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On Riesz mean for the coefficients of twisted Rankin-Selberg L-functions

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Abstract. Rankin obtained the asymptotic formula for the sum of coefficients of Rankin-Selberg L-series associated with a cusp form and a trivial character. Ivić-Matsumoto-Tanigawa studied the error term in it by using a mean value formula which is yielded from the Voronoï formula of the Riesz mean. In this paper, we consider more general Rankin-Selberg L-series. It is associated with two cusp forms and a non-trivial character mod d. Ivić-Matsumoto-Tanigawa's method cannot be applied directly to our case. We consider the sum of coefficients of twisted Rankin-Selberg L-series by a modification of their method.

1. Introduction.

Rankin [**10**] obtained an asymptotic formula for the sum of the square of the absolute value of Fourier coefficients of a cusp form of integral weight $\kappa \geq 1$. Ivić-Matsumoto-Tanigawa [**6**] suggested a way of improving the error term in Rankin's result. In this paper, we extend their method to the case of the sum of the product of Fourier coefficients of a cusp form and a twisted cusp form.

Let f and g be normalized Hecke eigen cusp forms of integral weight k and l respectively for $SL_2(\boldsymbol{Z})$ (we assume $k \geq l \geq 12$), and we use the symbols a_n and b_n for the n-th Fourier coefficients of f and g at ∞ respectively. The Fourier coefficients of f and g are known to be real. We fix a Dirichlet character χ mod d. The Rankin-Selberg L-function associated to f, g and χ is defined as the following Euler product:

$$\begin{aligned} L_{f\otimes g}(s,\chi) &= \prod_p (1-\alpha_p\beta_p\chi(p)p^{-s-(k+l)/2+1})^{-1}(1-\alpha_p\bar{\beta}_p\chi(p)p^{-s-(k+l)/2+1})^{-1} \\ &\quad \times (1-\bar{\alpha}_p\beta_p\chi(p)p^{-s-(k+l)/2+1})^{-1}(1-\bar{\alpha}_p\bar{\beta}_p\chi(p)p^{-s-(k+l)/2+1})^{-1}, \qquad (1.1) \end{aligned}$$

where α_p and β_p are complex numbers satisfying

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$$\alpha_p + \bar{\alpha}_p = a_p, \quad |\alpha_p| = p^{(k-1)/2},$$
$$\beta_p + \bar{\beta}_p = b_p \quad \text{and} \quad |\beta_p| = p^{(l-1)/2}.$$

(Here the bar symbol means the complex conjugate.) The existence of those numbers is implied by Deligne's result [**2**]. The above Euler product is absolutely convergent in $\Re(s) > 1$, and we know

$$L_{f\otimes g}(s,\chi) = L(2s,\chi^2)\sum_{n=1}^{\infty}\frac{a_n b_n \chi(n)}{n^{s+(k+l)/2-1}},$$

where $L(s,\chi^2)$ is the Dirichlet L-function associated with χ^2. We put

$$L_{f\otimes g}(s,\chi) = \sum_{n=1}^{\infty}\frac{c_n}{n^s}$$

in $\Re(s) > 1$, then it is clear that

$$c_n = n^{1-(k+l)/2}\chi(n)\sum_{m^2|n} a_{n/m^2} b_{n/m^2} m^{k+l-2}.$$

Deligne's result implies $c_n \ll n^{\varepsilon}$ for any $\varepsilon > 0$.

Rankin [**10**] and Ivić-Matsumoto-Tanigawa [**6**] investigated the sum of c_n in the case $f = g$ and χ is trivial. We call this case *the non-twisted case*. In this paper we are interested in the cases "$f \neq g$" or "$f = g$ and χ is not trivial". We call these cases *the twisted case*. In other words, in the non-twisted case $L_{f\otimes g}(s,\chi)$ has a pole at $s = 1$ and in the twisted case $L_{f\otimes g}(s,\chi)$ has no poles.

In the twisted case, the author [**4**] obtained

$$\sum_{n\le x} c_n \ll x^{5/3} d^{4/5+\varepsilon}, \tag{1.2}$$

when χ is a primitive character. This means that

$$\sum_{n\le x} a_n b_n \chi(n) \ll x^{(k+l)/2-2/5} d^{4/5+\varepsilon}.$$

This result is analogous to Rankin's result [**10**]. In the non-twisted case, Ivić-Matsumoto-Tanigawa [**6**] proposed a method of improving Rankin's result. If we apply directly Ivić-Matsumoto-Tanigawa's method [**6**] to the twisted case, we can see that the x-aspect of the estimate of $\sum_{n\le x} c_n$ is under the same situation as the non-twisted case, but the d-aspect of it is worse than (1.2). The aim of this paper is to propose a modification of Ivić-Matsumoto-Tanigawa's method [**6**] which improves the d-aspect. The new method introduced in the present paper yields the following two theorems.

When χ is a primitive character, we define the Riesz mean of c_n as follows:

$$D_\rho(x) = \Gamma(\rho+1)^{-1} \sum_{n \le x} c_n (x-n)^\rho, \tag{1.3}$$

where $x \ge 1$ and ρ is a real number.

THEOREM 1. *If* $D_1(x) \ll x^\alpha d^\beta$, *where* α *and* β *are real numbers satisfying* $d^\beta \le 2x^{1-\alpha/2}$, *then we have* $D_0(x) \ll x^{\alpha/2} d^{\beta/2}$.

THEOREM 2. *We put* $D_1^*(t) = D_1(t) - tL_{f\otimes g}(0,\chi)$. *Let* $x > 1$ *be a real number and* ε *be an arbitrary positive number satisfying* $1 > \varepsilon > 0$. *Then we have*

$$\int_0^x |D_1^*(t)|^2 \, dt = \frac{2d^3}{13(2\pi)^4} \sum_{n=1}^\infty \frac{|c_n|^2}{n^{7/4}} x^{13/4} + O(x^{3+\varepsilon} d^{4+\varepsilon}).$$

These theorems say that if we can obtain more detailed information on the error term in Theorem 2, then we have the possibility of improving the estimate in (1.2) with respect to both the x and the d-aspects. See the discussion at the end of the present paper.

Ivić-Matsumoto-Tanigawa's work is concerned only with the x-aspect. When we consider the d-aspect, their method is not suitable as it is. There are two novel points in the present paper. First, Ivić-Matsumoto-Tanigawa [**6**] proved a mean value formula for $\triangle_1(x) = D_1(x) - Q_1(x)$ ($Q_1(x)$ is defined below), but a direct generalization of this formula is not suitable for our purpose. We should study the mean value of a modification of $\triangle_1(x)$ (see Remark 1 and Remark 2 below).

Next we explain the second novel point. Ivić-Matsumoto-Tanigawa's method is based on two Voronoï formulas. They are called *the truncated Voronoï formula* and *the Voronoï formula of Meurman-type* (It means the type first introduced by Meurman [**8**]). In this paper, our method is also based on the same type of Voronoï formulas of $D_\rho(x)$ (see sections 3 and 5) as them. In order to obtain them, we have to consider the d-aspect carefully. To obtain the truncated Voronoï formula is not so difficult. However we are confronted with a difficult task of obtaining a Voronoï formula of Meurman-type. A good estimate of $D_1(x)$ is necessary to obtain it. When $d = 1$, we can estimate $D_1(x)$ by using the Voronoï formula (2.2) for $\rho = 2$ and the difference operator. This is easy. But in the case $d \ne 1$, we cannot obtain a good estimate of $D_1(x)$ with respect to the d-aspect when $d^2 \le x \le d^4/16\pi^4$ if we use the same argument as that in the non-twisted case.

