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The Gordian distance of handlebody-knots and Alexander biquandle colorings

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Abstract. We give lower bounds for the Gordian distance and the unknotting number of handlebody-knots by using Alexander biquandle colorings. We construct handlebody-knots with Gordian distance n and unknotting number n for any positive integer n.

1. Introduction.

The Gordian distance of two classical knots is the minimal number of crossing changes needed to be deformed each other. In particular, we call the Gordian distance of a classical knot and the trivial one the unknotting number of the classical knot. Clark, Elhamdadi, Saito and Yeatman [**2**] gave a lower bound for the Nakanishi index [**16**], which induced a lower bound for the unknotting number of classical knots. This is a generalization of the Przytycki's result [**17**]. In this paper, we give lower bounds for the Gordian distance and the unknotting number of handlebody-knots, which is a generalization of a classical knot with respect to a genus.

Ishii [**4**] introduced an enhanced constituent link of a spatial trivalent graph, and Ishii and Iwakiri [**6**] introduced an A-flow of a spatial graph, where A is an abelian group, to define colorings and invariants of handlebody-knots. Iwakiri [**12**] gave a lower bound for the unknotting number of handlebody-knots by using Alexander quandle colorings of its $\mathbb{Z}_2$ or $\mathbb{Z}_3$-flowed diagram. Ishii, Iwakiri, Jang and Oshiro [**7**] introduced a G-family of quandles, which is an extension of the above structures. Recently, Ishii and Nelson [**11**] introduced a G-family of biquandles, which is a biquandle version of a G-family of quandles.

In this paper, we extend the result in [**12**] in three directions. First, we extend from $\mathbb{Z}_2$, $\mathbb{Z}_3$-flows to any $\mathbb{Z}_m$-flow. Second, we extend from quandles to biquandles. Finally, we extend from unknotting numbers to Gordian distances. Thus we can determine the Gordian distance and the unknotting number of handlebody-knots more efficiently. We construct handlebody-knots with Gordian distance n and unknotting number n for any $n \in \mathbb{Z}_{>0}$ and note that one of them can not be obtained by using Alexander quandle colorings introduced in [**12**].

This paper is organized into seven sections. In Section 2, we recall the definition of a handlebody-knot and introduce the Gordian distance and the unknotting number of handlebody-knots. In Section 3, we recall the definition of a (bi)quandle and a G-family

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of (bi)quandles. In Section 4, we introduce a coloring of a diagram of a handlebody-knot by using a G-family of biquandles. In Section 5, we show that there are linear relationships for Alexander biquandle colorings of a diagram of a handlebody-knot. In Section 6, we give lower bounds for the Gordian distance and the unknotting number of handlebody-knots by using $\mathbb{Z}_m$-family of Alexander biquandles colorings. In section 7, we construct handlebody-knots with Gordian distance n and unknotting number n for any $n \in \mathbb{Z}_{>0}$. Moreover, we note that one of them can not be obtained by using Alexander quandle colorings with $\mathbb{Z}_2, \mathbb{Z}_3$-flows introduced in [**12**].

2. The Gordian distance of handlebody-knots.

A *handlebody-link*, which is introduced in [**4**], is the disjoint union of handlebodies embedded in the 3-sphere S^3. A *handlebody-knot* is a handlebody-link with one component. In this paper, we assume that every component of a handlebody-link is of genus at least 1. An S^1*-orientation* of a handlebody-link is an orientation of all genus 1 components of the handlebody-link, where an orientation of a solid torus is an orientation of its core S^1. Two S^1-oriented handlebody-links are *equivalent* if there exists an orientation-preserving self-homeomorphism of S^3 sending one to the other preserving the S^1-orientation.

A *spatial trivalent graph* is a graph whose vertices are valency 3 embedded in S^3. In this paper, a trivalent graph may have a circle component, which has no vertices. A *Y-orientation* of a spatial trivalent graph is a direction of all edges of the graph satisfying that every vertex of the graph is both the initial vertex of a directed edge and the terminal vertex of a directed edge (Figure 1). A vertex of a Y-oriented spatial trivalent graph can be allocated a sign; the vertex is said to be positive or negative, or to have sign $+1$ or -1. The standard convention is shown in Figure 1. For a Y-oriented spatial trivalent graph K and an S^1-oriented handlebody-link H, we say that K *represents* H if H is a regular neighborhood of K and the S^1-orientation of H agrees with the Y-orientation. Then any S^1-oriented handlebody-link can be represented by some Y-oriented spatial trivalent graph. We define a *diagram* of an S^1-oriented handlebody-link by a diagram of a Y-oriented spatial trivalent graph representing the handlebody-link. An S^1-oriented handlebody-link is *trivial* if it has a diagram with no crossings. Then the following theorem holds.

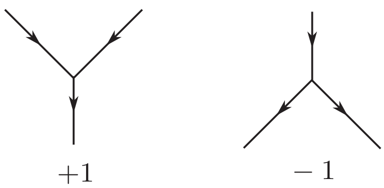


Figure 1. Y-orientations and signs.

Theorem 2.1 ([**5**]). *For a diagram D_i of an S^1-oriented handlebody-link H_i ($i = 1, 2$), H_1 and H_2 are equivalent if and only if D_1 and D_2 are related by a finite sequence*

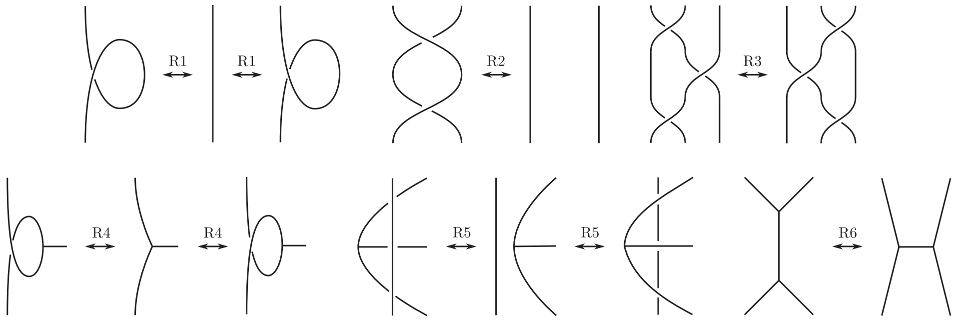


Figure 2. The Reidemeister moves for handlebody-links.

of R1–R6 moves depicted in Figure 2 preserving Y-orientations.

In this paper, for a diagram D of an S^1-oriented handlebody-link, we denote by $\mathcal{A}(D)$ and $\mathcal{SA}(D)$ the set of all arcs of D and the one of all semi-arcs of D respectively, where a semi-arc is a piece of a curve each of whose endpoints is a crossing or a vertex. An orientation of a (semi-)arc of D is also represented by the normal orientation obtained by rotating the usual orientation counterclockwise by $\pi/2$ on the diagram. For any $m \in \mathbb{Z}_{\geq 0}$, we put $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$.

A *crossing change* of an S^1-oriented handlebody-link H is that of a spatial trivalent graph representing H. This deformation can be realized by switching two handles depicted in Figure 3. It is easy to see that any two S^1-oriented handlebody-knots of the same genus can be related by a finite sequence of crossing changes. For any two S^1-oriented handlebody-knots H_1 and H_2 of the same genus, we define their *Gordian distance* $d(H_1, H_2)$ by the minimal number of crossing changes needed to be deformed each other. In particular, for any S^1-oriented handlebody-knot H and the S^1-oriented trivial handlebody-knot O of the same genus, we define $u(H) := d(H, O)$, which is called the *unknotting number* of H.

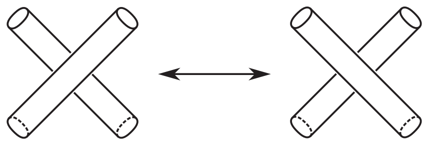


Figure 3. A crossing change of an S^1-oriented handlebody-link.

3. A biquandle and a G-family of biquandles.

We recall the definitions of a quandle and a biquandle.

Definition 3.1 ([**13**], [**14**]). A quandle is a non-empty set X with a binary operation $* : X \times X \to X$ satisfying the following axioms.

- For any $x \in X$, $x * x = x$.

- For any $x \in X$, the map $S_x : X \to X$ defined by $S_x(y) = y * x$ is a bijection.
- For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

Definition 3.2 ([**3**]). A biquandle is a non-empty set X with binary operations $\overline{*}, \underline{*} : X \times X \to X$ satisfying the following axioms.

- For any $x \in X$, $x \,\underline{*}\, x = x \,\overline{*}\, x$.
- For any $x \in X$, the map $\underline{S}_x : X \to X$ defined by $\underline{S}_x(y) = y \,\underline{*}\, x$ is a bijection.

  For any $x \in X$, the map $\overline{S}_x : X \to X$ defined by $\overline{S}_x(y) = y \,\overline{*}\, x$ is a bijection.

  The map $S : X \times X \to X \times X$ defined by $S(x, y) = (y \,\overline{*}\, x, x \,\underline{*}\, y)$ is a bijection.
- For any $x, y, z \in X$,

$$
\begin{aligned}
(x \,\underline{*}\, y) \,\underline{*}\, (z \,\underline{*}\, y) &= (x \,\underline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z), \\
(x \,\underline{*}\, y) \,\overline{*}\, (z \,\underline{*}\, y) &= (x \,\overline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z), \\
(x \,\overline{*}\, y) \,\overline{*}\, (z \,\overline{*}\, y) &= (x \,\overline{*}\, z) \,\overline{*}\, (y \,\underline{*}\, z).
\end{aligned}
$$

We define $\underline{*}^n x := \underline{S}_x^n$ and $\overline{*}^n x := \overline{S}_x^n$ for any $n \in \mathbb{Z}$. We note that $(X, *)$ is a quandle if and only if $(X, *, \overline{*})$ is a biquandle with $x \,\overline{*}\, y = x$. For any $m \in \mathbb{Z}_{\geq 0}$, a $\mathbb{Z}_m[s^{\pm 1}, t^{\pm 1}]$-module X is a biquandle with $a \,\underline{*}\, b = ta + (s - t)b$ and $a \,\overline{*}\, b = sa$, which we call an *Alexander biquandle*. When $s = 1$, an Alexander biquandle coincides with an Alexander quandle.

