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Notes on group actions on subfactors

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Abstract. We construct approximately inner actions of discrete amenable groups on strongly amenable subfactors of type II_1 with given invariants, and obtain classification results under some conditions. We also study the lifting of the relative χ group.

1. Introduction.

In the theory of subfactors initiated by V. F. R. Jones in [**15**], analysis of automorphisms and group actions on subfactors has been done by many people. In [**24**], P. Loi introduced the Loi invariant for automorphisms of subfactors and obtained some results on structure of subfactors of type III_λ, $0 < \lambda < 1$. In [**31**], S. Popa introduced the notion of proper outerness for automorphisms of subfactors (Choda and Kosaki introduced the same property in [**3**], and they call it strong outerness). With his classification of strongly amenable subfactors of type II_1 in [**32**], he classified properly outer actions of discrete amenable groups on strongly amenable subfactors of type II_1 by the Loi invariant, and solved the problem of classification of subfactors of type III_λ raised by Loi in [**24**]. On the other hand, related with orbifold construction, non-strongly outer automorphisms are studied in [**8**], [**10**], [**11**], [**12**], [**17**], [**18**], [**19**], [**20**], [**22**], [**25**], [**35**]. Also see [**9**, Chapter 15], in which almost all contents of the above referred papers are explained.

In [**26**], we classified approximately inner actions of discrete amenable groups on strongly amenable subfactors of type II_1 by the characteristic invariant and the ν invariant under some assumptions. Among these assumptions, the most important one is the triviality of the algebraic κ invariant. (See [**13**] and [**2**] for the original definition and properties of κ invariant. For a subfactor analogue of the κ invariant, see [**19**].) This result is a generalization of [**20**, Theorem 3.1]. When the algebraic κ invariant is trivial, we can classify approximately inner actions completely. Hence we have to investigate the case when the algebraic κ invariant is not trivial. In this case, we do not know whether there exist

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actions with given invariants or not. Hence what we should do first is to find a systematic way to construct actions with given invariants. We remark that if the algebraic κ invariant is trivial, then our characteristic invariant is exactly the same as the original one in [**14**], but if the algebraic κ invariant is not trivial, then our characteristic invariant may be different from the usual one, and this makes classification more difficult.

In this paper, we construct actions of discrete amenable groups on strongly amenable subfactors of type II_1 with given invariants, and classify actions under an extra assumption on the ν invariant. (We emphasize that we never assume the triviality of the algebraic κ invariant.) The most essential assumption in our theory is the extendability of the ν invariant to a homomorphism from a whole group. This assumption is similar to that of [**21**, Theorem 20]. In [**21**], Kawahigashi, Sutherland and Takesaki have classified actions of a discrete abelian group G on the injective type III_1 factor. The modular invariant ν appears as a cocycle conjugacy invariant, and this is a homomorphism from a subgroup of G to $\boldsymbol{R}$. Essential fact in their proof is that ν can be extended to a homomorphism of G due to the divisibility of $\boldsymbol{R}$. (Originally this idea was due to Connes. See [**4**, p. 466].)

In subfactor case, we can not expect such a property for the ν invariant generally. But if we assume the extendability of the ν invariant, our proof goes well as in the proof of [**21**, Theorem 20]. We remark that our results can be viewed as a generalization of [**20**, Theorem 4.1].

In appendix, we discuss liftings of $\chi_a(M,N)$ since we have to fix one lifting of $\chi_a(M,N)$ to define characteristic invariants.

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2. Main results.

Facts on automorphisms for subfactors are collected in [**26**, Section 2], and we will freely use notations there.

First we recall fundamental definitions and facts on automorphisms of subfactors. Let $N \subset M$ be a subfactor of type II_1 with finite index and $N \subset M \subset M_1 \subset M_2 \cdots$ the Jones tower. Then $\alpha \in \mathrm{Aut}(M,N)$ can be extended to an automorphism of M_k inductively by setting $\alpha(e_k) = e_k$, where e_k is the Jones projection in M_k.

