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On the family of pentagonal curves of genus 6 and associated modular forms on the ball

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Abstract. In this article we study the inverse of the period map for the family $\mathscr{F}$ of complex algebraic curves of genus 6 equipped with an automorphism of order 5 having 5 fixed points. This is a family with 2 parameters, and is fibred over a Del Pezzo surface. Our period map is essentially same as the Schwarz map for the Appell hypergeometric differential equation $F_1(3/5, 3/5, 2/5, 6/5)$.

This differential equation and the family $\mathscr{F}$ are studied by G. Shimura (1964), T. Terada (1983, 1985), P. Deligne and G. D. Mostow (1986) and T. Yamazaki and M. Yoshida (1984). Based on their results we give a representation of the inverse of the period map in terms of Riemann theta constants. This is the first variant of the work of H. Shiga (1981) and K. Matsumoto (1989, 2000) to the co-compact case.

0. Introduction.

We consider the configuration space

$$X^{\circ}(2,5) = ((\boldsymbol{P}^1)^5 - \Delta)/PGL_2(\boldsymbol{C})$$

where

$$\Delta = \{(\lambda_l) \in (\boldsymbol{P}^1)^5 : \lambda_i = \lambda_j \text{ for some } i \neq j\}.$$

Since every point $\lambda = (\lambda_1, \dots, \lambda_5) \in X^{\circ}(2,5)$ can be represented by $(0, \infty, 1, x, y)$, the space can be seen as an open set in $\boldsymbol{C}^2$:

$$X^{\circ}(2,5) \cong \Lambda = \{(s,t) \in \boldsymbol{C}^2 : st(s-1)(t-1)(s-t) \neq 0\}.$$

Let $\mathscr{F}$ be the family of algebraic curves

$$C(s,t) : y^5 = x(x-1)(x-s)(x-t) \tag{0.1}$$

of genus 6 parameterized by Λ. The curve $C(s,t)$ has also an expression of the form

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$$C_\lambda : y^5 = \prod_{i=1}^{5}(x - \lambda_i);$$

by using a general representative $\lambda = (\lambda_1, \dots, \lambda_5)$ of a point in $X^\circ(2,5)$.

The period

$$\eta(s,t) = \int_\gamma \frac{dz}{w^2}$$

of $C(s,t)$ satisfies the system of differential equations

$$(0.2) \quad \begin{cases} s(1-s)(\partial^2 u/\partial s^2) + t(1-s)(\partial^2 u/\partial s \partial t) + (6/5 - (11/5)s)(\partial u/\partial s) \\ \qquad - (3/5)t(\partial u/\partial t) - (9/5)u = 0, \\ t(1-t)(\partial^2 u/\partial t^2) + s(1-t)(\partial^2 u/\partial s \partial t) + (6/5 - 2t)(\partial u/\partial t) \\ \qquad - (2/5)s(\partial u/\partial s) - (6/5)u = 0. \end{cases}$$

This is the system for the Appell hypergeometric function $F_1(3/5, 3/5, 2/5, 6/5; s, t)$, and the dimension of the solution space at a generic point is equal to 3.

According to the works of T. Terada [**14**], P. Deligne and G. D. Mostow [**1**] and T. Yamazaki and M. Yoshida [**16**] the following properties are already known:

1. Let $\{\eta_1, \eta_2, \eta_3\}$ be a basis of the solutions of (0.2). The image of the Schwarz map $(s,t) \mapsto [\eta_1(s,t) : \eta_2(s,t) : \eta_3(s,t)] \in \boldsymbol{P}^2$ is an open dense subset of a 2-dimensional ball $\boldsymbol{B}_2$.
2. The monodromy group G is a congruence subgroup of the Picard modular group for $k = \boldsymbol{Q}(e^{2\pi i/5})$. The quotient space $\boldsymbol{B}_2/G$ is compact.
3. Let S_5 be the symmetric group of permutations of $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$, it causes a natural action on $X^\circ(2,5)$. There is a compactification $X(2,5)$ of $X^\circ(2,5)$ so that we have $S_5 \subset \mathrm{Aut}(X)$. Yoshida showed X is a Del Pezzo surface of degree 5.
4. The inverse of the Schawarz map is single valued.

Based on these established results we give an explicit expression of the inverse of the period map for $\mathscr{F}$. We obtain the results by the following steps.

We find a realization of the compactified configuration space $X(2,5)$ in $\boldsymbol{P}^{11}$ by using 12 extended crossratios. We construct the period matrix $\Omega = \Omega(\lambda)$ of C_λ, it gives an embedding ρ of the period domain for $\mathscr{F}$ into the Siegel space $\mathfrak{S}_6$. We fix a basis $\{\eta_1, \eta_2, \eta_3\}$ of the solutions for (0.2). It is a system of periods for appropriate 1-cycles $\gamma_1, \gamma_2, \gamma_3$ on C_λ coupled with the differential dz/w^2. By the method originally used by Picard we can determine the image of the Schwarz map from the Riemann period relation. That is given by $\boldsymbol{B}_2 = \{\eta \in \boldsymbol{P}^2 : {}^t\bar{\eta}H\eta < 0\}$ with $H = \mathrm{diag}(1, 1, (1-\sqrt{5})/2)$.

We give an explicit generator system of the monodromy group G for (0.2). The theta constants on $\mathfrak{S}_6$ with characteristics in $(\boldsymbol{Z}/10\boldsymbol{Z})^6$ determine a system of holomorphic functions on the period domain $\boldsymbol{B}_2$ via the embedding ρ. By using the transformation formula for the theta constants we can examine their behavior under the monodromy transformations. We know that many of them vanish on $\boldsymbol{B}_2$, and that there are essentially 13 theta constants those are not constantly zero there. One of them is the Riemann constant, and the rest give the 12 coordinates of the configuration space in $\boldsymbol{P}^{11}$ with the elevation to the 5-th power. This is our main theorem and the exact statement is given in Theorem 6.1.

Our method is essentially the same used in [**10**] and [**8**], but we could obtain the result applied to the co-compact case for the first time.

As a byproduct of the main theorem, in Theorem 6.3 we give an expression of the inverse Schawarz map of the Gauss hypergeometric differential equation for ${}_2F_1(1/5, 2/5, 4/5, x)$. In this case we have the arithmetic triangle group of co-compact type $\Delta(5,5,5)$ as the monodromy group, and it is the case mentioned by Shimura [**12**]. As another application, in Theorem 6.2 we give the explicit generator system for the graded ring of the automorphic forms with respect to the unitary group $U(2,1;\mathcal{O}_k)$ over $\mathcal{O}_k$.

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1. Realization of the configuration space $X(2,5)$.

Here we summarize the fundamental facts on $X(2,5)$. For precise arguments, see [**18**, Chapter V]. Let us consider ordered distinct five points on $\boldsymbol{P}^1$:

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in (\boldsymbol{P}^1)^5 - \Delta$$

where, Δ is given by

$$\Delta = \{(\lambda_l) \in (\boldsymbol{P}^1)^5 : \lambda_i = \lambda_j \text{ for some } i \neq j\}.$$

A projective transformation $g \in PGL_2(\boldsymbol{C})$ acts on $(\boldsymbol{P}^1)^5$ as

$$g \cdot (\lambda_1, \dots, \lambda_5) = (g(\lambda_1), \dots, g(\lambda_5)).$$

The configuration space $X^\circ(2,5)$ is defined by the quotient space

$$X^\circ(2,5) = ((\boldsymbol{P}^1)^5 - \Delta)/PGL_2(\boldsymbol{C}).$$

It has a compactification

$$X(2,5) = \overline{X^\circ(2,5)} = ((\boldsymbol{P}^1)^5 - \Delta')/PGL_2(\boldsymbol{C})$$

where

$$\Delta' = \{(\lambda_l) \in (\boldsymbol{P}^1)^5 : \lambda_i = \lambda_j = \lambda_k \text{ for some } i \neq j \neq k \neq i\}.$$

There exist ten lines on $X(2,5)$ of the form

$$L(ij) = \{(\lambda_l) \in (\boldsymbol{P}^1)^5 : \lambda_i = \lambda_j\}/PGL_2(\boldsymbol{C}) \cong \boldsymbol{P}^1.$$

By definition it holds $L(ij) \cap L(jk) = \phi$ $(i \neq j \neq k \neq i)$. Notice that $X(2,5) - X^{\circ}(2,5)$ is just the union of these ten lines, and that $X(2,5)$ is isomorphic to the blow-up of $\boldsymbol{P}^2$ at four points. We can see the blow down $\pi : X(2,5) \to \boldsymbol{P}^2$ by the following way: for a point on $X(2,5)$, we take the specialized representative $(\lambda_1, \lambda_2, \lambda_3, 0, \infty)$ and regard $[\lambda_1 : \lambda_2 : \lambda_3]$ as a point on $\boldsymbol{P}^2$. Then we obtain the following correspondence;

$$P_1 = [1:0:0] = \pi(L(15)), \quad P_2 = [0:1:0] = \pi(L(25)),$$
$$P_3 = [0:0:1] = \pi(L(35)), \quad P_4 = [1:1:1] = \pi(L(45)),$$

and

$$\pi(X^{\circ}(2,5)) = \{[\lambda_1 : \lambda_2 : \lambda_3] \in \boldsymbol{P}^2 : \lambda_i \neq \lambda_j \ (i \neq j), i, j = 1,2,3,4\}.$$

For five distinct numbers i, j, k, l, m in $\{1,2,3,4,5\}$, we define a divisor $D(ijklm)$ on $X(2,5)$ by

$$D(ijklm) = L(ij) + L(jk) + L(kl) + L(lm) + L(mi).$$

As easily shown, every $D(ijklm)$ is linearly equivalent to the divisor

$$3\pi^* H - L(15) - L(25) - L(35) - L(45),$$

where H is a line on $\boldsymbol{P}^2$. This is anti-canonical and very ample. In fact, we have the following proposition by direct calculations.

Proposition 1.1. *For twelve* $(ijklm)$, *set*

$$J(ijklm)(\lambda) = d_{ij}(\lambda)d_{jk}(\lambda)d_{kl}(\lambda)d_{lm}(\lambda)d_{mi}(\lambda) \quad (d_{ij}(\lambda) = a_j b_i - a_i b_j),$$

where $[a_i : b_i]$ *is the projective coordinate for* $\lambda_i \in \boldsymbol{P}^1$. *Then the equation* $J(ijklm) \cdot (\lambda) = 0$ *defines the divisor* $D(ijklm)$ *and the map*

$$J : X(2,5) \to \boldsymbol{P}^{11}, \quad J(\lambda) = [\cdots : J(ijklm)(\lambda) : \cdots]$$

is an embedding.

Remark 1.1. In terms of affine coordinates $\lambda_i = b_i/a_i$, we have

$$\frac{J(ijklm)(\lambda)}{J(pqrst)(\lambda)} = \frac{(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_l)(\lambda_l - \lambda_m)(\lambda_m - \lambda_i)}{(\lambda_p - \lambda_q)(\lambda_q - \lambda_r)(\lambda_r - \lambda_s)(\lambda_s - \lambda_t)(\lambda_t - \lambda_p)}.$$

We use this notation in following sections.

2. The family of pentagonal curves and the periods.

Let us consider the algebraic curve

$$C_\lambda : y^5 = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(x-\lambda_4)(x-\lambda_5),$$
$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in (\boldsymbol{P}^1)^5 - \Delta.$$

Set $\mathscr{F} = \{C_\lambda : \lambda \in X^\circ(2,5)\}$. We regard C_λ as a five sheeted cyclic covering over $\boldsymbol{P}^1$ branched at λ_i by the projection

$$\pi : C_\lambda \to \boldsymbol{P}^1, \quad (x, y) \mapsto x.$$

By the Hurwitz formula, the genus of C_λ is six. We have the following basis of $\mathrm{H}^0(C_\lambda, \Omega^1)$:

$$\text{(2.1)} \quad \varphi_1 = \frac{\mathrm{d}x}{y^2}, \quad \varphi_2 = \frac{\mathrm{d}x}{y^3}, \quad \varphi_3 = \frac{x\,\mathrm{d}x}{y^3}, \quad \varphi_4 = \frac{\mathrm{d}x}{y^4}, \quad \varphi_5 = \frac{x\,\mathrm{d}x}{y^4}, \quad \varphi_6 = \frac{x^2\,\mathrm{d}x}{y^4}.$$

Let ρ denotes the automorphism of order five:

$$\rho : C_\lambda \to C_\lambda, \quad (x, y) \mapsto (x, \zeta y) \quad (\zeta = \exp(2\pi\sqrt{-1}/5))$$

on C_λ.

