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On Meyer's function of hyperelliptic mapping class groups

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Abstract. In this paper, we consider Meyer's function of hyperelliptic mapping class groups of orientable closed surfaces and give certain explicit formulae for it. Moreover we study geometric aspects of Meyer's function, and relate it to the η-invariant of the signature operator and Morita's homomorphism, which is the core of the Casson invariant of integral homology 3-spheres.

0. Introduction.

Let Σ_g be an oriented closed surface of genus $g \geq 1$. The mapping class group Γ_g is defined to be the group of all isotopy classes of orientation preserving diffeomorphisms of Σ_g. By [**BH**] there exists the subgroup Δ_g of Γ_g which is the centralizer of the class of a hyperelliptic involution in Γ_g. This group fits in a nonsplit extension

$$0 \to \boldsymbol{Z}/2 \to \Delta_g \to \Gamma^{2g+2} \to 1,$$

where Γ^n denotes the mapping class group of S^2 leaving n points invariant. As is known, $\Delta_g = \Gamma_g$ if $g = 1, 2$ and $\Delta_g \neq \Gamma_g$ for $g \geq 3$. The group Δ_g is called the hyperelliptic mapping class group.

Our main object in the present paper is Meyer's signature cocycle [**Me**], which is a group 2-cocycle of the Siegel modular group $Sp(2g; \boldsymbol{Z})$. Topologically, this presents the signature of total spaces of surface bundles over a surface. Since the group Δ_g is acyclic over $\boldsymbol{Q}$ (cf. [**C**], [**K**]), the restriction of the pull-back of the signature cocycle via the classical representation

$$\rho : \Gamma_g \to Sp(2g; \boldsymbol{Z})$$

must be the coboundary of a unique rational 1-cochain of Δ_g. In this paper, we call it Meyer's function of genus g (see Remark 1.1 for precise definition).

In the case of genus one, Meyer gave an explicit formula of Meyer's function and the signature cocycle by using the Rademacher function (see also [**BG**], [**KM**] and [**S**]). It then seems to be natural problem that we try to generalize his results for surface of genus two. However, it is not so easy to describe Meyer's

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function explicitly for higher genera, because the representation ρ has non-trivial and huge kernel $\mathscr{I}_g$, which is called the Torelli group, in those cases.

On the one hand, geometric meanings of Meyer's function were studied by Atiyah in [**A**]. In fact, he related it to many invariants defined for each element of $\Delta_1 = \Gamma_1 \cong SL(2; \boldsymbol{Z})$ including Hirzebruch's signature defect, the logarithmic monodromy of Quillen's determinant line bundle, the Atiyah-Patodi-Singer η-invariant and its adiabatic limit.

The purpose of the present paper is twofold. First we give several formulae of Meyer's function on certain subgroups of Δ_g, which detect some information of it on the Torelli group. Secondly we study geometric aspects of Meyer's function under certain conditions. In fact, if we restrict ourselves to periodic automorphisms of Δ_g, we can regard Meyer's function as the η-invariant of the signature operator via mapping torus constructions. By this correspondence, we see that the η-invariant always takes its value in $(1/(2g+1))\boldsymbol{Z}$. On the other hand, Meyer's function can be interpreted as the Casson invariant of homology 3-spheres. From the theory of characteristic classes of surface bundles, due to Morita (see [**Mo2**], [**Mo3**]), the Casson invariant can be regarded as the secondary characteristic class associated to the first Morita-Mumford class $e_1 \in H^2(\Gamma_g; \boldsymbol{Q})$ (cf. [**Mo1**], [**Mu**]) through the correspondence between elements of Γ_g and 3-manifolds via the Heegaard splittings. In this point of view, the core of the Casson invariant is essentially represented by the homomorphism

$$d_0 : \mathscr{K}_g \to \boldsymbol{Q},$$

which we call Morita's homomorphism (see Section 5 for the definition). Here, $\mathscr{K}_g$ is the subgroup of Γ_g generated by all the Dehn twists along separating simple closed curves on Σ_g. Then Meyer's function coincides with Morita's homomorphism on $\Delta_g \cap \mathscr{K}_g$ (up to the factor $1/3$). Therefore, in principle, we can say that the Casson invariant of homology 3-spheres is determined by Meyer's function.

Now we describe the contents of this paper. In the next section, we review the definition of Meyer's signature cocycle and some results concerning genus one case. In Section 2, we derive several formulae of Meyer's function on certain subgroups of Δ_g. A relation between the η-invariant and Meyer's function will be discussed in Section 3. In Section 4 we recall the intersection cocycle of the mapping class group Γ_g, and relate the Casson invariant to Meyer's function in the last section.

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1. Definition and review of genus one case.

