finally, by letting $\varepsilon \searrow 0$, the assertion of Claim 5 follows. □

Proof of Theorem. For simplicity, set

$$\varphi(\xi) := \exp\Big\{\int_{\mathbb{R}\setminus\{0\}} \big(e^{\sqrt{-1}\xi t} - 1 - \sqrt{-1}\xi t\big)\nu_{1/a}(dt)\Big\}, \quad \xi \in \mathbb{R}.$$

It suffices to verify that

$$\lim_{m \to \infty} \mathbf{E}\Big[e^{\sqrt{-1}\xi(1/B_m)\sum_{j=j_0}^{m}(c_j/l(j))X_j}; S\Big] = \mathbf{P}(S)\varphi(\xi), \quad \xi \in \mathbb{R}, \quad S \in \mathcal{B}([0,1)).$$

First, the collection $\mathcal{M}$ of all sets S in $\mathcal{B}([0,1))$ satisfying the above is a monotone class over $[0,1)$. Next, $\bigcup_{n=1}^{\infty} \mathcal{B}_n$ is an algebra on $[0,1)$ and generates $\mathcal{B}([0,1))$ (cf. (18)), and is contained in $\mathcal{M}$ by Claims 2 and 3. Thus, by virtue of the monotone class theorem, $\mathcal{M} = \mathcal{B}([0,1))$, which is the desired result. □

Proof of Corollary 1. For simplicity, set $L(k) := l(j_0) + \cdots + l(k)$ $(k \geq j_0)$. First note that for $m \geq j_0$,

$$\sum_{i=1}^{l(j_0)+\cdots+l(m)} a_i \cos 2\pi n_i t = \sum_{j=j_0}^{m} \frac{c_j}{l(j)} \sum_{l=1}^{l(j)} \cos 2\pi N_j l t, \quad A_{L(m)}^2 = B_m^2.$$

For ${}^\forall N > l(j_0)$,

$${}^{\exists 1} m = m_N \geq j_0 \quad \text{s.t. } L(m) < N \leq L(m+1).$$

Then, by the preceding remark,

$$\mathbf{E}\Big[\Big(\frac{1}{A_N}\sum_{i=1}^{N} a_i \cos 2\pi n_i t - \frac{1}{B_m}\sum_{j=j_0}^{m} \frac{c_j}{l(j)} \sum_{l=1}^{l(j)} \cos 2\pi N_j l t\Big)^2\Big] = 2\Big(1 - \frac{B_m}{A_N}\Big).$$

Here

$$\begin{array}{ccccc} A_{L(m)} & < & A_N & \le & A_{L(m+1)} \\ \| & & & & \| \\ B_m & & & & B_{m+1} \end{array}$$

and thus $1 \le A_N/B_{m_N} \le B_{m_N+1}/B_{m_N}$. Since $\lim_{N\to\infty} m_N = \infty$, $\lim_{N\to\infty} A_N/B_{m_N} = 1$ by (3). This implies that

$$\lim_{N\to\infty}\mathbf{E}\Big[\Big(\frac{1}{A_N}\sum_{i=1}^{N} a_i \cos 2\pi n_i t - \frac{1}{B_{m_N}}\sum_{j=j_0}^{m_N} \frac{c_j}{l(j)} \sum_{l=1}^{l(j)} \cos 2\pi N_j l t\Big)^2\Big] = 0.$$

Now let $S \in \mathcal{B}([0,1))$ with $\mathbf{P}(S) > 0$. From the convergence above, it follows that

$$\frac{1}{A_N}\sum_{i=1}^{N} a_i \cos 2\pi n_i t - \frac{1}{B_{m_N}}\sum_{j=j_0}^{m_N} \frac{c_j}{l(j)} \sum_{l=1}^{l(j)} \cos 2\pi N_j l t \to 0 \quad \text{in } \mathbf{P}\big(\,\cdot \mid S\big) \quad \text{as } N \to \infty.$$

On the other hand, Theorem tells us that

$$\mathbf{P}\Bigg(\frac{1}{B_{m_N}}\sum_{j=j_0}^{m_N} \frac{c_j}{l(j)} \sum_{l=1}^{l(j)} \cos 2\pi N_j l t \in \cdot \;\Bigg|\; S\Bigg) \to \mu_{m_{1/a},0,\nu_{1/a}} \quad \text{as } N \to \infty.$$

Therefore $(1/A_N)\sum_{i=1}^{N} a_i \cos 2\pi n_i t$ is convergent in law to the same limit. □

3. Application examples of Theorem.

3.1. Berkes' counterexample.

Fix $A > 0$. Let $j_0 := \lceil 1/A \rceil \in \mathbb{N}$, and for $j \ge j_0$ set $l(j) := \lfloor Aj \rfloor \in \mathbb{N}$, $q(j) := aj^2 + 1 \in \mathbb{N}$. Here $a \in \mathbb{N}$ is taken so as to satisfy

$$\frac{4\pi l(j)^2}{2^{q(j+1)-q(j)}} \le \frac{1}{2^j} \quad (j \ge j_0). \tag{33}$$

And a strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ of natural numbers is defined in such a way that

$$\bigcup_{j=j_0}^{\infty} \big\{lN_j; 1 \le l \le l(j)\big\} = \big\{n_i; i = 1, 2, \dots\big\}.$$