Notation 2.1. Throughout this article ζ stands for $\exp(2\pi\sqrt{-1}/5)$.

Next, we construct a symplectic basis of $\mathrm{H}_1(C_\lambda, \boldsymbol{Z})$.

Let $\lambda^0 = (\lambda_1^0, \lambda_2^0, \lambda_3^0, \lambda_4^0, \lambda_5^0) \in X^\circ(2,5)$ be a real point with $\lambda_1^0 < \cdots < \lambda_5^0$, and let C_0 be the corresponding curve. Take a point $x_0 \in \boldsymbol{P}^1$ with $\mathrm{Im}(x_0) < 0$, and make a line segment l_i connecting x_0 and λ_i^0 $(i = 1, \dots, 5)$. Then $\Sigma = \boldsymbol{P}^1 - \bigcup l_i$ is simply connected and we can choose an isomorphism

$$v : \Sigma \times \boldsymbol{Z}/5\boldsymbol{Z} \to \pi^{-1}(\Sigma)$$

such that the fiber coordinate $k \in \boldsymbol{Z}/5\boldsymbol{Z}$ satisfies $v(x, k+1) = \rho(v(x,k))$. Let $\alpha(i,j)$ be the open interval $(\lambda_i^0, \lambda_j^0) \subset \boldsymbol{R} \subset \Sigma$. We obtain five arcs

$$\alpha_k^\circ(i,j) = v(\alpha(i,j), k) \quad (k = 1, \dots, 5)$$

in C_0. Let us consider 1-chains

$$\text{(2.2)} \qquad \alpha_k(i,j) = \alpha_k^\circ(i,j) \cup \{\pi^{-1}(\lambda_i^0), \pi^{-1}(\lambda_j^0)\} \quad (k = 1, \dots, 5)$$

with the common boundary $\partial\alpha_k(i,j) = \pi^{-1}(\lambda_j^0) - \pi^{-1}(\lambda_i^0)$. We define cycles $\gamma_1, \gamma_2, \gamma_3$ on C_0 combining these chains;

$$\text{(2.3)} \qquad \begin{cases} \gamma_1 = \alpha_1(1,2) + \alpha_2(2,1), \\ \gamma_2 = \alpha_1(3,4) + \alpha_2(4,3), \\ \gamma_3 = \alpha_1(1,3) + \alpha_2(3,4) + \alpha_3(4,2) + \alpha_2(2,1). \end{cases}$$

We set

$$(2.4)\quad \begin{cases} A_1=\gamma_1,\ \ A_2=\gamma_2,\ \ A_3=\gamma_3,\ \ A_4=\rho^2(\gamma_1),\ \ A_5=\rho^2(\gamma_2),\ \ A_6=\rho^4(\gamma_3),\\ B_1=\rho(\gamma_1)+\rho^3(\gamma_1),\quad B_2=\rho(\gamma_2)+\rho^3(\gamma_2),\quad B_3=\rho(\gamma_3)+\rho^2(\gamma_3),\\ B_4=\rho^3(\gamma_1),\quad B_5=\rho^3(\gamma_2),\quad B_6=\rho(\gamma_3).\end{cases}$$

The intersection numbers of these cycles are given by

$$A_i\cdot A_j=B_i\cdot B_j=0,\quad A_i\cdot B_j=\delta_{ij}.$$

So, $\{A_i,B_i\}$ is a symplectic basis of $\mathrm{H}_1(C_0,\boldsymbol{Z})$.

Let λ be a point on $X^\circ(2,5)$, and suppose an arc r from λ^0 to λ. Since the family $\mathscr{F}$ is locally trivial as a topological fiber space over $X^\circ(2,5)$, by using this trivialization along r, we obtain the systems $\{\alpha_k(i,j)(\lambda)\}$, $\{\gamma_i(\lambda)\}$ and the symplectic basis $\{A_i(\lambda),B_i(\lambda)\}$ on C_λ. We have the relation (2.4) between $\{\gamma_i(\lambda)\}$ and $\{A_i(\lambda),B_i(\lambda)\}$ also. We note that $\{A_i(\lambda),B_i(\lambda)\}$ depend on the homotopy class of r.

Now, we consider the period matrix of C_λ:

$$\Pi(\lambda)=\Pi=(Z_1,Z_2)=\begin{pmatrix}\int_{A_1}\varphi_1 & \cdots & \int_{A_6}\varphi_1 & \int_{B_1}\varphi_1 & \cdots & \int_{B_6}\varphi_1\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \int_{A_1}\varphi_6 & \cdots & \int_{A_6}\varphi_6 & \int_{B_1}\varphi_6 & \cdots & \int_{B_6}\varphi_6\end{pmatrix}.$$

The normalized period matrix $\Omega(\lambda)=\Omega=Z_1^{-1}Z_2$ belongs to the Siegel upper half space of degree 6:

$$\mathfrak{S}_6=\{\Omega\in GL_6(\boldsymbol{C}):{}^t\Omega=\Omega,\ \mathrm{Im}(\Omega)\ \text{is positive definite}\}.$$

The automorphism ρ acts on $\mathrm{H}^0(C_\lambda,\Omega^1)$ and $\mathrm{H}_1(C_\lambda,\boldsymbol{Z})$. So we have the representation matrices $R\in GL_6(\boldsymbol{C})$ and $M\in GL_{12}(\boldsymbol{Z})$ of ρ with respect to the bases $\{\varphi_i\}$ and $\{A_i,B_i\}$, respectively. It holds $R\Pi=\Pi M$ where $R=\mathrm{diag}(\zeta^3,\zeta^2,\zeta^2,\zeta,\zeta,\zeta)$. Put

$$(2.5)\qquad M=\begin{pmatrix}{}^tD & {}^tB\\ {}^tC & {}^tA\end{pmatrix},\quad \sigma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}.$$

Then the matrix σ belongs to the symplectic group

$$Sp_{12}(\boldsymbol{Z})=\{g\in GL_{12}(\boldsymbol{Z}):{}^tgJg=J\},\quad J=\begin{pmatrix}0 & I_6\\ -I_6 & 0\end{pmatrix}$$

and it holds

$$\Omega=(A\Omega+B)(C\Omega+D)^{-1}.$$

By the definition (2.4) of A_i,B_i, we have

(2.6) $\Pi = (a, b, c, R^2 a, R^2 b, R^4 c, (R+R^3)a, (R+R^3)b, (R+R^2)c, R^3 a, R^3 b, Rc),$

where we denote

$$a = {}^t\left(\int_{\gamma_1}\varphi_1, \dots, \int_{\gamma_1}\varphi_6\right), \quad b = {}^t\left(\int_{\gamma_2}\varphi_1, \dots, \int_{\gamma_2}\varphi_6\right), \quad c = {}^t\left(\int_{\gamma_3}\varphi_1, \dots, \int_{\gamma_3}\varphi_6\right).$$

According to (2.4),

$$\rho(A_1) = \rho(\gamma_1) = (\rho(\gamma_1) + \rho^3(\gamma_1)) - \rho^3(\gamma_1) = B_1 - B_4.$$

By the same way, we can describe $\rho(A_2), \dots, \rho(B_6)$ in terms of $\{A_i, B_i\}$. So we can determine M, and obtain

$$(2.7) \qquad \sigma = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Put

$$\eta(\lambda) = [\eta_1(\lambda) : \eta_2(\lambda) : \eta_3(\lambda)] \in \boldsymbol{P}^2, \ \ \eta_1(\lambda) = \int_{\gamma_1}\varphi_1, \ \ \eta_2(\lambda) = \int_{\gamma_2}\varphi_1, \ \ \eta_3(\lambda) = \int_{\gamma_3}\varphi_1.$$

These are multi-valued analytic functions of λ. The Riemann period relation for (2.6) induces

$$|\eta_1|^2 + |\eta_2|^2 + \frac{1-\sqrt{5}}{2}|\eta_3|^2 < 0,$$

namely, $\eta = (\eta_1, \eta_2, \eta_3)$ belongs to the complex ball

$$(2.8) \qquad \boldsymbol{B}_2 = \{\eta \in \boldsymbol{P}^2 : {}^t\bar{\eta}H\eta < 0\}, \quad H = \mathrm{diag}\left(1, 1, \frac{1-\sqrt{5}}{2}\right).$$

Next, we express Ω in terms of η explicitly. Write $a = (a_i)$, $b = (b_i)$ and $c = (c_i)$. Then, the Riemann bilinear relation $\Pi J {}^t\Pi = 0$ induces the following equations:

$$c_2 = -(\zeta^2+\zeta^3)(a_1a_2+b_1b_2)/c_1, \quad c_3 = -(\zeta^2+\zeta^3)(a_1a_3+b_1b_3)/c_1.$$

By substituting them for Z_1, Z_2 in Π we can proceed the calculation of $\Omega = Z_1^{-1}Z_2$ (using a computer). Hence we have the following:

Lemma 2.1. *Let $\Delta = \eta_1^2+\eta_2^2-\zeta^3(1+\zeta)\eta_3^2$. The period matrix $\Omega = (\Omega_{ij})$ is given by*

$$\begin{aligned}
\Omega_{11} &= (\zeta^3-1)(\eta_1^2+(1+\zeta^3)\eta_2^2+\eta_3^2)/\Delta, & \Omega_{44} &= -\zeta^2(\eta_1^2+\zeta^2\eta_2^2-(1+\zeta)\eta_3^2)/\Delta,\\
\Omega_{22} &= (\zeta^3-1)((1+\zeta^3)\eta_1^2+\eta_2^2+\eta_3^2)/\Delta, & \Omega_{55} &= -\zeta^2(\zeta^2\eta_1^2+\eta_2^2-(1+\zeta)\eta_3^2)/\Delta,\\
\Omega_{33} &= (\zeta^2-1)(\eta_1^2+\eta_2^2-\zeta^3\eta_3^2)/\Delta, & \Omega_{66} &= -\zeta^3(\eta_1^2+\eta_2^2-(1+\zeta^4)\eta_3^2)/\Delta,\\
\Omega_{12} &= (\zeta^3-\zeta)\eta_1\eta_2/\Delta, & \Omega_{45} &= (\zeta^4-\zeta^2)\eta_1\eta_2/\Delta,\\
\Omega_{15} &= (\zeta^4-\zeta)\eta_1\eta_2/\Delta, & \Omega_{24} &= (\zeta^4-\zeta)\eta_1\eta_2/\Delta,\\
\Omega_{13} &= (1-\zeta^2)\eta_1\eta_3/\Delta, & \Omega_{23} &= (1-\zeta^2)\eta_2\eta_3/\Delta,\\
\Omega_{46} &= (\zeta^4-\zeta)\eta_1\eta_3/\Delta, & \Omega_{56} &= (\zeta^4-\zeta)\eta_2\eta_3/\Delta,\\
\Omega_{16} &= (\zeta^3-\zeta)\eta_1\eta_3/\Delta, & \Omega_{26} &= (\zeta^3-\zeta)\eta_2\eta_3/\Delta,\\
\Omega_{34} &= (1-\zeta^3)\eta_1\eta_3/\Delta, & \Omega_{35} &= (1-\zeta^3)\eta_2\eta_3/\Delta,\\
\Omega_{14} &= \zeta^3((1+\zeta)\eta_1^2+(1+\zeta^3)\eta_2^2+\eta_3^2)/\Delta, & \Omega_{25} &= \zeta^3((1+\zeta^3)\eta_1^2+(1+\zeta)\eta_2^2+\eta_3^2)/\Delta,\\
\Omega_{36} &= (\zeta+\zeta^2)(\eta_1^2+\eta_2^2-\zeta^3(1+\zeta^2)\eta_3^2)/\Delta.
\end{aligned}$$

Now we define our period map

$$\Phi : X^\circ(2,5) \to \boldsymbol{B}_2, \quad \lambda \mapsto [\eta_1(\lambda) : \eta_2(\lambda) : \eta_3(\lambda)],$$

that is multi-valued analytic. The above Lemma says that the original period map $\lambda \mapsto \Omega(\lambda)$ factors as

$$X^\circ(2,5) \to \boldsymbol{B}_2 \to \mathfrak{S}_6.$$

Throughout this paper, we denote the matrix in Lemma 2.1 by $\Omega(\eta)$.