Definition 2.1 ([**24**, Section 5]). For α, we put

$$\Phi(\alpha) := \{\alpha|_{M' \cap M_k}\}_{k=1}^{\infty},$$

and we call $\Phi(\alpha)$ the Loi invariant.

Definition 2.2 ([**3**, Definition 1], [**32**, Definition 1.5.1]). An automorphism σ is said to be strongly outer if we have no non-zero $a \in \bigcup_k M_k$ satisfying $\sigma(x)a = ax$ for every $x \in M$. We denote the set of non-strongly outer automorphisms by $\mathrm{Cnt}_{\mathrm{r}}(M, N)$, which is a normal subgroup of $\mathrm{Aut}(M, N)$.

Throughout this paper, we always make the following assumptions on $N \subset M$.

(A1) $N \subset M$ is extremal,

(A2) $N \subset M$ and $M \subset M_1$ have the trivial normalizer,

(A3) $\mathrm{Ker}\,\Phi = \mathrm{Aut}(M, N)$,

(A4) $\chi_a(M, N) := (\mathrm{Ker}\,\Phi \cap \mathrm{Cnt}_{\mathrm{r}}(M, N))/(\mathrm{Int}(M, N))$ is a finite group,

(A5) there exists a lifting σ of $\chi_a(M, N)$ to $\mathrm{Aut}(M, N)$.

The assumption (A3) means that every action has the trivial Loi invariant. Note that we have many classes of subfactors satisfying the above assumptions, e.g., Jones subfactors with principal graph A_{2n+1} in [**15**], or subfactors coming from Hecke algebras in [**37**]. (Also see [**18**] and [**8**].) By [**12**, Corollary 2.2], $\chi_a(M, N)$ is always abelian, and hence $\chi_a(M, N)$ is a finite abelian group in our setting.

By [**26**, Theorem 3.1], we have a Connes-Radon-Nikodym type cocycle $u_{\alpha,\sigma} \in U(N)$ for every $\alpha \in \mathrm{Ker}\,\Phi$ and $\sigma \in \mathrm{Cnt}_{\mathrm{r}}(M, N)$. The algebraic κ invariant κ_a is defined by $\kappa_a(h, k) = u^*_{\sigma_k, \sigma_h}$ for $h, k \in \chi_a(M, N)$. We can easily verify that κ_a is a bicharacter of $\chi_a(M, N)$.

Next we recall the definition of cocycle conjugacy invariants for actions considered in [**26**]. Let $N \subset M$ be a subfactor of type II_1 with finite index, G a discrete group, and α an action of G on $N \subset M$. Then we get cocycle conjugacy invariants in the following way. The first invariant is a normal subgroup $H_\alpha \subset G$, which is the non-strongly outer part of α. Then we get a G-equivariant homomorphism ν_α from H_α to $\chi_a(M, N)$ by $\nu_\alpha(h) = [\alpha_h]$. This ν_α is the second cocycle conjugacy invariant, and we call this the ν invariant. By the assumption (A5), α_h has the form $\alpha_h = \mathrm{Ad}\, v_h \sigma_{\nu_\alpha(h)}$ for some unitary $v_h \in U(N)$. Then we get two scalars $\lambda_\alpha(g, h)$ and $\mu_\alpha(h, k)$ by the following equations for $g \in G$, $h, k \in H_\alpha$.

$$\alpha_g(v_{g^{-1}hg})u_{\alpha_g, \sigma_{\nu_\alpha(h)}} = \lambda_\alpha(g, h)v_h, \quad v_h\sigma_{\nu_\alpha(h)}(v_k) = \mu_\alpha(h, k)v_{hk}.$$

The pair $\lambda(g, h)$ and $\mu(h, k)$ satisfy the following relations for $h, k, l \in H$ and $g, g_1, g_2 \in G$.