First we briefly recall the definition of Meyer's signature cocycle (see [**Me**] for details). For $\alpha, \beta \in Sp(2g; \boldsymbol{Z}) \subset Sp(2g; \boldsymbol{R})$, let

$$V_{\alpha,\beta} = \{(u, v) \in \boldsymbol{R}^{2g} \oplus \boldsymbol{R}^{2g} \mid (\alpha^{-1} - 1)u + (\beta - 1)v = 0\},$$

where 1 denotes the identity matrix of size $2g \times 2g$. We define a pairing map

$$\langle\, , \rangle : V_{\alpha,\beta} \times V_{\alpha,\beta} \to \boldsymbol{Z}$$

by

$$\langle (u_1, v_1), (u_2, v_2) \rangle = (u_1 + v_1) \cdot (1 - \beta) v_2,$$

where $\cdot$ is the intersection number of $H_1(\Sigma_g; \boldsymbol{R}) \cong \boldsymbol{R}^{2g}$. It is easy to see that $\langle\, , \rangle$ is a symmetric bilinear form (possibly degenerate) on $V_{\alpha,\beta}$. We define a 2-cocycle $\tau_g(\alpha, \beta)$ to be the signature of $(V_{\alpha,\beta}, \langle\, , \rangle)$. Of course, we can consider τ_g as a 2-cocycle of Γ_g in terms of the representation ρ. Then it is known that the cohomology class represented by $-3\tau_g$ is equal to the first Morita-Mumford class $e_1 \in H^2(\Gamma_g; \boldsymbol{Z})$ when the genus of Σ_g is greater than one (see [**Mo2**]). This cohomology class e_1 is defined to be the Gysin image of the second power of the Euler class of the central extension

$$0 \to \boldsymbol{Z} \to \Gamma_{g,1} \to \Gamma_{g,*} \to 1.$$

Here $\Gamma_{g,1}$ is the mapping class group of $\Sigma_{g,1}$, a compact oriented surface of genus g with one boundary component, and $\Gamma_{g,*}$ denotes the one relative to a base point $* \in \Sigma_g$. The center $\boldsymbol{Z}$ of $\Gamma_{g,1}$ is generated by the Dehn twist along a simple closed curve which is parallel to the boundary of $\Sigma_{g,1}$. The Morita-Mumford class plays an important role in the stable cohomology classes of the mapping class group.

As is known, the rational cohomology of the group Δ_g vanishes in dimensions 1 and 2 (see [**C**] and [**K**] for example), so that the cocycle $\rho^*\tau_g|_{\Delta_g}$ must be the coboundary of a unique rational 1-cochain of Δ_g (in the following, we often omit ρ^* for simplicity). We call it Meyer's function of genus g and denote it by ϕ_g.

Remark 1.1. More precisely, we can show $(2g+1)\tau_g|_{\Delta_g} \in B^2(\Delta_g; \boldsymbol{Z})$. Thereby, Meyer's function is defined to be $\phi_g : \Delta_g \to (1/(2g+1))\boldsymbol{Z}$ such that $\delta\phi_g = \tau_g|_{\Delta_g}$.

The next properties follow easily from that of the signature cocycle:

Lemma 1.2. *For two elements $\varphi, \psi \in \Delta_g$, Meyer's function satisfies*
(1) $\phi_g(\varphi\psi) = \phi_g(\varphi) + \phi_g(\psi) - \tau_g(\varphi, \psi)$;

(2) $\phi_g(id) = 0$;
(3) $\phi_g(\varphi^{-1}) = -\phi_g(\varphi)$;
(4) $\phi_g(\varphi\psi\varphi^{-1}) = \phi_g(\psi)$.

REMARK 1.3. From Lemma 1.2 (1), we notice that the restriction of ϕ_g to the Torelli group $\Delta_g \cap \mathscr{I}_g$ is a homomorphism, where $\mathscr{I}_g = \operatorname{Ker} \rho$.

The group presentation of Δ_g was given by Birman and Hilden (see [**BH**]). By using its defining relations, we can determine the values of ϕ_g on generators of Δ_g.

PROPOSITION 1.4. *For the generators* $\zeta_1, \dots, \zeta_{2g+1}$ *of* Δ_g $(g \geq 2)$, *the values of Meyer's function* ϕ_g *are equal to* $(g+1)/(2g+1)$.

PROOF. Since each generator is mutually conjugate (in fact, $\zeta_{i+1} = \xi\zeta_i\xi^{-1}$ holds for $\xi = \zeta_1 \cdots \zeta_{2g+1}$), ϕ_g has the same values on generators, and we simply denote it by $\phi_g(\zeta)$. Direct computations using the relation (15.4) in [**BH**] show that

$$\begin{aligned} 0 = \phi_g(\zeta_1 \cdots \zeta_{2g+1}^2 \cdots \zeta_1) &= \phi_g(\xi) + \phi_g(\bar{\xi}) - \tau_g(\xi, \bar{\xi}) \\ &= 2\{(2g+1)\phi_g(\zeta) - 1\} - 2g \\ &= 2(2g+1)\phi_g(\zeta) - 2(g+1), \end{aligned}$$

where we put $\bar{\xi} = \zeta_{2g+1} \cdots \zeta_1$. As a result, we obtain $\phi_g(\zeta) = (g+1)/(2g+1)$ and the proof is completed. $\square$

By virtue of Lemma 1.2 and Proposition 1.4, if we present any element $\varphi \in \Delta_g$ in terms of the product of generators, we can evaluate $\phi_g(\varphi)$ explicitly.