3. The monodromy group and reflections.

The multi-valuedness of Φ induces a unitary representation of the fundamental group

$$\psi : \pi_1(X^\circ(2,5)) \to \Gamma = \{g \in GL_3(\boldsymbol{Z}[\zeta]) : {}^t\bar{g}Hg = H\}. \tag{3.1}$$

We call $\mathrm{Im}(\psi)$ the monodromy group of Φ. The group Γ acts on $\boldsymbol{B}_2$ (left action). The structures of our monodromy group is studied in [**16**]. Set

$$\Gamma(1-\zeta) = \{g \in \Gamma : g \equiv I_3 \mod 1-\zeta\}.$$

Theorem 3.1 (T. Yamazaki, M. Yoshida [**16**]). (1) *The monodromy group of the period map Φ coincides with $\Gamma(1-\zeta)$ and the quotient $\Gamma/(\pm I)\Gamma(1-\zeta)$ is isomorphic to the symmetric group S_5.*

(2) *The quotient $\boldsymbol{B}_2/\Gamma(1-\zeta)$ is biholomorphically equivalent to the blow up of $\boldsymbol{P}^2$ at four points.*

Remark 3.1 (see [**16**]). There are ten (-1)-curves on $\boldsymbol{B}_2/\Gamma(1-\zeta)$, and S_5 acts transitively on them.

According to [**14**] and [**16**], it is proved that Γ and $\Gamma(1-\zeta)$ are reflection groups and generator systems are given also. We expose those generator system in a form adapted for our calculation in the later sections.

Let us take the reference point $\lambda^0 \in X^\circ(2,5)$ again. Now we define the half way monodromy transformation g_{12} induced from the permutation of λ_1^0 and λ_2^0. Let us consider a continuous arc R_{12} in $X^\circ(2,5)$ starting from λ^0:

$$\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3^0, \lambda_4^0, \lambda_5^0), \quad (0 \le t \le 1) \tag{3.2}$$

such that (Figure 1)

$$\lambda_2(1) = \lambda_1^0, \quad \lambda_1(1) = \lambda_2^0, \quad 0 < \mathrm{Im}(\lambda_1(t)) < \mathrm{Im}(\lambda_2(t)) \quad (0 < t < 1).$$

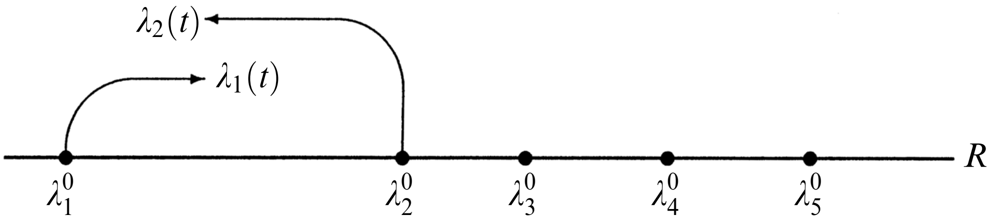

Figure 1.

Let $\eta(t) = \eta(\lambda(t))$ be the corresponding periods. Recall the definition (2.3). We get $\gamma_2(1) = \gamma_2(0), \gamma_3(1) = \gamma_3(0)$ and $\gamma_1(1) = -\rho(\gamma_1(0))$. Hence,

$$\begin{pmatrix} \eta_1(1) \\ \eta_2(1) \\ \eta_3(1) \end{pmatrix} = \begin{pmatrix} -\zeta^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1(0) \\ \eta_2(0) \\ \eta_3(0) \end{pmatrix}.$$

The matrix in the right hand side belongs to Γ, and we denote it by g_{12}. We define g_{ij} $(1 \le i < j \le 5)$ by the same manner. Set $h_{ij} = g_{ij}^2$.

Proposition 3.1 (see [**17**]). *The monodromy group of* Φ *is generated by*

$$h_{12}, \quad h_{13}, \quad h_{14}, \quad h_{23}, \quad h_{34}. \tag{3.3}$$

Let us define two reflections on $\boldsymbol{B}_2$ with respect to a root α as

$$\mathrm{T}_\alpha(\eta) = \eta - (1+\zeta^3)\frac{{}^t\bar{\alpha}H\eta}{{}^t\bar{\alpha}H\alpha}\alpha, \quad \mathrm{R}_\alpha(\eta) = \eta - (1+\zeta)\frac{{}^t\bar{\alpha}H\eta}{{}^t\bar{\alpha}H\alpha}\alpha.$$

Then we see that

Lemma 3.1. *Set*

$$\alpha_{12} = (1,0,0), \quad \alpha_{23} = (\zeta^3, 1, -(1+\zeta)), \quad \alpha_{34} = (0,1,0), \quad \alpha_{45} = (0,1,\zeta^3),$$
$$\alpha_{13} = (1,-1,1+\zeta), \quad \alpha_{14} = (1,\zeta^3,1+\zeta).$$

Then we have $g_{ij} = \mathrm{T}_{\alpha_{ij}}$, $h_{ij} = \mathrm{R}_{\alpha_{ij}}$, *and* g_{ij} *is of order ten,* h_{kl} *is of order five.*

Remark 3.2. The group Γ is generated by $\{g_{12}, g_{23}, g_{34}, g_{45}\}$ and $\pm I$.

The deformation of the curve $C_{\lambda(t)}$ along R_{12} in (3.2) induces a symplectic basis $\{A_i(t), B_i(t)\}$ on it. So $\{A_i(1), B_i(1)\}$ is again a symplectic basis on C_{λ^0}. Hence we obtain a symplectic transformation

$${}^t(B_1(1),\dots,B_6(1),A_1(1)\dots,A_6(1)) = \hat{g}_{12}{}^t(B_1(0),\dots,B_6(0),A_1(0)\dots,A_6(0)).$$

For $\hat{g}_{12} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we have $\Omega(\eta(1)) = (A\Omega(\eta(0)) + B)(C\Omega(\eta(0)) + D)^{-1}$. Recall R_{12} induces the change of cycles $(\gamma_1,\gamma_2,\gamma_3) \to (-\rho(\gamma_1),\gamma_2,\gamma_3)$. Together with (2.4), we obtain:

$$\hat{g}_{12} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \tag{3.4}$$

By the same consideration, we obtain the following:

$$(3.5)\quad \hat{g}_{23} = \begin{pmatrix} 1 & 1 & 0 & 1 & -1 & 1 & 2 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 0 & -2 & 2 & 0 & -1 & 1 & -2 \\ 1 & -1 & 2 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & -1 & 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & 1 & -1 & 1 & 2 & -2 & 0 & 1 & -1 & 2 \\ -1 & 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 1 & -2 & -1 & 1 & 1 & -1 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 2 & 0 & -1 & -2 & 0 & -1 & 1 & -1 & 0 & 2 \end{pmatrix},$$

$$(3.6)\quad \hat{g}_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(3.7)\quad \hat{g}_{45} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By sending g_{ij} to $\hat{g}_{ij}$, we define a homomorphism

$$\hat{}: \Gamma \to \mathrm{Aut}(H_1(C_{\lambda^0}, \boldsymbol{Z})) = Sp_{12}(\boldsymbol{Z}), \quad g \mapsto \hat{g}.$$

Remark 3.3. Since we have the relations

$$h_{12} = (g_{12})^2, \quad h_{23} = (g_{23})^2, \quad h_{34} = (g_{34})^2$$

$$h_{13} = (g_{23})^{-1}(g_{12})^2 g_{23}, \quad h_{14} = (g_{23}g_{34})^{-1}(g_{12})^2 g_{23}g_{34},$$

we can obtain $\hat{h}_{12}, \hat{h}_{13}, \hat{h}_{14}, \hat{h}_{23}, \hat{h}_{34}$.

4. Degenerate loci.

According to Theorem 3.1 and the result of T. Terada ([**14**]) the period map Φ induces a biholomorphic equivalence $X^\circ(2,5) \xrightarrow{\sim} \boldsymbol{B}_2^\circ/\Gamma(1-\zeta)$ ($\boldsymbol{B}_2^\circ = \mathrm{Im}\,\Phi$), moreover it has the unique extension

$$\tilde{\Phi} : X(2,5) \xrightarrow{\sim} \boldsymbol{B}_2/\Gamma(1-\zeta),$$

and $\bigcup L(ij) = X(2,5) - X^\circ(2,5)$ corresponds to $(\boldsymbol{B}_2 - \boldsymbol{B}_2^\circ)/\Gamma(1-\zeta)$. Let π be the projection $\boldsymbol{B}_2 \to \boldsymbol{B}_2/\Gamma(1-\zeta)$, and let $\ell(ij)$ denote $\pi^{-1}(\tilde{\Phi}(L(ij)))$.

Now we consider a degenerate curve

$$y^5 = (x-\lambda_1)^2(x-\lambda_3)(x-\lambda_4)(x-\lambda_5)$$

with $(\lambda_1, \lambda_1, \lambda_3, \lambda_4, \lambda_5) \in L(12)$, and putting $\lambda' = (\lambda_1, \lambda_3, \lambda_4, \lambda_5)$ we denote it by $C_{\lambda'}$. Let $\tilde{C}_{\lambda'}$ denote the non-singular model of $C_{\lambda'}$. It is a curve of genus 4. Set $\mathscr{F}_{12}$ be the totality of $\tilde{C}_{\lambda'}$. For the parameter $(\lambda^0)' = (\lambda_1^0, \lambda_3^0, \lambda_4^0, \lambda_5^0)$ the cycle γ_1 vanishes on $\tilde{C}_{(\lambda^0)'}$, but γ_2 and γ_3 are still alive. So we can define A_i, B_i $(i = 2,3,5,6)$ on $\tilde{C}_{\lambda'}$ by the same argument as for C_λ. Hence we obtain a basis $\{A_i, B_i\}$ $(i = 2,3,5,6)$ of $\mathrm{H}_1(\tilde{C}_{\lambda'}, \boldsymbol{Z})$. By putting $\lambda' = (0,1,t,\infty)$ the period

$$\int_\gamma x^{-4/5}(x-1)^{-2/5}(x-t)^{-2/5}\,dx, \quad (\gamma \in \mathrm{H}_1(\tilde{C}_{\lambda'}, \boldsymbol{Z})) \tag{4.1}$$

on $\tilde{C}_{\lambda'}$ gives a solution for the Gauss hypergeometric differential equation $E_{2,1}(1/5, 2/5, 4/5)$:

$$t(1-t)\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \left(\frac{4}{5} - \frac{8}{5}t\right)\frac{\mathrm{d}u}{\mathrm{d}t} - \frac{2}{5}u = 0. \tag{4.2}$$

The corresponding monodromy group is the triangle group $\Delta(5,5,5)$ (see [**13**], [**17**], [**18**, p. 138]). Set

$$\boldsymbol{B}_1 = \{\eta \in \boldsymbol{B}_2 : \eta_1 = 0\},$$

it is the mirror of the reflection g_{12}. By using the system $\{\gamma_2, \gamma_3\}$ we define a multi-valued map

$$\Phi_{12} : L(12) \to \boldsymbol{B}_1, \quad \lambda \mapsto [0 : \eta_2(\lambda) : \eta_3(\lambda)].$$