To overcome this trouble, we use a mean value formula which can be shown by using only the truncated Voronoï formula (see Lemma 2). This mean value formula is worse than the formula in Theorem 2, but it is enough to deduce an estimate of $D_1(x)$ which is necessary to obtain the Voronoï formula of Meurman-type. In Section 4, this argument will be explained.

Then, by using this estimate of $D_1(x)$, we can obtain the Voronoï formula of Meurman-type of $D_1(x)$ in the twisted case. By combining this formula with the truncated Voronoï formula, we obtain the above two theorems.

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2. The fundamental facts.

In this section, we introduce some facts about the Rankin-Selberg L-function and the Voronoï formula for the Riesz-mean of Rankin-Selberg series, which are defined in (1.1) and (1.3). Throughout this paper, ε is an arbitrarily small positive constant.

First, we mention the facts on the Rankin-Selberg L-function. The function $L_{f\otimes g}(s,\chi)$ can be continued analytically to the whole s-plane and holomorphic except for the simple pole at $s=1$ which appears when $f=g$ and χ is a principal character (see Li [**7**] and Ogg [**9**]). Li [**7**] proved the following functional equation when χ is a primitive character:

$$\Psi_{f\otimes g}(s,\chi)=C_\chi\Psi_{f\otimes g}(1-s,\bar{\chi}), \tag{2.1}$$

where

$$\Psi_{f\otimes g}(s,\chi)=\left(\frac{2\pi}{d}\right)^{-2\pi}\Gamma\left(s+\frac{k-l}{2}\right)\Gamma\left(s+\frac{k+l}{2}-1\right)L_{f\otimes g}(s,\chi)$$

and C_χ is a constant depending on χ with $|C_\chi|=1$. We know that $L_{f\otimes g}(s,\chi)$ has no zeros in $\Re(s)>1$ from the definition and $L_{f\otimes g}(s,\chi)$ has zeros at $s=-n$ ($n\in\mathbf{N}$, $n\geq(k-l)/2$) from the functional equation (2.1). These zeros are called *trivial zeros*.

Secondly, we mention some results on the Riesz mean of Rankin-Selberg series $D_\rho(x)$. Hafner [**3**] obtained the Voronoï formula for the Riesz mean of general L-functions with functional equations. The following Voronoï formula is obtained by applying his result to $D_\rho(x)$. We have

$$D_\rho(x)=Q_\rho(x)+\sum_{n=1}^{\infty}\frac{C_\chi\overline{c_n}(2\pi d^{-1})^2}{(16\pi^4nd^{-1})^{1+\rho}}f_\rho\left(\frac{16\pi^4xn}{d^4}\right), \tag{2.2}$$

where

$$Q_\rho(x) = \frac{1}{2\pi i}\int_C \frac{\Gamma(s)L_{f\otimes g}(s,\chi)x^{\rho+s}}{\Gamma(s+\rho+1)}ds \tag{2.3}$$

and

$$f_\rho(x) = \frac{1}{2\pi i}\int_{C_{0,b}} \frac{\Gamma(1-s)\Gamma(s+(k-l)/2)\Gamma(s+(k+l)/2-1)x^{1+\rho-s}}{\Gamma(2+\rho-s)\Gamma(1-s+(k-l)/2)\Gamma(-s+(k+l)/2)}ds. \tag{2.4}$$

Here the paths of integration C and $C_{0,b}$ conform to Hafner's notation; let R be a real number satisfying $R > (k+l)/2 - 1$, and the path C is the rectangle with vertices $b \pm iR$ and $1 - b \pm iR$ and has positive orientation and the path $C_{0,b}$ is the oriented polygonal path with vertices $-i\infty$, $-iR$, $b - iR$, $b + iR$, iR and $i\infty$, where $b > (k+l)/2 - 1$. From the definition, we see that $(d/dx)D_\rho(x) = D_{\rho-1}(x)$ and there are analogous relations for $Q_\rho(x)$ and $f_\rho(x)$.

The infinite sum on the right-hand side of (2.2) is absolutely convergent for $\rho > 3/2$ and it is convergent for $\rho > 1/2$ (see Hafner [**3**]). Moreover Hafner [**3**] showed the asymptotic expansion of $f_\rho(x)$ which holds for $x \geq 1$. From that expansion we know

$$f_\rho(x) = O(x^{(3+6\rho)/8}) + O(x^{1+\rho-b}) \tag{2.5}$$

for $x \geq 1$, where the last error term of (2.5) does not appear when $b > 1 + \rho$ and $\rho \in \boldsymbol{Z}$. Using Chandrasekharan-Narasimhan [**1**] and Hafner [**3**], we obtain

$$f_\rho(x) = \frac{1}{\sqrt{2\pi}} x^{(3+6\rho)/8} \sin\left(4x^{1/4} + \frac{3-2\rho}{4}\pi\right) + O(x^{(1+6\rho)/8}) + O(x^{1+\rho-b}) \tag{2.6}$$

for $x \geq 1$, where the last error term of (2.6) does not appear when $b > 1 + \rho$ and $\rho \in \boldsymbol{Z}$.

Lastly, we refer some facts for c_n. In Section 1, we already mentioned that $c_n \ll n^\varepsilon$ is implied by Deligne's result [**2**]. We can see $\sum_{n\leq M}|c_n| \ll M$ by using the Cauchy-Schwarz inequality and Rankin's result [**10**]. And we can also show the estimate

$$\sum_{x-y<n\leq x} |c_n| \ll y, \tag{2.7}$$

where $x > 0$, $x^\varepsilon < y \leq x$ and $0 < \varepsilon < 1/2$. In Lemma 1 of the author [**4**], this estimate was proved by using Theorem 1 of Shiu [**11**], the Cauchy-Schwarz inequality and applying the same method as that in the proof of Lemma 4 in Ivić-Matsumoto-Tanigawa [**6**].

3. The truncated Voronoï formulas for $D_\rho(x)$.

In this section, we prove the following proposition.

Proposition 1. *Let* $0 \le \rho \le 3/2$ *and* $1 > \varepsilon > 0$. *We put*

$$\varepsilon^* = \begin{cases} \varepsilon & 0 \le \rho \le 1 \\ \dfrac{1}{8} + \varepsilon & 1 < \rho \le 3/2. \end{cases}$$

Let $N \ge d^4$ *and we assume* N *is large enough compared with* k, l *and* ρ. *Then we have the following truncated Voronoï formula*:

$$\begin{aligned} D_\rho(x) &= \frac{x^\rho}{\Gamma(\rho+1)} L_{f\otimes g}(0,\chi) \\ &+ \frac{C_\chi}{(2\pi)^{\rho+1}} d^{\rho+1/2} x^{(3+6\rho)/8} \sum_{n \le N} \frac{\bar{c}_n}{n^{(5+2\rho)/8}} \sin\left(\frac{8\pi}{d}(xn)^{1/4} + \frac{3-2\rho}{4}\pi\right) \\ &+ O(x^{(1+6\rho)/8} d^{\rho+3/2} N^{(1-2\rho)/8} + x^{(1+3\rho)/4} d^{\rho+1} N^{(1-\rho)/4+\varepsilon}) \\ &+ O(x^{(3+3\rho)/4+\varepsilon} d^{\rho+1} N^{-(1+\rho)/4}) \\ &+ \begin{cases} O(x^\rho d^2 + x^{(1+3\rho)/4} d^{\rho+1} N^{(1-\rho)/4+\varepsilon^*} + x^{\rho-\varepsilon^*} d^{2+4\varepsilon^*}) & \text{if } d^4/16\pi^4 x \ge 1 \\ O(x^{(1+6\rho)/8} d^{\rho+3/2}) & \text{if } d^4/16\pi^4 x < 1 \end{cases} \\ &+ O(x^{\rho+\varepsilon}), \end{aligned}$$

where the last error term does not appear when $\rho = 0$.