Definition 3.3 ([**8**]). Let X be a biquandle. We define two families of binary operations $\underline{*}^{[n]}, \overline{*}^{[n]} : X \times X \to X (n \in \mathbb{Z})$ by the equalities

$$
\begin{aligned}
&a \,\underline{*}^{[0]}\, b = a, \ \ a \,\underline{*}^{[1]}\, b = a \,\underline{*}\, b, \ \ a \,\underline{*}^{[i+j]}\, b = (a \,\underline{*}^{[i]}\, b) \,\underline{*}^{[j]}\, (b \,\underline{*}^{[i]}\, b), \\
&a \,\overline{*}^{[0]}\, b = a, \ \ a \,\overline{*}^{[1]}\, b = a \,\overline{*}\, b, \ \ a \,\overline{*}^{[i+j]}\, b = (a \,\overline{*}^{[i]}\, b) \,\overline{*}^{[j]}\, (b \,\overline{*}^{[i]}\, b)
\end{aligned}
$$

for any $i, j \in \mathbb{Z}$.

Since $a = a \,\underline{*}^{[0]}\, b = (a \,\underline{*}^{[-1]}\, b) \,\underline{*}^{[1]}\, (b \,\underline{*}^{[-1]}\, b) = (a \,\underline{*}^{[-1]}\, b) \,\underline{*}\, (b \,\underline{*}^{[-1]}\, b)$, we have $a \underline{*}^{[-1]} b = a \underline{*}^{-1} (b \underline{*}^{[-1]} b)$ and $(b \underline{*}^{[-1]} b) \underline{*} (b \underline{*}^{[-1]} b) = b$. Then for an Alexander biquandle X, we have $a \,\underline{*}^{[n]}\, b = t^n a + (s^n - t^n)b$ and $a \,\overline{*}^{[n]}\, b = s^n a$ for any $a, b \in X$.

We define the *type* of a biquandle X by

$$
\operatorname{type} X = \min\{n > 0 \mid a \,\underline{*}^{[n]}\, b = a = a \,\overline{*}^{[n]}\, b \ (\forall a, b \in X)\}.
$$

Any finite biquandle is of finite type [**11**].

We also recall the definitions of a G-family of quandles and a G-family of biquandles.

Definition 3.4 ([**7**]). Let G be a group with the identity element e. A G-family of quandles is a non-empty set X with a family of binary operations $*^g : X \times X \to X$ $(g \in G)$ satisfying the following axioms.

- For any $x \in X$ and $g \in G$, $x *^g x = x$.
- For any $x, y \in X$ and $g, h \in G$, $x *^{gh} y = (x *^g y) *^h y$ and $x *^e y = x$.
- For any $x, y, z \in X$ and $g, h \in G$, $(x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z)$.

Definition 3.5 ([**8**], [**11**]). Let G be a group with the identity element e. A G-family of biquandles is a non-empty set X with two families of binary operations $\underline{*}^g, \overline{*}^g : X \times X \to X$ $(g \in G)$ satisfying the following axioms.

- For any $x \in X$ and $g \in G$,

$$x \,\underline{*}^g\, x = x \,\overline{*}^g\, x.$$

- For any $x, y \in X$ and $g, h \in G$,

$$x \,\underline{*}^{gh}\, y = (x \,\underline{*}^g\, y) \,\underline{*}^h\, (y \,\underline{*}^g\, y), \;\; x \,\underline{*}^e\, y = x,$$
$$x \,\overline{*}^{gh}\, y = (x \,\overline{*}^g\, y) \,\overline{*}^h\, (y \,\overline{*}^g\, y), \;\; x \,\overline{*}^e\, y = x.$$

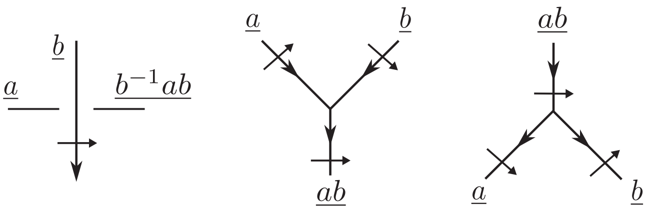
- For any $x, y, z \in X$ and $g, h \in G$,

$$(x \,\underline{*}^g\, y) \,\underline{*}^h\, (z \,\overline{*}^g\, y) = (x \,\underline{*}^h\, z) \,\underline{*}^{h^{-1}gh}\, (y \,\underline{*}^h\, z),$$
$$(x \,\overline{*}^g\, y) \,\underline{*}^h\, (z \,\overline{*}^g\, y) = (x \,\underline{*}^h\, z) \,\overline{*}^{h^{-1}gh}\, (y \,\underline{*}^h\, z),$$
$$(x \,\overline{*}^g\, y) \,\overline{*}^h\, (z \,\overline{*}^g\, y) = (x \,\overline{*}^h\, z) \,\overline{*}^{h^{-1}gh}\, (y \,\underline{*}^h\, z).$$

For a biquandle $(X, \underline{*}, \overline{*})$ with $\operatorname{type} X < \infty$, $(X, \{\underline{*}^{[n]}\}_{[n] \in \mathbb{Z}_{\operatorname{type} X}}, \{\overline{*}^{[n]}\}_{[n] \in \mathbb{Z}_{\operatorname{type} X}})$ is a $\mathbb{Z}_{\operatorname{type} X}$-family of biquandles [**11**]. In particular, when X is an Alexander biquandle, $(X, \{\underline{*}^{[n]}\}_{[n] \in \mathbb{Z}_{\operatorname{type} X}}, \{\overline{*}^{[n]}\}_{[n] \in \mathbb{Z}_{\operatorname{type} X}})$ is called a $\mathbb{Z}_{\operatorname{type} X}$-*family of Alexander biquandles.*

4. Colorings.

In this section, we introduce a coloring of a diagram of an S^1-oriented handlebody-link by a G-family of biquandles. Let G be a group and let D be a diagram of an S^1-oriented handlebody-link H. A *G-flow* of D is a map $\phi : \mathcal{A}(D) \to G$ satisfying



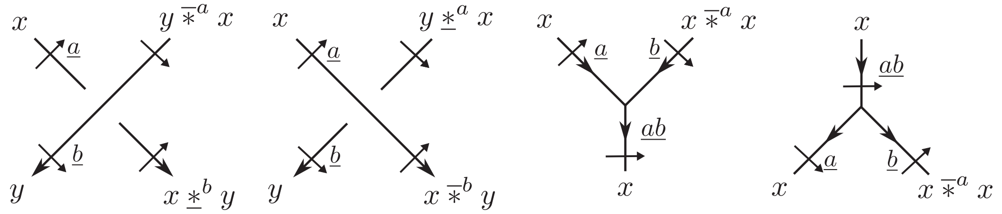
at each crossing and each vertex. In this paper, to avoid confusion, we often represent an element of G with an underline. We denote by (D, ϕ), which is called a *G-flowed diagram* of H, a diagram D given a G-flow ϕ and put $\operatorname{Flow}(D; G) := \{\phi \mid \phi : G\text{-flow of } D\}$. We can identify a G-flow ϕ with a homomorphism from the fundamental group $\pi_1(S^3 - H)$ to G.

Let G be a group and let D be a diagram of an S^1-oriented handlebody-link H. Let D' be a diagram obtained by applying one of Reidemeister moves to the diagram D once. For any G-flow ϕ of D, there is an unique G-flow ϕ' of D' which coincides with ϕ except near the point where the move applied. Therefore the number of G-flow of D, denoted by $\#\mathrm{Flow}(D;G)$, is an invariant of H. We call the G-flow ϕ' the *associated G-flow* of ϕ and the G-flowed diagram (D',ϕ') the *associated G-flowed diagram* of (D,ϕ).

For any $m \in \mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_m$-flow ϕ of a diagram D of an S^1-oriented handlebody-link H, we define $\gcd \phi := \gcd\{\phi(a), m \mid a \in \mathcal{A}(D)\} \in \mathbb{Z}_{\geq 0}$, where we regard $\phi(a)$ as an arbitrary element of $\mathbb{Z}$ which represents $\phi(a) \in \mathbb{Z}_m$. Then we have the following lemma in the same way as in [**9**].

Lemma 4.1. *For any $m \in \mathbb{Z}_{\geq 0}$, let (D,ϕ) be a $\mathbb{Z}_m$-flowed diagram of an S^1-oriented handlebody-link H and let (D',ϕ') be the associated $\mathbb{Z}_m$-flowed diagram of (D,ϕ). Then it follows that* $\gcd\phi = \gcd\phi'$.