It induces the restriction $\tilde{\Phi}|_{L(12)}$. By the same manner we obtain that $\tilde{\Phi}|_{L(ij)}$ is the mirror of the reflection g_{ij}. Suppose $\lambda \in L(12)$ and set $\eta = \eta(\lambda) = \Phi_{12}(\lambda)$. By putting $\eta_1 = 0$ in Lemma 2.1, we see that

$$(4.3) \quad \Omega(\eta) = \begin{pmatrix} \Omega_{11} & \Omega_{14} \\ \Omega_{41} & \Omega_{44} \end{pmatrix} \oplus \Omega'(\eta), \quad \begin{pmatrix} \Omega_{11} & \Omega_{14} \\ \Omega_{41} & \Omega_{44} \end{pmatrix} = \tau_0 = \begin{pmatrix} \zeta - 1 & \zeta^2 + \zeta^3 \\ \zeta^2 + \zeta^3 & -\zeta^4 \end{pmatrix}$$

with a certain element $\Omega'(\eta) \in \mathfrak{S}_4$. Moreover, in case $\eta_0 = [0 : 0 : 1] \in \ell(12) \cap \ell(34)$ we have

$$(4.4) \quad \Omega(\eta_0) = \begin{pmatrix} \Omega_{11} & \Omega_{14} \\ \Omega_{41} & \Omega_{44} \end{pmatrix} \oplus \begin{pmatrix} \Omega_{22} & \Omega_{25} \\ \Omega_{52} & \Omega_{55} \end{pmatrix} \oplus \begin{pmatrix} \Omega_{33} & \Omega_{36} \\ \Omega_{63} & \Omega_{66} \end{pmatrix} = \tau_0 \oplus \tau_0 \oplus \tau_0.$$

5. Theta functions.

5.1. Invariant theta characteristics.

We recall several basic facts on the Riemann theta function (see [**4**] and [**9**]). For a characteristic $(a, b) \in (\boldsymbol{R}^g)^2$, the theta function $\Theta_{(a,b)}(z, \Omega)$ on $\boldsymbol{C}^g \times \mathfrak{S}_g$ is defined by the series

$$\Theta_{(a,b)}(z, \Omega) = \sum_{n \in \boldsymbol{Z}^g} \exp[\pi\sqrt{-1}\,{}^t(n+a)\Omega(n+a) + 2\pi\sqrt{-1}\,{}^t(n+a)(z+b)].$$

This function satisfies the following relations

$$(5.1) \qquad \Theta_{(a,b)}(z+m, \Omega) = \exp(2\pi\sqrt{-1}\,{}^t ma)\Theta_{(a,b)}(z, \Omega),$$

$$(5.2) \quad \Theta_{(a,b)}(z+\Omega m, \Omega) = \exp(-\pi\sqrt{-1}\,{}^t m\Omega m - 2\pi\sqrt{-1}\,{}^t m(z+b))\Theta_{(a,b)}(z, \Omega)$$

for $m \in \boldsymbol{Z}^g$. For $n, m \in \boldsymbol{Z}^g$, we have

$$(5.3) \qquad \Theta_{(a+n, b+m)}(z, \Omega) = \exp(2\pi\sqrt{-1}\,{}^t am)\Theta_{(a,b)}(z, \Omega),$$

and it holds

$$(5.4) \qquad \Theta_{(-a,-b)}(z, \Omega) = \Theta_{(a,b)}(-z, \Omega).$$

The theta constant $\Theta_{(a,b)}(\Omega) = \Theta_{(a,b)}(0, \Omega)$ satisfies the following transformation formula (see [**4**, p176]) as function on $\mathfrak{S}_g$. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\boldsymbol{Z})$, set

(5.5) $$g\Omega = (A\Omega + B)(C\Omega + D)^{-1}$$

(5.6) $$g(a,b) = (Da - Cb, -Ba + Ab) + \frac{1}{2}((C\,{}^tD)_0, (A\,{}^tB)_0)$$

(5.7) $$\phi_{(a,b)}(g) = -\frac{1}{2}({}^ta\,{}^tDBa - 2\,{}^ta\,{}^tBCb + {}^tb\,{}^tCAb) + \frac{1}{2}({}^ta\,{}^tD - {}^tb\,{}^tC)(A\,{}^tB)_0$$

where $(A)_0$ stands for the diagonal vector of a matrix A. Then we have

(5.8) $$\Theta_{g(a,b)}(g\Omega) = \kappa(g)\exp(2\pi\sqrt{-1}\phi_{a,b}(g))\det(C\Omega + D)^{1/2}\Theta_{(a,b)}(\Omega)$$

where, $\kappa(g)$ is a certain 8-th root of 1 depending only on g.

Remark 5.1. By definition, we have

$$\Theta_{(a,b)}(z,\Omega) = \exp(\pi\sqrt{-1}\,{}^ta\Omega a + 2\pi\sqrt{-1}\,{}^ta(z+b))\Theta_{(0,0)}(z + \Omega a + b, \Omega),$$

so we often identify a characteristic $(a,b) \in (\boldsymbol{R}^g)^2$ with $\Omega a + b \in \boldsymbol{C}^g$. For $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\boldsymbol{Z})$, we have

$$\Omega'(Da - Cb) + (-Ba + Ab) = {}^t(C\Omega + D)^{-1}(\Omega a + b),$$

where $\Omega' = (A\Omega + B)(C\Omega + D)^{-1}$.

Henceforth we consider only the characteristics $(a,b) \in ((1/10)\boldsymbol{Z}^6)^2$.

Lemma 5.1. *Let σ be the matrix in* (2.7) *and write $a = (a_i)$, $b = (b_i)$.*
(1) *We have $\sigma(a,b) \equiv (a,b) \bmod \boldsymbol{Z}$ if and only if*

$$\begin{aligned} 5a_1 &\equiv \frac{1}{2}, \quad a_4 \equiv a_1, \quad b_1 \equiv -2a_1, \quad b_4 \equiv -a_1 \\ 5a_2 &\equiv \frac{1}{2}, \quad a_5 \equiv a_2, \quad b_2 \equiv -2a_2, \quad b_5 \equiv -a_2 \quad \bmod \boldsymbol{Z}. \\ 5a_3 &\equiv \frac{1}{2}, \quad a_6 \equiv a_3, \quad b_3 \equiv -2a_3, \quad b_6 \equiv -a_3 \end{aligned}$$

(2) *Let (a,b) be the characteristic with the above condition. Then we have*

$$\hat{g}(a,b) \equiv (a,b) \quad \bmod \boldsymbol{Z} \quad \textit{for all } g \in \Gamma(1-\zeta).$$

Proof. (1) Using the exact form (2.7) we can describe $\sigma(a,b)$. Then we deduce the assertion.

(2) The transformation $(a,b) \mapsto g(a,b)$ in (5.6) define a group action of the symplectic group on $(\boldsymbol{R}/\boldsymbol{Z})^{2g}$ (see [**4**]). We can check that the equality for every member of the generator system $\{h_{ij}\}$ of $\Gamma(1-\zeta)$. □

Notation 5.1. We denote an element $(a,b) \in ((1/10)\boldsymbol{Z}^{2n})^2$ of the form

$$a = \frac{1}{10}\,{}^t(a_1, a_2, \dots, a_n, a_1, a_2, \dots, a_n),$$

$$b = \frac{1}{10}\,{}^t(-2a_1, -2a_2, \dots, -2a_n, -a_1, -a_2, \dots, -a_n), \quad (a_1, \dots, a_n \in \boldsymbol{Z})$$

by $[[a_1, \dots, a_n]]$.

Definition 5.1. Let (a,b) be a characteristic satisfying the condition in Lemma 5.1 (1). Then we have $(a,b) \equiv [[a_1, a_2, a_3]] \pmod{\boldsymbol{Z}}$ with odd integers a_1, a_2, a_3. We call $[[a_1, a_2, a_3]]$ of this type a "σ-invariant" characteristic, and we set

$$\vartheta[[a_1, a_2, a_3]] = \{\eta \in \boldsymbol{B}_2 : \Theta_{[[a_1,a_2,a_3]]}(\Omega(\eta)) = 0\}.$$

Remark 5.2. By the transformation formula (5.8) and Lemma 5.1, we see that

$$\Theta_{[[a_1,a_2,a_3]]}(g\Omega(\eta)) = (\text{a unit function}) \times \Theta_{[[a_1,a_2,a_3]]}(\Omega(\eta))$$

for a σ-invariant characteristic $[[a_1, a_2, a_3]]$ and $g \in \Gamma(1-\zeta)$. Hence if we have $\eta \in \vartheta[[a_1, a_2, a_3]]$, then the $\Gamma(1-\zeta)$-orbit of η is contained in $\vartheta[[a_1, a_2, a_3]]$.

Lemma 5.2. *Let $[[a_1, a_2, a_3]]$ be a σ-invariant characteristic. If $2a_1^2 + 2a_2^2 + a_3^2 \notin 5\boldsymbol{Z}$, then $\Theta_{[[a_1,a_2,a_3]]}$ vanishes on $\boldsymbol{B}_2$.*

Proof. We apply the transformation formula (5.8) for $g = \sigma^4$. Using the explicit form of $g = \sigma^4 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we obtain

$$\phi_{[[a_1,a_2,a_3]]}(\sigma^4) = \frac{1}{40}(2a_1^2 + 2a_2^2 + a_3^2), \quad \det(C\Omega(\eta) + D) = 1$$

for all $\eta \in \boldsymbol{B}_2$. By (5.3), we may put

$$\Theta_{\sigma^4[[a_1,a_2,a_3]]}(\Omega) = \exp[2\pi\sqrt{-1}\,{}^t am]\Theta_{[[a_1,a_2,a_3]]}(\Omega)$$

for a certain $m \in \boldsymbol{Z}^6$. Returning to the explicit form of $\sigma^4[[a_1, a_2, a_3]]$ we should get m. We check that $\exp[2\pi\sqrt{-1}\,{}^t am] = 1$ by a computer aided calculation. Hence we have

$$\Theta_{[[a_1,a_2,a_3]]}(\Omega(\eta)) = \kappa(\sigma^4)\exp\left[\frac{1}{20}\pi\sqrt{-1}(2a_1^2 + 2a_2^2 + a_3^2)\right]\Theta_{[[a_1,a_2,a_3]]}(\Omega(\eta))$$

for all $\eta \in \boldsymbol{B}_2$. This implies our assertion since $\kappa(\sigma^4)$ is an 8-th root of 1. □

We consider odd integers a_1, a_2, a_3 modulo $10\boldsymbol{Z}$. There exist 25 representatives of the σ-invariant characteristic $[[a_1, a_2, a_3]]$ satisfying the condition $2a_1^2 + 2a_2^2 + a_3^2 \in 5\boldsymbol{Z}$:

$$\begin{aligned}&[[1,1,1]],[[1,1,9]],[[1,9,1]],[[9,1,1]],[[1,3,5]],[[1,7,5]],\\&[[3,1,5]],[[7,1,5]],[[3,3,3]],[[3,3,7]],[[3,7,3]],[[7,3,3]],\end{aligned} \tag{5.9}$$

and $[[5,5,5]]$ together with the inverses $[[9,9,9]] = -[[1,1,1]], \ldots.$

Remark 5.3. The characteristic $[[5,5,5]]$ is an odd half integer characteristic (see [**9**]), hence $\Theta_{[[5,5,5]]}(\Omega)$ vanishes identically.

By using the explicit form of $\hat{g}_{ij}$ in (3.4)–(3.7), we obtain $\hat{g}_{12}[[a_1, a_2, a_3]]$. Consequently we have

Lemma 5.3. *Let $[[a_1, a_2, a_3]]$ be a member of the system* (5.9). *The group Γ acts transitively on the set of twelve $\vartheta[[a_1, a_2, a_3]]$.*

5.2. The zero loci of twelve theta functions.

Here we state Riemann's theorem. Let C be an algebraic curve of genus g, let $\{A_i, B_i\}$ be a symplectic basis of $\mathrm{H}_1(C, \boldsymbol{Z})$ such that $A_i \cdot B_j = \delta_{ij}$, and let $\{\omega_i\}$ be the basis of $\mathrm{H}^0(C, \Omega^1)$ such that $\int_{A_i} \omega_j = \delta_{ij}$. Then $\Omega = (\int_{B_i} \omega_j)$ belongs to $\mathfrak{S}_g$. We denote ${}^t(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g)$ by $\int_\gamma \omega$.

Theorem 5.1 (see [**9**], p149). *Let us fix a point $P_0 \in C$. Then there is a vector $\Delta \in \boldsymbol{C}^g$, such that for all $z \in \boldsymbol{C}^g$, multi-valued function*

$$f(P) = \Theta_{(0,0)}\left(z + \int_{P_0}^{P} \omega, \Omega\right) \quad (P \in C)$$

on C either vanishes identically, or has g zeros $Q_1, \ldots, Q_g$ with

$$\sum_{i=1}^{g} \int_{P_0}^{Q_i} \omega \equiv -z + \Delta \quad mod\, \Omega \boldsymbol{Z}^g + \boldsymbol{Z}^g.$$

Remark 5.4 (see [**9**]). (1) The vector Δ in the theorem is called the Riemann constant, and depends on the symplectic basis $\{A_i, B_i\}$ and the base point P_0. Once $\{A_i, B_i\}$ and P_0 are fixed, then Δ is uniquely determined as a point of the Jacobian $J(C) = \boldsymbol{C}^g/(\Omega \boldsymbol{Z}^g + \boldsymbol{Z}^g)$ by the property of the theorem.

