J. Math. Soc. Japan
Vol. 55, No. 1, 2003

On the level by level equivalence between strong compactness and strongness

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(Received Jan. 4, 2000)
(Revised Jun. 13, 2001)

Abstract. We construct a model in which the least strongly compact cardinal κ is also the least strong cardinal, κ isn't 2^κ supercompact, and for any $\delta < \kappa$, if $\delta^{+\alpha}$ is regular, δ is $\delta^{+\alpha}$ strongly compact if and only if δ is $\delta + \alpha + 1$ strong.

1. Introduction and preliminaries.

In [**4**], Shelah and the author extended and generalized the work of Kimchi and Magidor [**13**] by proving the following theorem.

THEOREM 1. *Let* $V \models$ "ZFC + GCH + $\mathfrak{K} \neq \varnothing$ *is the class of supercompact cardinals*". *There is then a cardinal and cofinality preserving generic extension* $V[G] \models$ "ZFC + GCH + *If* $\kappa \leq \lambda$ *are cardinals*, *then* κ *is* λ *supercompact if and only if* κ *was* λ *supercompact in* V (*hence* $\mathfrak{K}$ *is the class of supercompact cardinals*) + *If* $\kappa \leq \lambda$ *are regular cardinals*, *then* κ *is* λ *strongly compact if and only if* κ *is* λ *supercompact*, *except possibly if* κ *is a measurable limit of cardinals* δ *which are* λ *supercompact*".

In the special case that there is only one supercompact cardinal in V and no cardinal is supercompact up to a measurable cardinal, Theorem 1 can be restated as follows.

THEOREM 2. *Let* $V \models$ "ZFC + GCH + κ_0 *is the unique supercompact cardinal*". *There is then a cardinal and cofinality preserving generic extension* $V[G] \models$ "ZFC + GCH + κ_0 *is the unique supercompact cardinal* + *If* $\kappa \leq \lambda$ *are regular cardinals*, *then* κ *is* λ *strongly compact if and only if* κ *is* λ *supercompact*".

We take this opportunity to make two remarks concerning Theorem 1 (and its restatement as Theorem 2). The first is that the above statements are the "modern" statements of these theorems, taking into account Hamkins' gap forcing

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2000 *Mathematics Subject Classifications*. 03E35, 03E55.
The author wishes to thank Joel Hamkins for helpful conversations on the subject matter of this paper. In addition, the author wishes to express his gratitude to the referee, for the many helpful comments, corrections, and suggestions that have been incorporated into this version of the paper.

work begun in [**8**] and extended in [**9**] and [**10**]. Also, the terminology we use in models which witness the conclusions of Theorems 1 and 2 is that there is a level by level equivalence or level by level correspondence between strong compactness and supercompactness, except, in the case of Theorem 1, at measurable limits. This exception occurs because of the result of Menas [**16**], which shows that measurable limits of λ supercompact or λ strongly compact cardinals are λ strongly compact but need not be λ supercompact.

In [**3**], Cummings and the author constructed a model containing a proper class of strongly compact cardinals in which the classes of strongly compact and strong cardinals coincide precisely. Specifically, the following theorem was proven.

Theorem 3. Con(ZFC + *There is a proper class of supercompact cardinals*) $\Rightarrow$ Con(ZFC + GCH + *There is a proper class of strongly compact cardinals* + *No strongly compact cardinal* κ *is* 2^κ *supercompact* + $\forall\kappa$[κ *is strongly compact if and only if* κ *is strong*]).

The purpose of this paper is to combine Theorems 2 and 3 to produce a model containing a non-supercompact cardinal κ which is both the least strongly compact and least strong cardinal so that, roughly speaking, at regular levels, there is a precise correspondence between the notions of strong compactness and strongness. Specifically, we prove the following theorem.

Theorem 4. *Let* $V \models$ "ZFC + κ *is supercompact* + *No cardinal* $\lambda > \kappa$ *is measurable* + *There is no pair of cardinals* $\delta < \lambda$ *so that* δ *is* λ *supercompact and* λ *is measurable*". *There is then a partial ordering* $\mathbf{P} \subseteq V$ *so that* $V^{\mathbf{P}} \models$ "ZFC + GCH + *No cardinal* $\lambda > \kappa$ *is measurable* + *There is no pair of cardinals* $\delta < \lambda$ *so that* δ *is* λ *supercompact and* λ *is measurable* + κ *is both the least strongly compact and least strong cardinal* + κ *isn't* 2^κ *supercompact* + *For* $\delta < \kappa$, *if* $\delta^{+\alpha}$ *is regular, then* δ *is* $\delta^{+\alpha}$ *strongly compact if and only if* δ *is* $\delta + \alpha + 1$ *strong*". *If we assume in addition that* GCH *and level by level equivalence between strong compactness and supercompactness hold in* V, *then* $\mathbf{P}$ *can be defined so that* $\mathbf{P} \in V$, $|\mathbf{P}| = \kappa$, *and forcing with* $\mathbf{P}$ *preserves cardinals and cofinalities.*

We digress briefly to mention that if $\alpha < \beta$ are ordinals, then $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$ and (α, β) are as in standard interval notation. Also, we are assuming familiarity with the large cardinal notions of measurability, strongness, superstrongness, strong compactness, and supercompactness. Interested readers may consult [**12**], [**15**], or [**17**] for further details. We mention only that unlike [**12**], we will say that the cardinal κ is λ strong for $\lambda > \kappa$ if there is $j : V \to M$ an elementary embedding having critical point κ so that $j(\kappa) > |V_\lambda|$ and $V_\lambda \subseteq M$. If $|V_\lambda|$ is regular, then we may assume that $M^\kappa \subseteq M$ as well. As always, κ is strong if κ is λ strong for every $\lambda > \kappa$. We will also say the cardinal κ is superstrong with target λ if there is $j : V \to M$ an elementary embedding having critical point κ so that $j(\kappa) = \lambda$, $M^\kappa \subseteq M$, and $V_\lambda \subseteq M$.

We note that we are also assuming some familiarity with the basics of extender technology and the transference of generic objects via elementary embeddings. The section on background material of [**7**] is extremely useful in this regard. Readers may also consult [**15**] for additional details concerning extenders.

