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The maximum of the 1-measurement of a metric measure space

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Abstract. For a metric measure space, we consider the set of distributions of 1-Lipschitz functions, which is called the 1-measurement. On the 1-measurement, we have the Lipschitz order relation introduced by M. Gromov. The aim of this paper is to study the maximum and maximal elements of the 1-measurement of a metric measure space with respect to the Lipschitz order. We present a necessary condition of a metric measure space for the existence of the maximum of the 1-measurement. We also consider a metric measure space that has the maximum of its 1-measurement.

1. Introduction.

In this paper, we study the maximum and the maximal elements of the 1-measurement of a metric measure space. Based on the measure concentration phenomenon, M. Gromov introduced various concepts and invariants in the metric measure space framework ([**3**]). Observable diameter is one of the most important invariants defined by him. It represents how much the measure of a metric measure space concentrates and it is defined by the 1-measurement. The 1-*measurement* of a metric measure space X is defined as

$$\mathcal{M}(X;1) := \{ f_* m_X \mid f : X \to \mathbb{R} : 1\text{-Lipschitz function} \},$$

where a 1-Lipschitz function is a Lipschitz continuous function with its Lipschitz constant less than or equal to one. The 1-measurement has a natural order relation called the Lipschitz order (Definition 2.4 and Remark 2.6).

From now on, we call a metric measure space an mm-space in short. We assume any mm-space X is equipped with a complete separable metric d_X and a Borel probability measure m_X. We additionally assume $X = \operatorname{supp} m_X$ unless otherwise stated, where $\operatorname{supp} m_X$ is the support of m_X.

We firstly treat the n-dimensional unit sphere $S^n(1)$ centered at the origin in $\mathbb{R}^{n+1}$ as an mm-space. A compact Riemannian manifold is considered as an mm-space with the Riemannian distance function and the normalized volume measure.

THEOREM 1.1 (Gromov [**2**, Section 9]). *The push-forward $\xi_* m_{S^n(1)}$ of the measure $m_{S^n(1)}$ by the distance function ξ from one point in $S^n(1)$ is the maximum of $\mathcal{M}(S^n(1);1)$.*

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M. Gromov proved this theorem using convexly derived measures but we do not use it in this paper and we give an alternative proof using Lévy's isoperimetric inequality (Theorem 2.11). He also stated that the isoperimetric inequality of $S^n(1)$ yields the existence of the maximum of $\mathcal{M}(S^n(1);1)$ without proof. We give a detailed proof of Theorem 1.1 in Section 3.1. As a corollary of Theorem 1.1, we see the normal law à la Lévy (Corollary 3.6). Theorem 1.1 can be thought as a finite-dimensional version of the normal law à la Lévy.

We obtain the following result for a general mm-space. Denote the diameter of X by $\operatorname{diam} X$.

Theorem 1.2. *Let (X, d_X, m_X) be an mm-space. Any measure $\mu \in \mathcal{M}(X;1)$ satisfying $\operatorname{diam}\operatorname{supp}\mu = \operatorname{diam} X < \infty$ is a maximal element of the 1-measurement $\mathcal{M}(X;1)$.*

Theorem 1.2 is simple but powerful to find a maximal element of the 1-measurement $\mathcal{M}(X;1)$. As a corollary of Theorem 1.2, we have the following.

Corollary 1.3. *Let an mm-space X satisfy $\operatorname{diam} X < \infty$ and a point $x_0 \in X$ satisfy $\sup_{x\in X} d_X(x,x_0) = \operatorname{diam} X$. Then, the push-forward $\xi_* m_X$ of m_X by the distance function ξ from the point x_0 is a maximal element of the 1-measurement $\mathcal{M}(X;1)$. In particular, if the maximum of the 1-measurement $\mathcal{M}(X;1)$ exists, then it is $\xi_* m_X$.*

In the case where two points $x_0, x_1 \in X$ satisfy $\sup_{x\in X} d_X(x,x_i) = \operatorname{diam} X < \infty$, $i = 0,1$, each push-forward $(\xi_i)_* m_X$ of m_X by the distance function ξ_i from the point x_i is a maximal element of the 1-measurement $\mathcal{M}(X;1)$. Therefore, if $(\xi_0)_* m_X$ and $(\xi_1)_* m_X$ are not isomorphic to each other, then the 1-measurement $\mathcal{M}(X;1)$ has no maximum because it has two different maximal elements. On the other hand, the push-forward by the distance function from one point does not depend on how to pick the point in a homogeneous space such as the flat torus T^n ($n \geq 2$) or the projective space $\mathbb{R}P^n$ ($n \geq 2$). However, $\mathcal{M}(T^n;1)$ and $\mathcal{M}(\mathbb{R}P^n;1)$ both have no maximum because of one of main theorems of this paper stated as follows.

Theorem 1.4. *Assume that the 1-measurement $\mathcal{M}(X;1)$ has its maximum. Then, for any two points $x, y \in X$ with $d_X(x,y) = \operatorname{diam} X < \infty$, we have*

$$d_X(x,z) + d_X(z,y) = \operatorname{diam} X \quad \textit{for any point } z \in X.$$

We prove Theorem 1.4 in Section 4.2. Theorem 1.4 is widely applicable not only for Riemannian manifolds but also for discrete spaces.

In the case where X is a compact Riemannian homogeneous space, by using Theorem 1.4, we see that the cut locus of every point consists of a single point if $\mathcal{M}(X;1)$ has its maximum. Such a Riemannian manifold is called a Wiedersehen manifold and is known to be isometric to a round sphere $S^n(r)$ of some radius $r > 0$ ([**6**]). Therefore, the following corollary follows.

Corollary 1.5. *Let X be a compact Riemannian homogeneous space. Then, the 1-measurement $\mathcal{M}(X;1)$ has its maximum if and only if X is isometric to a round sphere $S^n(r)$, $r > 0$.*

2. Preliminaries.

In this section, we enumerate some basics of mm-space and prepare for describing the maximum and maximal elements of the 1-measurement. We refer to [**3**], [**5**] for more details about this section.

2.1. Some basics of mm-space.

Definition 2.1 (mm-space). Let (X, d_X) be a complete separable metric space with a Borel probability measure m_X. We call such a triple (X, d_X, m_X) an *mm-space*. We sometimes say that X is an mm-space, for which the metric and measure of X are respectively indicated by d_X and m_X.