Now let us consider the genus one case. Then the mapping class group Γ_1 is identified with $SL(2; \boldsymbol{Z})$. Hereafter we fix the following group presentation of $SL(2; \boldsymbol{Z})$:

$$\langle \zeta_1, \zeta_2 \mid \zeta_1\zeta_2\zeta_1 = \zeta_2\zeta_1\zeta_2, \ (\zeta_1\zeta_2\zeta_1)^4 = 1 \rangle,$$

where

$$\zeta_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \zeta_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Each generator corresponds to the Dehn twist along a simple closed curve on Σ_1 which presents a symplectic basis of $H_1(\Sigma_1; \boldsymbol{Z})$ respectively.

In this case, there are some interesting formulae for ϕ_1. Here we recall the initiative result due to Meyer very briefly (see [**Me**]).

For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\boldsymbol{Z})$, the Rademacher function $\Psi : SL(2;\boldsymbol{Z}) \to \boldsymbol{Z}$ is defined by

$$\Psi(\alpha) = \begin{cases} \dfrac{a+d}{c} - 12\,\mathrm{sgn}\,c \cdot s(a,c) - 3\,\mathrm{sgn}\,c(a+d) & \text{if } c \neq 0 \\ \dfrac{b}{d} & \text{if } c = 0, \end{cases}$$

where $s(a,c)$ is the Dedekind sum (see [**RG**]) and $\mathrm{sgn}(n) = n/|n|$ if $n \neq 0$ and 0 if $n = 0$. Further we define a class function $\sigma : SL(2;\boldsymbol{Z}) \to \boldsymbol{Z}$ by

$$\sigma(\alpha) = \mathrm{Sign}\begin{pmatrix} -2c & a-d \\ a-d & 2b \end{pmatrix}.$$

Under the above preparations, we can summarize Meyer's theorem as follows.

Proposition 1.5 (Meyer). *For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\boldsymbol{Z})$, Meyer's function of genus one is described via*

$$\phi_1(\alpha) = -\frac{1}{3}\Psi(\alpha) + \sigma(\alpha) \cdot \frac{1}{2}(1 + \mathrm{sgn}(\mathrm{tr}\,\alpha)).$$

In particular, if α is hyperbolic (i.e. $|\mathrm{tr}\,\alpha| > 2$), then $\phi_1(\alpha) = -(1/3)\Psi(\alpha)$ holds.

Remark 1.6. We see from Proposition 1.5 that the values of ϕ_1 on generators of $SL(2;\boldsymbol{Z})$ are equal to $2/3$. Accordingly, Proposition 1.4 holds for $g \geq 1$.

Other formulae of ϕ_1 are investigated by several authors (see [**BG**], [**KM**], [**S**]). In the next section, we try to extend these results to certain subgroups of Δ_g.

2. Formulae of Meyer's function.

In this section, we give some formulae of Meyer's function on certain subgroups of the hyperelliptic mapping class group Δ_g.

First we consider the following central $\boldsymbol{Z}$-extension of $\Gamma_{1,*} = SL(2;\boldsymbol{Z})$:

$$0 \longrightarrow \boldsymbol{Z} \longrightarrow \Gamma_{1,1} \xrightarrow{\rho_{1,1}} SL(2;\boldsymbol{Z}) \longrightarrow 1,$$

where the center $\boldsymbol{Z}$ of $\Gamma_{1,1}$ is generated by the Dehn twist along a simple closed curve on $\Sigma_{1,1}$ which is parallel to the boundary (namely, $\psi_1 = (\zeta_1\zeta_2\zeta_1)^4$ generates the kernel $\boldsymbol{Z}$). Hence as a subgroup of Γ_2, the group $\Gamma_{1,1}$ has the presentation

$$\Gamma_{1,1} = \langle \zeta_1, \zeta_2 \mid \zeta_1\zeta_2\zeta_1 = \zeta_2\zeta_1\zeta_2 \rangle,$$

where we have used the same letters for generators by abuse of notation.