Returning from our digression, let us observe that assuming GCH, if δ is $\delta^{+\alpha}$ supercompact for any ordinal $\alpha \geq 0$, then δ is $\delta + \alpha + 1$ strong. The converse is not true, since by Lemma 2.1 of [**3**] and Proposition 26.11 of [**12**], if δ is 2^δ supercompact, then δ has a normal measure concentrating on cardinals γ which are superstrong with target δ. Hence, we have the following easy lemma as an immediate corollary.

Lemma 1.1. *If $\delta < \sigma$ are so that δ is 2^δ supercompact and σ is the least 2^σ supercompact cardinal above δ, then there are unboundedly in σ many cardinals $\eta \in (\delta, \sigma)$ so that η isn't η^+ supercompact yet η is η' strong for every $\eta' \in (\eta, \sigma)$.*

Thus, given the above discussion, it makes sense to use the equivalence found in the statement of Theorem 4 as our definition of what it means for there to be a level by level correspondence at regular levels between the notions of strong compactness and strongness.

We note that for reasons similar to those given in [**4**], if there is a supercompact cardinal κ in the universe, or even if there is a cardinal κ in the universe which is only both strongly compact and strong, there cannot be a precise level by level equivalence in the above sense between the notions of strong compactness and strongness. Specifically, if $\delta^{+\alpha}$ is singular, it won't necessarily be the case that δ is $\delta^{+\alpha}$ strongly compact if and only if δ is $\delta + \alpha + 1$ strong. The following lemma, an analogue to a result of Magidor which is found as Lemma 7 of [**4**], makes this precise. For simplicity in presentation, we assume GCH, and we leave it to readers to recast this lemma suitably in the context of the negation of GCH.

Lemma 1.2. *Suppose κ is strongly compact and strong. Then $B = \{\delta < \kappa : \delta$ is $\delta^{+\delta}$ strongly compact and δ isn't $\delta + \delta$ strong$\}$ is unbounded in κ.*

Proof. The proof of Lemma 1.2 is essentially a modification of the proof of Lemma 7 of [**4**] in the context of strongness. Since κ is a strong cardinal, by choosing $j(\kappa)$ to be minimal, let $j : V \to M$ be an elementary embedding witnessing the $\kappa + \kappa$ strongness of κ so that $M \vDash$ "κ isn't $\kappa + \kappa$ strong". By GCH and the fact that $V_{\kappa+\kappa} \subseteq M$, $M \vDash$ "κ is $\kappa^{+\gamma}$ strongly compact for all $\gamma < \kappa$".

Let $\mu \in V$ be a κ-additive measure over κ, and let $\gamma < \kappa$. Since $V_{\kappa+\kappa} \subseteq M$, we know $\mu \in M$ and $\kappa^{+\gamma}$ has the same meaning in both V and M. Since $M \models$ "κ is $\kappa^{+\gamma}$ strongly compact for all $\gamma < \kappa$", we can find a sequence $\langle \mu_\gamma : \gamma < \kappa \rangle \in M$ so that $M \models$ "μ_γ is a κ-additive, fine ultrafilter over $P_\kappa(\kappa^{+\gamma})$". Thus, we can define in M the collection μ^* of subsets of $P_\kappa(\kappa^{+\kappa})$ by $A \in \mu^*$ if and only if $\{\gamma < \kappa : A \cap P_\kappa(\kappa^{+\gamma}) \in \mu_\gamma\} \in \mu$. It is easily checked that μ^* defines in M a κ-additive, fine ultrafilter over $P_\kappa(\kappa^{+\kappa})$. Hence, $M \models$ "κ is $\kappa^{+\kappa}$ strongly compact yet κ isn't $\kappa + \kappa$ strong", so by reflection, the set B is unbounded in κ. This proves Lemma 1.2. $\square$

We remark that in the context of GCH, $\delta^{+\delta}$ is the least singular strong limit cardinal of cofinality δ above δ. Also, the proof of Lemma 1.2 has been written to show we may infer, e.g., that $\{\delta < \kappa : \delta$ is $\delta^{+\delta^+}$ strongly compact and δ isn't $\delta + \delta^+ = \delta^+$ strong$\}$ is unbounded in κ. This is done by letting μ be a uniform κ-additive measure over κ^+ (which is possible since κ is κ^+ strongly compact) and then following the appropriate modification of the proof just given. This is in analogy to the proof of Lemma 7 of [**4**].

We return now to a discussion of preliminary material. When forcing, $q \geq p$ will mean that q is stronger than p. If $\boldsymbol{P}$ is our partial ordering, $V^{\boldsymbol{P}}$ and $V[G]$ will be used interchangeably to denote the generic extension when forcing with $\boldsymbol{P}$.

The partial ordering $\boldsymbol{P}$ is κ-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages (choosing the trivial condition at stage 0), then player II has a strategy which ensures the game can always be continued. Note that if $\boldsymbol{P}$ is κ-strategically closed and $f : \kappa \to V$ is a function in $V^{\boldsymbol{P}}$, then $f \in V$. $\boldsymbol{P}$ is $\prec\kappa$-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, then player II has a strategy which ensures the game can always be continued.