We denote the Borel σ-algebra over X by $\mathcal{B}_X$. For any point $x \in X$, any two subsets $A, B \subset X$ and any real number $r > 0$, we define

$$\begin{aligned} d_X(x, A) &:= \inf_{y \in A} d_X(x, y), \\ d_X(A, B) &:= \inf_{x \in A,\, y \in B} d_X(x, y), \\ U_r(A) &:= \{\, y \in X \mid d_X(y, A) < r \,\}, \\ B_r(A) &:= \{\, y \in X \mid d_X(y, A) \le r \,\}. \end{aligned}$$

Let $p : X \to Y$ be a measurable map from a measure space (X, m_X) to a topological space Y. *The push-forward of* m_X *by the map* p is defined as $p_* m_X(A) := m_X(p^{-1}(A))$ for any $A \in \mathcal{B}_Y$.

Definition 2.2 (mm-isomorphism). Two mm-spaces X and Y are said to be *mm-isomorphic* to each other if there exists an isometry $f : \operatorname{supp} m_X \to \operatorname{supp} m_Y$ such that $f_* m_X = m_Y$, where $\operatorname{supp} m_X$ is the *support of* m_X. Such an isometry f is called an *mm-isomorphism*. The mm-isomorphism relation is an equivalence relation on the set of mm-spaces. Denote by $\mathcal{X}$ the set of mm-isomorphism classes of mm-spaces.

Note that X is mm-isomorphic to $(\operatorname{supp} m_X, d_X, m_X)$. We assume that any mm-space X satisfies

$$X = \operatorname{supp} m_X$$

unless otherwise stated.

We define the 1-measurement of an mm-space. This is the set of distributions of 1-Lipschitz functions.

Definition 2.3 (1-measurement). The 1-*measurement* $\mathcal{M}(X; 1)$ of an mm-space X is defined as

$$\mathcal{M}(X; 1) := \{ f_* m_X \mid f : X \to \mathbb{R} : \text{1-Lipschitz function} \}.$$

We give the definition of the Lipschitz order. We consider the maximum and maximal elements of the 1-measurement of an mm-space with respect to this order relation.

Definition 2.4 (Lipschitz order). Let X and Y be two mm-spaces. We say that X *dominates* Y and write $Y \prec X$ if there exists a 1-Lipschitz map $f : X \to Y$ satisfying

$$f_* m_X = m_Y.$$

We call the relation $\prec$ on $\mathcal{X}$ the *Lipschitz order*.

Proposition 2.5 ([**5**, Proposition 2.11]). *The Lipschitz order $\prec$ is a partial order relation on $\mathcal{X}$.*

Remark 2.6. Since an element μ of 1-measurement $\mathcal{M}(X;1)$ is a measure on the real line $\mathbb{R}$, the triple $(\mathbb{R}, d_{\mathbb{R}}, \mu)$ is an mm-space, where $d_{\mathbb{R}}$ is the Euclidean distance on $\mathbb{R}$. We define the Lipschitz order between two elements of $\mathcal{M}(X;1)$ by considering $\mu \in \mathcal{M}(X;1)$ as an mm-space in the above way. In this manner, we consider the maximum and maximal elements of the 1-measurement $\mathcal{M}(X;1)$ with respect to the Lipschitz order. For two measures $\mu, \nu \in \mathcal{M}(X;1)$, we write $\mu \prec \nu$ as $(\mathbb{R}, d_{\mathbb{R}}, \mu) \prec (\mathbb{R}, d_{\mathbb{R}}, \nu)$ for simplicity.

Remark 2.7. For a Borel probability measure μ on the real line $\mathbb{R}$, we immediately see that the measure μ is the maximum of the 1-measurement $\mathcal{M}((\mathbb{R}, d_{\mathbb{R}}, \mu);1)$.

2.2. Observable diameter and partial diameter.

Observable diameter is one of the most important invariants. We remark that this is defined by the 1-measurement.

Definition 2.8 (Partial diameter). Let X be an mm-space. For any real number $\alpha \in [0,1]$, we define the *partial diameter* $\operatorname{diam}(X;\alpha) = \operatorname{diam}(m_X;\alpha)$ *of* X as

$$\operatorname{diam}(X;\alpha) := \inf\{ \operatorname{diam} A \mid m_X(A) \geq \alpha,\ A \in \mathcal{B}_X \},$$

where the *diameter of* A is defined by $\operatorname{diam} A := \sup_{x,y \in A} d_X(x,y)$ for $A \neq \emptyset$ and $\operatorname{diam} \emptyset := 0$.

Definition 2.9 (Observable diameter). Let X be an mm-space. For any real number $\kappa \in [0,1]$, we define the *κ-observable diameter* $\operatorname{ObsDiam}(X;-\kappa)$ *of* X as

$$\operatorname{ObsDiam}(X;-\kappa) := \sup_{\mu \in \mathcal{M}(X;1)} \operatorname{diam}(\mu;1-\kappa).$$

Proposition 2.10 ([**5**, Proposition 2.18]). *Let X and Y be two mm-spaces and $\kappa \in [0,1]$ a real number. If $Y \prec X$, then we obtain*

$$\operatorname{diam}(Y;1-\kappa) \leq \operatorname{diam}(X;1-\kappa),$$
$$\operatorname{ObsDiam}(Y;-\kappa) \leq \operatorname{ObsDiam}(X;-\kappa).$$

In other words, the partial diameter and the κ-observable diameter are non-decreasing invariants with respect to the Lipschitz order. We obtain the value of the observable diameter if we know the maximum of the 1-measurement because the partial diameter is non-decreasing invariants.

2.3. Lévy's isoperimetric inequality.

Let $S^n(r)$ be the n-dimensional sphere of radius $r > 0$ centered at the origin in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. We assume the distance $d_{S^n(r)}(x, y)$ between two points x and y in $S^n(r)$ to be the geodesic distance and the measure $m_{S^n(r)}$ on $S^n(r)$ to be the Riemannian volume measure on $S^n(r)$ normalized as $m_{S^n(r)}(S^n(r)) = 1$. Then, $(S^n(r), d_{S^n(r)}, m_{S^n(r)})$ is an mm-space.