The first term on the right-hand side of this truncated Voronoï formula is zero when $k = l$, because $L_{f\otimes g}(0,\chi) = 0$ in this case. This vanishing fact follows from the functional equation (2.1).

Proof of Proposition 1. We put $c = 1 + \varepsilon$ and $T > 2 + (k+l)/2$. We have

$$D_\rho(x) = I_\rho(x,c) + O(x^{1+\rho+\varepsilon} T^{-1-\rho} + x^{\rho+\varepsilon}) \tag{3.1}$$

where

$$I_\rho(x,c) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\Gamma(s)}{\Gamma(s+\rho+1)} L_{f\otimes g}(s,\chi) x^{s+\rho}\, ds.$$

This is shown by a way similar to the proof of Perron's formula. The Phragmén-Lindelöf theorem implies $L_{f\otimes g}(s,\chi) \ll (d(|t|+1))^{2(1-\sigma+\varepsilon)}$ in $-\varepsilon \le \sigma \le 1 + \varepsilon$. By using this estimate and the residue theorem, we obtain

$$I_\rho(x,c) = \frac{x^\rho}{\Gamma(\rho+1)} L_{f\otimes g}(0,\chi) + J_\rho(x) \\ + O(x^{1+\rho+\varepsilon}T^{-1-\rho} + x^{\rho-\varepsilon}d^{2+4\varepsilon}T^{1-\rho+4\varepsilon}), \tag{3.2}$$

where

$$J_\rho(x) = \frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT} \frac{\Gamma(s)}{\Gamma(s+\rho+1)} L_{f\otimes g}(s,\chi)x^{s+\rho}\,ds.$$

By using the functional equation (2.1), we find

$$J_\rho(x) = \frac{C_\chi}{2\pi i}\left(\frac{2\pi}{d}\right)^2 x^{1+\rho}\sum_{n=1}^{\infty}\bar{c}_n K_\rho(n), \tag{3.3}$$

where

$$K_\rho(n) = \int_{-\varepsilon-iT}^{-\varepsilon+iT} \frac{\Gamma(s)\Gamma(1-s+(k-l)/2)\Gamma(-s+(k+l)/2)}{\Gamma(s+1+\rho)\Gamma(s+(k-l)/2)\Gamma(s+(k+l)/2-1)}\left(\frac{16\pi^4 xn}{d^4}\right)^{s-1} ds.$$

We divide the sum on the right-hand side of (3.3) into $n \le N$ and $n > N$. We put $T = 2\pi d^{-1}(x+(N+1/2))^{1/4}$. Using Stirling's estimate and the first derivative test (see Lemma 2.1 of Ivić [**5**]), we have

$$K_\rho(n) \ll \left(\frac{d^4}{nx}\right)^{1+\varepsilon^*} T^{1-\rho+4\varepsilon^*}\left(1+\left(\log\frac{n}{N+1/2}\right)^{-1}\right) + (nx)^{-1-\varepsilon}d^{4+4\varepsilon}T^{1-\rho+4\varepsilon} \tag{3.4}$$

for $n > N$. For obtaining (3.4), we need to move the path of integration of $K_\rho(n)$ to the line segment with vertices $-\varepsilon-1/8-iT$ and $-\varepsilon-1/8+iT$ in the case $1 < \rho \le 3/2$. This change of path produces the last term on the right-hand side of (3.4). In other words, the last error term of (3.4) does not appear in the case $0 \le \rho \le 1$. The estimate (3.4) implies

$$J_\rho(x) = \frac{C_\chi}{2\pi i}x^{1+\rho}\left(\frac{2\pi}{d}\right)^2 \sum_{n\le N}\bar{c}_n K_\rho(n) + O(x^{(1+3\rho)/4}d^{1+\rho}N^{(1-\rho)/4+\varepsilon}). \tag{3.5}$$

In the method of Ivić-Matsumoto-Tanigawa [**6**], moving the path of integration of $K_\rho(n)$ to a suitable path shows that $K_\rho(n)$ is expressed by $f_\rho(16\pi^4 xn)$. And they obtained the truncated Voronoï formula by using (2.6). Similarly, in the case $16\pi^4 xn/d^4 \ge 1$, we use the fact that $K_\rho(n)$ can be expressed by $f_\rho(16\pi^4 xn/d^4)$. We can use (2.6) for $f_\rho(16\pi^4 xn/d^4)$ if $16\pi^4 xn/d^4 \ge 1$. Hence, we divide the sum on the right-hand side of (3.5) into two parts $d^4/16\pi^4 x \le n < N$ and $n < d^4/16\pi^4 x$ in the case $d^4/16\pi^4 x > 1$.

First, we suppose $d^4/16\pi^4 x > 1$. We change the path of integration of $K_\rho(n)$ to the path L which is the oriented polygonal path with vertices $\mu - i\infty$, $\mu - iT$, $-\varepsilon - iT$, $-\varepsilon + iT$, $\mu + iT$ and $\mu + i\infty$, where $1/4 < \mu < 1$. Then we have

$$\begin{aligned} J_\rho(x) &= \frac{C_\chi}{2\pi i} x^{1+\rho} \left(\frac{2\pi}{d}\right)^2 \sum_{d^4/16\pi^4 x \le n \le N} \bar{c}_n \\ &\quad \times \int_L \frac{\Gamma(s)\Gamma(1-2+(k-l)/2)\Gamma(-s+(k+l)/2)}{\Gamma(s+\rho+1)\Gamma(s+(k-l)/2)\Gamma(s+(k+l)/2-1)} \left(\frac{16\pi^4 xn}{d^4}\right)^{s-1} ds \\ &\quad + \frac{C_\chi}{2\pi i} x^{1+\rho} \left(\frac{2\pi}{d}\right)^2 \sum_{n < d^4/16\pi^4 x} \bar{c}_n K_\rho(n) + O(x^{(1+3\rho)/4} d^{1+\rho} N^{(1-\rho)/4+\varepsilon}) \end{aligned} \tag{3.6}$$

by using the first derivative test. The residue theorem yields

$$\begin{aligned} J_\rho(x) &= \frac{C_\chi}{2\pi i} x^{1+\rho} \left(\frac{2\pi}{d}\right)^2 \sum_{d^4/16\pi^4 x \le n \le N} \bar{c}_n \\ &\quad \times \left(\frac{16\pi^4 xn}{d^4}\right)^{-1-\rho} \left\{ f_\rho\left(\frac{16\pi^4 xn}{d^4}\right) + Q_\rho^*\left(\frac{16\pi^4 xn}{d^4}\right) \right\} \\ &\quad + \frac{C_\chi}{2\pi i} x^{1+\rho} \left(\frac{2\pi}{d}\right)^2 \sum_{n < d^4/16\pi^4 x} \bar{c}_n K_\rho(n) + O(x^{(1+3\rho)/4} d^{1+\rho} N^{(1-\rho)/4+\varepsilon}), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} &Q_\rho^*(x) \\ &= \begin{cases} 0 & \text{if } k-l=0,2 \text{ or } \rho=0, \\ \displaystyle\sum_{m=1}^{K} \frac{(-1)^m}{m!} \frac{\Gamma(1+m+(k-l)/2)\Gamma(m+(k+l)/2)x^{\rho-m}}{\Gamma(1+\rho-m)\Gamma(-m+(k-l)/2)\Gamma(-m+(k+l)/2-1)} & \text{otherwise,} \end{cases} \end{aligned}$$

and

$$K = \begin{cases} \min\left\{\rho, \dfrac{k-l}{2} - 1\right\} & \text{if } \rho \in \boldsymbol{N}, \\ \dfrac{k-l}{2} - 1 & \text{if } \rho \notin \boldsymbol{N}. \end{cases}$$