(1) $\mu(h, k)\mu(hk, l) = \mu(k, l)\mu(h, kl)$,

(2) $\lambda(g_1 g_2, h) = \lambda(g_1, h)\lambda(g_2, g_1^{-1}hg_1)$,

(3) $\lambda(g,hk)\overline{\lambda(g,h)\lambda(g,k)} = \mu(h,k)\overline{\mu(g^{-1}hg, g^{-1}kg)}$

(4) $\lambda(h,k) = \mu(h,h^{-1}kh)\overline{\mu(k,h)\kappa_a(v(k),v(h))}$,

(5) $\lambda(e,h) = \lambda(g,e) = \mu(e,k) = \mu(h,e) = 1$.

The equation (4) shows the difference between the usual characteristic invariant and ours. This definition of λ and μ depends on the choice of v_h. To get rid of this dependency we have to define a suitable equivalence relation for (λ,μ). On this point see [**26**]. We denote the equivalence class of $(\lambda_\alpha,\mu_\alpha)$ by $\Lambda(\alpha) = [\lambda_\alpha,\mu_\alpha]$, and the set of $[\lambda,\mu]$ by $\Lambda(G,H|\kappa_a)$.

Conversely for a given normal subgroup $H \subset G$, $[\lambda,\mu] \in \Lambda(G,H|\kappa_a)$ and $v \in \mathrm{Hom}_G(H,\chi_a(M,N))$, we will construct an action α with $H_\alpha = H$, $\Lambda(\alpha) = [\lambda,\mu]$ and $v_\alpha = v$ in the following proposition.

Proposition 2.3. *Let $N \subset M$ be a strongly amenable subfactor of type II_1, G a discrete amenable group. Assume that v can be extended to a homomorphism from G. Then for every $[\lambda,\mu] \in \Lambda(G,H|\kappa_a)$ and v, there exists an action α of G with $H_\alpha = H$, $\Lambda(\alpha) = [\lambda,\mu]$ and $v_\alpha = v$.*

Proof. By the assumptions, we have an extension of v from G to $\chi_a(M,N)$, which we denote by v again. Hence $g \to \sigma_{v(g)}$ is an action of G on $N \subset M$. Let κ_a be the algebraic κ invariant for $N \subset M$, and set $\lambda'(g,n) := \kappa_a(v(n),v(g))\lambda(g,n)$. Then it is easy to verify that $[\lambda',\mu]$ is in $\Lambda(G,H)$, that is, $[\lambda',\mu]$ is a usual characteristic invariant. Let m be an action of G on the injective type II_1 factor R_0 with the characteristic invariant $[\lambda',\mu]$. Define an action α of G on $N \otimes R_0 \subset M \otimes R_0$ by $\alpha_g := \sigma_{v(g)} \otimes m_g$. Since $N \subset M$ is isomorphic to $N \otimes R_0 \subset M \otimes R_0$ by [**32**] (also see [**1**]), α can be regarded as an action of $N \subset M$. Then this α is a desired one. □

On classification of actions, we have the following result.

Theorem 2.4. *Let $N \subset M$, G be as in the previous proposition. Let α and β be approximately inner actions of G. Assume v_α can be extended to a homomorphism from G. Then α and β are stably conjugate if $H_\alpha = H_\beta$, $\Lambda(\alpha) = \Lambda(\beta)$ and $v_\alpha = v_\beta$ hold.*