Then

THEOREM 1 (cf. Berkes [**1**]). (i) $\liminf_{i\to\infty} \sqrt{i}\Big(\frac{n_{i+1}}{n_i} - 1\Big) = \frac{1}{\sqrt{2A}}$.
(ii) *For* ${}^\forall S \in \mathcal{B}([0,1))$ *with* $\mathbf{P}(S) > 0$,

$$\mathbf{P}\bigg(\sqrt{\frac{2}{N}}\sum_{i=1}^{N}\cos 2\pi n_i t \in \cdot \ \bigg|\ S\bigg) \to \mu_{m_A,0,\nu_A} \quad \textit{as } N \to \infty.$$

PROOF. We verify (ii) only. Since $l(j) \sim Aj$ as $j \to \infty$, (1) is fine. Let $c_j = l(j)$. Then

$$\begin{aligned} 2B_m^2 = \sum_{j=j_0}^{m} l(j) = \sum_{j=j_0}^{m} \big(Aj + O(1)\big) &= \frac{A}{2}\big(m(m+1) - j_0(j_0-1)\big) + O(m) \\ &\sim \frac{A}{2}m^2 \quad \text{as } m \to \infty, \end{aligned}$$

so (2) and (3) are fine. Since

$$\frac{B_m^2}{c_m^2} = \frac{B_m^2}{l(m)^2} \sim \frac{(A/4)m^2}{(Am)^2} = \frac{1}{4A} \quad \text{as } m \to \infty,$$

(4) holds with $a = 1/A$.

Since, by (33),

$$2^{q(j+1)-q(j)} \geq 4\pi l(j)^2 2^j = 4\pi \cdot 2^j \cdot l(j) \cdot l(j) > l(j),$$

$$\begin{aligned} \frac{1}{B_m}\sum_{j=j_0}^{m} c_j \frac{l(j)}{2^{q(j+1)-q(j)}} = \frac{1}{B_m}\sum_{j=j_0}^{m} \frac{l(j)^2}{2^{q(j+1)-q(j)}} &\leq \frac{1}{B_m}\sum_{j=j_0}^{m} \frac{1}{4\pi}\frac{1}{2^j} \\ &\leq \frac{1}{4\pi}\frac{1}{B_m} \to 0 \quad \text{as } m \to \infty, \end{aligned}$$

(6) and (7) are fine. Thus, the assertion of (ii) follows from Corollary 1. □

3.2. Fukuyama–Takahashi's counterexample.

Let $\{\lambda(i)\}_{i=1}^{\infty}$ be a positive sequence satisfying

$$\lim_{i\to\infty} \lambda(i) = \infty, \tag{34}$$

$$\lim_{i\to\infty} \big(\lambda(i+1) - \lambda(i)\big) = 0, \tag{35}$$

$$\sum_{i=1}^{\infty} \frac{1}{\lambda(i)^2} = \infty. \tag{36}$$

Note that by (36),

$$\sum_{i=1}^{\infty} \frac{1}{\lambda(i)} = \infty. \tag{37}$$

For $k \in \mathbb{N}$, we define $p(k) := \max\left\{j \in \mathbb{N}; \sum_{i=1}^{j} 1/\lambda(i) \le k\right\}$, where $\max\emptyset := 0$. By (37), $p(k) < \infty$. When $\left\{j \in \mathbb{N}; \sum_{i=1}^{j} 1/\lambda(i) \le k\right\} = \emptyset$, i.e., $1/\lambda(1) > k$, $p(k) = 0$, and thus $k \ge 1/\lambda(1) \underset{\text{iff}}{\Leftrightarrow} p(k) \ge 1$. For convenience, $p(0) := 0$. By definition, $p(k) \le p(k+1)$ $(k \ge 0)$ and

$$\sum_{i=1}^{p(k)} \frac{1}{\lambda(i)} \le k < \sum_{i=1}^{p(k)+1} \frac{1}{\lambda(i)}, \tag{38}$$

from which, it follows that $\lim_{k\to\infty} p(k) = \infty$.

For $k \in \mathbb{Z}_{\ge 0}$, we define $l(k) := p(k+1) - p(k)$. Since, by (34), $\lim_{k\to\infty} \lambda\big(p(k)+1\big) = \infty$,

$$^{\exists} j_0 \in \mathbb{N} \ \text{ s.t. } \begin{cases} j_0 \ge \dfrac{1}{\lambda(1)}, \\ \lambda\big(p(k)+1\big) \ge 1 \ \ (^{\forall} k \ge j_0). \end{cases}$$

Then, for $k \ge j_0$,

$$\sum_{i=1}^{p(k)+1} \frac{1}{\lambda(i)} = \sum_{i=1}^{p(k)} \frac{1}{\lambda(i)} + \frac{1}{\lambda(p(k)+1)} \le k+1$$

by (38). By the definition of $p(\cdot)$, $p(k) < p(k)+1 \le p(k+1)$, and hence $l(k) \ge 1$ $(k \ge j_0)$.