When $n < d^4/16\pi^4 x$, we have

$$\begin{aligned} K_\rho(n) &\ll n^{-1-\varepsilon^*} x^{-(3+\rho)/4} d^{3+\rho} N^{(1-\rho)/4+\varepsilon^*} + n^{-1-\varepsilon^*} x^{-1-\varepsilon^*} d^{4+4\varepsilon^*} \\ &\quad + n^{-9/8-\varepsilon} x^{-9/8-\varepsilon} d^{9/2+4\varepsilon} T^{3/2-\rho+4\varepsilon} \end{aligned}$$

by using the first derivative test. Here the last term on the right-hand side does not appear in the case $0 \le \rho \le 1$. These results and (2.6) imply

$$J_\rho(x) = \frac{C_\chi}{\sqrt{2\pi}}\left(\frac{2\pi}{d}\right)^{-\rho-1/2} x^{(3+6\rho)/8} \sum_{d^4/16\pi^4 x \le n \le N} \frac{\bar{c}_n}{n^{(5+2\rho)/8}} \sin\left(\frac{8\pi}{d}(xn)^{1/4} + \frac{3-2\rho}{4}\pi\right)$$
$$+ O(x^{(1+6\rho)/8} d^{3/2+\rho} N^{(1-2\rho)/8} + x^\rho d^2$$
$$+ x^{(1+3\rho)/4} d^{1+\rho} N^{(1-\rho)/4+\varepsilon^*} + x^{\rho-\varepsilon^*} d^{2+4\varepsilon^*}).$$

Secondly, we suppose $d^4/16\pi^4 x \le 1$. We can obtain (3.7) in the same way, but the sum for $n < d^4/16\pi^4 x$ does not appear. By using (2.6), we obtain

$$J_\rho(x) = \frac{C_\chi}{\sqrt{2\pi}}\left(\frac{2\pi}{d}\right)^{-\rho-1/2} x^{(3+6\rho)/8} \sum_{d^4/16\pi^4 x \le n \le N} \frac{\bar{c}_n}{n^{(5+2\rho)/8}} \sin\left(\frac{8\pi}{d}(xn)^{1/4} + \frac{3-2\rho}{4}\pi\right)$$
$$+ O(x^{(1+6\rho)/8} d^{3/2+\rho} N^{(1-2\rho)/8})$$
$$+ \begin{cases} O(x^\rho d^2 + x^{(1+3\rho)/4} d^{1+\rho} N^{(1-\rho)/4+\varepsilon^*} + x^{\rho-\varepsilon^*} d^{2+4\varepsilon^*}) & d^4/16\pi^4 x \ge 1 \\ O(x^{(1+3\rho)/4} d^{1+\rho} N^{(1-\rho)/4+\varepsilon}) & d^4/16\pi^4 x < 1. \end{cases}$$

We recall (3.1), (3.2) and $T = 2\pi d^{-1}(x + (N+1/2))^{1/4}$, then we obtain the truncated Voronoï formula for $D_\rho(x)$. □

Remark 1. Ivić-Matsumoto-Tanigawa's truncated Voronoï formula includes the term $Q_\rho(x)$ (see the results (1.5) and Lemma 2 in Ivić-Matsumoto-Tanigawa [**6**]). In our case, we do not separate the term $Q_\rho(x)$ from $J_\rho(x)$ when $d^4/16\pi^4 x > 1$. If we separate $Q_\rho(x)$ from $J_\rho(x)$ in this case, we get a truncated Voronoï formula with an error term which is worse than Proposition 1 with respect to d. In the case $d^4/16\pi^4 x \le 1$ we can obtain the following truncated Voronoï formula:

$$D_\rho(x) = Q_\rho(x) + \frac{C_\chi}{(2\pi)^{1+\rho}} x^{(3+6\rho)/8} d^{1/2+\rho} \sum_{n \le N} \frac{\bar{c}_n}{n^{(5+2\rho)/8}} \sin\left(\frac{8\pi}{d}(xn)^{1/4} + \frac{3-2\rho}{4}\pi\right)$$
$$+ O(x^{(1+6\rho)/8} d^{3/2+\rho} + x^{(3+3\rho)/4+\varepsilon} d^{1+\rho} N^{(1+\rho)/4} + x^\rho d^2)$$
$$+ O(x^{\rho+\varepsilon} + x^{(1+3\rho)/4} d^{1+\rho} N^{(1-\rho)/4+\varepsilon}) \tag{3.8}$$

for large N with $N > \max\{d^4, 16\pi^4\}$. This formula corresponds to Ivić-Matsumoto-Tanigawa's result. The way of obtaining it is as follows. From the definition of $Q_\rho^*(x)$ and (3.7), we find

$$C_\chi\left(\frac{2\pi}{d}\right)^{-2-4\rho}\sum_{n\le N}Q_\rho^*\left(\frac{16\pi^4xn}{d^4}\right)$$

$$=\begin{cases}0 \quad \text{if} \ \ k-l=0,2 \ \text{ or } \ \rho=0,\\ -\displaystyle\sum_{m=1}^{K}\frac{(-1)^m}{m!}\frac{\Gamma(1+m+(k-l)/2)\Gamma(m+(k+l)/2)x^{\rho-m}C_\chi(2\pi/d)^{-4m-2}}{\Gamma(1+\rho-m)\Gamma(-m+(k-l)/2)\Gamma(-m+(k+l)/2-1)}\\ \qquad\times\displaystyle\sum_{n>N}\frac{\bar{c}_n}{n^{1+m}}+\sum_{m=1}^{K}\frac{(-1)^m}{m!}\frac{x^{\rho-m}}{\Gamma(1+\rho-m)}L_{f\otimes g}(-m,\chi) \quad \text{otherwise},\end{cases}$$

$$=\begin{cases}0 \quad \text{if} \ \ k-l=0,2 \ \text{ or } \ \rho=0,\\ \displaystyle\sum_{m=1}^{K}\frac{(-1)^m}{m!}\frac{x^{\rho-m}}{\Gamma(1+\rho-m)}L_{f\otimes g}(-m,\chi)+O(x^{(1+6\rho)/8}d^{3/2+\rho}N^{-1}) \quad \text{otherwise},\end{cases}$$

$$=\begin{cases}Q_\rho(x) \quad \text{if} \ \ k=l,\\ Q_\rho(x)-\dfrac{x^\rho}{\Gamma(\rho+1)}L_{f\otimes g}(0,\chi) \quad \text{if} \ \ \rho=0 \ \text{ or } \ k\ne l,\\ Q_\rho(x)-\dfrac{x^\rho}{\Gamma(\rho+1)}L_{f\otimes g}(0,\chi)+O(x^{(1+6\rho)/8}d^{3/2+\rho}N^{-1}) \quad \text{otherwise}.\end{cases} \tag{3.9}$$

The formula (3.8) is shown from (2.6), (3.7) and (3.9), which holds in the case $d^4/16\pi^4x>1$.