Proof. Set $K := \chi_a(M,N)$. Let $\tilde{\alpha}$ be the extension of α on $N \rtimes_\sigma K \subset M \rtimes_\sigma K$ defined in [**26**], and $\tilde{\tilde{\alpha}}$ be the natural extension of $\tilde{\alpha}$ on $\tilde{\tilde{N}} \subset \tilde{\tilde{M}} := N \rtimes_\sigma K \rtimes_{\hat{\sigma}} \hat{K} \subset M \rtimes_\sigma K \rtimes_{\hat{\sigma}} \hat{K}$. Let w_k and v_p be the usual implementing unitaries of $\sigma, \hat{\sigma}$ in $M \rtimes_\sigma K$ and $M \rtimes_\sigma K \rtimes_{\hat{\sigma}} \hat{K}$ respectively. Then by the definition of $\tilde{\tilde{\alpha}}$, we have $\tilde{\tilde{\alpha}}_g(x) = \alpha_g(x)$, $\tilde{\tilde{\alpha}}_g(w_k) = u_{\alpha_g,\sigma_k} w_k$ and $\tilde{\tilde{\alpha}}_g(v_p) = v_p$ for $x \in M$, $k \in K$ and $p \in \hat{K}$. On the other hand, the second dual action $\hat{\hat{\sigma}}$ of σ satisfies $\hat{\hat{\sigma}}_k = \mathrm{id}$ on $M \rtimes_\sigma K$ and $\hat{\hat{\sigma}}_k(v_p) = \overline{\langle k,p \rangle} v_p$ for $p \in \hat{K}$.

The Takesaki duality theorem says that $\tilde{\tilde{N}} \subset \tilde{\tilde{M}}$ is isomorphic to $N \otimes B(l^2(K)) \subset M \otimes B(l^2(K))$ via an isomorphism Ψ satisfying the following.

$$(1) \quad (\Psi(\pi_{\hat{\sigma}} \circ \pi_\sigma(a))\xi)(k) = \sigma_k^{-1}(a)\xi(k),$$

$$(2) \quad (\Psi(\pi_{\hat{\sigma}}(w_l))\xi)(k) = \xi(l^{-1}k),$$

$$(3) \quad (\Psi(v_p)\xi)(k) = \overline{\langle k, p\rangle}\xi(k),$$

where π_σ is the embedding of M in $M \rtimes_\sigma K$, and $\pi_{\hat{\sigma}}$ is the embedding of $M \rtimes_\sigma K$ into $M \rtimes_\sigma K \rtimes_{\hat{\sigma}} \hat{K}$.

Define a unitary $c_g \in N \otimes B(l^2(K))$ by $(c_g\xi)(k) := u^*_{\alpha_g, \sigma_k^{-1}}\xi(k)$. Since c_g commutes with elements in $N' \otimes \mathbf{C1}$, c_g is indeed in $N \otimes B(l^2(K))$. Moreover since we have

$$\begin{aligned}
(c_g\alpha_g \otimes \mathrm{id}(c_h)\xi)(k) &= u^*_{\alpha_g, \sigma_k^{-1}}\alpha_g(u^*_{\alpha_h, \sigma_k^{-1}})\xi(k) \\
&= u^*_{\alpha_{gh}, \sigma_k^{-1}}\xi(k) \\
&= (c_{gh}\xi)(k),
\end{aligned}$$

c_g is an $\alpha \otimes \mathrm{id}$ cocycle. Then as in the argument in [**23**, Section 5], it is shown that $\Psi \circ \tilde{\tilde{\alpha}}_g \circ \Psi^{-1} = \mathrm{Ad}\, c_g(\alpha_g \otimes \mathrm{id})$ holds.

On the other hand we have $\Psi \circ \hat{\hat{\sigma}}_k \circ \Psi^{-1} = \sigma_k \otimes \mathrm{Ad}\, \rho_k^{-1}$, where ρ is the left regular representation of K.

Here we consider the Connes-Radon-Nikodym type cocycle for $\mathrm{Ad}\, c_g\alpha_g \otimes \mathrm{id}$ and $\sigma_k \otimes \mathrm{Ad}\, \rho_k^{-1}$. Take $0 \neq a \in M_n$ with $\sigma_k(x)a = ax$ for every $x \in M$. By [**26**, Theorem 3.1], $\alpha_g(a) = u_{\alpha_g, \sigma_k}a$ holds. It is obvious that $\sigma_k \otimes \mathrm{Ad}\, \rho_k^{-1}(x)(a \otimes \rho_k^{-1}) = (a \otimes \rho_k^{-1})x$ holds for every $M \otimes B(l^2(K))$. Here we have the following.