Let G be a group, X be a G-family of biquandles and let (D,ϕ) be a G-flowed diagram of an S^1-oriented handlebody-link H. An *X-coloring* of (D,ϕ) is a map $C : \mathcal{SA}(D,\phi) \to X$ satisfying



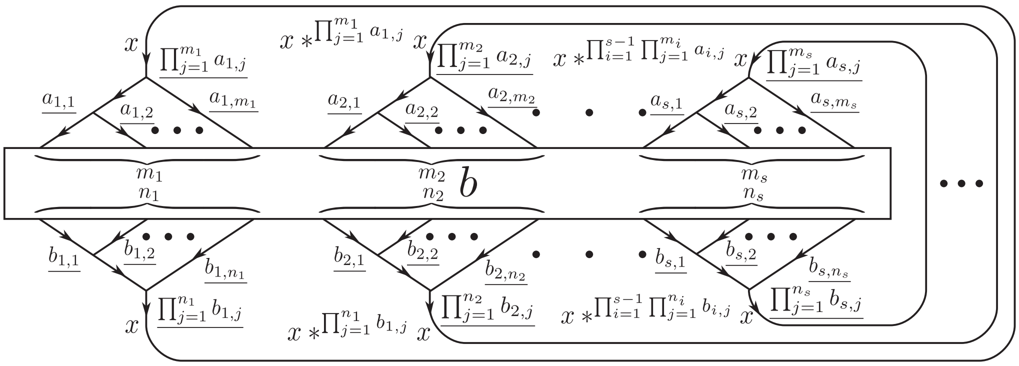
at each crossing and each vertex, where $\mathcal{SA}(D,\phi)$ is the set of all semi-arcs of (D,ϕ). We denote by $\mathrm{Col}_X(D,\phi)$ the set of all X-colorings of (D,ϕ). We note that $\mathrm{Col}_X(D,\phi)$ is a vector space over X when X is a $\mathbb{Z}_m$-family of Alexander biquandles and a field.

Proposition 4.2 ([**11**]). *Let X be a G-family of biquandles and let (D,ϕ) be a G-flowed diagram of an S^1-oriented handlebody-link H. Let (D',ϕ') be the associated G-flowed diagram of (D,ϕ). For any X-coloring C of (D,ϕ), there is an unique X-coloring C' of (D',ϕ') which coincides with C except near the point where the move applied.*

We call the X-coloring C' the *associated X-coloring* of C. By this proposition, we have $\#\mathrm{Col}_X(D,\phi) = \#\mathrm{Col}_X(D',\phi')$.

Proposition 4.3. *Let G be a group and let X be a G-family of biquandles. Then the following hold.*

1. *Let (D,ϕ) be a G-flowed diagram of an S^1-oriented handlebody-link. Then it follows that $\#\mathrm{Col}_X(D,\phi) \geq \#X$.*
2. *Let (O,ψ) be a G-flowed diagram of an S^1-oriented m-component trivial handlebody-link. Then it follows that $\#\mathrm{Col}_X(O,\psi) = (\#X)^m$.*



(D', ϕ')

Figure 4. A G-flowed diagram (D', ϕ') and its X-coloring.

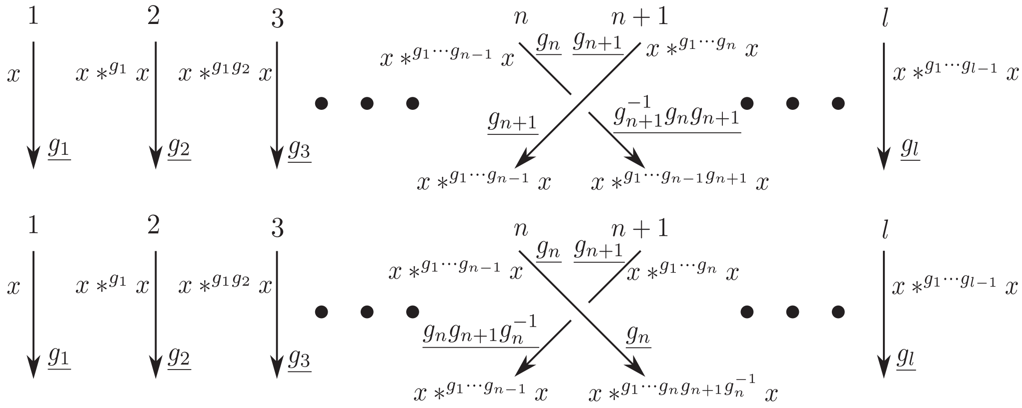


Figure 5. An X-coloring of (D', ϕ') in the part of b.

Proof. 1. By Theorem 2.1 and [**15**], we can deform (D, ϕ) into the G-flowed diagram (D', ϕ') depicted in Figure 4 by a finite sequence of Reidemeister moves preserving Y-orientations, where b is a classical l-braid, and $a_{i,1}, \ldots, a_{i,m_i}, b_{i,1}, \ldots, b_{i,n_i} \in G$ for any $i = 1, \ldots, s$. We note that $\prod_{j=1}^{m_i} a_{i,j} = \prod_{j=1}^{n_i} b_{i,j}$ for any $i = 1, \ldots, s$, and $x \underline{*}^g x = x \overline{*}^g x$ for any $x \in X$ and $g \in G$. By Proposition 4.2, it is sufficient to prove that $\#\mathrm{Col}_X(D', \phi') \geq \#X$. Here for any $x \in X$ and $g \in G$, we write $x *^g x$ for $x \underline{*}^g x$ and $x \overline{*}^g x$ simply. Then for any $x \in X$, the assignment of elements of X to each semi-arc of (D', ϕ') as shown in Figures 4 and 5 is an X-coloring, where each g_i represents an element of G in Figure 5. Therefore we have $\#\mathrm{Col}_X(D', \phi') \geq \#X$.

2. It is sufficient to prove that $\#\mathrm{Col}_X(O, \psi) = \#X$ when $m = 1$. Let (O_g, ψ_g) be a G-flowed diagram of an S^1-oriented trivial handlebody-knot of genus g. By Theorem 2.1, we can deform (O_g, ψ_g) into the G-flowed diagram (O'_g, ψ'_g) depicted in Figure 6 by a finite sequence of Reidemeister moves preserving Y-orientations, where $a_i \in G$ for any $i = 1, \ldots, g$, and e is the identity of G. By Proposition 4.2, it is sufficient

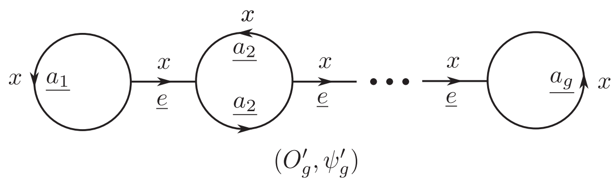


Figure 6. A G-flowed diagram (O'_g, ψ'_g) and its X-coloring.

to prove that $\#\mathrm{Col}_X(O'_g, \psi'_g) = \#X$. For any $x \in X$, the assignment of x to each semi-arc of (O'_g, ψ'_g) as shown in Figure 6 is an X-coloring. On the other hand, since any X-coloring of (O'_g, ψ'_g) is given by Figure 6 for some $x \in X$, we have $\#\mathrm{Col}_X(O'_g, \psi'_g) = \#X$. □

5. Linear relationships for Alexander biquandle colorings.

For any $\mathbb{Z}_m$-flowed diagram (D, ϕ) of an S^1-oriented handlebody-link, we define the *Alexander numbering* of (D, ϕ) by assigning elements of $\mathbb{Z}_m$ to each region of (D, ϕ) as shown in Figure 7, where the unbounded region is labeled 0. It is an extension of the Alexander numbering of a classical knot diagram [**1**]. It is easy to see that for any $\mathbb{Z}_m$-flowed diagram (D, ϕ) of an S^1-oriented handlebody-link, there uniquely exists the Alexander numbering of (D, ϕ). For example, a $\mathbb{Z}_m$-flowed diagram of the handlebody-knot 5_2 [**10**] with the Alexander numbering is depicted in Figure 8. For any semi-arc α of (D, ϕ), we denote by $\rho(\alpha)$ the Alexander number of the region which the normal orientation of α points to.

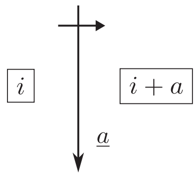


Figure 7. The Alexander numbering of (D, ϕ).

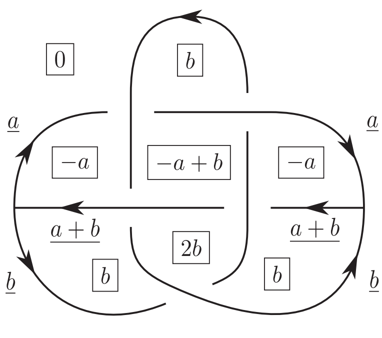


Figure 8. A $\mathbb{Z}_m$-flowed diagram of 5_2 with the Alexander numbering.

In the following, every component of a diagram of any S^1-oriented handlebody-link may have a crossing at least 1. Let (D, ϕ) be a $\mathbb{Z}_m$-flowed diagram of an S^1-oriented handlebody-link with the Alexander numbering and let X be a $\mathbb{Z}_m$-family of Alexander biquandles. We put $C(D, \phi) = \{c_1, \dots, c_n\}$ and $V(D, \phi) = \{\tau_1, \dots, \tau_{2k}\}$, where $C(D, \phi)$ and $V(D, \phi)$ are the set of all crossings of (D, ϕ) and the one of all vertices of (D, ϕ) respectively, where the sign of τ_i is 1 for any $i = 1, \dots, k$ and -1 for any $i = k+1, \dots, 2k$. Then we denote by x_i each semi-arc of (D, ϕ) as shown in Figure 9, which implies $\mathcal{SA}(D, \phi) = \{x_1, \dots, x_{2n+3k}\}$.

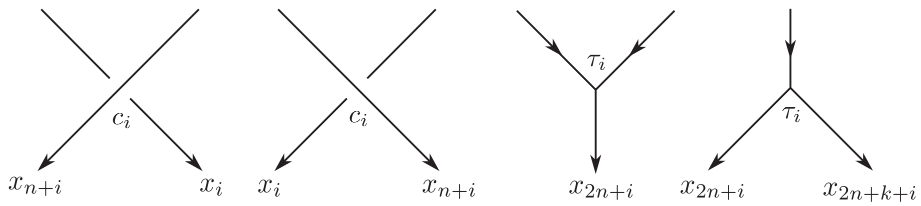


Figure 9. Semi-arcs x_i of (D, ϕ).