The above central extension also can be described explicitly via Maslov index $\mu : SL(2;\boldsymbol{R}) \times SL(2;\boldsymbol{R}) \to \boldsymbol{Z}$ (see [**LV**]). If we consider the universal covering $\widetilde{SL}(2;\boldsymbol{R}) \to SL(2;\boldsymbol{R})$, we can take the preimage $\widetilde{SL}(2;\boldsymbol{Z})$ of $SL(2;\boldsymbol{Z})$ in $\widetilde{SL}(2;\boldsymbol{R})$, because $\pi_1(SL(2;\boldsymbol{R})) \cong \boldsymbol{Z}$ is a central subgroup in $\widetilde{SL}(2;\boldsymbol{R})$. Then the group $\widetilde{SL}(2;\boldsymbol{Z})$ is isomorphic to $SL(2;\boldsymbol{Z}) \times \boldsymbol{Z}$ with the group structure given by

$$(\alpha, m) \cdot (\beta, n) = (\alpha\beta, m + n + \mu(\alpha, \beta)).$$

The identification of $\Gamma_{1,1}$ with $\widetilde{SL}(2;\boldsymbol{Z})$ is explicitly obtained via

$$\zeta_1 \mapsto \left(\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, 1 \right) \quad \text{and} \quad \zeta_2 \mapsto \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 0 \right).$$

Further we easily see that $\operatorname{Ker}(\widetilde{SL}(2;\boldsymbol{Z}) \to SL(2;\boldsymbol{Z})) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 4k \right) \mid k \in \boldsymbol{Z} \right\}$.

Thereby we can extend Proposition 1.5 as follows.

Proposition 2.1. *For an element $\varphi \in \Gamma_{1,1} \subset \Gamma_2$, Meyer's function of genus two is given by*

$$\phi_2|_{\Gamma_{1,1}}(\varphi) = -\frac{1}{15}\pi^{ab}(\varphi) + \rho_{1,1}^*\phi_1(\varphi),$$

where $\pi^{ab} : \Gamma_{1,1} \to \boldsymbol{Z}$ is the abelianization homomorphism (*namely, ζ_i corresponds to* 1) *and ϕ_1 is Meyer's function of genus one.*

Remark 2.2. Here we would like to emphasize that we can catch the information on the core (that is, the normal generator ψ_1) of the Torelli group $\mathscr{I}_2$ in the above formula, which cannot recover from the *Sp*-representation.

Proof. The restriction of τ_2 to the subgroup $\Gamma_{1,1}$ coincides with the pull-back $\rho_{1,1}^*\tau_1$, so that we have $\delta\phi_2|_{\Gamma_{1,1}} = \rho_{1,1}^*(\delta\phi_1) = \delta(\rho_{1,1}^*\phi_1)$. Accordingly there exists a homomorphism $f : \Gamma_{1,1} \to \boldsymbol{Z}$ uniquely such that

$$\phi_2|_{\Gamma_{1,1}}(\varphi) = f(\varphi) + \rho_{1,1}^*\phi_1(\varphi) \quad \text{for } \varphi \in \Gamma_{1,1}.$$

By the fact that $H_1(\Gamma_{1,1};\boldsymbol{Z}) \cong \boldsymbol{Z}$, we easily see that f is a rational multiple of the abelianization homomorphism $\pi^{ab} : \Gamma_{1,1} \to \boldsymbol{Z}$. Hence it is sufficient to evaluate the above equation on a generator of $\Gamma_{1,1}$ in order to determine the coefficient of π^{ab}. If we put $\varphi = \zeta_1$, then we have $\phi_2|_{\Gamma_{1,1}}(\varphi) = 3/5$, $\rho_{1,1}^*\phi_1(\varphi) = 2/3$ by means of Proposition 1.4 and clearly $\pi^{ab}(\varphi) = 1$. Consequently we can conclude $f = -(1/15)\pi^{ab}$ and obtain the desired formula. □

The next example is an extension of Sczech's formula on $SL(2;\boldsymbol{Z})$ to the group $\Gamma_{1,1}$ (see [**S**], Theorem 2).

Example 2.3. Put $\gamma_n = \left(\begin{pmatrix} 0 & -1 \\ 1 & p_0 \end{pmatrix}, q_0\right) \cdots \left(\begin{pmatrix} 0 & -1 \\ 1 & p_n \end{pmatrix}, q_n\right) \in \Gamma_{1,1} \cong \widetilde{SL}(2, \boldsymbol{Z})$, where $p_i \in \boldsymbol{Z}$ and $q_i \equiv -1 \pmod 4$ for each i. Then by virtue of Proposition 2.1, we obtain

$$\phi_2(\gamma_n) = -\frac{1}{5}\sum_{k=0}^{n}(2p_k + q_k) - \operatorname{Sign} P_n,$$

where $\operatorname{Sign} P_n$ denotes the signature of the matrix P_n whose (i,j)-entry is $-p_i$ if $i = j$, 1 if $|i - j| = 1$ or n, and 0 otherwise.