THEOREM 2.11 (Lévy's isoperimetric inequality [**1**], [**4**]). *For any closed subset $\Omega \subset S^n(1)$, we take a metric ball B_Ω of $S^n(1)$ with $m_{S^n(1)}(B_\Omega) = m_{S^n(1)}(\Omega)$. Then we have*

$$m_{S^n(1)}(U_r(\Omega)) \geq m_{S^n(1)}(U_r(B_\Omega))$$

for any $r > 0$.

2.4. Box distance.

In this subsection, we briefly describe the box distance which is needed in Subsection 3.2.

DEFINITION 2.12 (Parameter). Let $I := [0, 1)$ and let $\mathcal{L}^1$ be the one-dimensional Lebesgue measure on I. Let X be a topological space with a Borel probability measure m_X. A map $\varphi : I \to X$ is called a *parameter of X* if φ is a Borel measurable map such that

$$\varphi_* \mathcal{L}^1 = m_X.$$

DEFINITION 2.13 (Pseudo-metric). A *pseudo-metric ρ on a set S* is defined to be a function $\rho : S \times S \to [0, \infty)$ satisfying that, for any $x, y, z \in S$,

1. $\rho(x, x) = 0$,
2. $\rho(y, x) = \rho(x, y)$,
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

DEFINITION 2.14 (Box distance). For two pseudo-metrics ρ_1 and ρ_2 on $I := [0, 1)$, we define $\square(\rho_1, \rho_2)$ to be the infimum of $\varepsilon \geq 0$ satisfying that there exists a Borel subset $I_0 \subset I$ such that

1. $|\rho_1(s, t) - \rho_2(s, t)| \leq \varepsilon$ for any $s, t \in I_0$,
2. $\mathcal{L}^1(I_0) \geq 1 - \varepsilon$.

We define the *box distance $\square(X, Y)$ between two mm-spaces X and Y* to be the infimum of $\square(\varphi^* d_X, \psi^* d_Y)$, where $\varphi : I \to X$ and $\psi : I \to Y$ run over all parameters of X and Y, respectively, and where $\varphi^* d_X(s, t) := d_X(\varphi(s), \varphi(t))$ for $s, t \in I$.

THEOREM 2.15 ([**5**, Theorem 4.10]). *The box distance $\square$ is a metric on the set $\mathcal{X}$ of mm-isomorphism classes of mm-spaces.*

Proposition 2.16 ([**5**, Proposition 4.12]). *Let X be a complete separable metric space. For any two Borel probability measures μ and ν on X, we have*

$$\square((X,\mu),(X,\nu)) \le 2d_{\mathrm{P}}(\mu,\nu),$$

where d_{P} is the Prohorov distance (*see* [**5**, Definition 1.14]).

Theorem 2.17 ([**5**, Theorem 4.35]). *Let X, Y, X_n and Y_n be mm-spaces, $n = 1, 2, \dots$. If X_n and Y_n $\square$-converge to X and Y respectively as $n \to \infty$ and if $X_n \prec Y_n$ for any n, then $X \prec Y$.*

3. The maximum of the 1-measurement of n-dimensional sphere.

3.1. The maximum of the 1-measurement of n-dimensional sphere –The proof of Theorem 1.1–

The aim of this subsection is to prove Theorem 1.1. We prepare some lemmas for the proof.

Lemma 3.1. *Let X be an mm-space and $f : X \to \mathbb{R}$ a Borel measurable function. We define the function $F : \mathbb{R} \to [0,1]$ as $F(t) := f_* m_X((-\infty, t])$. If the function $F|_{\mathrm{Im}\, f} : \mathrm{Im}\, f \to [0,1]$ is bijective, then we have*

$$F_* f_* m_X((-\infty, a]) = a$$

for all $a \in [0,1]$.

Proof. For any $a \in [0,1]$, we see

$$\begin{aligned}
F_* f_* m_X((-\infty, a]) &= f_* m_X(F^{-1}((-\infty, a])) \\
&= f_* m_X(\{\, t \in \mathbb{R} \mid F(t) \le a \,\}) \\
&= f_* m_X(\{\, t \in \mathrm{Im}\, f \mid F|_{\mathrm{Im}\, f}(t) \le a \,\}) \\
&= f_* m_X(\{\, t \in \mathrm{Im}\, f \mid t \le (F|_{\mathrm{Im}\, f})^{-1}(a) \,\}) \\
&= f_* m_X(\{\, t \in \mathbb{R} \mid t \le (F|_{\mathrm{Im}\, f})^{-1}(a) \,\}) \\
&= f_* m_X((-\infty, (F|_{\mathrm{Im}\, f})^{-1}(a)]) \\
&= F((F|_{\mathrm{Im}\, f})^{-1}(a)) = a,
\end{aligned}$$

where we use the non-decreasing and bijective property of $F|_{\mathrm{Im}\, f}$ in the fourth equality. This completes the proof. $\square$

The following three lemmas are properties of generalized functions.

Lemma 3.2. *For a non-decreasing function $G : \mathbb{R} \to [0,1]$ with $G(t_0) = 0$ for some $t_0 \in \mathbb{R}$, we define $\tilde{G} : (0,1] \to \mathbb{R}$ by*

$$\tilde{G}(s) := \inf\{\, t \in \mathbb{R} \mid s \le G(t) \,\}.$$

Then, $\tilde{G}$ is non-decreasing and lower bounded on $(0,1]$. In particular, $\tilde{G}$ takes finite values on $(0,1]$.