4. The estimate on $D_1(x)$.

The truncated Voronoï formula and the Voronoï formula of Meurman-type (which means the type introduced by Meurman [**8**]) give a strong mean value result which is in Theorem 2. Before investigating the Voronoï formula of Meurman-type for $D_1(x)$, for a preparation we have to obtain an upper bound of $D_1(x)$.

Rankin's result [**10**] implies the estimate of $D_1(x)$ in the non-twisted case (see Ivić-Matsumoto-Tanigawa [**6**]). The same estimate can also be obtained by using another method introduced by Landau and Walfisz. (This method was used in the proof of (1.2). We also use this method in Lemma 1 in this paper.) However, in the present twisted case, we cannot obtain the desired estimate of $D_1(x)$ by using only the Landau-Walfisz method or (1.2). This is one of the complications in the twisted case.

We describe the outline of the story how to obtain an estimate of $D_1(x)$. Hafner's Voronoï formula will yield $D_1(x)\ll x^{6/5+\varepsilon}d^{8/5+\varepsilon}$, by using the Landau-

Walfisz method, except for the case $d^2 \le x \le d^4/16\pi^4$. In the excepted case, we will obtain the same estimate as above by using a certain mean value formula for $D_1(x)$ (see Lemma 2 below) which will be deduced from the truncated Voronoï formula.

We can say that the study of the estimate of $D_\rho(x)$ with respect to d becomes more difficult as ρ grows in the twisted case. Actually, we can obtain (1.2) by using only Hafner's Voronoï formula with the Landau-Walfisz method, but it is not sufficient to obtain a good estimate of $D_1(x)$.

First, we estimate $D_1(x)$ by using the Landau-Walfisz method.

Lemma 1. *We have* $D_1(x) \ll x^{6/5}d^{8/5+\varepsilon}$ *except for the case* $d^2 \le x < d^4/16\pi^4$. *In particular, we have* $D_1(x) \ll x^{6/5}d^{8/5}$ *in the case* $x \ll d^2$.

Proof of Lemma 1. We define the operator Δ_τ as follows;

$$\Delta_\tau(h(x)) = h(x+\tau) - h(x),$$

where $0 < \tau \le x$ and $h(x)$ is a function. From (2.2), we can see

$$\Delta_\tau(D_2(x)) = \Delta_\tau(Q_2(x)) + \sum_{n=1}^{\infty} \frac{4\pi^2 d^{-2} C_\chi \overline{c_n}}{(16\pi^4 d^{-4} n)^3} \Delta_\tau\left(f_2\left(\frac{16\pi^4 xn}{d^4}\right)\right). \tag{4.1}$$

The definitions of D_ρ and Q_ρ imply

$$\Delta_\tau(D_2(x)) = \int_x^{x+\tau} D_1(v)\,dv \tag{4.2}$$

$$= \tau D_1(x) + \frac{\tau^2}{2} D_0(x) + \int_x^{x+\tau} \sum_{x<n\le x} c_n(x-n)\,dv$$

$$= \tau D_1(x) + \frac{\tau^2}{2} D_0(x) + O(\tau^3) \tag{4.3}$$

and

$$\Delta_\tau(Q_2(x)) = \int_x^{x+\tau} Q_1(v)\,dv \tag{4.4}$$

$$= \begin{cases} 0 & \text{if } k = l, \\ 2^{-1}L_{f\otimes g}(0,\chi)(2x\tau+\tau^2) & \text{if } k-l=2, \\ 2^{-1}L_{f\otimes g}(0,\chi)(2x\tau+\tau^2) - \tau L_{f\otimes g}(-1,\chi) & \text{otherwise.} \end{cases} \tag{4.5}$$

As for the remaining part of (4.1), we can also obtain the expression of the same type as (4.2) and (4.4). By using the mean value theorem, we find that there is a real number ξ in $[x, x+\tau]$ satisfying

$$\Delta_\tau\left(f_2\left(\frac{16\pi^4 xn}{d^4}\right)\right) = \left(\frac{2\pi}{d}\right)^2 n\tau f_1\left(\frac{16\pi^4 \xi n}{d^4}\right). \tag{4.6}$$

The relations (4.3), (4.5) and (4.6) yield

$$\begin{aligned} D_1(x) &= -\frac{\tau}{2}D_0(x) + O(\tau^2) \\ &\quad + \sum_{n\le M}\frac{4\pi^2 d^{-2}C_\chi\overline{c_n}}{(16\pi^4 d^{-4}n)^2} f_1\left(\frac{16\pi^4\xi n}{d^4}\right) + \sum_{n>M}\frac{4\pi^2 d^{-2}C_\chi\overline{c_n}\tau^{-1}}{(16\pi^4 d^{-4}n)^3}\Delta_\tau\left(f_2\left(\frac{16\pi^4 xn}{d^4}\right)\right) \\ &\quad + \begin{cases} 0 & \text{if } k=l, \\ 2^{-1}L_{f\otimes g}(0,\chi)(2x+\tau) & \text{if } k-l=2, \\ 2^{-1}L_{f\otimes g}(0,\chi)(2x+\tau) - L_{f\otimes g}(-1,\chi) & \text{otherwise.} \end{cases} \end{aligned} \tag{4.7}$$

If $d^2 \le x$ and $x \ge d^4/16\pi^4$, we can apply (2.5) to (4.7) and we obtain

$$\sum_{n\le M}\frac{4\pi^2 d^{-2}C_\chi\overline{c_n}}{(16\pi^4 d^{-4}n)^2} f_1\left(\frac{16\pi^4\xi n}{d^4}\right) \ll d^{3/2}x^{9/8}M^{1/8}$$

and

$$\sum_{n>M}\frac{4\pi^2 d^{-2}C_\chi\overline{c_n}\tau^{-1}}{(16\pi^4 d^{-4}n)^3}\Delta_\tau\left(f_2\left(\frac{16\pi^4 xn}{d^4}\right)\right) \ll d^{5/2}x^{15/8}M^{-1/8}\tau^{-1}.$$

These estimates yield

$$\begin{aligned} D_1(x) &= -\frac{\tau}{2}D_0(x) + O(\tau^2 + \tau^{-1/2}x^{3/2}d^2) \\ &\quad + \begin{cases} 0 & \text{if } k=l, \\ 2^{-1}L_{f\otimes g}(0,\chi)(2x+\tau) & \text{if } k-l=2, \\ 2^{-1}L_{f\otimes g}(0,\chi)(2x+\tau) - L_{f\otimes g}(-1,\chi) & \text{otherwise,} \end{cases} \end{aligned} \tag{4.8}$$

where we put $M = x^3 d^4 \tau^{-4}$. Putting $\tau = x^{3/5}d^{4/5}$, we obtain the estimate in the statement of Lemma 1 by using (4.8) and (1.2) in the case $d^2 \le x$ and $x \ge d^4/16\pi^4$. If $x \ll d^2$, we can prove Lemma 1 by using (2.7). □

The following Lemma 2 is necessary for obtaining a good estimate on $D_1(x)$ in $d^2 \le x < d^4/16\pi^4$.