Moreover, it is known by Fukuyama–Takahashi [**3**] that

$$l(k) \to \infty \quad \text{as } k \to \infty, \tag{39}$$

$$\sum_{k=j_0}^{\infty} \frac{1}{l(k)} = \infty. \tag{40}$$

DEFINITION 5. We define a positive sequence $\{a_i\}_{i=1}^{\infty}$ by

$$a_i := \begin{cases} 1 & \text{if } 1 \le i \le p(j_0), \\ \dfrac{1}{l(j)} \sqrt{\dfrac{p(j_0)}{2} \displaystyle\prod_{k=j_0}^{j-1} \Big(1 + \frac{1}{2l(k)}\Big)} & \text{if } p(j) < i \le p(j+1) \ \ (j \ge j_0), \end{cases}$$

where $\prod_{k=j_0}^{j_0-1} \big(1 + 1/2l(k)\big) := 1$ and $\{A_N\}_{N=1}^{\infty}$ by

$$A_N = \sqrt{\frac{1}{2}\sum_{i=1}^{N} a_i^2}.$$

REMARK 2. $\{A_{p(j)}^2\}_{j=j_0}^{\infty}$ satisfies that $A_{p(j_0)}^2 = p(j_0)/2$, $A_{p(j+1)}^2 = A_{p(j)}^2\big(1 + 1/2l(j)\big)$ $(j \ge j_0)$.

PROOF. Since, by definition, $A^2_{p(j_0)} = p(j_0)/2$, $a_i = 1/l(j_0)\sqrt{p(j_0)/2}$ $(p(j_0) < i \le p(j_0+1))$,

$$\begin{aligned} A^2_{p(j_0+1)} &= \frac{1}{2}\sum_{i=1}^{p(j_0)} a_i^2 + \frac{1}{2}\sum_{p(j_0)<i\le p(j_0+1)} a_i^2 \\ &= A^2_{p(j_0)} + \frac{1}{2l(j_0)} A^2_{p(j_0)} = A^2_{p(j_0)}\Big(1+\frac{1}{2l(j_0)}\Big). \end{aligned}$$

Next suppose that the recursion formula

$$A^2_{p(j+1)} = A^2_{p(j)}\Big(1+\frac{1}{2l(j)}\Big) \tag{41}$$

holds for $j_0 \le j \le k$. Then

$$\begin{aligned} A^2_{p(k+2)} &= \frac{1}{2}\sum_{i=1}^{p(k+1)} a_i^2 + \frac{1}{2}\sum_{p(k+1)<i\le p(k+2)} a_i^2 \\ &= A^2_{p(k+1)} + \frac{1}{2l(k+1)} A^2_{p(k+1)} = A^2_{p(k+1)}\Big(1+\frac{1}{2l(k+1)}\Big). \end{aligned}$$

Thus, for $j = k+1$, (41) holds. Therefore, $\{A^2_{p(j)}\}$ satisfies the recursion formula (41) for $^\forall j \ge j_0$. $\square$

DEFINITION 6. After taking a strictly increasing sequence $\{q(j)\}_{j=j_0}^\infty$ of natural numbers so as to satisfy

$$\begin{cases} q(j_0) = p(j_0)+1, \\ 2^{q(j+1)-q(j)} \ge 2l(j) \vee \big(\pi A_{p(j)} l(j)^2 j^2\big) \quad (^\forall j \ge j_0), \end{cases}$$

we define a strictly increasing sequence $\{n_i\}_{i=1}^\infty$ of natural numbers by

$$n_i := \begin{cases} 2^i & \text{if } 1 \le i \le p(j_0), \\ 2^{q(j)}\big(i-p(j)\big) & \text{if } p(j) < i \le p(j+1) \ \ (j \ge j_0). \end{cases}$$

THEOREM 2 (cf. Fukuyama–Takahashi [**3**], Matsushita [**4**]). (i) $\lim_{N\to\infty} A_N = \infty$, $\limsup_{i\to\infty} \lambda(i)\dfrac{a_i}{A_i} = 1$.

(ii) $\liminf_{i\to\infty} \lambda(i)\Big(\dfrac{n_{i+1}}{n_i} - 1\Big) = 1$.

(iii) *For* $^\forall S \in \mathcal{B}([0,1))$ *with* $\mathbf{P}(S) > 0$,

$$\mathbf{P}\bigg(\frac{1}{A_N}\sum_{i=1}^N a_i \cos 2\pi n_i t \in \cdot \ \bigg|\ S\bigg) \to \mu_{m_{1/4},0,\nu_{1/4}} \quad \textit{as } N\to\infty.$$

PROOF. We show (iii) only.

First, by (39), (1) is fine. Let $c_j = A_{p(j)}$ $(j \geq j_0)$. Then, by (41),

$$B_m = \sqrt{\sum_{j=j_0}^{m} \frac{A_{p(j)}^2}{2l(j)}} = \sqrt{\sum_{j=j_0}^{m} \left(A_{p(j+1)}^2 - A_{p(j)}^2\right)} = \sqrt{A_{p(m+1)}^2 - A_{p(j_0)}^2},$$

$$A_{p(m+1)}^2 = \frac{p(j_0)}{2} \prod_{j=j_0}^{m} \frac{A_{p(j+1)}^2}{A_{p(j)}^2} = \frac{p(j_0)}{2} \prod_{j=j_0}^{m} \Big(1 + \frac{1}{2l(j)}\Big) \geq \frac{p(j_0)}{2} \sum_{j=j_0}^{m} \frac{1}{2l(j)},$$

$$\frac{A_{p(m+1)}^2}{A_{p(m)}^2} = 1 + \frac{1}{2l(m)}.$$

By (40) and (39), it follows that $B_m \to \infty$,

$$\frac{B_{m+1}}{B_m} = \sqrt{\frac{(A_{p(m+2)}^2/A_{p(m+1)}^2) - (A_{p(j_0)}^2/A_{p(m+1)}^2)}{1 - (A_{p(j_0)}^2/A_{p(m+1)}^2)}} \to 1, \quad \frac{B_m^2}{c_m^2} = \frac{A_{p(m+1)}^2}{A_{p(m)}^2} - \frac{A_{p(j_0)}^2}{A_{p(m)}^2} \to 1.$$

Thus, (2), (3) and (4) with $a = 4$ hold.