Proof. We take a real number $t_0 \in \mathbb{R}$ satisfying $G(t_0) = 0$. Fix a real number $s \in (0,1]$ and define $A := \{ t \in \mathbb{R} \mid s \leq G(t) \}$. For any element $t \in A$, we have $G(t_0) < s \leq G(t)$. Since G is non-decreasing, the inequality $t_0 < t$ follows. This implies that $t_0 \leq \tilde{G}(s)$. The function $\tilde{G}$ is a non-decreasing function on $(0,1]$ because we have $\{ t \in \mathbb{R} \mid s' \leq G(t) \} \supset \{ t \in \mathbb{R} \mid s \leq G(t) \}$ for any $0 < s' \leq s$. This completes the proof. □

Lemma 3.3. *Let $G : \mathbb{R} \to [0,1]$ be a non-decreasing and right continuous function such that $G(t_0) = 0$ for some $t_0 \in \mathbb{R}$. We define $\tilde{G} : [0,1] \to \mathbb{R}$ by*

$$\tilde{G}(s) := \begin{cases} \inf\{ t \in \mathbb{R} \mid s \leq G(t) \} & \text{if } s \in (0,1], \\ c & \text{if } s = 0, \end{cases}$$

where c is an arbitrary constant. Then, we have

$$G \circ \tilde{G}(s) \geq s, \qquad s \in [0,1], \tag{3.1}$$

$$\tilde{G} \circ G(t) \leq t, \qquad t \in \mathbb{R} \text{ with } G(t) > 0, \tag{3.2}$$

$$\tilde{G}^{-1}((-\infty, t]) \setminus \{0\} = (0, G(t)], \qquad t \in \mathbb{R}. \tag{3.3}$$

Proof. First we prove (3.1). If $s = 0$, we have (3.1) because $\operatorname{Im} G \subset [0,1]$. Fix a real number $s \in (0,1]$ and define $A := \{ t \in \mathbb{R} \mid s \leq G(t) \}$. By the definition of infimum, we have

$$G(t') \geq \inf_{t \in A} G(t)$$

for any $t' \in A$. For any $t' > \inf A$, we have $t' \in A$ because G is non-decreasing. By this, we have

$$\lim_{t' \to \inf A + 0} G(t') \geq \inf_{t \in A} G(t).$$

By the right continuity of G, we obtain

$$G(\inf A) \geq \inf_{t \in A} G(t).$$

Therefore, we have

$$\begin{aligned} G(\tilde{G}(s)) &= G(\inf A) \\ &\geq \inf_{t \in A} G(t) \\ &= \inf\{ G(t) \mid s \leq G(t) \} \\ &\geq s. \end{aligned}$$

Next we prove (3.2). We take any real number $t \in \mathbb{R}$ with $G(t) > 0$, then we have

$$\tilde{G}(G(t)) = \inf\{\, t' \in \mathbb{R} \mid G(t') \geq G(t) \,\} \leq t.$$

Finally we prove (3.3). Take any real number $s \in \tilde{G}^{-1}((-\infty, t]) \setminus \{0\}$. Since $\tilde{G}^{-1}(\mathbb{R}) = [0,1]$, we have $s \in (0,1]$. It follows from $\tilde{G}(s) \leq t$ and the non-decreasing property of G that $G \circ \tilde{G}(s) \leq G(t)$. This implies that $s \leq G(t)$ by (3.1) and we have $s \in (0, G(t)]$. Conversely, take any real number $s \in (0, G(t)]$. We obtain $\tilde{G}(s) \leq \tilde{G} \circ G(t)$ because $\tilde{G}$ is non-decreasing by Lemma 3.2. Then we have $\tilde{G}(s) \leq t$ by (3.2). This completes the proof. □

Remark 3.4. In Lemma 3.3, $\tilde{G}$ is a Borel measurable function. In fact, $\tilde{G}|_{(0,1]}$ is Borel measurable on $(0,1]$ in view of Lemma 3.2.

Lemma 3.5. *Let $f, g : X \to \mathbb{R}$ be two Borel measurable functions and define two functions $F, G : \mathbb{R} \to [0,1]$ as $F(t) := f_* m_X((-\infty, t])$, $G(t) := g_* m_X((-\infty, t])$. We assume that some real number t_0 satisfies $G(t_0) = 0$. We define $\tilde{G} : [0,1] \to \mathbb{R}$ by*

$$\tilde{G}(s) := \begin{cases} \inf\{\, t \in \mathbb{R} \mid s \leq G(t) \,\} & \text{if } s \in (0,1], \\ c & \text{if } s = 0, \end{cases}$$

where c is an arbitrary constant. We define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi := \tilde{G} \circ F$. If $F|_{\operatorname{Im} f} : \operatorname{Im} f \to [0,1]$ is bijective, then we have

$$\varphi_* f_* m_X = g_* m_X.$$

Proof. Take any real number $t \in \mathbb{R}$. we have

$$\begin{aligned} \varphi_* f_* m_X((-\infty, t]) &= \tilde{G}_* F_* f_* m_X((-\infty, t]) \\ &= F_* f_* m_X(\tilde{G}^{-1}((-\infty, t])) \\ &= F_* f_* m_X(\tilde{G}^{-1}((-\infty, t]) \setminus \{0\}) \\ &= F_* f_* m_X((0, G(t)]) \\ &= F_* f_* m_X((-\infty, G(t)]) \\ &= G(t) \\ &= g_* m_X((-\infty, t]). \end{aligned}$$

In the third and fourth equalities, we use $F_* f_* m_X((-\infty, 0]) = 0$ obtained by Lemma 3.1. We use (3.3) of Lemma 3.3 in the fourth equality. We have the sixth equality by Lemma 3.1. This completes the proof. □

Proof of Theorem 1.1. Take a point $\bar{x} \in S^n(1)$ and define $\xi : S^n(1) \to \mathbb{R}$ by $\xi(x) := d_{S^n(1)}(\bar{x}, x)$. Take any 1-Lipschitz function $g : S^n(1) \to \mathbb{R}$. We prove the existence of a 1-Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying

$$\varphi_* \xi_* m_{S^n(1)} = g_* m_{S^n(1)}$$

in the following. Put two functions $V, G : \mathbb{R} \to [0,1]$ as $V(t) := \xi_* m_{S^n(1)}((-\infty, t])$, $G(t) := g_* m_{S^n(1)}((-\infty, t])$. We define $\tilde{G} : [0,1] \to \mathbb{R}$ as

$$\tilde{G}(s) := \inf\{\, t \in \mathbb{R} \mid s \le G(t)\,\}$$

if $s \in (0,1]$, and

$$\tilde{G}(0) := \lim_{s \to +0} \tilde{G}(s), \tag{3.4}$$

if $s = 0$. We have $G(t_0) = 0$ for some t_0 because g has a lower bound. The existence of limit of (3.4) is guaranteed because G is non-decreasing and $\tilde{G}$ has a lower bound on $(0,1]$ by Lemma 3.2. Put $\varphi : \mathbb{R} \to \mathbb{R}$ as $\varphi := \tilde{G} \circ V$. Since $V|_{\mathrm{Im}\,\xi}$ is bijective, we apply Lemma 3.5 to obtain