Lemma 2. *We have*

$$\int_0^X |D_1(x)|^2\,dx = \frac{2d^3X^{13/4}}{13(2\pi)^4}\sum_{n=1}^\infty \frac{|c_n|^2}{n^{7/4}} + O(d^{7/2+\varepsilon}X^{25/8+\varepsilon} + d^{4+\varepsilon}X^{3+\varepsilon}).$$

Proof of Lemma 2. From Proposition 1, we know

$$D_1(x) = xL_{f\otimes g}(0,\chi) + \frac{C_\chi d^{3/2}x^{9/8}}{(2\pi)^2}\sum_{n\le N}\frac{\overline{c_n}}{n^{7/8}}\sin\left(\frac{8\pi(xn)^{1/4}}{d}+\frac{\pi}{4}\right)$$
$$+ O(d^{5/2}x^{7/8}N^{-1/8} + d^2xN^\varepsilon + d^2x^{3/2+\varepsilon}N^{-1/2} + x^{1+\varepsilon}), \qquad (4.9)$$

where $N \ge d^4$. Using the fact $L_{f\otimes g}(0,\chi) \ll d^{2+\varepsilon}$ which can be shown by the Phragmén-Lindelöf theorem and $N \ge d^4$, we can obtain

$$D_1(x) = \delta_N(x) + E_N^*(x),$$

where

$$\delta_N(x) = \frac{C_\chi d^{3/2}x^{9/8}}{(2\pi)^2}\sum_{n\le N}\frac{\overline{c_n}}{n^{7/8}}\sin\left(\frac{8\pi(xn)^{1/4}}{d}+\frac{\pi}{4}\right) \qquad (4.10)$$

and

$$E_N^*(x) = O(d^{5/2}x^{7/8}N^{-1/8} + d^2xN^\varepsilon + d^2x^{3/2+\varepsilon}N^{-1/2} + x^{1+\varepsilon}).$$

We put $N = Axd^4$, where A is a positive constant which is independent of x and d. Then we have

$$E_N^*(x) = O(d^{2+\varepsilon}x^{1+\varepsilon}). \qquad (4.11)$$

We write

$$\int_0^X |D_1(x)|^2\,dx = \sum_{j=1}^{\infty}\int_{2^{-j}X}^{2^{-j+1}X}|D_1(x)|^2\,dx \qquad (4.12)$$

and use

$$\int_Y^{2Y}|D_1(x)|^2\,dx \ll \int_Y^{2Y}|\delta_1(x)|^2\,dx + \int_Y^{2Y}|E_N^*(x)|^2\,dx$$
$$+ \left(\int_Y^{2Y}|\delta_1(x)|^2\,dx\right)^{1/2}\left(\int_Y^{2Y}|E_N^*(x)|^2\,dx\right)^{1/2} \qquad (4.13)$$

which is shown by using the Cauchy-Schwarz inequality. The same argument as in the proof of Theorem 13.5 of Ivić [**5**] yields

$$\int_Y^{2Y}|\delta_1(x)|^2\,dx = \frac{2d^3}{13(2\pi)^4}\sum_{n=1}^{\infty}\frac{|c_n|^2}{n^{7/4}}((2Y)^{13/4} - Y^{13/4}) + O(d^4Y^3N^\varepsilon). \qquad (4.14)$$

Therefore we obtain

$$\int_Y^{2Y} |D_1(x)|^2\,dx = \frac{2d^3}{13(2\pi)^4}\sum_{n=1}^{\infty}\frac{|c_n|^2}{n^{7/4}}((2Y)^{13/4} - Y^{13/4})$$
$$+ O(d^{7/2+\varepsilon}Y^{25/8+\varepsilon} + d^{4+\varepsilon}Y^{3+\varepsilon})$$

by using (4.11), (4.13) and (4.14). This result and (4.12) imply Lemma 2. □

Lemma 3. *In the case* $d^2 \le x < d^4/16\pi^4$, *we have*

$$D_1(x) \ll x^{6/5}d^{8/5+\varepsilon}.$$

Proof of Lemma 3. We have

$$D_1(x) = \frac{1}{H}\int_x^{x+H} D_1(t)\,dt - \frac{1}{H}\int_x^{x+H}\int_x^t D_0(u)\,dudt$$

and $D_0(u) \ll u^{3/5}d^{4/5+\varepsilon}$ from (1.2). These facts imply

$$|D_1(x)|^2 \le \left\{\frac{1}{H}\int_x^{x+H}|D_1(t)|\,dt + O(Hx^{3/5}d^{4/5+\varepsilon})\right\}^2$$
$$\le \left\{\left(\int_x^{x+H} H^{-2}\,dt\right)^{1/2}\left(\int_x^{x+H}|D_1(t)|^2\,dt\right)^{1/2} + O(Hx^{3/5}d^{4/5+\varepsilon})\right\}^2$$
$$\ll \frac{1}{H}\int_x^{x+H}|D_1(t)|^2\,dt + O(H^2x^{6/5}d^{8/5+\varepsilon}).$$

By using Lemma 2 and putting $H = x^{3/5}d^{4/5}$, we obtain

$$|D_1(x)|^2 \ll d^{27/10+\varepsilon}x^{101/40+\varepsilon} + d^{16/5+\varepsilon}x^{12/5+\varepsilon}.$$

Therefore, under the condition $d^2 \le x < d^4/16\pi^4$, we can see that $D_1(x) \ll x^{6/5}d^{8/5+\varepsilon}$. □

5. The Voronoï formula of Meurman-type for $D_1(x)$.

We have now obtained good estimates of $D_1(x)$ for any case, from Lemmas 1 and 3. By using these results, we prove the Voronoï formula of Meurman-type for $D_1(x)$ as follows:

Proposition 2. *Let* $x \ge 1$ *which is not an integer, and* M *is a positive number not less than* d^4. *Then we have*

$$D_1(x) = xL_{f\otimes g}(0,\chi) + \frac{C_\chi d^{3/2}x^{9/8}}{(2\pi)^2}\sum_{n\le M}\frac{\overline{c_n}}{n^{7/8}}\sin\left(\frac{8\pi(xn)^{1/4}}{d}+\frac{\pi}{4}\right)$$
$$+ O(d^2x^{3/2+\varepsilon}M^{-1/2}\|x\|^{-1} + d^{3/2+\varepsilon}x^{13/8}M^{-3/8+\varepsilon})$$
$$+ O(d^{21/10+\varepsilon}x^{11/8}M^{-17/40} + d^{23/10+\varepsilon}x^{9/8}M^{-11/40})$$
$$+ O(d^{31/10+\varepsilon}x^{9/8}M^{-27/40} + d^{5/2}x^{7/8} + d^2x^{13/8}M^{-1/2})$$
$$+\begin{cases} 0 & \text{if } d^4/16\pi^4 < 1 \text{ and } k-l = 0,2,\\ O(d^6) & \text{if } d^4/16\pi^4 < 1 \text{ and } k-l \ne 0,2,\\ O(d^2xM^\varepsilon) & \text{otherwise,}\end{cases}$$

where $\|x\|$ means the distance between x and the nearest integer from x. Note that $L_{f\otimes g}(0,\chi) = 0$ if $k = l$.