From the choice of $\{q(j)\}_{j=j_0}^{\infty}$ (cf. Definition 6), (6) is clear. As for (7), since

$$\begin{aligned}\frac{1}{B_m} \sum_{j=j_0}^{m} c_j \frac{l(j)}{2^{q(j+1)-q(j)}} &\leq \frac{1}{B_m} \sum_{j=j_0}^{m} \frac{A_{p(j)} l(j)}{\pi A_{p(j)} l(j)^2 j^2} \\ &\leq \Big(\frac{1}{\pi} \sum_{j=1}^{\infty} \frac{1}{j^2}\Big) \frac{1}{B_m} \to 0 \quad \text{as } m \to \infty,\end{aligned}$$

(7) is fine.

Therefore, by Corollary 1, it holds that for ${}^{\forall}S \in \mathcal{B}([0,1))$ with $\mathbf{P}(S) > 0$,

$$\mathbf{P}\bigg(\frac{1}{\widetilde{A}_N} \sum_{i=1}^{N} \widetilde{a}_i \cos 2\pi \widetilde{n}_i t \in \cdot \;\bigg|\; S\bigg) \to \mu_{m_{1/4},0,\nu_{1/4}} \quad \text{as } N \to \infty, \tag{42}$$

where

$$\widetilde{n}_i := \begin{cases} 2^{q(j_0)} i & \text{if } 1 \leq i \leq l(j_0), \\ 2^{q(j+1)}\Big(i - \big(l(j_0) + \cdots + l(j)\big)\Big) & \text{if } l(j_0) + \cdots + l(j) < i \leq l(j_0) + \cdots \\ & \qquad \cdots + l(j) + l(j+1) \quad (j \geq j_0), \end{cases}$$

$$\widetilde{a}_i := \begin{cases} \dfrac{A_{p(j_0)}}{l(j_0)} & \text{if } 1 \leq i \leq l(j_0), \\ \dfrac{A_{p(j+1)}}{l(j+1)} & \text{if } l(j_0) + \cdots + l(j) < i \leq l(j_0) + \cdots + l(j) + l(j+1) \quad (j \geq j_0), \end{cases}$$

$$\widetilde{A}_N := \sqrt{\frac{1}{2} \sum_{i=1}^{N} \widetilde{a}_i^2}.$$

Here note that $l(j_0)+\cdots+l(j)=p(j+1)-p(j_0)\ \ (j\ge j_0)$ and that by the definition of $\{a_i\}_{i=1}^{\infty}$ (cf. Definition 5), $a_i=A_{p(j)}/l(j)\ \ (p(j)<i\le p(j+1),\ \ j\ge j_0)$. Then

$$\begin{aligned}\widetilde{n}_i &= \begin{cases} 2^{q(j_0)}\big(i+p(j_0)-p(j_0)\big) & \text{if } p(j_0)<i+p(j_0)\le p(j_0+1), \\ 2^{q(j+1)}\big(i+p(j_0)-p(j+1)\big) & \text{if } p(j+1)<i+p(j_0)\le p(j+2) \quad (j\ge j_0) \end{cases} \\ &= n_{i+p(j_0)}, \\ \widetilde{a}_i &= \begin{cases} \dfrac{A_{p(j_0)}}{l(j_0)} & \text{if } p(j_0)<i+p(j_0)\le p(j_0+1), \\ \dfrac{A_{p(j+1)}}{l(j+1)} & \text{if } p(j+1)<i+p(j_0)\le p(j+2) \quad (j\ge j_0) \end{cases} \\ &= a_{i+p(j_0)}, \end{aligned}$$

so that

$$\widetilde{A}_N=\sqrt{\frac{1}{2}\sum_{i=p(j_0)+1}^{N+p(j_0)} a_i^2}=\sqrt{A_{N+p(j_0)}^2-A_{p(j_0)}^2},$$

$$\sum_{i=1}^{N}\widetilde{a}_i\cos 2\pi\widetilde{n}_i t=\sum_{i=p(j_0)+1}^{N+p(j_0)} a_i\cos 2\pi n_i t=\sum_{i=1}^{N+p(j_0)} a_i\cos 2\pi n_i t-\sum_{i=1}^{p(j_0)} a_i\cos 2\pi n_i t.$$

Substituting these into (42) yields that

$$\begin{aligned}&\mathbf{P}\left(\frac{1}{\sqrt{A_N^2-A_{p(j_0)}^2}}\left(\sum_{i=1}^{N} a_i\cos 2\pi n_i t-\sum_{i=1}^{p(j_0)} a_i\cos 2\pi n_i t\right)\in\cdot\ \middle|\ S\right) \\ &\quad\to\mu_{m_{1/4},0,\nu_{1/4}}\quad\text{as } N\to\infty.\end{aligned}$$