Suppose that $\kappa < \lambda$ are regular cardinals. A partial ordering $\boldsymbol{P}(\kappa, \lambda)$ that will be used in forcing iterations throughout the course of this paper is the partial ordering for adding a non-reflecting stationary set of ordinals of cofinality κ to λ. Specifically, $\boldsymbol{P}(\kappa, \lambda)$ is defined as $\{p$: For some $\alpha < \lambda$, $p : \alpha \to \{0, 1\}$ is a characteristic function of S_p, a subset of α not stationary at its supremum nor having any initial segment which is stationary at its supremum, so that $\beta \in S_p$ implies $\beta > \kappa$ and $\operatorname{cof}(\beta) = \kappa\}$, ordered by $q \geq p$ if and only if $q \supseteq p$ and $S_p = S_q \cap \sup(S_p)$, i.e., S_q is an end extension of S_p. It is well-known that for G V-generic over $\boldsymbol{P}(\kappa, \lambda)$ (see [**6**] or [**13**]), in $V[G]$, if we assume GCH holds in V, a non-reflecting stationary set $S = S[G] = \bigcup\{S_p : p \in G\} \subseteq \lambda$ of ordinals of cofinality κ has been introduced, the bounded subsets of λ are the same as those in V, and cardinals, cofinalities, and GCH have been preserved. It can be shown (see the proof of Lemma 4.15 given on page 436 of [**6**] or the proofs of Lemmas 1.1 and 1.3 of [**1**] and the remark immediately following the proof of Lemma 1.3 of [**1**]) that $\boldsymbol{P}(\kappa,\lambda)$ is $\prec\lambda$-strategically closed. And, whenever a forcing iteration $\boldsymbol{P}$ has as one of its components a partial ordering of the form $\boldsymbol{P}(\kappa,\lambda)$ for some regular κ and λ, we will say that λ is in the domain of $\boldsymbol{P}$.

2. The Proof of Theorem 4.

We turn now to the proof of Theorem 4.

Proof. Let $V \models$ "ZFC $+ \kappa$ is supercompact $+$ No cardinal $\lambda > \kappa$ is measurable $+$ There is no pair of cardinals $\delta < \lambda$ so that δ is λ supercompact and λ is measurable". Without loss of generality, by first doing a preliminary class forcing if necessary, we may also assume that $V \models$ GCH and that V is as in [**4**], i.e., for $\delta < \lambda < \kappa$, δ and λ both regular, $V \models$ "δ is λ strongly compact if and only if δ is λ supercompact".

Let $\langle \delta_\alpha : \alpha < \kappa \rangle$ enumerate the set A of measurable cardinals below κ having the properties that for every $\delta \in A$, there is some β so that $\delta^{+\beta}$ is regular, δ is $\delta + \beta + 1$ strong, and δ isn't $\delta^{+\beta}$ supercompact. By Lemma 1.1, A is unbounded in κ. Since these are the cardinals at which the desired level by level equivalence between strong compactness and strongness will fail in the generic extension, our goal will be to destroy every element of A. The partial ordering $\boldsymbol{P}$ that will do this is the Easton support iteration $\langle\langle \boldsymbol{P}_\alpha, \dot{\boldsymbol{Q}}_\alpha \rangle : \alpha < \kappa \rangle$ of length κ so that $\boldsymbol{P}_0$ is the partial ordering for adding a Cohen subset to ω and $\boldsymbol{P}_{\alpha+1} = \boldsymbol{P}_\alpha * \dot{\boldsymbol{Q}}_\alpha$, where $\dot{\boldsymbol{Q}}_\alpha$ is a term for $\boldsymbol{P}(\omega, \delta_\alpha)$. It is easily shown by an induction similar to the one given in Lemma 8 of [**4**] that $V^{\boldsymbol{P}} \models$ GCH and forcing with $\boldsymbol{P}$ preserves cardinals and cofinalities.

Lemma 2.1. *$V^{\boldsymbol{P}} \models$ "κ is strongly compact".*

Proof. The proof of Lemma 2.1 uses ideas of Magidor which, although unpublished by him, are contained in proofs found in [**2**], [**3**] and [**1**]. However, for comprehensibility, we give a complete proof of Lemma 2.1 below.

Let $\lambda > \kappa$ be regular, and by Proposition 2.7 of [**16**], let $k_1 : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ generated by an ultrafilter over $P_\kappa(\lambda)$ so that $M \models$ "κ isn't λ supercompact". Since GCH implies $M \models$ "κ is measurable", we may choose a normal ultrafilter of Mitchell order 0 over κ so that $k_2 : M \to N$ is an elementary embedding witnessing the measurability of κ definable in M with $N \models$ "κ isn't measurable". It is the case that if $k : V \to N$ is an elementary embedding with critical point κ and there is some $y \in N$ so that $k''\lambda \subseteq y$ and $N \models$ "$|y| < k(\kappa)$", then k witnesses the λ strong compactness of κ. Since $k_2(k_1''\lambda)$ works as such a y, $j = k_2 \circ k_1$ is an elementary embedding witnessing the λ strong compactness of κ. We show that j extends in $V^{\boldsymbol{P}}$ to $j : V^{\boldsymbol{P}} \to N^{j(\boldsymbol{P})}$. Since this extended embedding witnesses the λ strong compactness of κ in $V^{\boldsymbol{P}}$, this proves Lemma 2.1.

To do this, write $j(\boldsymbol{P})$ as $\boldsymbol{P} * \dot{\boldsymbol{Q}} * \dot{\boldsymbol{R}}$, where $\dot{\boldsymbol{Q}}$ is a term for the portion of $j(\boldsymbol{P})$ between κ and $k_2(\kappa)$ and $\dot{\boldsymbol{R}}$ is a term for the rest of $j(\boldsymbol{P})$, i.e., the part above $k_2(\kappa)$. Note that since $N \vDash$ "κ isn't measurable", $\kappa \notin \mathrm{dom}(\dot{\boldsymbol{Q}})$. Thus, the domain of $\dot{\boldsymbol{Q}}$ is composed of N-measurable cardinals in the interval $(\kappa, k_2(\kappa)]$ (that we can infer $k_2(\kappa) \in \mathrm{dom}(\dot{\boldsymbol{Q}})$ will be shown immediately after the construction of the generic object G_1), and the domain of $\dot{\boldsymbol{R}}$ is composed of N-measurable cardinals in the interval $(k_2(\kappa), k_2(k_1(\kappa)))$.

Let G_0 be V-generic over $\boldsymbol{P}$. We construct in $V[G_0]$ an $N[G_0]$-generic object G_1 over $\boldsymbol{Q}$ and an $N[G_0][G_1]$-generic object G_2 over $\boldsymbol{R}$. Since $\boldsymbol{P}$ is an Easton support iteration of length κ, a direct limit is taken at stage κ, and no forcing is done at κ, the construction of G_1 and G_2 automatically guarantees that $j''G_0 \subseteq G_0 * G_1 * G_2$. This means that $j : V \to N$ extends in $V[G_0]$ to $j : V[G_0] \to N[G_0][G_1][G_2]$.