Proof of Proposition 2. Hafner's Voronoï formula (2.2) implies

$$D_1(x) = Q_1(x) + \sum_{n=1}^{\infty}\frac{(2\pi d^{-1})^2 C_\chi \overline{c_n}}{(16\pi^4 d^{-4}n)^2} f_1\left(\frac{16\pi^4 xn}{d^4}\right). \tag{5.1}$$

First, we consider the case $d^4/16\pi^4 x > 1$. We divide the infinite sum of (5.1) into three parts which are the sums over $n < d^4/16\pi^4$, $d^4/16\pi^4 \le x < M$ and $M < n$. We move the path of integration of f_1 in the sum over $n < d^4/16\pi^4$. The new path is $C_{0,1+\varepsilon}$. The residue theorem and the definition of $Q_\rho(x)$ imply

$$Q_1(x) + \sum_{n<d^4/16\pi^4 x}\frac{C_\chi \bar{c}_n}{(2\pi d^{-1})^6 n^2} f_1\left(\frac{16\pi^4 xn}{d^4}\right)$$
$$=\begin{cases} S(k,l,\chi) \quad \text{if } k-l=0,\\ xL_{f\otimes g}(0,\chi) + S(k,l,\chi) \quad \text{if } k-l=2,\\ xL_{f\otimes g}(0,\chi) + S(k,l,\chi) \\ \quad + \displaystyle\sum_{n\ge d^4/16\pi^4 x}\frac{C_\chi \bar{c}_n d^6}{(2\pi)^6 n^2}\frac{\Gamma(2+(k-l)/2)\Gamma(1+(k+l)/2)}{\Gamma(-1+(k-l)/2)\Gamma(-2+(k+l)/2)} \quad \text{otherwise,}\end{cases} \tag{5.2}$$

where

$$S(k,l,\chi) = \sum_{n<d^4/16\pi^4 x}\frac{C_\chi \bar{c}_n d^6}{(2\pi)^6 n^2}\frac{1}{2\pi i}\int_{C_{0,1+\varepsilon}}\frac{\Gamma(1-s)\Gamma(1+(k-l)/2)\Gamma(s+(k+l)/2-1)}{\Gamma(3-s)\Gamma(1-s+(k-l)/2)\Gamma(-s+(k+l)/2)} \times \left(\frac{16\pi^4 xn}{d^4}\right)^{2-s} ds.$$

Secondly, we estimate the right-hand side of (5.2) and we obtain

$$D_1(x) = xL_{f\otimes g}(0,\chi) + \sum_{d^4/16\pi^4 x \le n} \frac{C_\chi \bar{c}_n}{n^2}\left(\frac{2\pi}{d}\right)^{-6} f_1\left(\frac{16\pi^4 xn}{d^4}\right)$$
$$+\begin{cases} 0 & \text{if } d^4/16\pi^4 x < 1 \text{ and } k-l = 0,2, \\ O(d^6) & \text{if } d^4/16\pi^4 x < 1 \text{ and } k-l \neq 0,2, \\ O(d^2 x M^{\varepsilon}) & \text{otherwise.} \end{cases} \tag{5.3}$$

Actually, in the proof of Proposition 1, we have already estimated each part of integrals in (5.2). Therefore we do not describe the details of the argument for obtaining (5.3). By using (2.6), we have

$$D_1(x) = \delta_M(x) + E_M^{**}(x),$$

where $\delta_M(x)$ is defined in (4.10) and

$$E_M^{**}(x) = \frac{C_\chi d^{3/2} x^{9/8}}{(2\pi)^2} \sum_{n>M} \frac{\bar{c}_n}{n^{7/8}} \sin\left(\frac{8\pi(xn)^{1/4}}{d} + \frac{\pi}{4}\right) + O(d^{5/2} x^{7/8})$$
$$+\begin{cases} 0 & \text{if } d^4/16\pi^4 x < 1 \text{ and } k-l = 0,2, \\ O(d^6) & \text{if } d^4/16\pi^4 x < 1 \text{ and } k-l \neq 0,2, \\ O(d^2 x M^{\varepsilon}) & \text{otherwise.} \end{cases} \tag{5.4}$$

We put

$$S = \sum_{n>M} \frac{\bar{c}_n}{n^{7/8}} \sin\left(\frac{8\pi(xn)^{1/4}}{d} + \frac{\pi}{4}\right).$$

Then we obtain

$$S \ll \left|\int_M^\infty \overline{D_0(t)}\left(-\frac{7}{8t^{15/8}} \sin\left(\frac{8\pi(xn)^{1/4}}{d} + \frac{\pi}{4}\right) + \frac{2\pi x^{1/4}}{dt^{13/8}} \cos\left(\frac{8\pi(xn)^{1/4}}{d} + \frac{\pi}{4}\right)\right) dt\right|$$
$$+ M^{-11/40} d^{4/5+\varepsilon} \tag{5.5}$$

by using (1.2) and partial summation. We apply integration by parts to (5.5) with $(d/dx)D_1(x) = D_0(x)$. We already know the estimate of $D_1(x)$. Using this, we obtain

$$S \ll \left|\int_M^\infty \overline{D_1(t)} \frac{x^{1/2} t^{-19/8}}{d^2} \sin\left(\frac{8\pi(xn)^{1/4}}{d} + \frac{\pi}{4}\right) dt\right|$$
$$+ M^{-11/40} d^{4/5+\varepsilon} + M^{-27/40} d^{8/5+\varepsilon} + M^{-17/40} x^{1/4} d^{3/5+\varepsilon}. \tag{5.6}$$

We put

$$S_1 = \int_M^\infty \overline{D_1(t)} t^{-19/8} \sin\left(\frac{8\pi(xn)^{1/4}}{d} + \frac{\pi}{4}\right) dt, \tag{5.7}$$

and we obtain

$$S_1 \ll \left| \int_M^\infty \overline{D_1(t)} t^{-19/8} \exp\left(\frac{8\pi(xn)^{1/4}}{d}\right)\right| + \left| \int_M^\infty \overline{D_1(t)} t^{-19/8} \exp\left(-\frac{8\pi(xn)^{1/4}}{d}\right)\right|. \tag{5.8}$$

We can investigate the right-hand side of (5.8) by the same argument as in Lemma 5 of Ivić-Matsumoto-Tanigawa [**6**]. In their argument, they use their truncated Voronoï formula. In this paper, we have obtained the corresponding formula in Proposition 1. We use it with $N = Yd^4$. Then we find

$$S_1 \ll d^{5/2} M^{-1/2} + d^{5/2} x^{-1/8+\varepsilon} \|x\|^{-1} M^{-1/2} + d^{2+\varepsilon} M^{-3/8+\varepsilon}. \tag{5.9}$$

The estimates (5.6) and (5.9) yield Proposition 2. □

When $d^4/16\pi^4 x \le 1$, we can obtain the following formula. This formula corresponds to Ivić-Matsumoto-Tanigawa's result.