We denote by u_i, v_i, v_i', w_i, α_i, β_i and γ_i the semi-arcs incident to a crossing c_i or a vertex τ_i as shown in Figure 10. We put $\phi_i := \phi(u_i) = \phi(w_i)$, $\psi_i := \phi(v_i) = \phi(v_i')$, $\eta_i := \phi(\alpha_i)$ and $\theta_i := \phi(\beta_i)$. We denote by $\epsilon_{c_i} \in \{\pm 1\}$ and $\epsilon_{\tau_i} \in \{\pm 1\}$ the signs of a crossing c_i and a vertex τ_i respectively (see Figure 10).

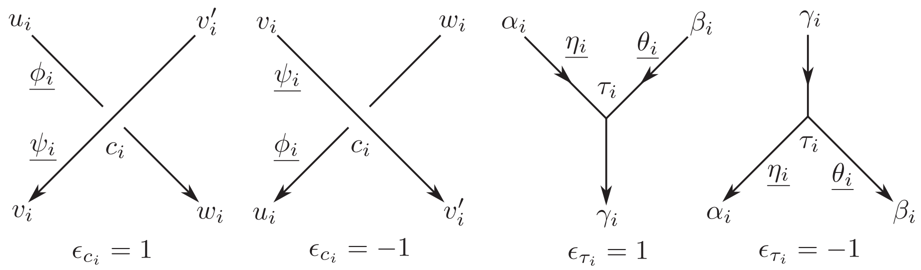


Figure 10. Notations.

For any semi-arcs $y, y' \in \mathcal{SA}(D, \phi)$, we put

$$\delta(y, y') := \begin{cases} 1 & (y = y'), \\ 0 & (y \neq y'). \end{cases}$$

Then we define a matrix $A(D, \phi; X) = (a_{i,j}) \in M(2n + 4k, 2n + 3k; X)$ by

$$a_{i,j} = \begin{cases} \delta(u_i, x_j) t^{\psi_i} + \delta(v_i, x_j)(s^{\psi_i} - t^{\psi_i}) - \delta(w_i, x_j) & (1 \leq i \leq n), \\ -\delta(v_{i-n}, x_j) s^{\phi_{i-n}} + \delta(v_{i-n}', x_j) & (n + 1 \leq i \leq 2n), \\ \delta(\alpha_{i-2n}, x_j) - \delta(\gamma_{i-2n}, x_j) & (2n + 1 \leq i \leq 2n + 2k), \\ \delta(\beta_{i-2n-2k}, x_j) - \delta(\gamma_{i-2n-2k}, x_j) s^{\eta_{i-2n-2k}} & (2n + 2k + 1 \leq i \leq 2n + 4k). \end{cases}$$

We note that $A(D,\phi;X)$ is determined up to permuting of rows and columns of the matrix, and it follows that

$$\mathrm{Col}_X(D,\phi)=\left\{\begin{pmatrix} z_1\\ z_2\\ \vdots\\ z_{2n+3k}\end{pmatrix}\in X^{2n+3k}\,\middle|\,A(D,\phi;X)\begin{pmatrix} z_1\\ z_2\\ \vdots\\ z_{2n+3k}\end{pmatrix}=\mathbf{0}\right\}.$$

For example, let (E,ψ) be the $\mathbb{Z}_m$-flowed diagram of the handlebody-knot depicted in Figure 11. Then we have

$$A(E,\psi;X)=\begin{pmatrix} -1 & 0 & s^a-t^a & t^a & 0 & 0 & 0\\ 0 & -1 & 0 & s^b-t^b & 0 & t^b & 0\\ 0 & 1 & -s^b & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -s^a & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 1 & 0\\ 1 & 0 & 0 & 0 & -s^a & 0 & 0\\ 0 & 0 & 0 & 0 & -s^a & 0 & 1\end{pmatrix}.$$

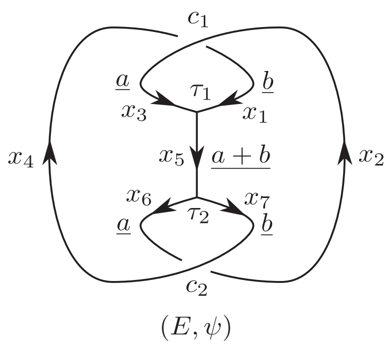


Figure 11. A $\mathbb{Z}_m$-flowed diagram (E,ψ).

Then we have the following proposition.

Proposition 5.1. *Let (D,ϕ) be a $\mathbb{Z}_m$-flowed diagram of an S^1-oriented handlebody-link with the Alexander numbering and let X be a $\mathbb{Z}_m$-family of Alexander biquandles. Let $\boldsymbol{a}_i$ be the i-th row of $A(D,\phi;X)$. Then it follows that*

$$\sum_{i=1}^{n}\epsilon_{c_i}t^{-\rho(w_i)}(s^{\phi_i}-t^{\phi_i})\boldsymbol{a}_i+\sum_{i=1}^{n}\epsilon_{c_i}t^{-\rho(v'_i)}(s^{\psi_i}-t^{\psi_i})\boldsymbol{a}_{n+i}$$
$$+\sum_{i=1}^{2k}\epsilon_{\tau_i}t^{-\rho(\alpha_i)}(s^{\eta_i}-t^{\eta_i})\boldsymbol{a}_{2n+i}+\sum_{i=1}^{2k}\epsilon_{\tau_i}t^{-\rho(\beta_i)}(s^{\theta_i}-t^{\theta_i})\boldsymbol{a}_{2n+2k+i}=\mathbf{0}.$$

Proof. For any semi-arc y incident to a crossing or a vertex σ, we put

$$\epsilon(y;\sigma) := \begin{cases} 1 & \text{if the orientation of } y \text{ points to } \sigma, \\ -1 & \text{otherwise.} \end{cases}$$

We set $(a_{i,j}) := A(D,\phi;X)$. It is sufficient to prove that for any $j = 1, 2, \dots, 2n+3k$,

$$\sum_{i=1}^{n} \epsilon_{c_i} t^{-\rho(w_i)}(s^{\phi_i} - t^{\phi_i})a_{i,j} + \sum_{i=1}^{n} \epsilon_{c_i} t^{-\rho(v_i')}(s^{\psi_i} - t^{\psi_i})a_{n+i,j}$$
$$+ \sum_{i=1}^{2k} \epsilon_{\tau_i} t^{-\rho(\alpha_i)}(s^{\eta_i} - t^{\eta_i})a_{2n+i,j} + \sum_{i=1}^{2k} \epsilon_{\tau_i} t^{-\rho(\beta_i)}(s^{\theta_i} - t^{\theta_i})a_{2n+2k+i,j} = 0.$$

For the first term, we have

$$\begin{aligned}
&\epsilon_{c_i} t^{-\rho(w_i)}(s^{\phi_i} - t^{\phi_i})\delta(u_i,x_j)t^{\psi_i} = \delta(u_i,x_j)\epsilon(u_i;c_i)t^{-\rho(u_i)}(s^{\phi(u_i)} - t^{\phi(u_i)}), \\
&\epsilon_{c_i} t^{-\rho(w_i)}(s^{\phi_i} - t^{\phi_i})\delta(v_i,x_j)(s^{\psi_i} - t^{\psi_i}) \\
&= \epsilon_{c_i} t^{-\rho(w_i)} s^{\phi_i}\delta(v_i,x_j)(s^{\psi_i} - t^{\psi_i}) - \epsilon_{c_i} t^{-\rho(w_i)} t^{\phi_i}\delta(v_i,x_j)(s^{\psi_i} - t^{\psi_i}) \\
&= \epsilon_{c_i} t^{-\rho(w_i)}\delta(v_i,x_j)(s^{\psi_i} - t^{\psi_i})s^{\phi_i} + \delta(v_i,x_j)\epsilon(v_i;c_i)t^{-\rho(v_i)}(s^{\phi(v_i)} - t^{\phi(v_i)}), \qquad (1) \\
&\epsilon_{c_i} t^{-\rho(w_i)}(s^{\phi_i} - t^{\phi_i})(-\delta(w_i,x_j)) = \delta(w_i,x_j)\epsilon(w_i;c_i)t^{-\rho(w_i)}(s^{\phi(w_i)} - t^{\phi(w_i)}).
\end{aligned}$$

For the second term, we have

$$\begin{aligned}
&\epsilon_{c_i} t^{-\rho(v_i')}(s^{\psi_i} - t^{\psi_i})(-\delta(v_i,x_j)s^{\phi_i}) = -\epsilon_{c_i} t^{-\rho(v_i')}\delta(v_i,x_j)(s^{\psi_i} - t^{\psi_i})s^{\phi_i}, \qquad (2) \\
&\epsilon_{c_i} t^{-\rho(v_i')}(s^{\psi_i} - t^{\psi_i})\delta(v_i',x_j) = \delta(v_i',x_j)\epsilon(v_i';c_i)t^{-\rho(v_i')}(s^{\phi(v_i')} - t^{\phi(v_i')}).
\end{aligned}$$

For the third term, we have

$$\begin{aligned}
&\epsilon_{\tau_i} t^{-\rho(\alpha_i)}(s^{\eta_i} - t^{\eta_i})\delta(\alpha_i,x_j) = \delta(\alpha_i,x_j)\epsilon(\alpha_i;\tau_i)t^{-\rho(\alpha_i)}(s^{\phi(\alpha_i)} - t^{\phi(\alpha_i)}), \\
&\epsilon_{\tau_i} t^{-\rho(\alpha_i)}(s^{\eta_i} - t^{\eta_i})(-\delta(\gamma_i,x_j)) = \delta(\gamma_i,x_j)\epsilon(\gamma_i;\tau_i)t^{-\rho(\gamma_i)}t^{\theta_i}(s^{\eta_i} - t^{\eta_i}). \qquad (3)
\end{aligned}$$