To build G_1, note that since k_2 is generated by an ultrafilter $\mathscr{U}$ over κ and since in both V and M, $2^\kappa = \kappa^+$, $|k_2(\kappa^+)| = |k_2(2^\kappa)| = |\{f : f : \kappa \to \kappa^+$ is a function$\}| = |[\kappa^+]^\kappa| = \kappa^+$. Thus, as $N[G_0] \vDash$ "$|\wp(\boldsymbol{Q})| = k_2(2^\kappa)$", we can let $\langle D_\alpha : \alpha < \kappa^+ \rangle$ enumerate in $V[G_0]$ the dense open subsets of $\boldsymbol{Q}$ present in $N[G_0]$. Since the κ closure of N with respect to either M or V implies the least element of the domain of $\boldsymbol{Q}$ is $>\kappa^+$, the definition of $\boldsymbol{Q}$ as the Easton support iteration which adds a non-reflecting stationary set of ordinals of cofinality ω to certain $N[G_0]$-measurable cardinals in the interval $(\kappa, k_2(\kappa)]$ implies, by arguments found on page 115 of the proof of Lemma 8 of [**4**] (to handle successor stages) in tandem with arguments found in the proof of Theorem 2.5 of [**5**] (to handle limit stages), that $N[G_0] \vDash$ "$\boldsymbol{Q}$ is $\prec\kappa^+$-strategically closed". By the fact the standard arguments show that forcing with the κ-c.c. partial ordering $\boldsymbol{P}$ preserves that $N[G_0]$ remains κ-closed with respect to either $M[G_0]$ or $V[G_0]$, $\boldsymbol{Q}$ is $\prec\kappa^+$-strategically closed in both $M[G_0]$ and $V[G_0]$.

We can now construct G_1 in either $M[G_0]$ or $V[G_0]$ as follows. Player I picks $p_\alpha \in D_\alpha$ extending $\sup(\langle q_\beta : \beta < \alpha \rangle)$ and player II responds by picking $q_\alpha \geq p_\alpha$ (so $q_\alpha \in D_\alpha$). By the $\prec\kappa^+$-strategic closure of $\boldsymbol{Q}$ in both $M[G_0]$ and $V[G_0]$, player II has a winning strategy for this game, so $\langle q_\alpha : \alpha < \kappa^+ \rangle$ can be taken as an increasing sequence of conditions with $q_\alpha \in D_\alpha$ for $\alpha < \kappa^+$. Clearly, $G_1 = \{p \in \boldsymbol{Q} : \exists \alpha < \kappa^+ [q_\alpha \geq p]\}$ is our $N[G_0]$-generic object over $\boldsymbol{Q}$.

It remains to construct in $V[G_0]$ the desired $N[G_0][G_1]$-generic object G_2 over $\boldsymbol{R}$. To do this, we first note that by GCH, the fact $\lambda > \kappa$, Lemma 2.1 of [**3**], and Proposition 26.11 of [**12**], $M \vDash$ "κ is superstrong with target $k_1(\kappa) > \lambda$", so $M \vDash$ "κ is δ strong for all cardinals $\delta \in [\kappa, k_1(\kappa))$". Thus, since by the choice of k_1, $M \vDash$ "κ isn't λ supercompact", we can write $k_1(\boldsymbol{P})$ as $\boldsymbol{P} * \dot{\boldsymbol{S}} * \dot{\boldsymbol{T}}$, where $\Vdash_{\boldsymbol{P}}$ "$\dot{\boldsymbol{S}} = \dot{\boldsymbol{P}}(\omega, \kappa)$", and $\dot{\boldsymbol{T}}$ is a term for the rest of $k_1(\boldsymbol{P})$. Also, since $\kappa \in \operatorname{dom}(\boldsymbol{P} * \dot{\boldsymbol{S}})$, $k_2(\kappa) \in \operatorname{dom}(k_2(\boldsymbol{P} * \dot{\boldsymbol{S}}))$, i.e., $k_2(\kappa) \in \operatorname{dom}(\boldsymbol{P} * \dot{\boldsymbol{Q}})$, i.e., $k_2(\kappa) \in \operatorname{dom}(\dot{\boldsymbol{Q}})$.

Note now that $M \vDash$ "No cardinal $\gamma \in (\kappa, \lambda]$ is measurable". This is since $M^\lambda \subseteq M$ and $V \vDash$ "No cardinal above κ is measurable". Thus, the domain of $\dot{\boldsymbol{T}}$ is composed of M-measurable cardinals in the interval $(\lambda, k_1(\kappa))$, which implies that in M, $\Vdash_{\boldsymbol{P} * \dot{\boldsymbol{S}}}$ "$\dot{\boldsymbol{T}}$ is $\prec\lambda^+$-strategically closed". Further, since $V \vDash \mathrm{GCH}$ and λ is regular, $|[\lambda]^{<\kappa}| = \lambda$ and $2^\lambda = \lambda^+$. Therefore, as k_1 is generated by an ultrafilter $\mathscr{U}$ over $P_\kappa(\lambda)$, $|k_1(\kappa^+)| = |k_1(2^\kappa)| = |2^{k_1(\kappa)}| = |\{f : f : P_\kappa(\lambda) \to \kappa^+ \text{ is a function}\}| = |[\kappa^+]^\lambda| = \lambda^+$.

Work until otherwise specified in M. Consider the "term forcing" partial ordering $\boldsymbol{T}^*$ (see [7], Section 1.2.5, page 8) associated with $\dot{\boldsymbol{T}}$, i.e., $\tau \in \boldsymbol{T}^*$ if and only if τ is a term in the forcing language with respect to $\boldsymbol{P} * \dot{\boldsymbol{S}}$ and $\Vdash_{\boldsymbol{P} * \dot{\boldsymbol{S}}}$ "$\tau \in \dot{\boldsymbol{T}}$", ordered by $\tau \geq \sigma$ if and only if $\Vdash_{\boldsymbol{P} * \dot{\boldsymbol{S}}}$ "$\tau \geq \sigma$". Although $\boldsymbol{T}^*$ as defined is technically a proper class, it is possible to restrict the terms appearing in it to a sufficiently large set-sized collection, with the additional crucial property that any term τ forced to be in $\dot{\boldsymbol{T}}$ is also forced to be equal to an element of $\boldsymbol{T}^*$. As we will show below, this can be done in such a way that $M \vDash$ "$|\boldsymbol{T}^*| = k_1(\kappa)$".