(2) If we take P_0 such that the divisor $(2g-2)P_0$ is linearly equivalent to the canonical divisor, then we have $\Delta \in (1/2)\Omega \boldsymbol{Z} + (1/2)\boldsymbol{Z}$.

Corollary 5.1 (see [**9**]). *Under the same situation as the theorem, $\Theta_{a,b}(\Omega) = 0$ if and only if there exist $Q_1, \ldots, Q_g \in C$ such that*

$$\Delta - (\Omega a + b) \equiv \sum_{i=1}^{g-1} \int_{P_0}^{Q_i} \omega.$$

Now, let us return to our case. Let $\lambda^0 \in X^\circ(2,5)$ and C_0 be as in Section 2 and $\omega_1, \dots, \omega_6$ be the basis of $\mathrm{H}^0(C_0, \Omega^1)$ such that $\int_{A_i} \omega_j = \delta_{ij}$. We denote the ramified points over $\lambda_i^0 \in \boldsymbol{P}^1$ by $P_i \in C_0$. Let us take a base point P_0 arbitrarily among $\{P_1, \dots, P_5\}$ and let Δ_0 be the Riemann constant with respect to $\{A_i, B_i\}$ and P_0.

Lemma 5.4. *The Riemann constant Δ_0 corresponds to the characteristic* $[[5,5,5]]$.

Proof. The divisor of the holomorphic 1-form $(x - \lambda_i^0)^2\, dx/y^4$ is $10P_i$. Hence Δ_0 is a half integer characteristic (see Remark 5.4). For $z = \Omega a + b$ $(a, b \in \boldsymbol{R}^6)$ and $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, applying (5.8) we have

$$\Theta_{\sigma(\Delta_0 - z)}(\Omega) = (\text{a unit function}) \times \Theta_{\Delta_0 - z}(\Omega), \tag{5.10}$$

since $\sigma\Omega = \Omega$. By (5.6) and Remark 5.1, we have

$$\sigma(\Delta_0 - z) = \sigma\Delta_0 - {}^t(C\Omega + D)^{-1} z.$$

Hence it holds

$$\Theta_{\sigma\Delta_0 - {}^t(C\Omega + D)^{-1} z}(\Omega) = 0 \Leftrightarrow \Theta_{\Delta_0 - z}(\Omega) = 0$$

$$\Leftrightarrow z \equiv \sum_{i=1}^{5} \int_{P_0}^{Q_i} \omega \quad \text{for } {}^\exists Q_1, \dots, Q_5 \in C_0$$

by Corollary 5.1. Namely, putting $w = {}^t(C\Omega + D)^{-1} z$ we have

$$\Theta_{\sigma\Delta_0 - w}(\Omega) = 0 \Leftrightarrow {}^t(C\Omega + D) w \equiv \sum_{i=1}^{5} \int_{P_0}^{Q_i} \omega \quad \text{for } {}^\exists Q_1, \dots, Q_5 \in C_0.$$

Let us recall that σ is the symplectic representation matrix of ρ with respect to the basis $\{A_i, B_i\}$ of $\mathrm{H}_1(C_0, \boldsymbol{Z})$. And we have

$$(I \quad \Omega) \begin{pmatrix} {}^tD & {}^tB \\ {}^tC & {}^tA \end{pmatrix} = ({}^t(C\Omega + D) \quad {}^t(A\Omega + B)) = {}^t(C\Omega + D)(I \quad \Omega),$$

so ${}^t(C\Omega + D)$ is the representation matrix of ρ with respect to the basis $\{\omega_1, \dots, \omega_6\}$ of $\mathrm{H}^0(C_0, \Omega^1)$. Hence it holds

$$\Theta_{\sigma\Delta_0 - w}(\Omega) = 0 \Leftrightarrow w \equiv \sum_{i=1}^{5} \int_{P_0}^{Q_i} (\rho^{-1})^* \omega \equiv \sum_{i=1}^{5} \int_{\rho^{-1}(P_0)}^{\rho^{-1}(Q_i)} \omega \equiv \sum_{i=1}^{5} \int_{P_0}^{\rho^{-1}(Q_i)} \omega.$$

Recalling Remark 5.4 (1), this implies that $\sigma\Delta_0$ is the Riemann constant, that is $\sigma\Delta_0 \equiv \Delta_0$. Hence we have $\Delta_0 \equiv [[5,5,5]]$, since $[[5,5,5]]$ is the unique σ-invariant half integer characteristic. □

Next, let us consider the oriented arcs $\alpha_k(i,j)$ defined by (2.2) and the integrals $\int_{\alpha_k(i,j)} \omega$.

Lemma 5.5. *The integral $\int_{\alpha_k(i,j)} \omega$ is a five torsion point on $\boldsymbol{C}^6/(\Omega\boldsymbol{Z}^6+\boldsymbol{Z}^6)$. In an explicit way, we have*

$$\int_{\alpha_k(1,2)} \omega \equiv [[6,0,0]], \int_{\alpha_k(1,3)} \omega \equiv [[8,2,6]], \int_{\alpha_k(1,4)} \omega \equiv [[8,8,6]], \int_{\alpha_k(1,5)} \omega \equiv [[8,0,8]]$$

$$\mathit{mod}\, \Omega\boldsymbol{Z}^6+\boldsymbol{Z}^6$$

under the identification referred in Remark 5.1 (*Note that any $\alpha_k(i,j)$ is written as a combination of $\alpha_k(1,2)$, $\alpha_k(1,3)$, $\alpha_k(1,4)$ and $\alpha_k(1,5)$*).

Proof. Since $D_{ij} = \alpha_i(1,5) - \alpha_j(1,5)$ is a cycle, we see that $\int_{\alpha_i(1,5)} \omega \equiv \int_{\alpha_j(1,5)} \omega \mod \Omega\boldsymbol{Z}^6+\boldsymbol{Z}^6$. And we have

$$\int_{D_{12}+D_{15}} \varphi_1 = \int_{2\alpha_1(1,5)-\alpha_2(1,5)-\alpha_5(1,5)} \varphi_1 = (2-\zeta^2-\zeta^3)\int_{\alpha_1(1,5)} \varphi_1.$$

By the same calculation, we see that

$$\int_{\alpha_1(1,5)} \varphi_k = \begin{cases} \dfrac{1}{5}(2-\zeta-\zeta^4)\displaystyle\int_{D_{12}+D_{15}} \varphi_k & (k=1,2,3) \\ \dfrac{1}{5}(2-\zeta^2-\zeta^3)\displaystyle\int_{D_{12}+D_{15}} \varphi_k & (k=4,5,6) \end{cases}$$
$$= \frac{1}{5}\int_{[2-\rho^2-\rho^3](D_{12}+D_{15})} \varphi_k.$$

Calculating intersection numbers, we have the following equality

$$[2-\rho^2-\rho^3](D_{12}+D_{15}) = 2A_1+2A_3+A_4+A_6+4B_1+4B_3-B_4-B_6$$

as homology classes. Hence it holds

$$\int_{\alpha_1(1,5)} \omega \equiv \frac{1}{5}\int_{2A_1+2A_3+A_4+A_6+4B_1+4B_3-B_4-B_6} \omega$$
$$\equiv \frac{1}{10}\int_{-6A_1-6A_3-8A_4-8A_6+8B_1+8B_3+8B_4+8B_6} \omega \equiv [[8,0,8]].$$

By the same way, we obtain the results for $\alpha_k(1,2)$, $\alpha_k(1,3)$ and $\alpha_k(1,4)$. □

Let C_λ $(\lambda \in X^\circ(2,5))$ be any element of our family $\mathscr{F}$. We defined in Section 2 the systems $\{\alpha_k(i,j)(\lambda)\}$, $\{\gamma_i(\lambda)\}$ and $\{A_i(\lambda), B_i(\lambda)\}$ on C_λ depending on the arc r. The point P_0 has always the same meaning. So Lemma 5.4 and 5.5 are true for C_λ using these notations. Let $\Delta \equiv [[5,5,5]]$ denote the Riemann constant on C_λ.

Now, recall that $\boldsymbol{B}_2^\circ$ and $\ell(ij)$ stand for $\Phi(X^\circ(2,5))$ and $\pi^{-1}(\tilde{\Phi}(L(ij)))$, respectively (see Section 4).

Proposition 5.1. *Let* $[[a_1,a_2,a_3]]$ *be a* σ*-invariant characteristic in* (5.9). *Then we have* $\vartheta[[a_1,a_2,a_3]] \cap \boldsymbol{B}_2^\circ = \phi$.

Proof. Let us consider a curve $C = C_\lambda$ $(\lambda \in X^\circ(2,5))$ and its period $\Omega = \Omega_\lambda$. We assume that $\Theta_{[[1,1,1]]}(\Omega) = 0$. According to Corollary 5.1, there exist points $Q_1, \dots, Q_5 \in C$ such that

$$\sum_{i=1}^{5} \int_{P_5}^{Q_i} \omega \equiv \Delta - [[1,1,1]] \equiv [[4,4,4]].$$

On the other hand, by Lemma 5.5, we have

$$\int_{P_4}^{P_3} \omega \equiv [[0,4,0]], \qquad \int_{P_5}^{P_1} \omega \equiv [[2,0,2]].$$

Hence it holds

$$\sum_{i=1}^{5} \int_{P_5}^{Q_i} \omega \equiv 2\int_{P_5}^{P_1} \omega + \int_{P_4}^{P_3} \omega.$$

By Abel's theorem, the divisor $\sum_{i=1}^{g} Q_i$ is linearly equivalent to the divisor $D = 2P_1 + P_3 - P_4 + 3P_5$, and we have

$$\dim \mathrm{H}^0(C, \mathcal{O}(D)) = \dim \mathrm{H}^0\left(C, \mathcal{O}\left(\sum_{i=1}^{g} Q_i\right)\right) \geq 1. \tag{5.11}$$

For the effective divisor $D' = D + P_4$, we have

$$\dim \mathrm{H}^0(C, \mathcal{O}(D')) = \dim \mathrm{H}^0(C, \Omega^1(-D')) + 1$$

by the Riemann-Roch. We claim that $\dim \mathrm{H}^0(C, \Omega^1(-D')) = 0$. In fact, the basis $\{\varphi_i\}$ is written as

$$\varphi_1 = y^2\varphi, \quad \varphi_2 = y\varphi, \quad \varphi_3 = xy\varphi, \quad \varphi_4 = \varphi, \quad \varphi_5 = x\varphi, \quad \varphi_6 = x^2\varphi$$
$$\left(\varphi = 5\frac{dy}{f'(x)}, \quad f(x) = \prod_{i=1}^{5}(x - \lambda_i)\right),$$

and we have following vanishing orders;

$$\mathrm{ord}_{P_i}(y) = 1, \quad \mathrm{ord}_{P_i}(x - \lambda_j) = 5\delta_{ij}, \quad \mathrm{ord}_{P_i}(\varphi) = 0 \quad (i, j = 1, \dots, 5).$$

Because any holomorphic 1-form is written in the form

$$(\text{quadratic polynomial of } x, y) \times \varphi,$$

we see that there is no holomorphic 1-form ξ such that

$$\mathrm{ord}_{P_1}(\xi) \geq 2, \quad \mathrm{ord}_{P_3}(\xi) \geq 1, \quad \mathrm{ord}_{P_5}(\xi) \geq 3.$$

Hence we have $\dim \mathrm{H}^0(C, \mathcal{O}(D')) = 1$, that is, $\mathrm{H}^0(C, \mathcal{O}(D'))$ contains only constant functions. This contradicts (5.11), since $\mathrm{H}^0(C, \mathcal{O}(D)) \subset \mathrm{H}^0(C, \mathcal{O}(D'))$ and D is not effective. So we have $\vartheta[[1,1,1]] \cap \boldsymbol{B}_2^\circ = \phi$. Now the assertion follows from Lemma 5.3. □

Hence $\vartheta[[a_1, a_2, a_3]]$ is a union of several $\ell(ij)$'s.