For the last term, we have

$$\begin{aligned}
&\epsilon_{\tau_i} t^{-\rho(\beta_i)}(s^{\theta_i} - t^{\theta_i})\delta(\beta_i,x_j) = \delta(\beta_i,x_j)\epsilon(\beta_i;\tau_i)t^{-\rho(\beta_i)}(s^{\phi(\beta_i)} - t^{\phi(\beta_i)}), \\
&\epsilon_{\tau_i} t^{-\rho(\beta_i)}(s^{\theta_i} - t^{\theta_i})(-\delta(\gamma_i,x_j)s^{\eta_i}) = \delta(\gamma_i,x_j)\epsilon(\gamma_i;\tau_i)t^{-\rho(\gamma_i)}(s^{\theta_i} - t^{\theta_i})s^{\eta_i}. \qquad (4)
\end{aligned}$$

We note that

$$\begin{aligned}
(1) + (2) &= \delta(v_i,x_j)\epsilon(v_i;c_i)t^{-\rho(v_i)}(s^{\phi(v_i)} - t^{\phi(v_i)}), \\
(3) + (4) &= \delta(\gamma_i,x_j)\epsilon(\gamma_i;\tau_i)t^{-\rho(\gamma_i)}(s^{\phi(\gamma_i)} - t^{\phi(\gamma_i)}).
\end{aligned}$$

Therefore for any $j = 1, 2, \dots, 2n+3k$, it follows that

$$\sum_{i=1}^{n} \epsilon_{c_i} t^{-\rho(w_i)}(s^{\phi_i} - t^{\phi_i})a_{i,j} + \sum_{i=1}^{n} \epsilon_{c_i} t^{-\rho(v_i')}(s^{\psi_i} - t^{\psi_i})a_{n+i,j}$$

$$
\begin{aligned}
&+\sum_{i=1}^{2k}\epsilon_{\tau_i}t^{-\rho(\alpha_i)}(s^{\eta_i}-t^{\eta_i})a_{2n+i,j}+\sum_{i=1}^{2k}\epsilon_{\tau_i}t^{-\rho(\beta_i)}(s^{\theta_i}-t^{\theta_i})a_{2n+2k+i,j}\\
&=\sum_{i=1}^{n}(\delta(u_i,x_j)\epsilon(u_i;c_i)t^{-\rho(u_i)}(s^{\phi(u_i)}-t^{\phi(u_i)})\\
&\qquad+\delta(v_i,x_j)\epsilon(v_i;c_i)t^{-\rho(v_i)}(s^{\phi(v_i)}-t^{\phi(v_i)})\\
&\qquad+\delta(v_i',x_j)\epsilon(v_i';c_i)t^{-\rho(v_i')}(s^{\phi(v_i')}-t^{\phi(v_i')})\\
&\qquad+\delta(w_i,x_j)\epsilon(w_i;c_i)t^{-\rho(w_i)}(s^{\phi(w_i)}-t^{\phi(w_i)}))\\
&\quad+\sum_{i=1}^{2k}(\delta(\alpha_i,x_j)\epsilon(\alpha_i;\tau_i)t^{-\rho(\alpha_i)}(s^{\phi(\alpha_i)}-t^{\phi(\alpha_i)})\\
&\qquad+\delta(\beta_i,x_j)\epsilon(\beta_i;\tau_i)t^{-\rho(\beta_i)}(s^{\phi(\beta_i)}-t^{\phi(\beta_i)})\\
&\qquad+\delta(\gamma_i,x_j)\epsilon(\gamma_i;\tau_i)t^{-\rho(\gamma_i)}(s^{\phi(\gamma_i)}-t^{\phi(\gamma_i)}))\\
&=t^{-\rho(x_j)}(s^{\phi(x_j)}-t^{\phi(x_j)})-t^{-\rho(x_j)}(s^{\phi(x_j)}-t^{\phi(x_j)})\\
&=0.
\end{aligned}
$$
□

Let X be an Alexander biquandle and let $m=\operatorname{type}X$. Then X is also a $\mathbb{Z}_m$-family of Alexander biquandles. Let D be an oriented classical link diagram. We can regard D as a $\mathbb{Z}_m$-flowed diagram $(D,\phi_{(1)})$ of an S^1-oriented handlebody-link whose components are of genus 1, where $\phi_{(1)}$ is the constant map to 1. Hence we can regard an X-coloring of D as an X-coloring of $(D,\phi_{(1)})$. We define a matrix $A(D;X)\in M(2n,2n;X)$ by $A(D;X)=A(D,\phi_{(1)};X)$, where n is the number of crossings of D. Then the set of all X-colorings of D, denoted by $\operatorname{Col}_X(D)$, is given by

$$
\operatorname{Col}_X(D)=\left\{\begin{pmatrix}z_1\\z_2\\\vdots\\z_{2n}\end{pmatrix}\in X^{2n}\,\middle|\,A(D;X)\begin{pmatrix}z_1\\z_2\\\vdots\\z_{2n}\end{pmatrix}=\mathbf{0}\right\}.
$$

Therefore we obtain the following corollary.

COROLLARY 5.2. *Let D be a diagram of an oriented classical link with the Alexander numbering and let X be an Alexander biquandle. Let $\boldsymbol{a}_i$ be the i-th row of $A(D;X)$. Then it follows that*

$$
\sum_{i=1}^{n}\epsilon_{c_i}t^{-\rho(w_i)}(s-t)\boldsymbol{a}_i+\sum_{i=1}^{n}\epsilon_{c_i}t^{-\rho(v_i')}(s-t)\boldsymbol{a}_{n+i}=\mathbf{0}.
$$

6. Main theorem.

In this section, we give lower bounds for the Gordian distance and the unknotting number of S^1-oriented handlebody-knots.

THEOREM 6.1. *Let H_i be an S^1-oriented handlebody-knot of genus g and let D_i be* *a diagram of* H_i $(i=1,2)$*. Let* $X=\mathbb{Z}_p[t^{\pm 1}]/(f(t))$ *which is a* $\mathbb{Z}_m$*-family of Alexander biquandles, where* p *is a prime number,* $s \in \mathbb{Z}_p[t^{\pm 1}]$ *and* $f(t) \in \mathbb{Z}_p[t^{\pm 1}]$ *is an irreducible polynomial. Then it follows that*

$$\max_{\phi_1 \in \mathrm{Flow}(D_1;\mathbb{Z}_m)} \min_{\substack{\phi_2 \in \mathrm{Flow}(D_2;\mathbb{Z}_m) \\ \gcd \phi_1 = \gcd \phi_2}} |\dim \mathrm{Col}_X(D_1,\phi_1) - \dim \mathrm{Col}_X(D_2,\phi_2)| \leq d(H_1,H_2).$$

Proof. Let (D,ϕ) be a $\mathbb{Z}_m$-flowed diagram of an S^1-oriented handlebody-knot and let $C(D,\phi)=\{c_1,\dots,c_n\}$ and $V(D,\phi)=\{\tau_1,\dots,\tau_{2k}\}$. Let $(\overline{D},\overline{\phi})$ be the $\mathbb{Z}_m$-flowed diagram of an S^1-oriented handlebody-knot which is obtained from (D,ϕ) by the crossing change at c_1 and let $C(\overline{D},\overline{\phi})=\{\overline{c}_1,\dots,\overline{c}_n\}$ and $V(\overline{D},\overline{\phi})=\{\overline{\tau}_1,\dots,\overline{\tau}_{2k}\}$, where $\overline{\phi}$, $\overline{c}_i$ and $\overline{\tau}_i$ originate from ϕ, c_i and τ_i naturally and respectively (see Figure 12). In the following, we show that

$$|\dim \mathrm{Col}_X(D,\phi) - \dim \mathrm{Col}_X(\overline{D},\overline{\phi})| \leq 1,$$

that is,

$$|\mathrm{rank}\, A(D,\phi;X) - \mathrm{rank}\, A(\overline{D},\overline{\phi};X)| \leq 1.$$

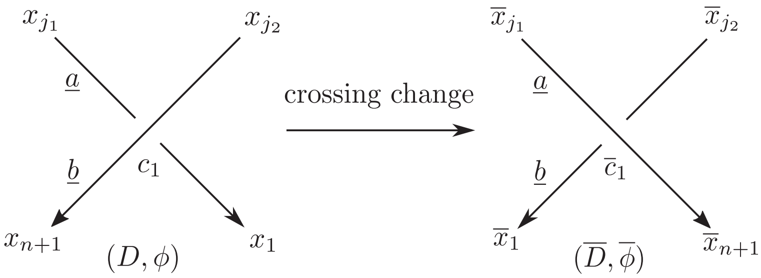


Figure 12. The crossing change at c_1.

We may assume that c_1 is a positive crossing and $\overline{c}_1$ is a negative crossing. We denote by $\overline{x}_i$ each semi-arc of $(\overline{D},\overline{\phi})$ in the same way as in Figure 9 with respect to $\overline{c}_i$ or $\overline{\tau}_i$, and so are $\overline{v}'_i$, $\overline{w}_i$, $\overline{\alpha}_i$, $\overline{\beta}_i$, $\overline{\phi}_i$, $\overline{\psi}_i$, $\overline{\eta}_i$, $\overline{\theta}_i$, $\overline{\epsilon}_{c_i}$ and $\overline{\epsilon}_{\tau_i}$ (see Figure 10). We denote by x_{j_1} and x_{j_2} the semi-arcs which point to the crossing c_1 of (D,ϕ) as shown in Figure 12, and we put $a := \phi_1 = \overline{\psi}_1$ and $b := \psi_1 = \overline{\phi}_1$. We note that $\mathrm{Col}_X(D,\phi)$ and $\mathrm{Col}_X(\overline{D},\overline{\phi})$ are vector spaces over X since X is a $\mathbb{Z}_m$-family of Alexander biquandles and a field.