Clearly, $\boldsymbol{T}^* \in M$. Also, since $\Vdash_{\boldsymbol{P} * \dot{\boldsymbol{S}}}$ "$\dot{\boldsymbol{T}}$ is $\prec\lambda^+$-strategically closed", it can easily be verified that $\boldsymbol{T}^*$ itself is $\prec\lambda^+$-strategically closed in M (if $\Vdash_{\boldsymbol{P} * \dot{\boldsymbol{S}}}$ "$\dot{\mathscr{S}}$ is a strategy for player II" and $\langle \tau_\alpha : \alpha < \beta \rangle$ is a play of the game with player II making a move at stage β, then player II chooses a term τ_β so that $\Vdash_{\boldsymbol{P} * \dot{\boldsymbol{S}}}$ "$\tau_\beta \geq \tau_\alpha$ for every $\alpha < \beta$, $\tau_\beta \in \dot{\boldsymbol{T}}$, and τ_β was chosen according to $\dot{\mathscr{S}}$") and, since $M^\lambda \subseteq M$, in V as well.

Observe that $M \vDash$ "$k_1(\kappa)$ is measurable and $|\boldsymbol{P} * \dot{\boldsymbol{S}}| < k_1(\kappa)$" and $\Vdash_{\boldsymbol{P} * \dot{\boldsymbol{S}}}$ "$\dot{\boldsymbol{T}}$ is an Easton support iteration of length $k_1(\kappa)$ and $|\dot{\boldsymbol{T}}| = k_1(\kappa)$". We can thus let $\dot{f}$ be a term so that $\Vdash_{\boldsymbol{P} * \dot{\boldsymbol{S}}}$ "$\dot{f} : k_1(\kappa) \to \dot{\boldsymbol{T}}$ is a bijection". Since $M \vDash$ "$|\boldsymbol{P} * \dot{\boldsymbol{S}}| < k_1(\kappa)$", for each $\alpha < k_1(\kappa)$, let $S_\alpha = \{r^\alpha_\beta : \beta < \eta^\alpha < k_1(\kappa)\}$ be a maximal incompatible set of elements of $\boldsymbol{P} * \dot{\boldsymbol{S}}$ so that for some term τ^α_β, $r^\alpha_\beta \Vdash$ "$\tau^\alpha_\beta = \dot{f}(\alpha)$". Define $T_\alpha = \{\tau^\alpha_\beta : \beta < \eta^\alpha\}$ and $T = \bigcup_{\alpha < k_1(\kappa)} T_\alpha$. Clearly, $|T| = k_1(\kappa)$, so we can let $\langle \tau_\alpha : \alpha < k_1(\kappa) \rangle$ enumerate the members of T. $\langle \tau_\alpha : \alpha < k_1(\kappa) \rangle$ is so that if $\Vdash_{\boldsymbol{P} * \dot{\boldsymbol{S}}}$ "$\tau \in \dot{\boldsymbol{T}}$", then for some $\alpha < k_1(\kappa)$, $\Vdash_{\boldsymbol{P} * \dot{\boldsymbol{S}}}$ "$\tau = \tau_\alpha$". Therefore, we can restrict the set of terms we choose so that we can assume that in M, $|\boldsymbol{T}^*| = k_1(\kappa)$. Since $M \vDash$ "$2^{k_1(\kappa)} = (k_1(\kappa))^+ = k_1(\kappa^+)$", this means we can let $\langle D_\alpha : \alpha < \lambda^+ \rangle$ enumerate in V the dense open subsets of $\boldsymbol{T}^*$ found in M and argue as we did earlier to construct in V an M-generic object H_2 over $\boldsymbol{T}^*$.

Note now that since N is given by an ultrapower of M via a normal ultrafilter $\mathscr{U} \in M$ over κ, Fact 2 of Section 1.2.2 of [7] tells us that $k_2'' H_2$ generates an N-generic object G^*_2 over $k_2(\boldsymbol{T}^*)$. By elementariness, $k_2(\boldsymbol{T}^*)$ is the term forcing in N defined with respect to $k_2(k_1(\boldsymbol{P}) \restriction (\kappa + 1)) = k_2(\boldsymbol{P} * \dot{\boldsymbol{S}}) = \boldsymbol{P} * \dot{\boldsymbol{Q}}$.

Therefore, since $j(\boldsymbol{P}) = k_2(k_1(\boldsymbol{P})) = \boldsymbol{P} * \dot{\boldsymbol{Q}} * \dot{\boldsymbol{R}}$, G_2^* is N-generic over $k_2(\boldsymbol{T}^*)$, and $G_0 * G_1$ is $k_2(\boldsymbol{P} * \dot{\boldsymbol{S}})$-generic over N, Fact 1 of Section 1.2.5 of [**7**] tells us that for $G_2 = \{i_{G_0 * G_1}(\tau) : \tau \in G_2^*\}$, G_2 is $N[G_0][G_1]$-generic over $\boldsymbol{R}$. Thus, in $V[G_0]$, $j : V \to N$ extends to $j : V[G_0] \to N[G_0][G_1][G_2]$. As $V^{\boldsymbol{P}} \vDash$ "κ is λ strongly compact", this completes the proof of Lemma 2.1. $\square$

LEMMA 2.2. *$V^{\boldsymbol{P}} \vDash$ "κ is strong".*

PROOF. The proof of Lemma 2.2 is similar to the proof of Lemma 2.5 of [**3**]. We use for the proof of this lemma notation and terminology from the introductory section of [**7**]. Fix $\lambda_0 \geq 0$ so that $\kappa^{+\lambda_0}$ is regular. Let $\lambda = \lambda_0 + 1$. Let $j : V \to M$ be an elementary embedding witnessing the λ strongness of κ generated by a $(\kappa, \kappa^{+\lambda})$-extender of width κ with $j(\kappa)$ minimal so that $M \vDash$ "κ isn't $\kappa + \lambda$ strong", and let $i : V \to N$ be the elementary embedding witnessing the measurability of κ generated by the normal ultrafilter $\mathscr{U} = \{x \subseteq \kappa : \kappa \in j(x)\}$. We then have the commutative diagram

$$\begin{tikzcd}
V \arrow[rr, "j"] \arrow[dr, "i"'] & & M \\
& N \arrow[ur, "k"'] &
\end{tikzcd}$$

where $j = k \circ i$ and the critical point of k is above κ.