Remark 2. For any large number M with $M \ge d^4$ and any non-integral $x > 1$, we have

$$\begin{aligned} D_1(x) = Q_1(x) &+ \frac{C_\chi d^{3/2} x^{9/8}}{(2\pi)^2} \sum_{n \le M} \frac{\overline{c_n}}{n^{7/8}} \sin\left(\frac{8\pi(xn)^{1/4}}{d} + \frac{\pi}{4}\right) \\ &+ O(d^2 x^{13/8} M^{-1/2} + d^2 x^{3/2+\varepsilon} M^{-1/2} \|x\|^{-1}) \\ &+ O(d^{3/2+\varepsilon} x^{13/8} M^{-3/8+\varepsilon} + d^{5/2} x^{7/8} + d^{23/10+\varepsilon} x^{9/8} M^{-11/40}) \\ &+ O(d^{31/10+\varepsilon} x^{9/8} M^{-27/40} + d^{21/10+\varepsilon} x^{11/8} M^{-17/40}) \end{aligned} \tag{5.10}$$

when $d^4/16\pi^4 x \le 1$. In this case, we can see

$$\begin{aligned} D_1(x) = Q_1(x) + \delta_M(x) &+ \frac{C_\chi d^{3/2} x^{9/8}}{(2\pi)^2} \sum_{n \le M} \frac{\overline{c_n}}{n^{7/8}} \sin\left(\frac{8\pi(xn)^{1/4}}{d} + \frac{\pi}{4}\right) \\ &+ O(d^{5/2} x^{7/8}) \end{aligned} \tag{5.11}$$

from (2.2) and (2.6). This implies (5.10) by the same way as in the proof of Proposition 2 when $d^4/16\pi^4 x < 1$. However, the error term in (5.11) is worse than Proposition 2 with respect to d in the case $d^4/16\pi^4 x \ge 1$.

6. Completion of the proofs of theorems.

Now we proceed to the final stage of the proofs of our theorems. The remaining part of the proof is quite similar to that in Ivić-Matsumoto-Tanigawa [**6**], hence it is sufficient to give a brief sketch.

We can prove Theorem 1 easily by the same argument as in Ivić-Matsumoto-Tanigawa [**6**]. As for Ivić-Matsumoto-Tanigawa's notations H_1 and H_2, they put $H_1 = H_2 = x^{\alpha/2}$, but we put $H_1 = H_2 = x^{\alpha/2}d^{\beta/2}$ in our case.

For the proof of Theorem 2, we recall the argument in the proof of Lemma 2. We note that the important steps are the evaluations of $\int_Y^{2Y} |\delta_N(t)|^2\, dt$ and $\int_Y^{2Y} |E_N(t)|^2\, dt$, where $\delta_N(t)$ is defined in (4.10) and we put $E_N(t) = D_1^*(t) - \delta_N(t)$. The former have been investigated and the result is (4.14). The latter can be estimated by using the truncated Voronoï formula for $D_1(x)$ (see Proposition 1) and the Voronoï formula of Meurman-type for $D_1(x)$ (see Proposition 2) with $N = M = AX^s d^4$, where A is a constant which is independent of x and d. The argument for estimating the latter is the same as that in Ivić-Matsumoto-Tanigawa [**6**]. Then we obtain Theorem 2.

We conclude this paper with the following remarks which correspond to Theorem 3 of Ivić-Matsumoto-Tanigawa [**6**]. We put $D_0^*(t) = D_0(t) - L_{f\otimes g}(0,\chi)$, then we have $(d/dx)D_1^*(x) = D_0^*(x)$. From this relation we have

$$D_1^*(x) = \frac{1}{H}\int_x^{x+H} D_1^*(t)\, dt - \frac{1}{H}\int_x^{x+H}\int_x^t D_0^*(u)\, dudt, \tag{6.1}$$

where $0 < H \ll x$. We know $D_0^*(u) \ll u^{3/5}d^{4/5+\varepsilon}$ from (1.2) if $d^2 \ll x$, because $L_{f\otimes g}(0,\chi) \ll d^{2+\varepsilon}$ which was already mentioned. Then we find that

$$D_1^*(x) = \frac{1}{H}\int_x^{x+H} D_1^*(t)\, dt - \frac{1}{H}\int_x^{x+H}\int_x^t D_0(u)\, dudt + \frac{H}{2}L_{f\otimes g}(0,\chi). \tag{6.2}$$

This equation implies

$$\begin{aligned} &\left|D_1^*(x) - \frac{H}{2}L_{f\otimes g}(0,\chi)\right|^2 \\ &\quad \le \left\{\left(\int_x^{x+H} H^{-2}\, dt\right)^{1/2}\left(\int_x^{x+H} |D_1^*(t)|^2\, dt\right)^{1/2} + O(Hx^{3/5}d^{4/5+\varepsilon})\right\}^2 \\ &\quad \ll \frac{1}{H}\int_x^{x+H} |D_1^*(t)|^2\, dt + O(H^2x^{6/5}d^{8/5+\varepsilon}) \end{aligned} \tag{6.3}$$

by using the Cauchy-Schwarz inequality and (1.2). Theorem 2 yields

$$\left|D_1^*(x) - \frac{H}{2}L_{f\otimes g}(0,\chi)\right|^2$$
$$\ll \frac{1}{H}\left\{\frac{2d^3}{13(2\pi)^4}((x+H)^{13/4} - x^{13/4})\sum_{n=1}^{\infty}\frac{|c_n|^2}{n^{7/4}} + O(x^{3+\varepsilon}d^{4+\varepsilon})\right\} + H^2x^{6/5}d^{8/5+\varepsilon}$$
$$\ll x^{9/4}d^3 + H^{-1}x^{3+\varepsilon}d^{4+\varepsilon} + H^2x^{6/5}d^{8/5+\varepsilon}. \tag{6.4}$$

We put $H = x^{3/5}d^{4/5}$ (here, the relation $H \ll x$ implies $d^2 \ll x$), and we obtain

$$\left|D_1^*(x) - \frac{H}{2}L_{f\otimes g}(0,\chi)\right|^2 \ll x^{12/5+\varepsilon}d^{16/5+\varepsilon}.$$

This yields

$$D_1(x) \ll x^{6/5+\varepsilon}d^{8/5+\varepsilon},$$

when $d^2 \ll x$. This result combined with Theorem 1 does not improve (1.2), similar to the case of Ivić-Matsumoto-Tanigawa [**6**]. However, the argument analogous to them gives us the following observation by using Theorem 2. If one can obtain

$$\int_0^x |D_1^*(t)|^2\,dt = \frac{2d^3}{13(2\pi)^4}\sum_{n=1}^{\infty}\frac{|c_n|^2}{n^{7/4}}x^{13/4} + F(x),$$

and

$$F(x) = d^4x^3P(\log xd) + O(x^\alpha d^\beta), \tag{6.5}$$

where $P(x)$ is a polynomial of degree m and α is a real number satisfying $\alpha < 3$, then one can see

$$D_1^*(x) \ll x^{9/4}d^3 + x^2d^4(\log xd)^m + H^{-1}x^\alpha d^\beta + H^2x^{6/5}d^{8/5+\varepsilon}$$

under the condition $d^2 \ll x$ by using the same method as (6.1)–(6.4). In fact, the condition $d^2 \ll x$ and $L_{f\otimes g}(0,\chi) \ll d^{2+\varepsilon}$ imply $L_{f\otimes g}(0,\chi) \ll x^{3/5}d^{4/5+\varepsilon}$. Then this estimate and (1.2) imply $D_0^*(x) \ll x^{3/5}d^{4/5+\varepsilon}$. Therefore we have the above estimate.

We put $H = x^{(5\alpha-6)/15}d^{(5\beta-8)/15}$ (here the relation $H \ll x$ implies $d^{(5\beta-8)/15} \ll x^{(21-5\alpha)/15}$), and we obtain

$$D_1(x) \ll x^{9/8}d^{3/2} + x^{1+\varepsilon}d^{2+\varepsilon} + x^{(3+5\alpha)/15}d^{(4+5\beta)/15}, \tag{6.6}$$

under the condition $d^{(5\beta-8)/15} \ll x^{(21-5\alpha)/15}$ and $d^2 \ll x$. This estimate suggests a possibility of an improvement of (1.2) with respect to both x and d.

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