LEMMA 5.6. *Let* η_0 *be the point* $[0:0:1] \in \boldsymbol{B}_2$, *and let* $[[a_1, a_2, a_3]]$ *be a member of* (5.9). *If* $a_1, a_2, a_3 \in \{1, 9\}$, *then we have* $\Theta_{[[a_1, a_2, a_3]]}(\Omega(\eta_0)) \neq 0$.

PROOF. We have

(5.12)

$$\Theta_{[[a_1, a_2, a_3]]}(\Omega(\eta_0)) = \Theta_{[[a_1]]}(\tau_0)\Theta_{[[a_2]]}(\tau_0)\Theta_{[[a_3]]}(\tau_0), \quad \tau_0 = \begin{pmatrix} \zeta - 1 & \zeta^2 + \zeta^3 \\ \zeta^2 + \zeta^3 & -\zeta^4 \end{pmatrix}$$

(see (4.4), Notation (5.1)). So our assertion is reduced to the inequality $\Theta_{[[1]]}(\tau_0) \neq 0$, since $\Theta_{[[9]]}$ is a constant multiple of $\Theta_{[[1]]}$. Set $(a, b) = [[1]]$, $n = {}^t(n_1, n_2)$ and

$$f(n_1, n_2) = \exp[\pi\sqrt{-1}({}^t(n + a)\tau_0(n + a) + 2{}^t(n + a)b)].$$

By definition, $\Theta_{[[1]]}(\tau_0) = \sum_{n_1, n_2 \in \boldsymbol{Z}} f(n_1, n_2)$. For simplicity, we denote $n + a$ by $m = (m_1, m_2)$. By elementary calculation, we see that

$$|f(n_1, n_2)| = \exp\left[-\pi \sin\left(\frac{2\pi}{5}\right)\{m_1^2 + (3 - \sqrt{5})m_1 m_2 + m_2^2\}\right].$$

In case $m_1 m_2 > 0$, we have

$$|f(n_1, n_2)| < \exp\left[-\pi \sin\left(\frac{2\pi}{5}\right)\{m_1^2 + m_2^2\}\right].$$

In case $m_1 m_2 < 0$, we have

$$\begin{aligned}|f(n_1,n_2)| &< \exp\left[-\pi\sin\left(\frac{2\pi}{5}\right)\{m_1^2+m_1m_2+m_2^2\}\right]\\ &= \exp\left[-\pi\sin\left(\frac{2\pi}{5}\right)\left\{\frac{1}{2}(m_1^2+m_2^2)+\frac{1}{2}(m_1+m_2)^2\right\}\right]\\ &< \exp\left[-\frac{\pi}{2}\sin\left(\frac{2\pi}{5}\right)\{m_1^2+m_2^2\}\right].\end{aligned}$$

Consequently,

$$|f(n_1,n_2)| < \alpha^{m_1^2+m_2^2}, \quad \left(\alpha = \exp\left[-\frac{\pi}{2}\sin\left(\frac{2\pi}{5}\right)\right]\right)$$

for any $n_1, n_2 \in \boldsymbol{Z}$. Set

$$D_1 = \{(n_1,n_2) \in \boldsymbol{Z}^2 : -10 \le n_1, n_2 \le 10\}, \quad D_2 = \boldsymbol{Z}^2 - D_1,$$

and consider the summations

$$S_1 = \sum_{D_1} f(n_1,n_2), \quad S_2 = \sum_{D_2} f(n_1,n_2).$$

Using a computer, we can evaluate $|S_1|$ and $|S_2|$. We have an approximate value

$$|S_1| \fallingdotseq 1.13746\ldots,$$

by *Mathematica*. On the other hand, we have

$$|S_2| < \sum_{D_2}|f(n_1,n_2)| < \sum_{D_2}\alpha^{m_1^2+m_2^2}.$$

The last term is very small. For example,

$$\sum_{n_1\ge 10, n_2\ge 0}\alpha^{m_1^2+m_2^2} < \left(\sum_{n_1\ge 10}\alpha^{n_1}\right)\left(\sum_{n_2\ge 0}\alpha^{n_2}\right) = \left(\frac{\alpha^{10}}{1-\alpha}\right)\left(\frac{1}{1-\alpha}\right) \fallingdotseq 5.40545\times 10^{-7},$$

and the same calculations shows $|S_1| \gg |S_2|$. This implies $\Theta_{[[1]]}(\tau_0) = S_1 + S_2 \neq 0$. □

Lemma 5.7. (1) *If we have* $a_1 \equiv 3, 7 \mod 10$, *then* $\Theta_{[[a_1,a_2,a_3]]}$ *vanishes on* $\ell(12)$.

(2) *If we have* $a_2 \equiv 3, 7 \mod 10$, *then* $\Theta_{[[a_1,a_2,a_3]]}$ *vanishes on* $\ell(34)$.

Proof. Set $g = \hat{g}_{12} = \begin{pmatrix} A & B \\ C & D\end{pmatrix}$ and set $\Omega = \Omega(\eta)$ with $\eta = [0:\eta_2:\eta_3] \in \boldsymbol{B}_2$. By the computation same as the one in the proof of Lemma 5.2, we have

$$g\Omega = \Omega,\ \det(C\Omega + D) = \zeta,\ \phi_{[[a_1,a_2,a_3]]}(g) = \frac{1}{40}a_1^2,\ \Theta_{g[[a_1,a_2,a_3]]}(\Omega) = \Theta_{[[a_1,a_2,a_3]]}(\Omega).$$

Hence it holds

$$\Theta_{[[a_1,a_2,a_3]]}(\Omega)^8 = \exp\left[\frac{2}{5}\pi\sqrt{-1}(a_1^2 - 1)\right]\Theta_{[[a_1,a_2,a_3]]}(\Omega)^8.$$

Therefore $\Theta_{[[a_1,a_2,a_3]]}(\Omega(\eta))$ vanishes on the mirror of g_{12} provided $a_1 \equiv 3, 7 \bmod 10$. This implies assertion (1). The assertion (2) follows from the same argument with $g = g_{34}$ and $\eta = [\eta_1 : 0 : \eta_3] \in \boldsymbol{B}_2$. □

Proposition 5.2. *We have Table* 1 *for the vanishing loci of twelve theta constants coming from the system* (5.9). *In the table, "v" implies that* $\Theta_{[[a_1,a_2,a_3]]}$ *vanishes along the corresponding divisor* $\ell(ij)$, *and the blank implies* $\Theta_{[[a_1,a_2,a_3]]}$ *is not identically zero there. For example,* $\Theta_{[[1,1,1]]}$ *vanishes on* $\ell(13)$ *and is not identically zero on* $\ell(12)$.

$[[a_1,a_2,a_3]]$	$\ell(12)$	$\ell(13)$	$\ell(14)$	$\ell(15)$	$\ell(23)$	$\ell(24)$	$\ell(25)$	$\ell(34)$	$\ell(35)$	$\ell(45)$
$[[1,1,1]]$		v		v	v	v				v
$[[1,1,9]]$		v	v			v	v		v	
$[[1,9,1]]$			v	v	v	v			v	
$[[9,1,1]]$		v	v		v		v			v
$[[1,3,5]]$			v	v	v		v	v		
$[[1,7,5]]$		v		v		v	v	v		
$[[3,1,5]]$	v		v		v				v	v
$[[7,1,5]]$	v	v				v			v	v
$[[3,3,3]]$	v		v				v	v	v	
$[[3,3,7]]$	v			v	v			v		v
$[[3,7,3]]$	v	v					v	v		v
$[[7,3,3]]$	v			v		v		v	v	

Table 1.

Proof. By Lemma 5.7,

$$\Theta_{[[3,1,5]]},\quad \Theta_{[[7,1,5]]},\quad \Theta_{[[3,3,3]]},\quad \Theta_{[[3,3,7]]},\quad \Theta_{[[3,7,3]]},\quad \Theta_{[[7,3,3]]}$$

vanish on $\ell(12)$, and

$$\Theta_{[[1,3,5]]},\quad \Theta_{[[1,7,5]]},\quad \Theta_{[[3,3,3]]},\quad \Theta_{[[3,3,7]]},\quad \Theta_{[[3,7,3]]},\quad \Theta_{[[7,3,3]]}$$

vanish on $\ell(34)$. By Lemma 5.6,

$$\Theta_{[[1,1,1]]},\quad \Theta_{[[1,1,9]]},\quad \Theta_{[[1,9,1]]},\quad \Theta_{[[9,1,1]]}$$

are not identically zero on $\ell(12)$ and on $\ell(34)$, since $\eta_0 = [0:0:1] \in \ell(12) \cap \ell(34)$. The result is obtained by applying the transformation formula (5.8) for the above theta constants and $\hat{g}_{ij}$. For example, we have

$$\Theta_{\hat{g}_{12}\hat{g}_{45}[[a_1,a_2,a_3]]}(\hat{g}_{12}\hat{g}_{45}\Omega) = (\text{a unit function}) \times \Theta_{[[a_1,a_2,a_3]]}(\Omega).$$

Since $\hat{g}_{12}\hat{g}_{45}[[1,3,5]] \equiv [[9,9,1]]$ and $g_{12}g_{45}(\ell(12)) = \ell(12)$, we see that $\Theta_{[[1,3,5]]}$ is not identically zero on $\ell(12)$. □

5.3. Automorphic factor.

We study the automorphic factor appeared in the transformation formula (5.8) with respect to $\Gamma(1-\zeta)$ and $\Omega = \Omega(\eta)$. Let Q be the diagonal matrix $\mathrm{diag}(1,1,-\zeta^3(1+\zeta))$. We denote ${}^t\eta Q \eta$ by $\langle \eta, \eta \rangle$. Set

$$F_g(\eta) = \frac{\langle g\eta, g\eta \rangle}{\langle \eta, \eta \rangle}$$

for $g \in \Gamma$ and $\eta \in \boldsymbol{B}_2$. Obviously, we have the following lemma.

Lemma 5.8. *$F_g(\eta)$ satisfies the cocycle condition with respect to Γ. That is,*

$$F_{g_1 g_2}(\eta) = F_{g_1}(g_2\eta) F_{g_2}(\eta), \quad g_1, g_2 \in \Gamma.$$

Proposition 5.3. *There exists a non trivial character*

$$\chi : \Gamma \to \mu_5 = \{1, \zeta, \dots, \zeta^4\}$$

such that

$$\det(C\Omega(\eta) + D) = \chi(g) F_g(\eta) \quad (\eta \in \boldsymbol{B}_2)$$

for $g \in \Gamma$, where the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is the symplectic representation $\hat{g}$ of g.

Proof. According to the case by case calculation, we have

$$\det(C\Omega(\eta) + D) = \zeta^3 F_g(\eta) \quad (\eta \in \boldsymbol{B}_2)$$

for $g = g_{12}, g_{23}, g_{34}, g_{45}$. Since $\det(C\Omega(\eta) + D)/F_g(\eta)$ satisfies the cocycle condition, we obtain the result. □

Now let $(a,b) = [[a_1,a_2,a_3]]$ be an invariant characteristic, and set $(a_g, b_g) = \hat{g}(a,b)$ for $g \in \Gamma(1-\zeta)$. Since $(a_g, b_g) \equiv (a,b) \bmod \boldsymbol{Z}$, we have

$$\Theta_{\hat{g}(a,b)}(\Omega) = \Theta_{(a_g,b_g)}(\Omega) = \Theta_{(a_g-a+a,\, b_g-b+b)}(\Omega) = \exp[2\pi\sqrt{-1}\,{}^t a(b_g - b)]\Theta_{(a,b)}(\Omega)$$

by (5.3). Set

$$\phi'_{[[a_1,a_2,a_3]]}(\hat{g}) = \phi_{[[a_1,a_2,a_3]]}(\hat{g}) - {}^t a(b_g - b).$$

Then we can write the transformation formula (5.8) as

(5.13)

$$\Theta_{[[a_1,a_2,a_3]]}(\Omega(g\eta)) = \kappa(\hat{g})\exp(2\pi\sqrt{-1}\phi'_{[[a_1,a_2,a_3]]}(\hat{g}))[\chi(g)F_g(\eta)]^{1/2}\Theta_{[[a_1,a_2,a_3]]}(\Omega(\eta)),$$

where $\kappa(\hat{g})$ is a 8-th root of 1 depending only on $\hat{g}$.