Observe that $M \vDash$ "No cardinal $\rho \in (\kappa, \kappa^{+\lambda}]$ is measurable". This is since $V_{\kappa+\lambda} \subseteq M$ and $V \vDash$ "There are no measurable cardinals above κ". This means in M, the least measurable cardinal $\kappa_0 > \kappa$ in the domain of $j(\boldsymbol{P})$ is so that $\kappa_0 > \kappa^{+\lambda}$. In addition, it is the case that $\kappa \notin \mathrm{dom}(j(\boldsymbol{P}))$. This is since, by choice of λ, $M \vDash$ "For every $\delta < \lambda_0$ so that $\kappa^{+\delta}$ is regular, κ is $\kappa^{+\delta}$ supercompact and $\kappa + \delta + 1$ strong". As $M \vDash$ "κ isn't $\kappa + \lambda$ strong", there are no other degrees of either supercompactness or strongness that could affect whether κ is an element of $\mathrm{dom}(j(\boldsymbol{P}))$.

Define now $f : \kappa \to \kappa$ by $f(\alpha) =$ The least measurable cardinal above α. We then have $\kappa < \kappa^{+\lambda} < j(f)(\kappa) < \kappa_0$. This last inequality is since the least measurable cardinal δ above any α isn't $\delta + 2$ strong, and by GCH in both V and M, δ isn't $2^\delta = \delta^+$ supercompact either. Thus, δ is both $\delta^{+0} = \delta$ supercompact and $\delta + 0 + 1 = \delta + 1$ strong and shows no further degrees of either supercompactness or strongness.

Note that $M = \{j(g)(a) : a \in [\kappa^{+\lambda}]^{<\omega},\ \mathrm{dom}(g) = [\kappa]^{|a|},\ g : [\kappa]^{|a|} \to V\} = \{k(i(g))(a) : a \in [\kappa^{+\lambda}]^{<\omega},\ \mathrm{dom}(g) = [\kappa]^{|a|},\ g : [\kappa]^{|a|} \to V\}$. By defining $\gamma = i(f)(\kappa)$, we have $k(\gamma) = k(i(f)(\kappa)) = j(f)(\kappa) > \kappa^{+\lambda}$. This means $j(g)(a) = k(i(g))(a) = k(i(g) \restriction [\gamma]^{|a|})(a)$, i.e., $M = \{k(h)(a) : a \in [\kappa^{+\lambda}]^{<\omega},\ h \in N,\ \mathrm{dom}(h) = [\gamma]^{|a|},\ h : [\gamma]^{|a|} \to N\}$. By elementariness, we must have $N \vDash$ "$\kappa \notin \mathrm{dom}(i(\boldsymbol{P}))$ and $\kappa < \gamma = i(f)(\kappa) < \delta_0 =$ The least element of the domain of $i(\boldsymbol{P}) - \kappa$", since $M \vDash$ "$k(\kappa) = \kappa$ isn't in the domain of $j(\boldsymbol{P})$ and $k(\kappa) = \kappa < k(\gamma) = k(i(f)(\kappa)) = j(f)(\kappa) < k(\delta_0) = \kappa_0$". Therefore, k is generated by an N-extender of width $\gamma \in (\kappa, \delta_0)$.

Write $i(\boldsymbol{P}) = \boldsymbol{P} * \dot{\boldsymbol{Q}}^0$, where $\dot{\boldsymbol{Q}}^0$ is a term for the portion of $i(\boldsymbol{P})$ whose domain is composed of ordinals in the interval $[\kappa, i(\kappa))$. By our previous work, the domain of $\dot{\boldsymbol{Q}}^0$ is actually composed of ordinals in the interval $(\kappa, i(\kappa))$, or more precisely, of ordinals in the interval $[\delta_0, i(\kappa))$. This means that if G_0 is once again V-generic over $\boldsymbol{P}$, the argument from Lemma 2.1 for the construction of the generic object G_1 can be applied here as well to construct in $V[G_0]$ an $N[G_0]$-generic object G_1^* over $\boldsymbol{Q}^0$. Since $i''G_0 \subseteq G_0 * G_1^*$, i extends in $V[G_0]$ to $i : V[G_0] \to N[G_0][G_1^*]$, and since $k''G_0 = G_0$ and $k(\kappa) = \kappa$, k extends in $V[G_0]$ to $k : N[G_0] \to M[G_0]$. By Fact 3 of Section 1.2.2 of [**7**], $k : N[G_0] \to M[G_0]$ is also generated by an extender of width $\gamma \in (\kappa, \delta_0)$.