$$\varphi_* \xi_* m_{S^n(1)} = g_* m_{S^n(1)}.$$

Let us prove that φ is a 1-Lipschitz function. If $t \le 0$, we have $\varphi(t) = \tilde{G}(0)$ by $V(t) = 0$. We obtain

$$\lim_{t \to +0} \varphi(t) = \lim_{t \to +0} \tilde{G} \circ V(t) = \tilde{G}(0)$$

because $\tilde{G}$ is continuous at 0 and $\lim_{t \to +0} V(t) = 0$. By this, we prove φ is a 1-Lipschitz function in the case where $t > 0$. The function φ is non-decreasing since the two functions $\tilde{G}$ and V are both non-decreasing. Thus, it is sufficient to prove that $\varphi(t+\varepsilon) \le \varphi(t) + \varepsilon$ for any $\varepsilon > 0$. Fix $t > 0$ and take any $\varepsilon > 0$. We have

$$\begin{aligned} m_{S^n(1)}(B_t(\bar{x})) &= \xi_* m_{S^n(1)}((-\infty, t]) \\ &= V(t) \\ &\le (G \circ \tilde{G})(V(t)) \\ &= G \circ \varphi(t) \\ &= m_{S^n(1)}(g^{-1}((-\infty, \varphi(t)])), \end{aligned}$$

where we use (3.1) of Lemma 3.3 in the inequality on the third line. We obtain

$$m_{S^n(1)}(B_{t+\varepsilon}(\bar{x})) \le m_{S^n(1)}(B_\varepsilon(g^{-1}((-\infty, \varphi(t)])))$$

by applying Theorem 2.11 (Lévy's isoperimetric inequality). We use this inequality to obtain

$$\begin{aligned} V(t+\varepsilon) &= \xi_* m_{S^n(1)}((-\infty, t+\varepsilon]) \\ &= m_{S^n(1)}(B_{t+\varepsilon}(\bar{x})) \\ &\le m_{S^n(1)}(B_\varepsilon(g^{-1}((-\infty, \varphi(t)]))) \\ &\le m_{S^n(1)}(g^{-1}(B_\varepsilon((-\infty, \varphi(t)]))) \\ &= g_* m_{S^n(1)}((-\infty, \varphi(t) + \varepsilon]) \end{aligned}$$

$$= G(\varphi(t) + \varepsilon),$$

where we have the inequality on the fourth line because g is a 1-Lipschitz function. Therefore, we have

$$\begin{aligned}\varphi(t+\varepsilon) &= \tilde{G} \circ V(t+\varepsilon) \\ &\leq \tilde{G} \circ G(\varphi(t) + \varepsilon) \\ &\leq \varphi(t) + \varepsilon,\end{aligned}$$

where we use (3.2) of Lemma 3.3 in the inequality of the third line. This completes the proof. □

3.2. The relation between the normal law à la Lévy and Theorem 1.1.

The aim of this section is to prove Corollary 3.6 by Theorem 1.1. It is one of the important applications of Theorem 1.1.

Corollary 3.6 (Normal law à la Lévy [**3**], [**5**]). *Let $f_n : S^n(\sqrt{n}) \to \mathbb{R}, n = 1, 2, \ldots,$ be 1-Lipschitz functions. Assume that a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ satisfies that the push-forward $(f_{n_i})_* m_{S^{n_i}(\sqrt{n_i})}$ converges weakly to a Borel probability measure σ. Then we have*

$$(\mathbb{R}, d_{\mathbb{R}}, \sigma) \prec (\mathbb{R}, d_{\mathbb{R}}, \gamma^1).$$

Here γ^1 is the one-dimensional standard Gaussian distribution on $\mathbb{R}$, i.e., $\gamma^1(dr) = (1/\sqrt{2\pi})e^{-r^2/2}$.

We prepare some lemmas to prove Corollary 3.6. We use the following three lemmas to prove $\square((\mathbb{R}, d_{\mathbb{R}}, (\xi_n)_* m_{S^n(\sqrt{n})}), (\mathbb{R}, d_{\mathbb{R}}, \gamma^1)) \to 0$.

Lemma 3.7. *For any real number $r \in \mathbb{R}$, we have*

$$\cos^{n-1} \frac{r}{\sqrt{n}} \to e^{-r^2/2} \quad \text{as } n \to \infty.$$

Proof. If $r = 0$, then the lemma is trivial. Assume $r \neq 0$. We first prove $\liminf_{n\to\infty} \cos^{n-1}(r/\sqrt{n}) \geq e^{-r^2/2}$. We use $\cos x \geq 1 - x^2/2$ for any $x \in [-\pi/2, \pi/2]$. For some positive integer $N \in \mathbb{N}$, we have $r \in [-(\pi/2)\sqrt{n}, (\pi/2)\sqrt{n}]$ for any positive integer $n \geq N$. Then, we have

$$\cos^{n-1} \frac{r}{\sqrt{n}} \geq \left(1 - \frac{1}{2}\left(\frac{r}{\sqrt{n}}\right)^2\right)^{n-1}$$

for any positive integer $n \geq N$. We obtain $\liminf_{n\to\infty} \cos^{n-1}(r/\sqrt{n}) \geq e^{-r^2/2}$ because we have

$$\left(1 - \frac{1}{2}\left(\frac{r}{\sqrt{n}}\right)^2\right)^{n-1} = \left(1 - \frac{r^2}{2n}\right)^{(-2n/r^2)\cdot(-r^2/2)-1} \to e^{-r^2/2} \quad \text{as } n \to \infty.$$