LEMMA 5.9. *Let g be in $\Gamma(1-\zeta)$. Then, the values*

$$[\exp(2\pi\sqrt{-1}\phi'_{[[a_1,a_2,a_3]]}(\hat{g}))]^5$$

are the same for all twelve characteristics $[[a_1,a_2,a_3]]$ in (5.9).

PROOF. By direct calculation, we have

$$5\phi'_{[[a_1,a_2,a_3]]}(\hat{h}_{12}) \equiv \frac{1}{8}, \quad 5\phi'_{[[a_1,a_2,a_3]]}(\hat{h}_{13}) \equiv \frac{3}{4}, \quad 5\phi'_{[[a_1,a_2,a_3]]}(\hat{h}_{14}) \equiv \frac{1}{2},$$

$$5\phi'_{[[a_1,a_2,a_3]]}(\hat{h}_{23}) \equiv \frac{1}{2}, \quad 5\phi'_{[[a_1,a_2,a_3]]}(\hat{h}_{34}) \equiv \frac{3}{4} \pmod{\boldsymbol{Z}}$$

for the twelve $[[a_1,a_2,a_3]]$. According to Lemma 5.8, the equality (5.13) shows that

$$\kappa(\hat{g})\exp[2\pi\sqrt{-1}\phi'_{[[a_1,a_2,a_3]]}(\hat{g})]$$

is a character on $\Gamma(1-\zeta)$. So we obtain the result for all $g \in \Gamma(1-\zeta)$. □

COROLLARY 5.2. *Let $[[a_1,a_2,a_3]]$ and $[[b_1,b_2,b_3]]$ be in (5.9). Then, the function*

$$\frac{\Theta_{[[a_1,a_2,a_3]]}(\Omega(\eta))^5}{\Theta_{[[b_1,b_2,b_3]]}(\Omega(\eta))^5}$$

is well-defined as a meromorphic function on $\boldsymbol{B}_2/\Gamma(1-\zeta)$.

Let $\Omega = \Omega_\lambda$ be the period matrix of a curve C_λ ($\lambda \in X^\circ(2,5)$), P_0 be a ramified point of $C \to \boldsymbol{P}^1$.

PROPOSITION 5.4. *Let $[[a_1,a_2,a_3]]$ and $[[b_1,b_2,b_3]]$ be in (5.9). The function*

$$f(P) = \frac{\Theta_{[[a_1,a_2,a_3]]}(\int_{P_0}^{P}\omega, \Omega)^5}{\Theta_{[[b_1,b_2,b_3]]}(\int_{P_0}^{P}\omega, \Omega)^5} \quad (P \in C_\lambda)$$

is a single-valued meromorphic function on C_λ, where the paths of integrations in the numerator and the denominator are chosen to be the same.

PROOF. Note that Proposition 5.1 asserts

$$\Theta_{[[a_1,a_2,a_3]]}\left(\int_{P_0}^{P_0}\omega,\Omega\right) = \text{const.}\times\Theta_{[[a_1,a_2,a_3]]}(0,\Omega)\neq 0,$$

where the constant depends on the path of integration. So the numerator is not identically zero, and the same argument goes for the denominator. By the assumption we have

$$[[a_1,a_2,a_3]]-[[b_1,b_2,b_3]]\in\left(\frac{1}{5}\boldsymbol{Z}^6\right)^2.$$

By using the formula (5.1) and (5.2) we can check that

$$\frac{\Theta_{[[a_1,a_2,a_3]]}(\int_{P_0}^{P}\omega+\Omega m+n,\Omega)^5}{\Theta_{[[b_1,b_2,b_3]]}(\int_{P_0}^{P}\omega+\Omega m+n,\Omega)^5}=\frac{\Theta_{[[a_1,a_2,a_3]]}(\int_{P_0}^{P}\omega,\Omega)^5}{\Theta_{[[b_1,b_2,b_3]]}(\int_{P_0}^{P}\omega,\Omega)^5}$$

for $m,n\in\boldsymbol{Z}^6$. This implies single-valuedness of f. □

Let us consider the meromorphic function

$$f(P)=\frac{\Theta_{[[1,1,1]]}(\int_{P_1}^{P}\omega,\Omega)^5}{\Theta_{[[3,3,7]]}(\int_{P_1}^{P}\omega,\Omega)^5}$$

on C_λ. By Lemma 5.5, we have

$$\Delta-[[1,1,1]]\equiv[[4,4,4]]\equiv 2\int_{P_1}^{P_2}\omega+3\int_{P_1}^{P_3}\omega+\int_{P_1}^{P_4}\omega,$$

$$\Delta-[[3,3,7]]\equiv[[2,2,8]]\equiv 3\int_{P_1}^{P_2}\omega+2\int_{P_1}^{P_3}\omega+\int_{P_1}^{P_4}\omega.$$

By Corollary 5.1, the zero divisor of $\Theta_{[[1,1,1]]}(\int_{P_1}^{P}\omega,\Omega)$ and $\Theta_{[[3,3,7]]}(\int_{P_1}^{P}\omega,\Omega)$ are $2P_2+3P_3+P_4$ and $3P_2+2P_3+P_4$ respectively. Hence we can write

$$f(P)=c\frac{x(P)-\lambda_3}{x(P)-\lambda_2},$$

where $x(P)$ is the coordinate function $x\in\boldsymbol{C}[x,y]/(y^5-\prod(x-\lambda_i))$ and $c\neq 0$ is a certain constant. By Lemma 5.5,

$$\int_{P_1}^{P_1}\omega\equiv[[0,0,0]],\qquad\int_{P_1}^{P_5}\omega\equiv[[8,0,8]].$$

Substitutes $P=P_1,P_5$ in the above form, then we obtain

$$\frac{\Theta_{[[1,1,1]]}([[0,0,0]],\Omega)^5}{\Theta_{[[3,3,7]]}([[0,0,0]],\Omega)^5}=c\frac{\lambda_1-\lambda_3}{\lambda_1-\lambda_2},\quad \frac{\Theta_{[[1,1,1]]}([[8,0,8]],\Omega)^5}{\Theta_{[[3,3,7]]}([[8,0,8]],\Omega)^5}=c\frac{\lambda_5-\lambda_3}{\lambda_5-\lambda_2}.$$

Set $[[8,0,8]]=\Omega\varepsilon'+\varepsilon''$. By elementary and patient calculation, we have

$$\Theta_{[[1,1,1]]}([[8,0,8]],\Omega)^5=-\zeta^2\exp[-5\pi\sqrt{-1}\,{}^t\varepsilon'\Omega\varepsilon'-10\pi\sqrt{-1}\,{}^t\varepsilon'\varepsilon'']\Theta_{[[1,9,1]]}(\Omega)^5$$

$$\Theta_{[[3,3,7]]}([[8,0,8]],\Omega)^5=\exp[-5\pi\sqrt{-1}\,{}^t\varepsilon'\Omega\varepsilon'-10\pi\sqrt{-1}\,{}^t\varepsilon'\varepsilon'']\Theta_{[[1,3,5]]}(\Omega)^5.$$

Eliminating c, we have the following equality

$$\frac{\Theta_{[[1,1,1]]}(\Omega)^5\Theta_{[[1,3,5]]}(\Omega)^5}{\Theta_{[[3,3,7]]}(\Omega)^5\Theta_{[[1,9,1]]}(\Omega)^5}=-\zeta^2\frac{(\lambda_1-\lambda_3)(\lambda_5-\lambda_2)}{(\lambda_1-\lambda_2)(\lambda_5-\lambda_3)}.$$

Note that we can regard the above equality as that of meromorphic functions on $\boldsymbol{B}_2/\Gamma(1-\zeta)\cong X(2,5)$. By the above equality and Proposition 5.2, we see that

1. The vanishing order of $\Theta_{[[1,1,1]]}(\Omega(\eta))^5$ on $\tilde{\Phi}(L(13))$ is 1,
2. The vanishing order of $\Theta_{[[1,3,5]]}(\Omega(\eta))^5$ on $\tilde{\Phi}(L(25))$ is 1,
3. The vanishing order of $\Theta_{[[3,3,7]]}(\Omega(\eta))^5$ on $\tilde{\Phi}(L(12))$ is 1,
4. The vanishing order of $\Theta_{[[1,9,1]]}(\Omega(\eta))^5$ on $\tilde{\Phi}(L(35))$ is 1.

Because Γ acts transitively on the set of σ-invariant characteristics (see Lemma 5.3), we obtain the following result.

Proposition 5.5. *Let* $[[a_1,a_2,a_3]]$ *be a* σ*-invariant characteristic. If the multi-valued function* $\Theta_{[[a_1,a_2,a_3]]}(\Omega(\eta))^5$ *on* $\boldsymbol{B}_2/\Gamma(1-\zeta)$ *vanishes identically on* $\tilde{\Phi}(L(ij))=\ell(ij)/\Gamma(1-\zeta)$, *then the vanishing order is* 1.

6. Conclusion.

Now we state our results.

—The Schwarz inverse for the Appell HGDE $F_1(3/5,3/5,2/5,6/5)$**—**

Recall the embedding $J: X(2,5)\to\boldsymbol{P}^{11}$ in Proposition 1.1 and the extended period map $\tilde{\Phi}$ in Section 4.

Theorem 6.1. *We have a commutative diagram*:

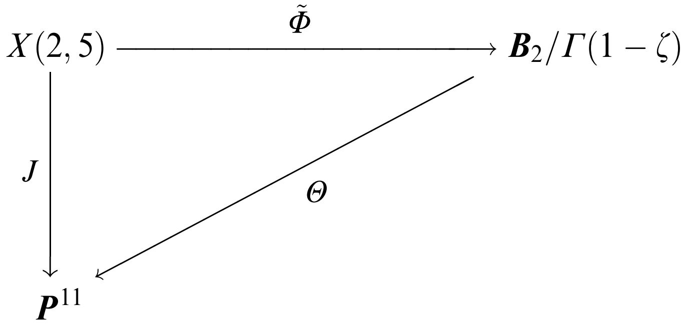

Figure 2.

by putting

$$(6.1)\qquad \Theta = \begin{bmatrix} \Theta_{[[1,1,1]]}(\Omega(\eta))^5 \\ \Theta_{[[1,1,9]]}(\Omega(\eta))^5 \\ \Theta_{[[1,9,1]]}(\Omega(\eta))^5 \\ \Theta_{[[9,1,1]]}(\Omega(\eta))^5 \\ \Theta_{[[1,3,5]]}(\Omega(\eta))^5 \\ \Theta_{[[1,7,5]]}(\Omega(\eta))^5 \\ \Theta_{[[3,3,3]]}(\Omega(\eta))^5 \\ \Theta_{[[3,3,7]]}(\Omega(\eta))^5 \\ \Theta_{[[3,7,3]]}(\Omega(\eta))^5 \\ \Theta_{[[7,3,3]]}(\Omega(\eta))^5 \\ \Theta_{[[7,1,5]]}(\Omega(\eta))^5 \\ \Theta_{[[3,1,5]]}(\Omega(\eta))^5 \end{bmatrix}, \quad J = \begin{bmatrix} c_1 J(13245)(\lambda) \\ c_2 J(13524)(\lambda) \\ c_3 J(15324)(\lambda) \\ c_4 J(13254)(\lambda) \\ c_5 J(15234)(\lambda) \\ c_6 J(13425)(\lambda) \\ d_1 J(12534)(\lambda) \\ d_2 J(12345)(\lambda) \\ d_3 J(13452)(\lambda) \\ d_4 J(15342)(\lambda) \\ d_5 J(12453)(\lambda) \\ d_6 J(12354)(\lambda) \end{bmatrix},$$

with constants

$$[c_1 : \cdots : c_6 : d_1 : \cdots : d_6] = [1 : -1 : 1 : 1 : \zeta^3 : \zeta^3 : -\zeta : \zeta : \zeta : -\zeta : -1 : -1] \in \boldsymbol{P}^{11}.$$

Moreover the map Θ is an embedding.