In analogy to the preceding paragraph, write $j(\boldsymbol{P}) = \boldsymbol{P} * \dot{\boldsymbol{Q}}^1$. By the last sentence of the preceding paragraph and the fact δ_0 is the least ordinal in the domain of $\dot{\boldsymbol{Q}}^0$, we can use Fact 2 of Section 1.2.2 of [**7**] to infer that $H = \{p \in \boldsymbol{Q}^1 : \exists q \in k''G_1^*[q \geq p]\}$ is $M[G_0]$-generic over $k(\boldsymbol{Q}^1)$. Thus, k extends in $V[G_0]$ to $k : N[G_0][G_1^*] \to M[G_0][H]$, and we get the new commutative diagram.

$$\begin{tikzcd} V[G_0] \arrow[rr, "j"] \arrow[dr, "i"'] & & M[G_0][H] \\ & N[G_0][G_1^*] \arrow[ur, "k"'] & \end{tikzcd}$$

As in the proofs of Theorems 1.6 and 3.6 of [**11**], since $V_{\kappa+\lambda} \subseteq M$ and $G_0 \in M[G_0][H]$, we can deduce that $(V[G_0])_{\kappa+\lambda} \subseteq M[G_0][H]$, i.e., that j remains a $\kappa + \lambda$ strong embedding. Since λ was arbitrary, this proves Lemma 2.2. $\square$

We note that the proof of Lemma 2.2 can be simplified by choosing λ to be so that $\lambda = \aleph_\lambda$. This is what was done in the proof of Lemma 2.5 of [**3**]. The proof of Lemma 2.2 was given in the above form, however, with an eye towards the proof of Lemma 2.3. Also, in Lemma 2.2, if $\lambda_0 = 0$, then $M \vDash$ "κ isn't measurable", and the proof reduces to showing that measurability is preserved.

LEMMA 2.3. *$V^{\boldsymbol{P}} \vDash$ "For $\delta < \kappa$, if $\delta^{+\alpha}$ is regular, then δ is $\delta^{+\alpha}$ strongly compact if and only if δ is $\delta + \alpha + 1$ strong".*

PROOF. The proof of Lemma 2.3 has as its key ingredient Hamkins' gap forcing work begun in [**8**] and extended in [**9**] and [**10**]. Let $\delta < \kappa$ and α be so that $\delta^{+\alpha}$ is regular in $V^{\boldsymbol{P}}$ and δ is measurable in $V^{\boldsymbol{P}}$. Fix λ for the rest of this proof as the least V-measurable cardinal above δ. Write $\boldsymbol{P} = \boldsymbol{P}^0 * \dot{\boldsymbol{P}}^1$, where the domain of $\boldsymbol{P}^0$ consists of ordinals below δ, and the domain of $\boldsymbol{P}^1$ consists of ordinals at and above δ.

We first show that $\delta \notin \mathrm{dom}(\boldsymbol{P})$. To see this, assume otherwise, and write $\boldsymbol{P} = \boldsymbol{P}_{\beta+1} * \dot{\boldsymbol{Q}}$, where β is so that $\delta = \delta_\beta$. By the definition of $\boldsymbol{P}$, $\Vdash_{\boldsymbol{P}_{\beta+1}}$ "δ isn't measurable". And, since λ is the least V-measurable above δ, the nature of the definition of $\boldsymbol{P}$ implies that $\Vdash_{\boldsymbol{P}_{\beta+1}}$ "$\dot{\boldsymbol{Q}}$ is λ-strategically closed". Hence, $V^{\boldsymbol{P}_{\beta+1} * \dot{\boldsymbol{Q}}} = V^{\boldsymbol{P}} \vDash$ "δ isn't measurable", a contradiction.

Assume now $V^{\boldsymbol{P}} \vDash$ "δ is $\delta^{+\alpha}$ strongly compact". Recall that $\boldsymbol{P}$ is cardinal preserving, and note that $\boldsymbol{P}$ is defined in a manner so that for every V-cardinal γ, any subset of γ in $V^{\boldsymbol{P}}$ of size below γ has a name of size below γ. Also, we can write $\boldsymbol{P} = \boldsymbol{P}_0 * \dot{\boldsymbol{Q}}'$, where $|\boldsymbol{P}_0| = \omega$ and $\Vdash_{\boldsymbol{P}_0}$ "$\dot{\boldsymbol{Q}}'$ is $\aleph_1$-strategically closed". Thus, $\boldsymbol{P}$ is a "mild gap forcing with respect to δ in the sense of [**9**] and [**10**] with a gap at $\aleph_1$", so by the results of [**9**] and [**10**], $V \vDash$ "δ is $\delta^{+\alpha}$ strongly compact". Therefore, by the nature of V, $V \vDash$ "δ is $\delta^{+\alpha}$ supercompact" as well, so by the proof of Lemma 2.2, $V^{\boldsymbol{P}^0} \vDash$ "δ is $\delta + \alpha + 1$ strong". Again by the choice of V, $V \vDash$ "δ isn't λ supercompact and therefore $\delta^{+\alpha} < \lambda$". Hence, the nature of the definition of $\boldsymbol{P}$, coupled with the fact $\delta \notin \mathrm{dom}(\boldsymbol{P})$, imply that $\Vdash_{\boldsymbol{P}^0}$ "$\dot{\boldsymbol{P}}^1$ is λ-strategically closed" and $V^{\boldsymbol{P}^0 * \dot{\boldsymbol{P}}^1} = V^{\boldsymbol{P}} \vDash$ "δ is $\delta + \alpha + 1$ strong".

For the other direction, assume $V^{\boldsymbol{P}} \vDash$ "δ is $\delta + \alpha + 1$ strong". By the results of [**8**], [**9**] and [**10**], $V \vDash$ "δ is $\delta + \alpha + 1$ strong" also. It must then be the case that $V \vDash$ "δ is $\delta^{+\alpha}$ supercompact", for otherwise, we reach the contradiction that $\delta \in \mathrm{dom}(\boldsymbol{P})$. Hence, since $V \vDash$ "δ is $\delta^{+\alpha}$ supercompact and $\delta^{+\alpha} < \lambda$", the argument employed in the proof of Lemma 2.1 can be used here as well to allow us to infer that $V^{\boldsymbol{P}^0} \vDash$ "δ is $\delta^{+\alpha}$ strongly compact". The analysis given in the preceding paragraphs and the strategic closure properties of $\boldsymbol{P}^1$ then yield that $V^{\boldsymbol{P}^0 * \dot{\boldsymbol{P}}^1} = V^{\boldsymbol{P}} \vDash$ "δ is $\delta^{+\alpha}$ strongly compact". This completes the proof of Lemma 2.3. □

LEMMA 2.4. *$V^{\boldsymbol{P}} \vDash$ "κ is both the least strongly compact and least strong cardinal".*