Finally, by noting that as $N\to\infty$,

$$\frac{1}{\sqrt{A_N^2-A_{p(j_0)}^2}}\sum_{i=1}^{p(j_0)} a_i\cos 2\pi n_i t\to 0,\quad \sqrt{A_N^2-A_{p(j_0)}^2}\sim A_N,$$

the preceding convergence becomes the convergence required of (iii). □

3.3. Takahashi's counterexample.

Fix $c>0$, $0<\alpha\le 1/2$. For $j\in\mathbb{N}$, let $p(j):=\lfloor j^{1/\alpha}\rfloor\in\mathbb{N}$. Since

$$\begin{aligned}(j+1)^{1/\alpha}-j^{1/\alpha}&=\frac{1}{\alpha}(j+\theta)^{(1/\alpha)-1}\quad\text{for some } \theta\in(0,1) \\ &\ge 2(j+\theta)\ge 2j,\end{aligned}$$

it is observed that

$$p(j+1)-p(j)=\lfloor (j+1)^{1/\alpha}-j^{1/\alpha}+j^{1/\alpha}\rfloor - p(j) \geq \lfloor 2j+j^{1/\alpha}\rfloor - \lfloor j^{1/\alpha}\rfloor = 2j,$$

so that $p(j+1)-p(j)-1>0$. Let

$$l(j):=\Big\lfloor \frac{p^\alpha(j)}{c}\Big\rfloor \wedge \big(p(j+1)-p(j)-1\big)\in \mathbb{Z}_{\geq 0}, \quad j\in\mathbb{N}.$$

By the definition of $p(\cdot)$, it is seen that

$$\frac{p^\alpha(j)}{j}=\Big(1-\frac{\{j^{1/\alpha}\}}{j^{1/\alpha}}\Big)^\alpha \to 1 \quad \text{as } j\to\infty, \tag{43}$$

$$\begin{aligned}
&p(j+1)-p(j)\\
&\quad=\frac{1}{\alpha}j^{(1/\alpha)-1}\bigg(\Big(1+\frac{\theta_j}{j}\Big)^{(1/\alpha)-1}-\frac{\alpha\{(j+1)^{1/\alpha}\}}{j^{(1/\alpha)-1}}+\frac{\alpha\{j^{1/\alpha}\}}{j^{(1/\alpha)-1}}\bigg) \quad \text{for some } \theta_j\in(0,1)\\
&\quad\sim \frac{1}{\alpha}j^{(1/\alpha)-1} \quad \text{as } j\to\infty.
\end{aligned} \tag{44}$$

From these, it follows that

$$\lim_{j\to\infty}\frac{l(j)}{j}=\beta(\alpha):=\begin{cases}\dfrac{1}{c} & \text{if } 0<\alpha<\dfrac{1}{2},\\[2mm] 2\wedge\Big(\dfrac{1}{c}\Big) & \text{if } \alpha=\dfrac{1}{2}.\end{cases} \tag{45}$$

Set $k_0:=\min\big\{j; l(j)\geq 1\big\}\in\mathbb{N}$. Since $p(j+1)-p(j)-1\geq 1$ and

$$\begin{aligned}
&\Big\lfloor\frac{p^\alpha(j)}{c}\Big\rfloor \geq 1 \Leftrightarrow \frac{p^\alpha(j)}{c}\geq 1 \Leftrightarrow p(j)\geq c^{1/\alpha} \Leftrightarrow \lfloor j^{1/\alpha}\rfloor \geq c^{1/\alpha}\\
&\Leftrightarrow \lfloor j^{1/\alpha}\rfloor \geq \lceil c^{1/\alpha}\rceil \Leftrightarrow j^{1/\alpha}\geq \lceil c^{1/\alpha}\rceil \Leftrightarrow j\geq \lceil c^{1/\alpha}\rceil^\alpha \Leftrightarrow j\geq \big\lceil \lceil c^{1/\alpha}\rceil^\alpha\big\rceil,
\end{aligned}$$

it is observed that

$$l(j)\geq 1 \Leftrightarrow \Big\lfloor \frac{p^\alpha(j)}{c}\Big\rfloor \geq 1 \Leftrightarrow j\geq \big\lceil \lceil c^{1/\alpha}\rceil^\alpha\big\rceil,$$

so that $k_0=\big\lceil \lceil c^{1/\alpha}\rceil^\alpha\big\rceil$. Note that when $0<c\leq 1$,

$$0<c^{1/\alpha}\leq 1 \Rightarrow \lceil c^{1/\alpha}\rceil = 1 \Rightarrow k_0=\big\lceil \lceil c^{1/\alpha}\rceil^\alpha\big\rceil = 1;$$

when $c>1$,

$$c^{1/\alpha}>1 \Rightarrow \lceil c^{1/\alpha}\rceil \geq 2 \Rightarrow k_0=\big\lceil \lceil c^{1/\alpha}\rceil^\alpha\big\rceil \geq \lceil 2^\alpha\rceil \geq 2.$$

DEFINITION 7. We inductively define a strictly increasing sequence $\{n_i\}_{i=1}^\infty$ of natural numbers:

(i) In the case when $c>1$.