$$\begin{aligned}
(\mathrm{Ad}\, c_g(\alpha_g \otimes \mathrm{id})(a \otimes \rho_k^{-1})\xi)(l) &= (c_g(\alpha_g(a) \otimes \rho_k^{-1})c_g^*\xi)(l) \\
&= u^*_{\alpha_g, \sigma_l^{-1}}\alpha_g(a)(c_g^*\xi)(kl) \\
&= u^*_{\alpha_g, \sigma_l^{-1}}u_{\alpha_g, \sigma_k}au_{\alpha_g, \sigma_{kl}^{-1}}\xi(kl) \\
&= u^*_{\alpha_g, \sigma_l^{-1}}u_{\alpha_g, \sigma_k}\sigma_k(u_{\alpha_g, \sigma_{kl}^{-1}})a\xi(kl) \\
&= u^*_{\alpha_g, \sigma_l^{-1}}u_{\alpha_g, \sigma_l^{-1}}a\xi(kl) \\
&= (a \otimes \rho_k^{-1}\xi)(l).
\end{aligned}$$

By [**26**, Theorem 3.1], the above equality implies $u_{\mathrm{Ad}\, c_g(\alpha \otimes \mathrm{id}), \sigma_k \otimes \mathrm{Ad}\, \rho_k^{-1}} = 1$ for every $g \in G$ and $k \in K$. Hence by replacing α and β if necessary, we may assume that $u_{\alpha_g,\sigma_k} = 1$ and $u_{\beta_g,\sigma_k} = 1$ hold for every $g \in G$ and $k \in K$. This especially implies $\alpha_g \sigma_k = \sigma_k \alpha_g$ and $\beta_g \sigma_k = \sigma_k \beta_g$.

Define two new actions $\bar{\alpha}$ and $\bar{\beta}$ by $\bar{\alpha}_g := \alpha_g \sigma_{\nu(g)}^{-1}$ and $\bar{\beta}_g := \beta_g \sigma_{\nu(g)}^{-1}$. Since α and β commute with $\sigma, \bar{\alpha}$ and $\bar{\beta}$ are indeed actions of G. By the construction of $\bar{\alpha}$ and $\bar{\beta}$, it is easy to see $H_\alpha = \bar{\alpha}^{-1}(\mathrm{Cnt}(M,N)) = \bar{\alpha}^{-1}(\mathrm{Int}(M,N)) = \bar{\beta}^{-1}(\mathrm{Cnt}(M,N)) = \bar{\beta}^{-1}(\mathrm{Int}(M,N))$.

Next we compute $\Lambda(\bar{\alpha})$. For $m \in H$, take $v_m \in U(N)$ with $\alpha_m = \mathrm{Ad}\, v_m \sigma_{\nu(m)}$. In this case, we have $\mathrm{Ad}\, \bar{\alpha}_m = \mathrm{Ad}\, v_m$ for $m \in H$. Moreover since

$$\begin{aligned} 1 &= u_{\alpha_h,\sigma_k} \\ &= u_{\mathrm{Ad}\, v_m \sigma_{\nu(m)},\sigma_k} \\ &= \mathrm{Ad}\, v_m (u_{\sigma_{\nu(m)},\sigma_k}) u_{\mathrm{Ad}\, v_m,\sigma_k} \\ &= \overline{\kappa_a(k,\nu(m))} v_m \sigma_k(v_m^*) \end{aligned}$$

holds, we have $\sigma_k(v_m) = \overline{\kappa_a(k,\nu(m))} v_m$.