We next prove $\limsup_{n\to\infty} \cos^{n-1}(r/\sqrt{n}) \le e^{-r^2/2}$. Take any real number $\varepsilon \in (0,1/2)$. Since $\lim_{x\to 0}(1-\cos x)/x^2 = 1/2$, there exists some $\delta > 0$ such that $\cos x \le 1-(1/2-\varepsilon)x^2$ for any $x \in (-\delta,\delta)$. We take some positive integer $N \in \mathbb{N}$ satisfying $|r/\sqrt{N}| < \delta$. For any positive integer $n \ge N$, we have

$$\cos^{n-1}\frac{r}{\sqrt{n}} \le \left(1-\frac{1-2\varepsilon}{2}\left(\frac{r}{\sqrt{n}}\right)^2\right)^{n-1}.$$

Since we have

$$\begin{aligned}\left(1-\frac{1-2\varepsilon}{2}\left(\frac{r}{\sqrt{n}}\right)^2\right)^{n-1} &= \left(1-\frac{1-2\varepsilon}{2}\cdot\frac{r^2}{n}\right)^{(-2n/((1-2\varepsilon)r^2))\cdot(-(1-2\varepsilon)r^2/2)-1} \\ &\to e^{(-r^2/2)\cdot(1-2\varepsilon)} \quad \text{as } n\to\infty\end{aligned}$$

and $e^{(-r^2/2)\cdot(1-2\varepsilon)} \to e^{-r^2/2}$ as $\varepsilon \to +0$, we obtain $\limsup_{n\to\infty}\cos^{n-1}(r/\sqrt{n}) \le e^{-r^2/2}$. This completes the proof. □

To use Lebesgue's dominated convergence theorem in the proof of Lemma 3.8 below, we prove the following inequality.

LEMMA 3.8. *For any integer* $n \ge 2$ *and any real number* $r \in [-(\pi/2)\sqrt{n}, (\pi/2)\sqrt{n}\,]$, *we have*

$$\cos^{n-1}\frac{r}{\sqrt{n}} \le e^{-r^2/4}.$$

PROOF. Take any integer $n \ge 2$. This lemma is clear if $r = \pm(\pi/2)\sqrt{n}$. Then, we prove the lemma in the case $r \in (-(\pi/2)\sqrt{n}, (\pi/2)\sqrt{n})$. By the symmetry, we may assume $r \ge 0$. Setting

$$f(r) := -\frac{r^2}{4} - (n-1)\log\cos\frac{r}{\sqrt{n}},$$

we have

$$\begin{aligned} f'(r) &= -\frac{r}{2} + (n-1)\cdot\frac{1}{\sqrt{n}}\tan\frac{r}{\sqrt{n}} \\ &= -\frac{\sqrt{n}}{2}\cdot\frac{r}{\sqrt{n}} + \left(\sqrt{n}-\frac{1}{\sqrt{n}}\right)\tan\frac{r}{\sqrt{n}} \\ &\ge \frac{\sqrt{n}}{2}\left(\tan\frac{r}{\sqrt{n}} - \frac{r}{\sqrt{n}}\right) \ge 0, \end{aligned}$$

where we use $n \ge 2$ in the first inequality and $r/\sqrt{n} \in [0,\pi/2)$ in the second inequality. Since $f(0) = 0$, we obtain $f(r) \ge 0$ for any $r \in [0,(\pi/2)\sqrt{n})$. This completes the proof. □

Lemma 3.9. *Fix a point* $\bar{x}_n \in S^n(\sqrt{n})$, *and define* $\xi_n : S^n(\sqrt{n}) \to \mathbb{R}$ *as* $\xi_n(x) := d_{S^n(\sqrt{n})}(x, \bar{x}_n)$ *for* $x \in S^n(\sqrt{n})$. *Then we have*

$$\frac{d((\xi_n - \sqrt{n}(\pi/2))_* m_{S^n(\sqrt{n})})}{d\mathcal{L}^1}(r) \to \frac{d\gamma^1}{d\mathcal{L}^1}(r) \text{ as } n \to \infty$$

for any $r \in \mathbb{R}$, *where we define* γ^1 *as*

$$\frac{d\gamma^1}{d\mathcal{L}^1}(r) := \frac{1}{\sqrt{2\pi}} e^{-r^2/2}.$$

In particular, we have

$$\left(\xi_n - \sqrt{n}\frac{\pi}{2}\right)_* m_{S^n(\sqrt{n})} \to \gamma^1 \quad \text{as } n \to \infty \text{ weakly.}$$

Proof. Let $\chi_{[-(\pi/2)\sqrt{n}, (\pi/2)\sqrt{n}]}$ be the indicator function of the subset $[-(\pi/2)\sqrt{n}, (\pi/2)\sqrt{n}]$. We have

$$\frac{d((\xi_n - \sqrt{n}(\pi/2))_* m_{S^n(\sqrt{n})})}{d\mathcal{L}^1}(r) = \chi_{[-(\pi/2)\sqrt{n}, (\pi/2)\sqrt{n}]} \cdot \frac{\cos^{n-1}(r/\sqrt{n})}{\int_{-(\pi/2)\sqrt{n}}^{(\pi/2)\sqrt{n}} \cos^{n-1}(t/\sqrt{n}) dt}$$

for any real number $r \in \mathbb{R}$. We obtain

$$\chi_{[-(\pi/2)\sqrt{n}, (\pi/2)\sqrt{n}]} \cdot \frac{\cos^{n-1}(r/\sqrt{n})}{\int_{-(\pi/2)\sqrt{n}}^{(\pi/2)\sqrt{n}} \cos^{n-1}(t/\sqrt{n}) dt} \to \frac{e^{-r^2/2}}{\int_{\mathbb{R}} e^{-t^2/2} dt} \quad \text{as } n \to \infty$$

because of Lebesgue's dominated convergence theorem, Lemmas 3.7 and 3.8. This completes the proof. □