Since $k_0\geq 2$ by the preceding remark, $p(k_0)=\lfloor k_0^{1/\alpha}\rfloor \geq \lfloor 2^{1/\alpha}\rfloor \geq \lfloor 2^2\rfloor = 4$. First we define $\{n_i\}_{1\leq i<p(k_0)}$ by the following recursion formula:

$$\begin{cases} n_1 := 1, \\ n_{i+1} := \Big\lfloor n_i\Big(1+\dfrac{c}{i^\alpha}\Big)+1\Big\rfloor \quad (1 \le i \le p(k_0)-2). \end{cases}$$

Next, when $\{n_i\}_{1\le i<p(j)}$, where $j \ge k_0$, is defined in advance, we define $\{n_i\}_{p(j)\le i<p(j+1)}$ in the following way:

$$q(j) := \begin{cases} \min\Big\{m \in \mathbb{N}; 2^m > n_{p(k_0)-1}\Big(1+\dfrac{c}{(p(k_0)-1)^\alpha}\Big)\Big\} & \text{if } j = k_0, \\ \min\Big\{m \in \mathbb{N}; 2^m > n_{p(j)-1}\Big(1+\dfrac{c}{(p(j)-1)^\alpha}\Big) \vee \big(n_{p(j-1)}j^3\big)\Big\} & \text{if } j > k_0, \end{cases}$$

$$n_i := \begin{cases} 2^{q(j)}\big(i-p(j)+1\big) & \text{if } p(j) \le i \le p(j)+l(j), \\ \Big\lfloor n_{i-1}\Big(1+\dfrac{c}{p(j)^\alpha}\Big)+1\Big\rfloor & \text{if } l(j) < p(j+1)-p(j)-1 \text{ and} \\ & \quad p(j)+l(j) < i \le p(j+1)-1. \end{cases}$$

(ii) In the case when $0 < c \le 1$.

Since $k_0 = 1$, $l(j) \ge 1$ $({}^\forall j \ge 1)$. Let $n_1 = n_{p(1)} := 1$. When $\{n_i\}_{1\le i\le p(j)}$, where $j \ge 1$, is defined in advance, we define $\{n_i\}_{p(j)<i\le p(j+1)}$ by the following recursion formula:

$$n_i := \begin{cases} n_{p(j)}\big(i-p(j)+1\big) & \text{if } p(j) < i \le p(j)+l(j), \\ \Big\lfloor n_{i-1}\Big(1+\dfrac{c}{p(j)^\alpha}\Big)+1\Big\rfloor & \text{if } l(j) < p(j+1)-p(j)-1 \text{ and} \\ & \quad p(j)+l(j)+1 \le i \le p(j+1)-1, \\ 2^{q(j+1)} & \text{if } i = p(j+1), \end{cases}$$

where

$$q(j+1) := \min\Big\{m \in \mathbb{N}; 2^m > n_{p(j+1)-1}\Big(1+\frac{c}{(p(j+1)-1)^\alpha}\Big) \vee \big(n_{p(j)}(j+1)^3\big)\Big\}.$$

Definition 8. We define a positive sequence $\{a_i\}_{i=1}^\infty$ by

$$a_i := \begin{cases} 1 & \text{if } \ i \in \displaystyle\sum_{j=k_0}^\infty \big[p(j), p(j)+l(j)\big] \cap \mathbb{N}, \\ \dfrac{1}{i^2} & \text{otherwise} \end{cases}$$

and $\{A_N\}_{N=1}^\infty$ by

$$A_N = \sqrt{\frac{1}{2}\sum_{i=1}^N a_i^2}.$$

Theorem 3 (cf. Takahashi [**7**], Matsushita [**4**]). (i) $\dfrac{n_{i+1}}{n_i} \ge 1 + \dfrac{c}{i^\alpha}$ $({}^\forall i \ge 1)$.

(ii) $\#\{i; a_i = 1\} = \infty$, $A_N \sim \sqrt{\dfrac{\beta(\alpha)}{4}} N^\alpha$ *as* $N \to \infty$.

(iii) *For* ${}^\forall S \in \mathcal{B}([0,1))$ *with* $\mathbf{P}(S) > 0$,

$$\mathbf{P}\left(\frac{1}{A_N}\sum_{i=1}^N a_i \cos 2\pi n_i t \in \cdot \;\middle|\; S\right) \to \mu_{m_{\beta(\alpha)},0,\nu_{\beta(\alpha)}} \quad \textit{as } N \to \infty.$$

Proof. We show (ii) and (iii).

The first part of (ii) is clear from the definition of $\{a_i\}$. As for the second part of (ii), first, set

$$j_0 := k_0 \vee 2 = \begin{cases} 2 & \text{if } \ 0 < c \le 1, \\ k_0 & \text{if } \ c > 1. \end{cases}$$

For $m \ge j_0$,

$$A^2_{p(m+1)-1} - A^2_{p(m)} = \frac{1}{2} l(m) + \frac{1}{2} \sum_{p(m)+l(m)<i<p(m+1)} \frac{1}{i^4},$$

$$A^2_{p(m+1)-1} = \frac{1}{2} \sum_{1 \le i < p(j_0)} a_i^2 + \frac{1}{2} \sum_{j=j_0}^m \bigl(l(j)+1\bigr) + \frac{1}{2} \sum_{j=j_0}^m \sum_{p(j)+l(j)<i<p(j+1)} \frac{1}{i^4},$$

from which, it follows that

$$\begin{aligned} \max_{p(m)\le N<p(m+1)} \bigl|A^2_{p(m+1)-1} - A^2_N\bigr| &= A^2_{p(m+1)-1} - A^2_{p(m)} \\ &= \frac{1}{2} l(m) + \frac{1}{2} \sum_{p(m)+l(m)<i<p(m+1)} \frac{1}{i^4}, \end{aligned} \tag{46}$$

$$\begin{aligned} \left|A^2_{p(m+1)-1} - \frac{1}{2}\sum_{j=j_0}^m \bigl(l(j)+1\bigr)\right| &\le \frac{1}{2} \sum_{1 \le i < p(j_0)} a_i^2 \\ &\quad + \frac{1}{2} \sum_{j=j_0}^\infty \sum_{p(j)+l(j)<i<p(j+1)} \frac{1}{i^4}. \end{aligned} \tag{47}$$