Now we consider the commutative diagram below:

$$\begin{tikzcd}
0 \arrow[r] & \boldsymbol{Z} \arrow[r] \arrow[d] & \Gamma_{1*1} \arrow[r, "\rho_{1*1}"] \arrow[d] & SL(2;\boldsymbol{Z}) \times SL(2;\boldsymbol{Z}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathscr{I}_2 \arrow[r] & \Gamma_2 \arrow[r, "\rho"] & Sp(4;\boldsymbol{Z}) \arrow[r] & 1,
\end{tikzcd}$$

where the upper sequence is a central $\boldsymbol{Z}$-extension of $SL(2;\boldsymbol{Z}) \times SL(2;\boldsymbol{Z})$ and $\operatorname{Ker}\rho_{1*1}$ is generated by $\psi_1 = (\zeta_1\zeta_2\zeta_1)^4 = (\zeta_5\zeta_4\zeta_5)^4$ as before. The group Γ_{1*1} is conceptually the mapping class group of the surface $\Sigma_{1,1} \cup \Sigma_{1,1}$ which is pasted along the boundary curves of each $\Sigma_{1,1}$.

As a consequence of Proposition 2.1, we obtain the following formula of Meyer's function of genus two on Γ_{1*1}.

Theorem 2.4. *For an element $\varphi \in \Gamma_{1*1} \subset \Gamma_2$, Meyer's function of genus two is described via*

$$\phi_2(\varphi) = -\frac{1}{15}\pi^{ab}(\varphi) + \rho_{1*1}^*(\phi_1 \times \phi_1)(\varphi),$$

where $\pi^{ab} : \Gamma_{1*1} \to \boldsymbol{Z}$ is the abelianization homomorphism and $\phi_1 \times \phi_1$ is the composite function $SL(2;\boldsymbol{Z}) \times SL(2;\boldsymbol{Z}) \to (1/3)\boldsymbol{Z} \times (1/3)\boldsymbol{Z} \to (1/3)\boldsymbol{Z}$ given by $(\alpha, \beta) \mapsto \phi_1(\alpha) + \phi_1(\beta)$.

Remark 2.5. For the subgroup $\Delta_{h,1}$ of $\Gamma_{h,1}$, which consists of all the inverse images of $\Delta_h \subset \Gamma_h$ under the natural projection $\Gamma_{h,1} \to \Gamma_h$, we can also give a similar formula as in Proposition 2.1. To be more precise, the difference between Meyer's function ϕ_h and the restriction of ϕ_g $(h < g)$ to the group $\Delta_{h,1}$ is described by a rational multiple of the abelianization homomorphism. In fact, we see from easy calculation that the coefficient is equal to $-(g-h)/((2h+1)(2g+1))$. Further, we naturally notice that an analogue of Theorem 2.4 holds for the amalgam of $\Delta_{h,1}$ and $\Delta_{h',1}$ along $\boldsymbol{Z}$ (this infinite cyclic group is generated by the Dehn twist along a simple closed curve on $\Sigma_{h+h'}$ which is the common boundary of two subsurfaces $\Sigma_{h,1}$ and $\Sigma_{h',1}$).

EXAMPLE 2.6. Let $\psi_h \in \Delta_g$ $(1 \leq h \leq g-1)$ be a BSCC-map of genus h (namely, a Dehn twist along a bounding simple closed curve on Σ_g which is invariant under the action of a hyperelliptic involution and separates Σ_g into two subsurfaces of genera h and $g-h$). Then ψ_h is presented by $\psi_h = \varphi(\zeta_1 \cdots \zeta_{2h})^{4h+2}\varphi^{-1}$ for some element $\varphi \in \Delta_g$. Thus it is easy to see from Remark 2.5 and Lemma 1.2 (4) that the value of Meyer's function on ψ_h is given by

$$\phi_g(\psi_h) = -\frac{4}{2g+1}h(g-h),$$

because the element $(\zeta_1 \cdots \zeta_{2h})^{4h+2}$ is in the kernel of the natural map $\Delta_{h,1} \to \Delta_h$.

3. Meyer's function and the η-invariant.

The purpose of the following sections is to study geometric meanings of Meyer's function. The results obtained here can be regarded as a generalization of Atiyah's paper [**A**].

As for the definition of the η-invariant of the signature operator, see the original paper [**APS**]. In short, it measures the extent to which the Hirzebruch signature formula fails for a non-closed 4-dimensional Riemannian manifold whose metric is a product near its boundary.

Let $\varphi \in \Gamma_g$ be of finite order and M_φ be the mapping torus corresponding to φ. Namely, it is the identification space $\Sigma_g \times [0,1]/(p,0) \sim (\varphi(p),1)$. We endow M_φ with the metric which is induced from the product of the standard metric on S^1 and a φ-invariant metric of Σ_g.

If we restrict ourselves to the hyperelliptic mapping class group Δ_g, we obtain the following theorem.