Proof of Corollary 3.6. Take any 1-Lipschitz functions $f_n : S^n(\sqrt{n}) \to \mathbb{R}$, $n = 1, 2, \ldots$. We may assume $\square((\mathbb{R}, d_{\mathbb{R}}, (f_{n_i})_* m_{S^{n_i}(\sqrt{n_i})}), (\mathbb{R}, d_{\mathbb{R}}, \sigma)) \to 0 \quad \text{as } i \to \infty$ because of Proposition 2.16. Take a point $\bar{x}_n \in S^n(\sqrt{n})$ and define $\xi_n(x) := d_{S^n(\sqrt{n})}(x, \bar{x}_n)$. By applying Theorem 1.1, we have $(\mathbb{R}, d_{\mathbb{R}}, (f_n)_* m_{S^n(\sqrt{n})}) \prec (\mathbb{R}, d_{\mathbb{R}}, (\xi_n)_* m_{S^n(\sqrt{n})})$ for any positive integer $n \in \mathbb{N}$. Since we have $\square((\mathbb{R}, d_{\mathbb{R}}, (\xi_n)_* m_{S^n(\sqrt{n})}), (\mathbb{R}, d_{\mathbb{R}}, \gamma^1)) \to 0$ by Lemma 3.9, we obtain $(\mathbb{R}, d_{\mathbb{R}}, \sigma) \prec (\mathbb{R}, d_{\mathbb{R}}, \gamma^1)$ by Theorem 2.17. This completes the proof. □

4. A necessary condition for the existence of the maximum of the 1-measurement.

The aim of this section is to prove Theorem 1.2 and Theorem 1.4. Theorem 1.4 gives a necessary condition for the existence of the maximum of the 1-measurement.

4.1. Maximal elements of the 1-measurement.

In this subsection, we prove Theorem 1.2. We need the following lemma for the proof of Theorem 1.2.

LEMMA 4.1. *Let μ, ν be two Borel probability measures on $\mathbb{R}$. We assume $\operatorname{diam}\operatorname{supp}\mu = \operatorname{diam}\operatorname{supp}\nu < \infty$ and $\mu \prec \nu$. Then, two mm-spaces $(\mathbb{R}, d_{\mathbb{R}}, \mu)$ and $(\mathbb{R}, d_{\mathbb{R}}, \nu)$ are mm-isomorphic to each other.*

PROOF. Since $\mu \prec \nu$, there exists a 1-Lipschitz function $\varphi : \operatorname{supp}\nu \to \operatorname{supp}\mu$ such that $\varphi_*\nu = \mu$. Put $c := \operatorname{diam}\operatorname{supp}\mu = \operatorname{diam}\operatorname{supp}\nu$, $y_0 := \min\operatorname{supp}\mu$ and $y_1 := \max\operatorname{supp}\mu$. We have $c = y_1 - y_0$. Since φ is surjective, there exists $x_i \in \operatorname{supp}\nu$ such that $\varphi(x_i) = y_i$ for each $i = 0, 1$. We have

$$c = y_1 - y_0 = \varphi(x_1) - \varphi(x_0) \le |x_1 - x_0| \le c$$

because φ is a 1-Lipschitz function. Therefore we obtain $|x_1 - x_0| = c$. In particular, the point x_0 is the maximum or the minimum of $\operatorname{supp}\nu$. We prove $\varphi(x) = y_0 + |x - x_0|$ for any $x \in \operatorname{supp}\nu$ in the following. If this is true, then we see that φ is an isometry and the proof of the lemma is completed. Let us first prove it in the case where $x_0 \le x_1$. We have $x_0 = \min\operatorname{supp}\nu,\ x_1 = \max\operatorname{supp}\nu$ because $x_1 = x_0 + c$. We obtain

$$\varphi(x) - y_0 = \varphi(x) - \varphi(x_0) \le x - x_0$$

for any $x \in \operatorname{supp}\nu$. This implies that $\varphi(x) \le y_0 + |x - x_0|$. We have

$$\begin{aligned}(y_0 + c) - \varphi(x) &= y_1 - \varphi(x)\\ &= \varphi(x_1) - \varphi(x)\\ &\le x_1 - x\\ &= x_0 + c - x.\end{aligned}$$

We also have $\varphi(x) \ge y_0 + |x - x_0|$. We next prove it in the case where $x_0 \ge x_1$ similarly. In fact, we have $\varphi(x) \le |x - x_0| + y_0$ because

$$\varphi(x) - y_0 = \varphi(x) - \varphi(x_0) \le |x - x_0|$$

and we have $\varphi(x) \ge |x - x_0| + y_0$ because

$$\begin{aligned}x_0 - x_1 + y_0 - \varphi(x) &= c + y_0 - \varphi(x)\\ &= y_1 - \varphi(x)\\ &= \varphi(x_1) - \varphi(x)\\ &\le x - x_1.\end{aligned}$$

This completes the proof. □

PROOF OF THEOREM 1.2. Take a measure $\mu \in \mathcal{M}(X;1)$ with $\operatorname{diam}\operatorname{supp}\mu = \operatorname{diam} X < \infty$ and a measure $\nu \in \mathcal{M}(X;1)$ with $\mu \prec \nu$. We have $(\mathbb{R}, d_{\mathbb{R}}, \nu) \prec (X, d_X, m_X)$ because $\nu \in \mathcal{M}(X;1)$. By Proposition 2.10, we have $\operatorname{diam}\operatorname{supp}\nu = \operatorname{diam}(\nu;1) \le \operatorname{diam}(X;1) = \operatorname{diam}\operatorname{supp} m_X = \operatorname{diam} X$. By $\mu \prec \nu$ and Proposition 2.10, we also have $\operatorname{diam}\operatorname{supp}\mu = \operatorname{diam}(\mu;1) \le \operatorname{diam}(\nu;1) = \operatorname{diam}\operatorname{supp}\nu$. Two mm-spaces $(\mathbb{R}, d_{\mathbb{R}}, \mu)$ and

$(\mathbb{R}, d_{\mathbb{R}}, \nu)$ are mm-isomorphic to each other because of $\operatorname{diam} \operatorname{supp} \mu = \operatorname{diam} \operatorname{supp} \nu$ and Lemma 4.1. This completes the proof. $\square$

Proof of Corollary 1.3. Let a point $x_0 \in X$ satisfy $\sup_{x \in X} d_X(x, x_0) = \operatorname{diam} X$ and ξ be the distance function from the point x_0. We take some sequence $\{x_n\}_{n=1}^{\infty}$ of X satisfying $d_X(x_n, x_0) \to \operatorname{diam} X$ as $n \to \infty$. By Theorem 1.2, it is sufficient to prove that $\operatorname{diam} \operatorname{supp} \xi_* m_X = \operatorname{diam} X$. We have $\operatorname{supp} \xi_* m_X = \overline{\xi(\operatorname{supp} m_X)}$ because ξ is continuous, where $\overline{\xi(\operatorname{supp} m_X)}$ is the closure of $\xi(\operatorname{supp} m_X)$. Using this equality, we obtain $\operatorname{diam} \operatorname{supp} \xi_* m_X = \operatorname{diam} \xi(\operatorname{supp} m_X) \geq |\xi(x_n) - \xi(x_0)| = d_X(x_n, x_0) \to \operatorname{diam} X$ as $n \to \infty$. Remark that we always assume $X = \operatorname{supp} m_X$. On the other hand, by the 1-Lipschitz continuity of ξ, we have $\operatorname{diam} \operatorname{supp} \xi_* m_X = \operatorname{diam} \xi(\operatorname{supp} m_X) \leq \operatorname{diam} X$. $\square$

4.2. A necessary condition for the existence of the maximum of the 1-measurement.

In this subsection, we prove Theorem 1.4 by using Theorem 1.2.