PROOF. Let $\beta < \kappa$ be fixed but arbitrary. Write $\boldsymbol{P} = \boldsymbol{P}_{\beta+1} * \dot{\boldsymbol{Q}}$. By the definition of $\boldsymbol{P}$, $\Vdash_{\boldsymbol{P}_{\beta+1}}$ "δ_β contains a non-reflecting stationary set of ordinals of cofinality ω", so by the fact $\Vdash_{\boldsymbol{P}_{\beta+1}}$ "$\dot{\boldsymbol{Q}}$ is λ'_β-strategically closed for λ'_β the least inaccessible cardinal above δ_β", $V^{\boldsymbol{P}_{\beta+1} * \dot{\boldsymbol{Q}}} = V^{\boldsymbol{P}} \vDash$ "δ_β contains a non-reflecting stationary set of ordinals of cofinality ω". By Theorem 4.8 of [**17**], $V^{\boldsymbol{P}} \models$ "There are no strongly compact cardinals in the interval $[\omega, \delta_\beta]$". Since β can be any ordinal below κ and the set A used in the definition of $\boldsymbol{P}$ is unbounded in κ, $V^{\boldsymbol{P}} \models$ "No cardinal $\gamma < \kappa$ is strongly compact", so by Lemma 2.1, $V^{\boldsymbol{P}} \models$ "κ is the least strongly compact cardinal".

By Lemma 2.2, the proof of Lemma 2.4 will thus be complete once we have shown that $V^{\boldsymbol{P}} \models$ "No cardinal $\gamma < \kappa$ is strong". To see that this is the case, let $\gamma < \kappa$ be a cardinal, and by the results of the preceding paragraph, let $\alpha > \gamma$ be so that in $V^{\boldsymbol{P}}$, $\gamma^{+\alpha}$ is regular and γ isn't $\gamma^{+\alpha}$ strongly compact. By Lemma 2.3, in $V^{\boldsymbol{P}}$, γ isn't $\gamma + \alpha + 1$ strong, i.e., $V^{\boldsymbol{P}} \models$ "γ isn't a strong cardinal". This completes the proof of Lemma 2.4. $\square$

As in Lemma 2.6 of [**3**], Lemma 2.1 of [**3**] implies that since $V^{\boldsymbol{P}} \models$ "κ is the least strong cardinal", $V^{\boldsymbol{P}} \models$ "κ isn't 2^κ supercompact". And, by the definition of $\boldsymbol{P}$, $|\boldsymbol{P}| = \kappa$. Thus, since the results of [**8**], [**9**] and [**10**] and the Lévy-Solovay results [**14**] tell us that $V^{\boldsymbol{P}} \models$ "No cardinal $\lambda > \kappa$ is measurable + There is no pair of cardinals $\delta < \lambda$ so that δ is λ supercompact and λ is measurable", Lemmas 2.1–2.4 complete the proof of Theorem 4. $\square$

3. Concluding remarks.

In conclusion to this paper, we ask if Theorem 4 can be extended to hold for more than one strongly compact cardinal, or even, as is the case in [**3**], to hold for a proper class of strongly compact cardinals. A difficulty in doing this using the methods of this paper lies in the fact that Magidor's argument employed in the proofs of Lemmas 2.1 and 2.3 for the preservation of strong compactness requires a limit on the forcing to add non-reflecting stationary sets of ordinals of the appropriate cofinality above the cardinal whose strong compactness, or degree of strong compactness, is to be preserved. Since no such limit would be possible if the proof methods given in this paper are followed to try to extend Theorem 4 to the situation where the first two strongly compact cardinals are the first two strong cardinals and level by level equivalence between strong compactness and strongness holds, it is unclear if there is any way of using the techniques of this paper to prove the desired, most general result.

References

[1] A. Apter, Strong Compactness, Measurability, and the Class of Supercompact Cardinals, Fund. Math., **167** (2001), 65–78.

[2] A. Apter and J. Cummings, Identity Crises and Strong Compactness, J. Symbolic Logic, **65** (2000), 1895–1910.

[3] A. Apter and J. Cummings, Identity Crises and Strong Compactness II: Strong Cardinals, Arch. Math. Logic, **40** (2001), 25–38.
[4] A. Apter and S. Shelah, On the Strong Equality between Supercompactness and Strong Compactness, Trans. Amer. Math. Soc., **349** (1997), 103–128.
[5] J. Baumgartner, Iterated Forcing, in: A. Mathias, editor, Surveys in Set Theory, Cambridge Univ. Press, Cambridge, England, 1983, 1–59.
[6] J. Burgess, Forcing, in: J. Barwise, editor, Handbook of Mathematical Logic, North-Holland, Amsterdam, 1977, 403–452.
[7] J. Cummings, A Model in which GCH Holds at Successors but Fails at Limits, Trans. Amer. Math. Soc., **329** (1992), 1–39.
[8] J. D. Hamkins, Destruction or Preservation As You Like It, Ann. Pure Appl. Logic, **91** (1998), 191–229.
[9] J. D. Hamkins, Gap Forcing, Israel J. Math., **125** (2001), 237–252.
[10] J. D. Hamkins, Gap Forcing: Generalizing the Lévy-Solovay Theorem, Bull. Symbolic Logic, **5** (1999), 264–272.
[11] J. D. Hamkins, The Lottery Preparation, Ann. Pure App. Logic, **101** (2000), 103–146.
[12] A. Kanamori, The Higher Infinite, Springer-Verlag, Berlin-New York, 1994.
[13] Y. Kimchi and M. Magidor, The Independence between the Concepts of Compactness and Supercompactness, circulated manuscript.
[14] A. Lévy and R. Solovay, Measurable Cardinals and the Continuum Hypothesis, Israel J. Math., **5** (1967), 234–248.
[15] D. A. Martin and J. Steel, A Proof of Projective Determinacy, J. Amer. Math. Soc., **2** (1989), 71–125.
[16] T. Menas, On Strong Compactness and Supercompactness, Ann. Math. Logic, **7** (1974), 327–359.
[17] R. Solovay, W. Reinhardt and A. Kanamori, Strong Axioms of Infinity and Elementary Embeddings, Ann. Math. Logic, **13** (1978), 73–116.

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