Let $\boldsymbol{a}_i$, $\overline{\boldsymbol{a}}_i$ and $\hat{\boldsymbol{a}}_i$ be the i-th rows of $A(D,\phi;X)$, $A(\overline{D},\overline{\phi};X)$ and $\hat{A}(\overline{D},\overline{\phi};X)$ respectively, where $\hat{A}(\overline{D},\overline{\phi};X)$ is the matrix obtained by permuting the first column and the $(n+1)$-th column of $A(\overline{D},\overline{\phi};X)$. We put $(a_{i,j}) := A(D,\phi;X)$, $(\overline{a}_{i,j}) := A(\overline{D},\overline{\phi};X)$ and $(\hat{a}_{i,j}) := \hat{A}(\overline{D},\overline{\phi};X)$. Then we have $\boldsymbol{a}_i = \hat{\boldsymbol{a}}_i$ when $i \neq 1, n+1$. We note that $\mathrm{rank}\, A(\overline{D},\overline{\phi};X) = \mathrm{rank}\, \hat{A}(\overline{D},\overline{\phi};X)$ and

$$\boldsymbol{a}_1 = (-1,0,\dots,0,\overset{\substack{j_1\\ \vee}}{t^b},0,\dots,0,\overset{\substack{n+1\\ \vee}}{s^b - t^b},0,\dots,0),$$

$$\boldsymbol{a}_{n+1} = (0, \ldots, 0, \overset{\overset{j_2}{\vee}}{1}, 0, \ldots, 0, \overset{\overset{n+1}{\vee}}{-s^a}, 0, \ldots, 0),$$

$$\overline{\boldsymbol{a}}_1 = (t^a, 0, \ldots, 0, \overset{\overset{j_1}{\vee}}{s^a - t^a}, 0, \ldots, 0, \overset{\overset{j_2}{\vee}}{-1}, 0, \ldots, 0),$$

$$\overline{\boldsymbol{a}}_{n+1} = (0, \ldots, 0, \overset{\overset{j_1}{\vee}}{-s^b}, 0, \ldots, 0, \overset{\overset{n+1}{\vee}}{1}, 0, \ldots, 0),$$

$$\hat{\boldsymbol{a}}_1 = (0, \ldots, 0, \overset{\overset{j_1}{\vee}}{s^a - t^a}, 0, \ldots, 0, \overset{\overset{j_2}{\vee}}{-1}, 0, \ldots, 0, \overset{\overset{n+1}{\vee}}{t^a}, 0, \ldots, 0),$$

$$\hat{\boldsymbol{a}}_{n+1} = (1, 0, \ldots, 0, \overset{\overset{j_1}{\vee}}{-s^b}, 0, \ldots, 0).$$

By Proposition 5.1, we obtain

$$\sum_{i=1}^{n} \epsilon_{c_i} t^{-\rho(w_i)} (s^{\phi_i} - t^{\phi_i}) \boldsymbol{a}_i + \sum_{i=1}^{n} \epsilon_{c_i} t^{-\rho(v_i')} (s^{\psi_i} - t^{\psi_i}) \boldsymbol{a}_{n+i}$$
$$+ \sum_{i=1}^{2k} \epsilon_{\tau_i} t^{-\rho(\alpha_i)} (s^{\eta_i} - t^{\eta_i}) \boldsymbol{a}_{2n+i} + \sum_{i=1}^{2k} \epsilon_{\tau_i} t^{-\rho(\beta_i)} (s^{\theta_i} - t^{\theta_i}) \boldsymbol{a}_{2n+2k+i} = \boldsymbol{0}$$

and

$$\sum_{i=1}^{n} \overline{\epsilon}_{c_i} t^{-\rho(\overline{w}_i)} (s^{\overline{\phi}_i} - t^{\overline{\phi}_i}) \overline{\boldsymbol{a}}_i + \sum_{i=1}^{n} \overline{\epsilon}_{c_i} t^{-\rho(\overline{v}_i')} (s^{\overline{\psi}_i} - t^{\overline{\psi}_i}) \overline{\boldsymbol{a}}_{n+i}$$
$$+ \sum_{i=1}^{2k} \overline{\epsilon}_{\tau_i} t^{-\rho(\overline{\alpha}_i)} (s^{\overline{\eta}_i} - t^{\overline{\eta}_i}) \overline{\boldsymbol{a}}_{2n+i} + \sum_{i=1}^{2k} \overline{\epsilon}_{\tau_i} t^{-\rho(\overline{\beta}_i)} (s^{\overline{\theta}_i} - t^{\overline{\theta}_i}) \overline{\boldsymbol{a}}_{2n+2k+i}$$
$$= \sum_{i=1}^{n} \overline{\epsilon}_{c_i} t^{-\rho(\overline{w}_i)} (s^{\overline{\phi}_i} - t^{\overline{\phi}_i}) \hat{\boldsymbol{a}}_i + \sum_{i=1}^{n} \overline{\epsilon}_{c_i} t^{-\rho(\overline{v}_i')} (s^{\overline{\psi}_i} - t^{\overline{\psi}_i}) \hat{\boldsymbol{a}}_{n+i}$$
$$+ \sum_{i=1}^{2k} \overline{\epsilon}_{\tau_i} t^{-\rho(\overline{\alpha}_i)} (s^{\overline{\eta}_i} - t^{\overline{\eta}_i}) \hat{\boldsymbol{a}}_{2n+i} + \sum_{i=1}^{2k} \overline{\epsilon}_{\tau_i} t^{-\rho(\overline{\beta}_i)} (s^{\overline{\theta}_i} - t^{\overline{\theta}_i}) \hat{\boldsymbol{a}}_{2n+2k+i} = \boldsymbol{0}.$$

If $\epsilon_{c_1} t^{-\rho(w_1)}(s^{\phi_1} - t^{\phi_1}) = 0$, we have $s^{\phi_1} - t^{\phi_1} = s^a - t^a = 0$, which implies that $\boldsymbol{a}_{n+1} = -\hat{\boldsymbol{a}}_1$. Hence it follows that

$$|\operatorname{rank} A(D, \phi; X) - \operatorname{rank} A(\overline{D}, \overline{\phi}; X)| = |\operatorname{rank} A(D, \phi; X) - \operatorname{rank} \hat{A}(\overline{D}, \overline{\phi}; X)| \leq 1.$$

If $\overline{\epsilon}_{c_1} t^{-\rho(\overline{w}_1)}(s^{\overline{\phi}_1} - t^{\overline{\phi}_1}) = 0$, we have $s^{\overline{\phi}_1} - t^{\overline{\phi}_1} = s^b - t^b = 0$, which implies that $\boldsymbol{a}_1 = -\hat{\boldsymbol{a}}_{n+1}$. Hence it follows that

$$|\operatorname{rank} A(D, \phi; X) - \operatorname{rank} A(\overline{D}, \overline{\phi}; X)| = |\operatorname{rank} A(D, \phi; X) - \operatorname{rank} \hat{A}(\overline{D}, \overline{\phi}; X)| \leq 1.$$

If $\epsilon_{c_1} t^{-\rho(w_1)}(s^{\phi_1} - t^{\phi_1}) \neq 0$ and $\overline{\epsilon}_{c_1} t^{-\rho(\overline{w}_1)}(s^{\overline{\phi}_1} - t^{\overline{\phi}_1}) \neq 0$, we can represent $\boldsymbol{a}_1$ and $\overline{\boldsymbol{a}}_1$ as linear combinations of $\boldsymbol{a}_2, \ldots, \boldsymbol{a}_{2n+4k}$ and $\overline{\boldsymbol{a}}_2, \ldots, \overline{\boldsymbol{a}}_{2n+4k}$ respectively. Hence it follows that

$$\operatorname{rank} A(D,\phi;X) = \operatorname{rank}\begin{pmatrix} \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_{2n+4k} \end{pmatrix},\ \operatorname{rank} A(\overline{D},\overline{\phi};X) = \operatorname{rank}\begin{pmatrix} \overline{\boldsymbol{a}}_2 \\ \vdots \\ \overline{\boldsymbol{a}}_{2n+4k} \end{pmatrix},$$

which implies that

$$\begin{aligned} |\operatorname{rank} A(D,\phi;X) - \operatorname{rank} A(\overline{D},\overline{\phi};X)| &= \left|\operatorname{rank}\begin{pmatrix} \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_{2n+4k} \end{pmatrix} - \operatorname{rank}\begin{pmatrix} \overline{\boldsymbol{a}}_2 \\ \vdots \\ \overline{\boldsymbol{a}}_{2n+4k} \end{pmatrix}\right| \\ &= \left|\operatorname{rank}\begin{pmatrix} \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_{2n+4k} \end{pmatrix} - \operatorname{rank}\begin{pmatrix} \hat{\boldsymbol{a}}_2 \\ \vdots \\ \hat{\boldsymbol{a}}_{2n+4k} \end{pmatrix}\right| \\ &\leq 1. \end{aligned}$$

Consequently, if we can deform H_1 into H_2 by crossing changes at l crossings, then for any $\mathbb{Z}_m$-flowed diagram (D_1,ϕ_1) of H_1, there exists a $\mathbb{Z}_m$-flowed diagram (D_2,ϕ_2) of H_2 satisfying $\gcd\phi_1 = \gcd\phi_2$ and

$$|\dim \operatorname{Col}_X(D_1,\phi_1) - \dim \operatorname{Col}_X(D_2,\phi_2)| \leq l$$

by Lemma 4.1. Therefore it follows that

$$\max_{\phi_1\in \operatorname{Flow}(D_1;\mathbb{Z}_m)} \min_{\substack{\phi_2\in \operatorname{Flow}(D_2;\mathbb{Z}_m)\\ \gcd\phi_1=\gcd\phi_2}} |\dim \operatorname{Col}_X(D_1,\phi_1) - \dim \operatorname{Col}_X(D_2,\phi_2)| \leq d(H_1,H_2). \qquad \square$$

By Proposition 4.3 and Theorem 6.1, the following corollary holds immediately.