Proof of Theorem 1.4. Without loss of generality, we may assume $\operatorname{diam} X > 0$. Otherwise, the assertion is trivial. We prove the contrapositive proposition of the theorem. Take three points $x_0, x_1, x_2 \in X$ satisfying $d_X(x_0, x_1) = \operatorname{diam} X$ and $d_X(x_0, x_2) + d_X(x_2, x_1) > d_X(x_0, x_1)$. Put $r_i := d_X(x_i, x_2), i = 0, 1$, $R := \operatorname{diam} X$ and $D := (r_0 + r_1 - R)/2 > 0$. We have $r_i - D > 0,\ i = 0, 1$ and $(r_0 - D) + (r_1 - D) = R$. By the symmetry, we may assume $r_1 \leq r_0$. Put a function $\xi : X \to \mathbb{R}$ as $\xi(x) := d_X(x, x_0)$ for $x \in X$ and define a function $\zeta : X \to \mathbb{R}$ by

$$\zeta(x) := \begin{cases} d_X(x, x_0) & \text{if } x \in U_{r_0 - D}(x_0), \\ R - d_X(x, x_1) & \text{if } x \in U_{r_1 - D}(x_1), \\ r_0 - D & \text{otherwise} \end{cases}$$

for $x \in X$. Let us prove that ζ is a 1-Lipschitz function. For any two points $x \in U_{r_0 - D}(x_0)$ and $y \in U_{r_1 - D}(x_1)$, we have

$$\begin{aligned} |\zeta(x) - \zeta(y)| &= |d_X(x, x_0) - R + d_X(y, x_1)| \\ &= R - d_X(x, x_0) - d_X(y, x_1) \\ &= d_X(x_0, x_1) - d_X(x, x_0) - d_X(y, x_1) \\ &\leq d_X(x, y). \end{aligned}$$

For $x \in U_{r_0 - D}(x_0)$ and $y \in U_{r_0 - D}(x_0)^c \cap U_{r_1 - D}(x_1)^c$, we have

$$\begin{aligned} |\zeta(x) - \zeta(y)| &= |d_X(x, x_0) - (r_0 - D)| \\ &= -d_X(x, x_0) + r_0 - D \\ &\leq -d_X(x, x_0) + d_X(x_0, y) \\ &\leq d_X(x, y). \end{aligned}$$

For $x \in U_{r_1 - D}(x_1)$ and $y \in U_{r_0 - D}(x_0)^c \cap U_{r_1 - D}(x_1)^c$, we have

$$\begin{aligned}|\zeta(x)-\zeta(y)| &= |(R-d_X(x,x_1))-(r_0-D)| \\ &= |(R-d_X(x,x_1))-(R-r_1+D)| \\ &= |r_1-D-d_X(x,x_1)| \\ &= r_1-D-d_X(x,x_1) \\ &\leq d_X(y,x_1)-d_X(x,x_1) \\ &\leq d_X(x,y).\end{aligned}$$

Thus, the function ζ is a 1-Lipschitz function.

We prove $\operatorname{diam}\operatorname{supp}\xi_*m_X = \operatorname{diam}\operatorname{supp}\zeta_*m_X = \operatorname{diam}X$ in the same way as the proof of Corollary 1.3. In fact, we have $\operatorname{diam}\operatorname{supp}\xi_*m_X = \operatorname{diam}\xi(\operatorname{supp}m_X) \geq |\xi(x_1)-\xi(x_0)| = \operatorname{diam}X$ and $\operatorname{diam}\operatorname{supp}\zeta_*m_X \geq |\zeta(x_1)-\zeta(x_0)| = \operatorname{diam}X$. Therefore, two measures ξ_*m_X and ζ_*m_X are both maximal elements by Theorem 1.2. Let us prove that two measures ξ_*m_X and ζ_*m_X are not mm-isomorphic to each other. It is sufficient to prove that $\xi_*m_X \neq \zeta_*m_X$ and $(R-\xi)_*m_X \neq \zeta_*m_X$. We prove those by contradiction. We first assume $\xi_*m_X = \zeta_*m_X$. Then we have

$$\begin{aligned}m_X(B_{r_0-D}(x_0)\sqcup U_D(x_2)) &\leq m_X(U_{r_1-D}(x_1)^c) \\ &= \zeta_*m_X([0,r_0-D]) \\ &= \xi_*m_X([0,r_0-D]) \\ &= m_X(B_{r_0-D}(x_0)).\end{aligned}$$

This inequality contradicts $m_X(U_D(x_2))>0$. We next assume $(R-\xi)_*m_X = \zeta_*m_X$ and then we have

$$\begin{aligned}m_X(U_{r_1-D}(x_1))\sqcup U_D(x_2)) &\leq m_X(B_{r_0-D}(x_0)^c) \\ &= \xi_*m_X((r_0-D,R]) \\ &= (R-\xi)_*m_X([0,r_1-D)) \\ &= \zeta_*m_X([0,r_1-D)) \\ &= \xi_*m_X([0,r_1-D)) \\ &= (R-\xi)_*m_X((r_0-D,R]) \\ &= \zeta_*m_X((r_0-D,R]) \\ &= m_X(U_{r_1-D}(x_1)),\end{aligned}$$

where we use $r_1-D \leq r_0-D$ in the equality on the fifth line. This inequality contradicts $m_X(U_D(x_2))>0$. This completes the proof. □

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