THEOREM 3.1. *Let Δ_g be the hyperelliptic mapping class group of genus g. Then*

$$\eta(M_\varphi) = \phi_g(\varphi)$$

holds for any element $\varphi \in \Delta_g$ of finite order. In particular, $\eta(M_\varphi)$ is a topological invariant on M_φ (that is, independent of the choice of a metric).

PROOF. Let m be the order of φ. Using the properties of Meyer's function several times (see Lemma 1.2), it follows that

$$0 = \phi_g(\varphi^m) = m\phi_g(\varphi) - \sum_{k=1}^{m-1} \tau_g(\varphi, \varphi^k).$$

We therefore have

$$\phi_g(\varphi) = \frac{1}{m} \sum_{k=1}^{m-1} \tau_g(\varphi, \varphi^k),$$

and this coincides with the η-invariant of the mapping torus M_φ by virtue of our previous results (see [**Mf**]). The second assertion is clear from the first one and this completes the proof of Theorem 3.1. □

The corollary stated below is an easy consequence of the definition of Meyer's function (see Remark 1.1).

COROLLARY 3.2. *Let $\varphi \in \Gamma_g$ be of finite order. If $\varphi \in \Delta_g$, namely φ commutes with a hyperelliptic involution, then*

$$\eta(M_\varphi) \in \frac{1}{2g+1} \boldsymbol{Z}$$

holds, where $(1/(2g+1))\boldsymbol{Z}$ denotes the additive group $\{n/(2g+1) \in \boldsymbol{Q} \mid n \in \boldsymbol{Z}\}$.

EXAMPLE 3.3. Let $\varphi \in \Gamma_3$ be of order 3 so that the quotient orbifold of Σ_3 by its cyclic action is homeomorphic to $S^2(3,3,3,3,3)$. Then direct computation shows that the η-invariant of corresponding mapping torus is given by

$$\eta(M_\varphi) = -\frac{2}{3} \notin \frac{1}{7} \boldsymbol{Z}.$$

Hence, Corollary 3.2 implies that φ cannot be realized as an automorphism of a hyperelliptic Riemann surface.

4. Intersection cocycle on the hyperelliptic mapping class group.

In this section, we quickly review the intersection cocycle which represents the first Morita-Mumford class $e_1 \in H^2(\Gamma_g; \boldsymbol{Q})$. See [**Mo4**], [**Mo5**] for details.

Let $\mathscr{I}_g$ denote the Torelli group of Σ_g (i.e. $\mathscr{I}_g = \operatorname{Ker} \rho$). Then by virtue of fundamental results of Johnson ([**J1**], [**J2**]), there exists the following exact sequence:

$$1 \to \mathscr{K}_g \to \mathscr{I}_g \xrightarrow{\tau} \Lambda^3 H/H \to 1,$$

where $\mathscr{K}_g$ is the subgroup of Γ_g generated by all the Dehn twists along separating simple closed curves on Σ_g and $\Lambda^3 H$ is the third exterior power of $H = H_1(\Sigma_g; \boldsymbol{Z})$. The map τ is now called Johnson's homomorphism. After Johnson's work, this homomorphism is extended to the whole mapping class group Γ_g

as a crossed homomorphism by Morita [**Mo4**]. More precisely, there exists a map $\tilde{k} : \Gamma_g \to (1/2)\Lambda^3 H/H$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{I}_g & \xrightarrow{\tau} & \Lambda^3 H/H \\ \big\downarrow & & \big\downarrow \\ \Gamma_g & \xrightarrow{\tilde{k}} & \dfrac{1}{2}\Lambda^3 H/H \end{array}$$

where $(1/2)\Lambda^3 H = \Lambda^3 H \otimes (1/2)\boldsymbol{Z}$. It should be noted that $\tilde{k}$ is defined as an extension of Johnson's homomorphism uniquely up to coboundaries. Namely it is defined as an element of $H^1(\Gamma_g; (1/2)\Lambda^3 H/H)$ and further serves a generator of the cohomology group (modulo possibly torsions). By using this crossed homomorphism, Morita explicitly described a group 2-cocycle of Γ_g representing e_1 (see [**Mo5**]).

PROPOSITION 4.1 (Morita). *There exists a uniquely defined $Sp(2g; \boldsymbol{Z})$-equivariant homomorphism $C : \Lambda^3 H/H \otimes \Lambda^3 H/H \to \boldsymbol{Q}$ such that the cohomology class of the 2-cocycle $c(\varphi, \psi) = C(\tilde{k}(\varphi), \varphi_* \tilde{k}(\psi))$ $(\varphi, \psi \in \Gamma_g)$ represents the first Morita-Mumford class $e_1 \in H^2(\Gamma_g; \boldsymbol{Q})$.*

We shall call the above cocycle c the intersection cocycle. It should be remarked that c is defined once we fix a crossed homomorphism $\tilde{k} : \Gamma_g \to (1/2)\Lambda^3 H/H$, which represents a generator of $H^1(\Gamma_g; (1/2)\Lambda^3 H/H)$ (modulo possibly torsions). On the other hand, if we choose a suitable crossed homomorphism, that is, modify a given crossed homomorphism by a coboundary, we can show the following.