COROLLARY 6.2. *Let H be an S^1-oriented handlebody-knot and let D be a diagram of H. Let $X = \mathbb{Z}_p[t^{\pm 1}]/(f(t))$ which is a $\mathbb{Z}_m$-family of Alexander biquandles, where p is a prime number, $s \in \mathbb{Z}_p[t^{\pm 1}]$ and $f(t) \in \mathbb{Z}_p[t^{\pm 1}]$ is an irreducible polynomial. Then it follows that*

$$\max_{\phi\in \operatorname{Flow}(D;\mathbb{Z}_m)} \dim \operatorname{Col}_X(D,\phi) - 1 \leq u(H).$$

7. Examples.

In this section, we give some examples. In Example 7.1, we give a handlebody-knot with unknotting number 2, and in Remark 7.2, we note that it can not be obtained by using Alexander quandle colorings with $\mathbb{Z}_2,\mathbb{Z}_3$-flows introduced in [**12**]. In Example 7.3, we give three handlebody-knots with unknotting number n for any $n \in \mathbb{Z}_{>0}$. In Example 7.4, we give two handlebody-knots with their Gordian distance n for any $n \in \mathbb{Z}_{>0}$.

EXAMPLE 7.1. Let H be the handlebody-knot represented by the $\mathbb{Z}_{10}$-flowed diagram (D,ϕ) depicted in Figure 13. Then we show that $u(H) = 2$.

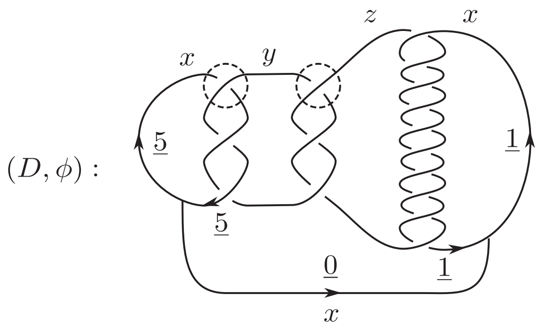


Figure 13. A $\mathbb{Z}_{10}$-flowed diagram (D,ϕ) of H.

Let $s = 1 \in \mathbb{Z}_3[t^{\pm 1}]$ and let $f(t) = t^4 + 2t^3 + t^2 + 2t + 1 \in \mathbb{Z}_3[t^{\pm 1}]$, which is an irreducible polynomial. Then $X := \mathbb{Z}_3[t^{\pm 1}]/(f(t))$ is a $\mathbb{Z}_{10}$-family of Alexander biquandles. Then for any $x, y, z \in X$, the assignment of them to each semi-arc of (D,ϕ) as shown in Figure 13 is an X-coloring of (D,ϕ), which implies $\dim \mathrm{Col}_X(D,\phi) \geq 3$. By Corollary 6.2, we obtain $2 \leq u(H)$. On the other hand, we can deform H into a trivial handlebody-knot by the crossing changes at two crossings surrounded by dotted circles depicted in Figure 13. Therefore it follows that $u(H) = 2$.

Remark 7.2. We show that the result in Example 7.1 can not be obtained by using Alexander quandle colorings with $\mathbb{Z}_2, \mathbb{Z}_3$-flows introduced in [**12**].

Let H be the handlebody-knot represented by the $\mathbb{Z}_m$-flowed diagram $(D, \phi(a,b))$ depicted in Figure 14 for any $m = 2, 3$ and $a, b \in \mathbb{Z}_m$. Let p be a prime number, $s = 1 \in \mathbb{Z}_p[t^{\pm 1}]$, $f(t)$ be an irreducible polynomial in $\mathbb{Z}_p[t^{\pm 1}]$ and let $X = \mathbb{Z}_p[t^{\pm 1}]/(f(t))$ which is a $\mathbb{Z}_m$-family of Alexander (bi)quandles. We note that $\mathrm{Col}_X(D, \phi(a,b))$ is generated by $x, y, z \in X$ as shown in Figure 14 for any $m = 2, 3$ and $a, b \in \mathbb{Z}_m$. If $(a,b) = (1,0)$, x, y and z need to satisfy the following relations:

$$\begin{aligned}
&(t^2 - t + 1)x - (t^2 - t + 1)y = 0,\\
&-t(t^2 - t + 1)x + t^{-1}(t+1)(t-1)(t^2 - t + 1)y + t^{-1}(t^2 - t + 1)z = 0,\\
&-t^{-1}(t-1)(t^2 - t + 1)x + t^{-2}(t^2 - t - 1)(t^2 - t + 1)y + t^{-2}(t^2 - t + 1)z = 0,\\
&((t^3 + t^2 - 1)(t^2 - t + 1) - t)x - ((t^3 + t^2 - 1)(t^2 - t + 1) - t)z = 0,
\end{aligned}$$

that is,

$$M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$M = \begin{pmatrix} t^2 - t + 1 & -(t^2 - t + 1) & 0 \\ -t(t^2 - t + 1) & t^{-1}(t+1)(t-1)(t^2 - t + 1) & t^{-1}(t^2 - t + 1) \\ -t^{-1}(t-1)(t^2 - t + 1) & t^{-2}(t^2 - t - 1)(t^2 - t + 1) & t^{-2}(t^2 - t + 1) \\ (t^3 + t^2 - 1)(t^2 - t + 1) - t & 0 & -(t^3 + t^2 - 1)(t^2 - t + 1) + t \end{pmatrix}.$$

$(D, \phi(a, b))$:

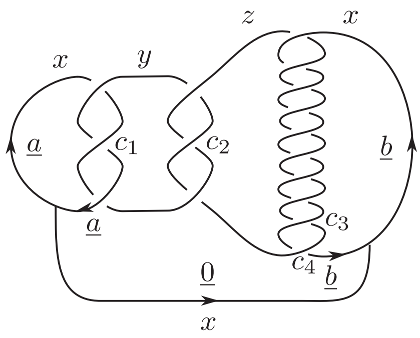


Figure 14. A $\mathbb{Z}_m$-flowed diagram $(D, \phi(a, b))$ of H.

These relations are obtained from crossings c_1, c_2, c_3 and c_4 as shown in Figure 14. When $t^2 - t + 1 \neq 0$ in X, it is clearly that rank $M \geq 1$. When $t^2 - t + 1 = 0$ in X, we have

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -t & 0 & t \end{pmatrix},$$

which implies that rank $M = 1$. Hence we have $\dim \mathrm{Col}_X(D, \phi(1, 0)) = 3 - \mathrm{rank}\, M \leq 2$. Therefore we can not obtain $2 \leq u(H)$.

We can prove the remaining cases in the same way.

Example 7.3. Let A_n, B_n and C_n be the handlebody-knots represented by the $\mathbb{Z}_8$-flowed diagram (D_{A_n}, ϕ_{A_n}), the $\mathbb{Z}_{24}$-flowed diagram (D_{B_n}, ϕ_{B_n}) and the $\mathbb{Z}_8$-flowed diagram (D_{C_n}, ϕ_{C_n}) depicted in Figures 15, 16 and 17 respectively for any $n \in \mathbb{Z}_{>0}$. Then we show that $u(A_n) = u(B_n) = u(C_n) = n$.

1. Let $s = t + 1 \in \mathbb{Z}_3[t^{\pm 1}]$ and let $f(t) = t^2 + t + 2 \in \mathbb{Z}_3[t^{\pm 1}]$, which is an irreducible polynomial. Then $X := \mathbb{Z}_3[t^{\pm 1}]/(f(t))$ is a $\mathbb{Z}_8$-family of Alexander biquandles. Then for any $x_0, x_1, \ldots, x_n \in X$, the assignment of them to each semi-arc of (D_{A_n}, ϕ_{A_n}) as shown in Figure 15 is an X-coloring of (D_{A_n}, ϕ_{A_n}), which implies $\dim \mathrm{Col}_X(D_{A_n}, \phi_{A_n}) \geq n+1$. By Corollary 6.2, we obtain $n \leq u(A_n)$. On the other hand, we can deform A_n into a trivial handlebody-knot by the crossing changes at n crossings surrounded by dotted circles depicted in Figure 15. Therefore it follows that $u(A_n) = n$.

2. Let $s = t^2 + 1 \in \mathbb{Z}_5[t^{\pm 1}]$ and let $f(t) = t^2 + 2t + 4 \in \mathbb{Z}_5[t^{\pm 1}]$, which is an irreducible polynomial. Then $X := \mathbb{Z}_5[t^{\pm 1}]/(f(t))$ is a $\mathbb{Z}_{24}$-family of Alexander biquandles. Then for any $x_0, x_1, \ldots, x_n \in X$, the assignment of them to each semi-arc of (D_{B_n}, ϕ_{B_n}) as shown in Figure 16 is an X-coloring of (D_{B_n}, ϕ_{B_n}), which implies $\dim \mathrm{Col}_X(D_{B_n}, \phi_{B_n}) \geq n+1$. By Corollary 6.2, we obtain $n \leq u(B_n)$. On the other hand, we can deform B_n into a trivial handlebody-knot by the crossing changes at n crossings surrounded by dotted circles depicted in Figure 16. Therefore it follows that $u(B_n) = n$.