Next it is verified that

$$\frac{1}{2} \sum_{j=j_0}^m \bigl(l(j)+1\bigr) \sim \frac{\beta(\alpha)}{4} m^2 \quad \text{as } m \to \infty. \tag{48}$$

Indeed, since $\varepsilon(j) = (l(j)+1)/j - \beta(\alpha) \to 0$ as $j \to \infty$ by (45),

$$\left|\frac{1}{2m^2} \sum_{j=j_0}^m \bigl(l(j)+1\bigr) - \frac{\beta(\alpha)}{4}\right|$$

$$
\begin{aligned}
&= \left| \frac{\beta(\alpha)}{2m^2} \sum_{j=j_0}^{m} j - \frac{\beta(\alpha)}{4} + \frac{1}{2m^2} \sum_{j=j_0}^{m} \varepsilon(j)j \right| \\
&\le \frac{\beta(\alpha)}{4}\Big(\frac{1}{m} + \frac{j_0(j_0-1)}{m^2}\Big) + \frac{1}{2m^2}\Big| \sum_{j_0 \le j < m_0} \varepsilon(j)j \Big| + \Big(\max_{m_0 \le j \le m} |\varepsilon(j)|\Big)\frac{1}{4}\Big(1+\frac{1}{m}\Big) \\
&\to 0 \quad \text{as } m \to \infty \text{ first and } m_0 \to \infty \text{ second.}
\end{aligned}
$$

Using (48) in (47) yields that $A^2_{p(m+1)-1} \sim (\beta(\alpha)/4)m^2$ as $m \to \infty$. By combining this with (46), it is observed that

$$
\begin{aligned}
&\max_{p(m) \le N < p(m+1)} \Big| \frac{\beta(\alpha)}{4} - \frac{A_N^2}{m^2} \Big| \\
&\le \Big| \frac{\beta(\alpha)}{4} - \frac{A^2_{p(m+1)-1}}{m^2} \Big| + \frac{1}{m^2} \max_{p(m) \le N < p(m+1)} \big| A^2_{p(m+1)-1} - A_N^2 \big| \to 0 \quad \text{as } m \to \infty.
\end{aligned}
$$

Now, for $N \ge p(j_0)$,

$$
{}^{\exists 1} m = m_N \ge j_0 \ \text{ s.t. } \ p(m) \le N < p(m+1).
$$

Since $\lim_{N \to \infty} m_N = \infty$, (43) implies that

$$
p(m_N) \sim m_N^{1/\alpha}, \quad p\big(m_N + 1\big) \sim \big(m_N + 1\big)^{1/\alpha} \sim m_N^{1/\alpha} \quad \text{as } N \to \infty,
$$

and thus $N \sim m_N^{1/\alpha}$ as $N \to \infty$. On the other hand, since, by the preceding convergence,

$$
\frac{A_N}{m_N} \to \sqrt{\frac{\beta(\alpha)}{4}} \quad \text{as } N \to \infty, \tag{49}
$$

combining them, we have

$$
A_N \sim \sqrt{\frac{\beta(\alpha)}{4}} N^{\alpha} \quad \text{as } N \to \infty.
$$

(iii) Take $\{l(j)+1\}_{j=j_0}^{\infty}$ as a sequence of natural numbers and $\{l(j)+1\}_{j=j_0}^{\infty}$ as a positive sequence. By (45), (1) is all right. Since, by (48),

$$
B_m = \sqrt{\frac{1}{2} \sum_{j=j_0}^{m} \big(l(j)+1\big)} \sim \sqrt{\frac{\beta(\alpha)}{4}} m \quad \text{as } m \to \infty, \tag{50}
$$

(2) and (3) are fine. As for (4), since

$$
\frac{B_m^2}{(l(m)+1)^2} \sim \frac{(\beta(\alpha)/4)m^2}{(\beta(\alpha)m)^2} = \frac{1}{4\beta(\alpha)},
$$

(4) is all right with $a = 1/\beta(\alpha)$.

By the definition of $\{q(j)\}_{j=j_0}^{\infty}$ (cf. Definition 7),

$$\begin{array}{l} 2^{q(j)} = n_{p(j)} < \underset{\substack{\| \\ 2^{q(j)}\cdot 2}}{n_{p(j)+1}} < \underset{\substack{\| \\ 2^{q(j)}\cdot 3}}{n_{p(j)+2}} < \cdots < \underset{\substack{\| \\ 2^{q(j)}(l(j)+1)}}{n_{p(j)+l(j)}} \le n_{p(j+1)-1} < 2^{q(j+1)}, \\ 2^{q(j+1)} > 2^{q(j)}(j+1)^3. \end{array}$$

From these, it follows that $2^{q(j+1)-q(j)} > l(j)+1$ and that

$$\begin{aligned} \sum_{j=j_0}^{m} \big(l(j)+1\big)\frac{l(j)+1}{2^{q(j+1)-q(j)}} \le \sum_{j=j_0}^{m} \frac{(l(j)+1)^2}{(j+1)^3} &\le \Big(\sup_{j\ge j_0}\frac{l(j)+1}{j+1}\Big)^2 \sum_{j=j_0}^{m}\frac{1}{j+1} \\ &\le \Big(\sup_{j\ge j_0}\frac{l(j)+1}{j+1}\Big)^2 \log(m+1), \end{aligned}$$

so that

$$\frac{1}{B_m}\sum_{j=j_0}^{m}\big(l(j)+1\big)\frac{l(j)+1}{2^{q(j+1)-q(j)}} \to 0 \quad \text{as } m\to\infty.$$

Thus, (6) and (7) are all right.