PROPOSITION 4.2. *There exists a crossed homomorphism $\tilde{k}_0 : \Gamma_g \to (1/2)\Lambda^3 H/H$ so that the restriction of it to the hyperelliptic mapping class group Δ_g is zero map.*

PROOF. By the fact that the cohomology groups $H^*(\Delta_g; H_{\boldsymbol{Q}})$ vanish for the coefficient $H_{\boldsymbol{Q}} = H \otimes \boldsymbol{Q}$ (see [**K**]), we easily see that

$$H^1(\Delta_g; \Lambda^3 H_{\boldsymbol{Q}}) \to H^1(\Delta_g; \Lambda^3 H_{\boldsymbol{Q}}/H_{\boldsymbol{Q}})$$

is an isomorphism. Moreover we find $H^1(\Delta_g; H_{\boldsymbol{Q}}^{\otimes 3}) = 0$ by means of the same argument as in [**K**] using the Lyndon-Hochschild-Serre spectral sequence of the extension in Introduction, so that we can conclude $H^1(\Delta_g; \Lambda^3 H_{\boldsymbol{Q}}) = 0$. Hence it follows that the extended Johnson's homomorphism $[\tilde{k}] \in H^1(\Gamma_g; (1/2)\Lambda^3 H/H)$ is at most torsion on Δ_g. Namely, $n\tilde{k}|_{\Delta_g} = \delta u$ holds for some positive integer n and $u \in (1/2)\Lambda^3 H/H$. If we define a new crossed homomorphism by $\tilde{k}_0 = n\tilde{k} - \delta u$, then it becomes a desired one. The proof is over. $\square$

REMARK 4.3. In the case of genus two, it follows directly that $\tilde{k} \equiv 0$, because the inclusion $H \to \Lambda^3 H$ defined by $u \mapsto u \wedge \omega_0$, where ω_0 is the symplectic class, is an isomorphism in this case.

If we restrict the above crossed homomorphism $\tilde{k}_0$ to the group $\Delta_g \cap \mathscr{I}_g$, then we have $\tilde{k}_0|_{\Delta_g \cap \mathscr{I}_g} = n\tau|_{\Delta_g} = 0$. Since the Torelli group $\mathscr{I}_g$ is torsion free, Johnson's result (that is, the previous exact sequence) implies that

COROLLARY 4.4. $\Delta_g \cap \mathscr{I}_g = \Delta_g \cap \mathscr{K}_g$.

We call this group the hyperelliptic Torelli group and denote it by $\mathscr{J}_g$. It would be interesting to study the structure of $\mathscr{J}_g$ more deeply. For instance, it is conjectured that $\mathscr{J}_g$ is the normal closure of BSCC-maps $(\zeta_1 \cdots \zeta_{2h})^{4h+2}$ $(1 \leq h \leq [g/2])$ and not finitely generated (cf. [**P**], [**McM**]).

5. Meyer's function and the Casson invariant.

The Casson invariant is an integer valued invariant defined for oriented integral homology 3-spheres (see [**AM**]). Roughly speaking, it counts the number (with signs) of conjugacy classes of irreducible representations of the fundamental group of an oriented homology 3-sphere into the Lie group $SU(2)$.

Now we have obtained two canonical cocycles, one is the signature cocycle τ_g and the other is the intersection cocycle c, for the first Morita-Mumford class $e_1 \in H^2(\Gamma_g; \boldsymbol{Q})$. The difference between these two cocycles is a coboundary of the mapping class group Γ_g. Since Γ_g is perfect for $g \geq 3$ and $H_1(\Gamma_2; \boldsymbol{Z}) = \boldsymbol{Z}/10$, we have a uniquely defined mapping $d : \Gamma_g \to \boldsymbol{Q}$ such that

$$\delta d = c + 3\tau_g,$$

provided we fix a crossed homomorphism. If we restrict the mapping d to the subgroup $\mathscr{K}_g$, then it is a homomorphism there (further, independent of the choice of a crossed homomorphism) and essentially represents the Casson invariant under the correspondence between elements of the Torelli group and homology 3-spheres via Heegaard splittings (see [**Mo2**], [**Mo3**]). We denote it by

$$d_0 : \mathscr{K}_g \to \boldsymbol{Q}$$

and call it Morita's homomorphism. Then we have

THEOREM 5.1. *Let $\mathscr{J}_g$ be the hyperelliptic Torelli group of genus g. Then Meyer's function essentially coincides with Morita's homomorphism on $\mathscr{J}_g$. To be more precise,*

$$\phi_g(\varphi) = \frac{1}{3} d_0(\varphi)$$

holds for any element $\varphi \in \mathscr{J}_g$.