Proof. By Proposition 5.2 and Proposition 5.5, the zero divisor of the i-th component of Θ coincides with that of the i-th component of J via the isomorphism $\tilde{\Phi}$. Hence the assertion is obvious except determination of the ratios of constants c_i, d_i.

The ratios are obtained by elementary (but complicated) calculation. Here we omit it, for the details see [**5**]. □

Let K_X be the canonical class of $X = X(2,5)$.

Corollary 6.1. *We have an isomorphism of $\boldsymbol{C}$-algebras*

$$\boldsymbol{C}[\Theta_{[[a_1,a_2,a_3]]}(\Omega(\eta))^5] \cong \bigoplus_{n=0}^{\infty} \mathrm{H}^0(X, \mathcal{O}_X(-nK_X)),$$

where the left hand side is the $\boldsymbol{C}$-algebra of the functions on $\boldsymbol{B}_2$ generated by the twelve theta functions in Theorem 6.1. *Especially the $\boldsymbol{C}$-vector space spanned by $\{\Theta_{[[a_1,a_2,a_3]]}(\Omega(\eta))^5\}$ coincides with $\mathrm{H}^0(X, \mathcal{O}_X(-K_X))$.*

Proof. The map J is essentially the anti-canonical map (see Section 1). Hence the assertion follows from Theorem 6.1. □

Remark 6.1. By the Riemann-Roch theorem, we obtain

$$\dim \mathrm{H}^0(X, \mathcal{O}_X(-nK_X)) = \frac{1}{2}(-nK_X)\cdot(-nK_X - K_X) + 1$$
$$= \frac{5}{2}n(n+1) + 1,$$

since $(-K_X)\cdot(-K_X) = 5$. So we have $\dim \mathrm{H}^0(X, \mathcal{O}_X(-K_X)) = 6$, and twelve theta constants $\{\Theta_{[[a_1,a_2,a_3]]}(\Omega(\eta))^5\}$ satisfy 6 independent linear equations. It is known that the image of X in $\boldsymbol{P}^5$ by the anti-canonical map is determined by the system of quadratic equations (see [**2**, Chapter 5]).

—The graded ring of Automorphic forms—

Recall the automorphic factor $F_g(\eta)$ in Lemma 5.8. We consider the automorphic function $f(\eta)$ on $\boldsymbol{B}_2$ in the sense that we have

$$f(g\eta) = F_g(\eta)^k f(\eta) \quad \text{for } g \in \Gamma(1-\zeta), \tag{6.2}$$

where k is a non negative integer. Let us denote the vector space of holomorphic functions satisfying (6.2) by $A_k(F_g)$.

Proposition 6.1. *Let* $[[a_1,a_2,a_3]]$ *and* $[[b_1,b_2,b_3]]$ *be members of the system in* (5.9), *then it holds*

$$\Theta_{[[a_1,a_2,a_3]]}(\Omega(\eta))^5\Theta_{[[b_1,b_2,b_3]]}(\Omega(\eta))^5 \in A_5(F_g).$$

Proof. By (5.13) and Lemma 5.9, we have

$$\begin{aligned}\Theta_{[[a_1,a_2,a_3]]}&(\Omega(g\eta))^5\Theta_{[[b_1,b_2,b_3]]}(\Omega(g\eta))^5 \\ &= \kappa(g)^{10}\exp(2\pi\sqrt{-1}\phi'_{[[a_1,a_2,a_3]]}(g))^{10}F_g(\eta)^5 \\ &\quad \times \Theta_{[[a_1,a_2,a_3]]}(\Omega(\eta))^5\Theta_{[[b_1,b_2,b_3]]}(\Omega(g\eta))^5\end{aligned} \tag{6.3}$$

for $g \in \Gamma(1-\zeta)$. We must show

$$[\kappa(g)\exp(2\pi\sqrt{-1}\phi'_{[[a_1,a_2,a_3]]}(g))]^{10} = 1 \tag{6.4}$$

for $g \in \Gamma(1-\zeta)$. Let η_0 be the point

$$(\text{the mirror of } h_{12}) \cap (\text{the mirror of } h_{34}) = [0:0:1] \in \boldsymbol{B}_2.$$

Then η_0 is fixed by h_{12} and h_{34}. Moreover, $F_g(\eta) = \zeta$ for h_{12} and h_{34}. So we have

$$\Theta_{[[1,1,1]]}(\Omega(\eta_0))^{10} = [\kappa(g)\exp(2\pi\sqrt{-1}\phi'_{[[1,1,1]]}(g))]^{10}\Theta_{[[1,1,1]]}(\Omega(\eta_0))^{10} \quad (g = h_{12}, h_{34})$$

by (6.3). Since $\Theta_{[[1,1,1]]}(\Omega(\eta_0)) \neq 0$ (see Proposition 5.6), we obtain (6.4) for h_{12} and h_{34}. By the same way, we see that (6.4) holds for any member h_{ij} of the generator system of $\Gamma(1-\zeta)$. Hence it holds for all $g \in \Gamma(1-\zeta)$. □

THEOREM 6.2. (1) *We have an isomorphism of the* $\boldsymbol{C}$*-algebras*:

$$\bigoplus_{n=0}^{\infty} A_{5n}(F_g) = \boldsymbol{C}[\Theta_{[[a_1,a_2,a_3]]}(\Omega(\eta))^5 \Theta_{[[b_1,b_2,b_3]]}(\Omega(\eta))^5]$$

$$\cong \bigoplus_{n=0}^{\infty} \mathrm{H}^0(X, \mathcal{O}_X(-2nK_X)).$$

(2) $A_n(F_g) = \{0\}$ *for* $n \in \boldsymbol{N}$, $n \equiv 1,2,3,4 \mod 5$.

PROOF. By Proposition 6.1, $f \in A_5(F_g)$ defines a meromorphic function

$$\frac{f(\eta)}{\Theta_{[[1,1,1]]}(\Omega(\eta))^{10}}$$

on $\boldsymbol{B}_2/\Gamma(1-\zeta)$. So, by Theorem 6.1, we have the isomorphism of $\boldsymbol{C}$-vector spaces:

$$A_{5n}(F_g) \cong \mathrm{H}^0(X, \mathcal{O}_X(-2nK_X)) \quad \text{for } n \in \boldsymbol{N}.$$

Hence we have the assertion (1).

Next let us recall that X is the blow up of $\boldsymbol{P}^2$ at 4 points. We denote this blow up by $\pi : X \to \boldsymbol{P}^2$. Then the Neron-Severi group NS(X) has a system of free generator E_1, E_2, E_3, E_4 and π^*H, where $\{E_i\}$ are the exceptional curves with respect to π, and H is a general line on $\boldsymbol{P}^2$. For $n \notin 5\boldsymbol{Z}$, there is no divisor D on X such that $5D = -2nK_X$, since $-K_X = 3\pi^*H - E_1 - E_2 - E_3 - E_4$. This implies the assertion (2), since

$$A_n(F_g)^5 \subset A_{5n}(F_g) \cong \mathrm{H}^0(X, \mathcal{O}_X(-2nK_X)).$$ □

—The Schwarz inverse for the Gauss HGDE $E_{2,1}(1/5, 2/5, 4/5)$**—**

Let us consider the 1-dimensional disk

$$\boldsymbol{B}_1 = \{\eta \in \boldsymbol{B}_2 : \eta_1 = 0\},$$

and the degenerate period map

$$\Phi_{12} : L(12) \cong \boldsymbol{P}^1 \to \boldsymbol{B}_1, \quad t \mapsto \left[0 : \int_{\gamma_2} \omega : \int_{\gamma_3} \omega\right],$$

$$\omega = x^{-4/5}(x-1)^{-2/5}(x-t)^{-2/5}\,dx,$$

as in Section 4 (the parameter λ is specialized as $(\lambda_1, \dots, \lambda_5) = (0, 0, 1, t, \infty)$). Set

$$\Gamma(1-\zeta)_1 = \{g \in \Gamma(1-\zeta) : g(\boldsymbol{B}_1) = \boldsymbol{B}_1\}.$$

As we mentioned in Section 4, this is the triangle group $\Delta(5,5,5)$ up to the center. Recall those are the Schwarz map and the monodromy group for Gauss hypergeometric differential equation $E_{2,1}(1/5, 2/5, 4/5)$ (see (4.2)).

Theorem 6.3. *The map*

$$\Theta_{12} : \boldsymbol{B}_1/\Gamma(1-\zeta)_1 \to \boldsymbol{P}^1, \quad \eta \mapsto [\Theta_{[[1,1,1]]}(\Omega(\eta))^5 : -\Theta_{[[1,1,9]]}(\Omega(\eta))^5]$$

is an isomorphism, and this is the inverse map of the Schwarz map

$$\Phi_{12} : \boldsymbol{P}^1 \to \boldsymbol{B}_1/\Gamma(1-\zeta)_1, \quad [1:t] \mapsto \left[0 : \int_{\gamma_2} \omega : \int_{\gamma_3} \omega\right].$$

Proof. By Theorem 6.1, the restriction of the meromorphic function

$$\frac{\Theta_{[[1,1,9]]}(\Omega(\eta))^5}{\Theta_{[[1,1,1]]}(\Omega(\eta))^5}$$

on $L(12)$ is of degree 1. In fact, $L(12) \cap L(13) = L(12) \cap L(14) = L(12) \cap L(15) = \phi$, so the numerator vanishes at only $L(12) \cap L(35)$ with order 1, the denominator vanishes at only $L(12) \cap L(45)$ with order 1, and $L(12) \cap L(35) \neq L(12) \cap L(45)$ (see Section 1). Hence the map Θ_{12} is an isomorphism. Moreover, by Theorem 6.1, we have the equality

$$\frac{\Theta_{[[1,1,9]]}(\Omega(\eta))^5}{\Theta_{[[1,1,1]]}(\Omega(\eta))^5} = -\frac{(\lambda_1-\lambda_3)(\lambda_3-\lambda_5)(\lambda_5-\lambda_2)(\lambda_2-\lambda_4)(\lambda_4-\lambda_1)}{(\lambda_1-\lambda_3)(\lambda_3-\lambda_2)(\lambda_2-\lambda_4)(\lambda_4-\lambda_5)(\lambda_5-\lambda_1)},$$

on $\boldsymbol{B}_2/\Gamma(1-\zeta) \cong X(2,5)$, and this induces the equality

$$\frac{\Theta_{[[1,1,9]]}(\Omega(\eta))^5}{\Theta_{[[1,1,1]]}(\Omega(\eta))^5} = -\frac{(\lambda_3-\lambda_5)(\lambda_4-\lambda_1)}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_5)}$$

on $L(12)$. Putting $(\lambda_1, \lambda_3, \lambda_4, \lambda_5) = (0, 1, t, \infty)$, we obtain

$$\frac{(\lambda_3-\lambda_5)(\lambda_4-\lambda_1)}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_5)} = t. \qquad \square$$

Let us consider a holomorphic function f on $\boldsymbol{B}_1$ satisfying the condition:

$$f(g\eta) = F_g(\eta)^k f(\eta) \quad \text{for } g \in \Gamma(1-\zeta)_1,$$

and we denote the $\boldsymbol{C}$-vector space of such functions by $M_k(F_g)$.

Corollary 6.2. (1) *We have an isomorphism of* $\boldsymbol{C}$*-algebras*:

$$\bigoplus_{n=0}^{\infty} M_{5n}(F_g)$$

$$= \boldsymbol{C}[\Theta_{[[1,1,1]]}(\Omega(\eta))^{10}, \Theta_{[[1,1,1]]}(\Omega(\eta))^{5}\Theta_{[[1,1,9]]}(\Omega(\eta))^{5}, \Theta_{[[1,1,9]]}(\Omega(\eta))^{10}]$$

$$\cong \boldsymbol{C}[x_0^2, x_0x_1, x_1^2]$$

$$\cong \bigoplus_{n=0}^{\infty} \mathrm{H}^0(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(-nK_{\boldsymbol{P}^1})),$$

where $[x_0 : x_1]$ *is homogeneous coordinates of* $\boldsymbol{P}^1$.

(2) $M_n(F_g) = \{0\}$ *for* $n \in \boldsymbol{N}$, $n \equiv 1, 2, 3, 4$ *mod* 5.

Proof. The assertion (1) is a direct consequence of Corollary 6.2 and Theorem 6.3. The assertion (2) is obtained by the same argument as the proof of Theorem 6.2. □

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