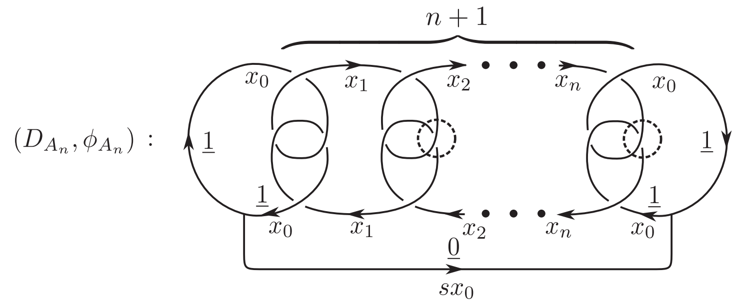


Figure 15. A $\mathbb{Z}_8$-flowed diagram (D_{A_n}, ϕ_{A_n}) of A_n.

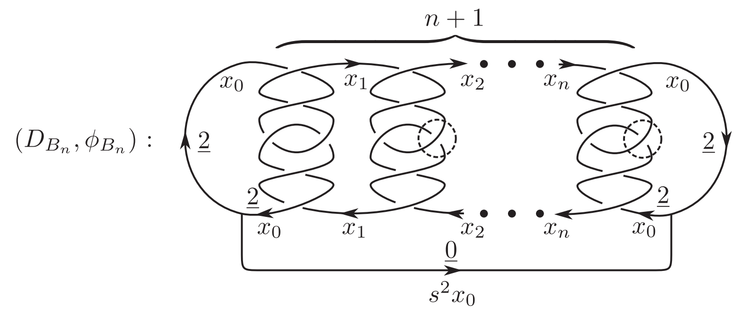


Figure 16. A $\mathbb{Z}_{24}$-flowed diagram (D_{B_n}, ϕ_{B_n}) of B_n.

3. Let $s = 2t - 1 \in \mathbb{Z}_3[t^{\pm 1}]$ and let $f(t) = t^2 + t + 2 \in \mathbb{Z}_3[t^{\pm 1}]$, which is an irreducible polynomial. Then $X := \mathbb{Z}_3[t^{\pm 1}]/(f(t))$ is a $\mathbb{Z}_8$-family of Alexander biquandles. Then for any $x_0, x_1, \ldots, x_n \in X$, the assignment of them to each semi-arc of (D_{C_n}, ϕ_{C_n}) as shown in Figure 17 is an X-coloring of (D_{C_n}, ϕ_{C_n}), which implies $\dim \mathrm{Col}_X(D_{C_n}, \phi_{C_n}) \geq n+1$. By Corollary 6.2, we obtain $n \leq u(C_n)$. On the other hand, we can deform C_n into a trivial handlebody-knot by the crossing changes at n crossings surrounded by dotted circles depicted in Figure 17. Therefore it follows that $u(C_n) = n$.

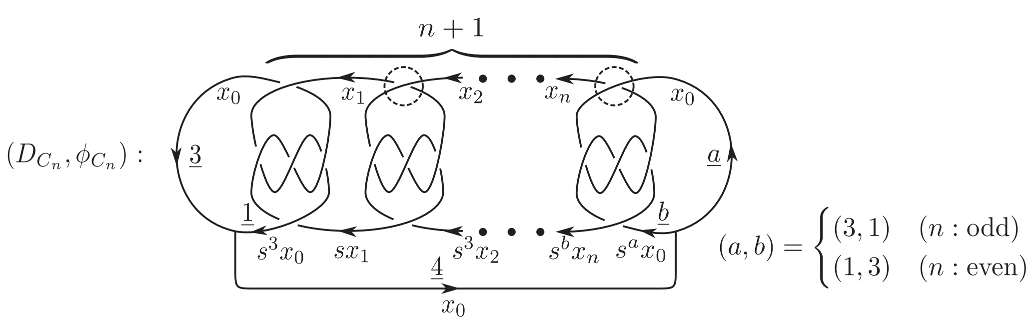


Figure 17. A $\mathbb{Z}_8$-flowed diagram (D_{C_n}, ϕ_{C_n}) of C_n.

EXAMPLE 7.4. Let H_n and H'_n be the handlebody-knots represented by the $\mathbb{Z}_3$-

flowed diagrams (D_n, ϕ_n) and $(D'_n, \phi'_n(a,b))$ respectively depicted in Figure 18 for any $n \in \mathbb{Z}_{>0}$ and $a, b \in \mathbb{Z}_3$. Then we show that $d(H_n, H'_n) = n$.

Let $s = 1 \in \mathbb{Z}_2[t^{\pm 1}]$ and let $f(t) = t^2 + t + 1 \in \mathbb{Z}_2[t^{\pm 1}]$, which is an irreducible polynomial. Then $X := \mathbb{Z}_2[t^{\pm 1}]/(f(t))$ is a $\mathbb{Z}_3$-family of Alexander (bi)quandles. Then for any $x_0, x_1, \ldots, x_n, y_1, \ldots, y_n \in X$, the assignment of them to each semi-arc of (D_n, ϕ_n) as shown in Figure 18 is an X-coloring of (D_n, ϕ_n), which implies $\dim \mathrm{Col}_X(D_n, \phi_n) \geq 2n + 1$.

On the other hand, we note that $\mathrm{Col}_X(D'_n, \phi'_n(a,b))$ is generated by $x_0, x_1, x'_1, \ldots, x_n, x'_n, y_1, y'_1, \ldots, y_n, y'_n \in X$ as shown in Figure 18 for any $a, b \in \mathbb{Z}_3$. If $(a,b) = (0,0)$, it is easy to see that $\dim \mathrm{Col}_X(D'_n, \phi'_n(a,b)) = 1$. If $(a,b) = (1,1), (1,2), (2,1), (2,2)$, we obtain that $x_i = x'_i = y_i = y'_i$ for any $i = 1, 2, \ldots, n$, which implies $\dim \mathrm{Col}_X(D'_n, \phi'_n(a,b)) \leq n + 1$. If $(a,b) = (0,1), (0,2)$, we have

$$\begin{aligned}
&x_0 = x_1 = x_2, \\
&x_{i+2} = x'_i \ (i = 1, 2, \ldots, n-2), \\
&x'_i = \begin{cases} x_i \underline{*}^{b}\, y'_i \ (i : \text{odd}), \\ x_i \underline{*}^{-b}\, y'_i \ (i : \text{even}), \end{cases} \\
&x_n = x'_{n-1}, \\
&y_i = y'_i \ (i = 1, 2, \ldots, n).
\end{aligned}$$

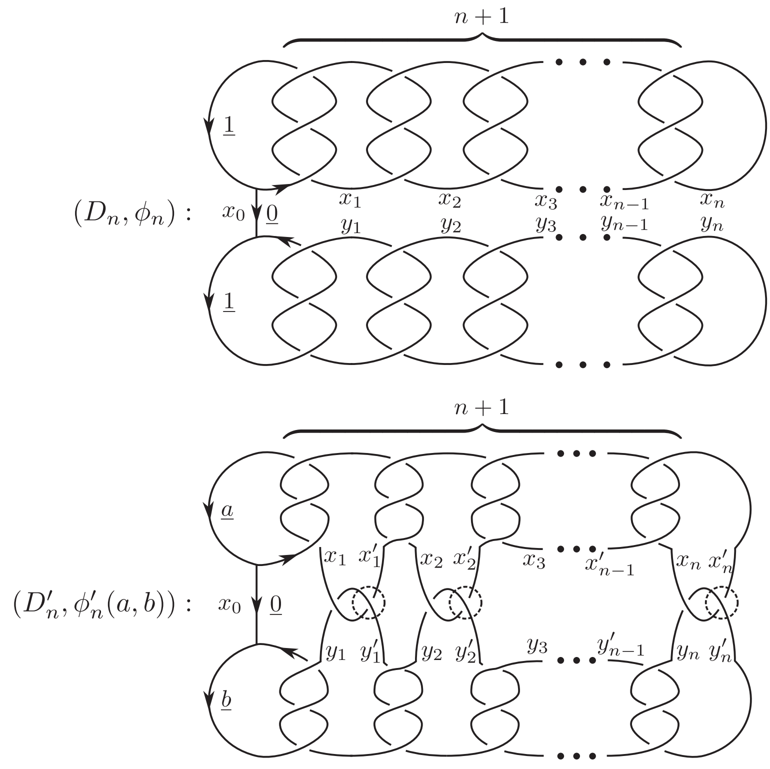


Figure 18. $\mathbb{Z}_3$-flowed diagrams (D_n, ϕ_n) and (D'_n, ϕ'_n) of H_n and H'_n.

Hence $\mathrm{Col}_X(D'_n, \phi'_n(a,b))$ is generated by $x_0, y_1, \ldots, y_n \in X$, which implies $\dim \mathrm{Col}_X(D'_n, \phi'_n(a,b)) \leq n+1$. If $(a,b) = (1,0), (2,0)$, in the same way as when $(a,b) = (0,1), (0,2)$, $\mathrm{Col}_X(D'_n, \phi'_n(a,b))$ is generated by $x_0, x_1, \ldots, x_n \in X$, which implies $\dim \mathrm{Col}_X(D'_n, \phi'_n(a,b)) \leq n+1$. Hence for any $a, b \in \mathbb{Z}_3$, $\dim \mathrm{Col}_X(D'_n, \phi'_n(a,b)) \leq n+1$, which implies that

$$\dim \mathrm{Col}_X(D_n, \phi_n) - \dim \mathrm{Col}_X(D'_n, \phi'_n(a,b)) \geq n.$$

By Theorem 6.1, it follows that $n \leq d(H_n, H'_n)$.

Finally, we can deform H'_n into H_n by the crossing changes at n crossings surrounded by dotted circles depicted in Figure 18. Therefore it follows that $d(H_n, H'_n) = n$.

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