Proof. First of all, let us take a crossed homomorphism $\tilde{k}_0$ in Proposition 4.2 and consider the intersection cocycle c_0 corresponding to $\tilde{k}_0$. If we restrict ourselves to the group Δ_g, it follows that

$$\delta d|_{\Delta_g} = c_0|_{\Delta_g} + 3\tau_g|_{\Delta_g} = 3\delta\phi_g.$$

We therefore obtain $d|_{\Delta_g} = 3\phi_g$ because $H^1(\Delta_g; \boldsymbol{Z}) = 0$. Further if we restrict this equation to the group $\mathscr{I}_g$, it does not depend on the choice of the crossed homomorphism $\tilde{k}_0$. Hence we have $3\phi_g \equiv d_0$ on $\mathscr{I}_g$ as desired. This completes the proof. □

The corollary stated below is immediate from Example 2.6 and Theorem 5.1.

Corollary 5.2. *Let $\psi_h \in \mathscr{I}_g$ be a BSCC-map of genus h. Then the value of Morita's homomorphism on ψ_h is given by*

$$d_0(\psi_h) = -\frac{12}{2g+1} h(g-h).$$

Remark 5.3. It has been shown by Morita that Corollary 5.2 actually holds for the whole group $\mathscr{K}_g$. Our method here gives a new proof of Morita's result by means of Meyer's function.

References

[AM] S. Akbulut and J. McCarthy, Casson's invariant for oriented homology 3-spheres, An Exposition Mathematical Notes, **36**, Princeton Univ. Press, 1990.

[A] M. F. Atiyah, The logarithm of the Dedekind η-function, Math. Ann., **278** (1987), 335–380.

[APS] M. F. Atiyah, V. K. Patodi and I. M. Singer, Spectral asymmetry and Riemannian geometry I, Math. Proc. Cambridge Philos. Soc., **77** (1975), 43–69.

[BG] J. Barge and E. Ghys, Cocycles d'Euler et de Maslov, Math. Ann., **294** (1992), 235–265.

[BH] J. Birman and H. Hilden, On mapping class groups of closed surfaces as covering spaces, Advances in the theory of Riemann surfaces, **66**, Princeton Univ. Press, 1971, 81–115.

[C] F. R. Cohen, Homology of mapping class groups for surfaces of low genus, Contemp. Math., **58** (1987), 21–30.

[J1] D. Johnson, An abelian quotient of the mapping class group $\mathscr{I}_g$, Math. Ann., **249** (1980), 225–242.

[J2] D. Johnson, The structure of the Torelli group II and III, Topology, **24** (1985), 113–144.

[K] N. Kawazumi, Homology of hyperelliptic mapping class groups for surfaces, Topology Appl., **76** (1997), 203–216.

[KM] R. Kirby and P. Melvin, Dedekind sums, μ-invariants and the signature cocycle, Math. Ann., **299** (1994), 231–267.

[LV] G. Lion and M. Vergne, The Weil representation, Maslov index and theta series, Progr. Math., **6** (1980).

[McM] D. McCullough and A. Miller, The genus 2 Torelli group is not finitely generated, Topology Appl., **22** (1986), 43–49.

[Me] W. Meyer, Die Signatur von Flächenbündeln, Math. Ann., **201** (1973), 239–264.

[Mf] T. Morifuji, The η-invariant of mapping tori with finite monodromies, Topology Appl., **75** (1997), 41–49.
[Mo1] S. Morita, Characteristic classes of surface bundles, Invent. Math., **90** (1987), 551–577.
[Mo2] S. Morita, Casson's invariant for homology 3-spheres and characteristic classes of surface bundles I, Topology, **28** (1989), 305–323.
[Mo3] S. Morita, On the structure of the Torelli group and the Casson invariant, Topology, **30** (1991), 603–621.
[Mo4] S. Morita, The extension of Johnson's homomorphism from the Torelli group to the mapping class group, Invent. Math., **111** (1993), 197–224.
[Mo5] S. Morita, A linear representation of the mapping class group of orientable surfaces and characteristic classes of surface bundles, Proc. 37th Taniguchi symp. Topology and Teichmüller spaces, 1996 (ed. by S. Kojima), 159–186.
[Mu] D. Mumford, Towards an enumerative geometry of the moduli space of curves, Arithmetic and Geometry, Progr. Math., **36** (1983), 271–328.
[P] J. Powell, Two theorems on the mapping class group of surfaces, Proc. Amer. Math. Soc., **68** (1978), 347–350.
[RG] H. Rademacher and E. Grosswald, Dedekind sums, Carus Math. Monogr., **16** (Math. Assoc. America, 1972).
[S] R. Sczech, Dedekind sums and signatures of intersection forms, Math. Ann., **299** (1994), 269–274.

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