Therefore, by Theorem, it holds that for $^\forall S \in \mathcal{B}([0,1))$ with $\mathbf{P}(S)>0$,

$$\mathbf{P}\Big(\frac{1}{B_m}\sum_{j=j_0}^{m}\sum_{l=1}^{l(j)+1}\cos 2\pi N_j l t \in \cdot \;\Big|\; S\Big) \to \mu_{m_{\beta(\alpha)},0,\nu_{\beta(\alpha)}} \quad \text{as } m\to\infty.$$

Here, since, by the definition of $\{n_i\}_{i=1}^\infty$ and $\{a_i\}_{i=1}^\infty$ (cf. Definitions 7 and 8),

$$\sum_{j=j_0}^{m}\sum_{p(j)\le i\le p(j)+l(j)} a_i\cos 2\pi n_i t = \sum_{j=j_0}^{m}\sum_{l=1}^{l(j)+1}\cos 2\pi N_j l t,$$

the convergence above is rewritten as

$$\mathbf{P}\Big(\frac{1}{B_m}\sum_{j=j_0}^{m}\sum_{p(j)\le i\le p(j)+l(j)} a_i\cos 2\pi n_i t \in \cdot \;\Big|\; S\Big) \to \mu_{m_{\beta(\alpha)},0,\nu_{\beta(\alpha)}} \quad \text{as } m\to\infty. \tag{51}$$

For $N \ge p(j_0)$, take $m = m_N \ge j_0$ such that $p(m)\le N < p(m+1)$. Then

$$\begin{aligned} &\mathbf{E}\Big[\Big(\frac{1}{A_N}\sum_{i=1}^{N} a_i\cos 2\pi n_i t - \frac{1}{B_m}\sum_{j=j_0}^{m}\sum_{p(j)\le i\le p(j)+l(j)} a_i\cos 2\pi n_i t\Big)^2\Big] \\ &\quad \le 2\mathbf{E}\Big[\Big(-\frac{1}{A_N}\sum_{N<i<p(m+1)} a_i\cos 2\pi n_i t\Big)^2\Big] \\ &\qquad + 2\mathbf{E}\Big[\Big(\frac{1}{A_N}\sum_{1\le i<p(j_0)} a_i\cos 2\pi n_i t \end{aligned}$$

$$
\begin{aligned}
&\quad + \Big(\frac{1}{A_N} - \frac{1}{B_m}\Big) \sum_{j=j_0}^{m} \sum_{p(j)\le i\le p(j)+l(j)} a_i \cos 2\pi n_i t \\
&\quad + \frac{1}{A_N} \sum_{j=j_0}^{m} \sum_{p(j)+l(j)<i<p(j+1)} a_i \cos 2\pi n_i t\Big)^2\Big] \\
&= \frac{2}{A_N^2}\big(A_{p(m+1)-1}^2 - A_N^2\big) + 2\bigg(\frac{A_{p(j_0)-1}^2}{A_N^2} + \Big(\frac{1}{A_N} - \frac{1}{B_m}\Big)^2 \frac{1}{2} \sum_{j=j_0}^{m} \big(l(j)+1\big) \\
&\qquad\qquad\qquad\qquad + \frac{1}{A_N^2}\cdot\frac{1}{2} \sum_{j=j_0}^{m} \sum_{p(j)+l(j)<i<p(j+1)} \frac{1}{i^4}\bigg) \\
&\le \Big(\frac{m}{A_N}\Big)^2 \bigg(\frac{1}{m}\frac{l(m)}{m} + \frac{1}{m^2} \sum_{p(m)+l(m)<i<p(m+1)} \frac{1}{i^4}\bigg) \\
&\quad + \frac{1}{A_N^2}\bigg(2A_{p(j_0)-1}^2 + \sum_{j=j_0}^{\infty} \sum_{p(j)+l(j)<i<p(j+1)} \frac{1}{i^4}\bigg) \\
&\quad + 2\Big(\frac{m}{A_N} - \frac{m}{B_m}\Big)^2 \frac{1}{m^2}\frac{1}{2} \sum_{j=j_0}^{m} \big(l(j)+1\big) \quad \big(\text{cf. (46)}\big) \\
&\to 0 \quad \text{as } N\to\infty \quad \big(\text{by (45), (48), (49) and (50)}\big).
\end{aligned}
$$

By the same reasoning as in the proof of Corollary 1, it follows that

$$
\mathbf{P}\bigg(\frac{1}{A_N}\sum_{i=1}^{N} a_i \cos 2\pi n_i t \in \cdot \;\bigg|\; S\bigg) \to \mu_{m_{\beta(\alpha)},0,\nu_{\beta(\alpha)}} \quad \text{as } N\to\infty. \qquad \square
$$

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