First we compute $\lambda_{\bar{\alpha}}$. Since we have $u_{\alpha_g,\sigma_k} = 1$, $\alpha_g(v_{g^{-1}ng}) = \lambda_\alpha(g,n) v_n$ holds by the definition of λ_α. Then we get

$$\begin{aligned} \bar{\alpha}_g(v_{g^{-1}ng}) &= \alpha_g \sigma_{\nu(g)}^{-1}(v_{g^{-1}ng}) \\ &= \overline{\kappa_a(\nu(g)^{-1},\nu(n))} \alpha_g(v_{g^{-1}ng}) \\ &= \kappa_a(\nu(g),\nu(n)) \lambda_\alpha(g,n) v_n, \end{aligned}$$

and $\lambda_{\bar{\alpha}}(g,n) = \kappa_a(\nu(g),\nu(n)) \lambda_\alpha(g,n)$ holds. Next we compute $\mu_{\bar{\alpha}}$. By the definition of μ_α, we have $v_m \sigma_{\nu(m)}(v_n) = \mu_\alpha(m,n) v_{mn}$, $m,n \in H$. Hence we get $v_m v_n = \kappa_a(\nu(m),\nu(n)) \mu_\alpha(m,n) v_{mn}$ and consequently $\mu_{\bar{\alpha}}(m,n) = \kappa_a(\nu(m),\nu(n)) \mu_\alpha(m,n)$.

Similar computation is valid for $\bar{\beta}$, and by the assumption $\Lambda(\alpha) = \Lambda(\beta)$, we get $\Lambda(\bar{\alpha}) = \Lambda(\bar{\beta})$. Hence $\bar{\alpha}$ and $\bar{\beta}$ are cocycle conjugate by [**26**, Theorem 5.1]. Then $\bar{\alpha}$ and $\bar{\beta}$ are stably conjugate, and hence there exists an automorphism $\theta \in \mathrm{Aut}(M \otimes B(l^2(G)), N \otimes B(l^2(G)))$ with $\theta \circ (\bar{\beta}_g \otimes \mathrm{Ad}\, \varrho_g) \circ \theta^{-1} = \bar{\alpha}_g$, where ϱ is the right regular representation of G.

To prove the main theorem, we need the following proposition where $N \subset M$ can be an arbitrary subfactor of finite index.

Proposition 2.5. *Let σ be a non-strongly outer automorphism. Take $0 \neq a \in M_n$ such that $\sigma(x)a = ax$ holds for every $x \in M$. Then $v \in M$ is in M^σ if and only if $va = av$ holds.*

Proof. First assume that $v \in M^\sigma$. Then we have $va = \sigma(v)a = av$. Conversely assume that $va = av$ holds. Then $\sigma(v)a = av = va$ holds. Hence we have $\sigma(v)aa^* = vaa^*$. Here aa^* is in $M' \cap M_n$. Let E be the minimal conditional expectation from M_n onto M. Then we get $\sigma(v)E(aa^*) = E(\sigma(v)aa^*) = E(vaa^*) = vE(aa^*)$, and $E(aa^*) \in M \cap M' = \boldsymbol{C}$. Since a is not zero, $E(aa^*)$ is a non-zero scalar. Hence v is in M^σ. $\square$

Remark. The above proposition can be regarded as a subfactor-analogue of the characterization of the centralizer of type III factors. Namely let M be a type III factor, ϕ a faithful normal state of M. Then $a \in M$ is in M_ϕ if and only if $[\phi, a] = 0$.

We continue the proof of Theorem 2.4. Since an outer action of a finite group is stable, we can find a unitary $w \in N \otimes B(l^2(G))$ such that $w^*\sigma_k \otimes \mathrm{id}(w) = u_{\theta,\sigma_k\otimes\mathrm{id}}$. Hence $\theta\circ\sigma_k\otimes\mathrm{id}\circ\theta^{-1} = \mathrm{Ad}\, u_{\theta,\sigma_k\otimes\mathrm{id}}\circ\sigma_k\otimes\mathrm{id} = \mathrm{Ad}\, w^*\circ\sigma_k\otimes\mathrm{id}\circ\mathrm{Ad}\, w$ holds. If we can prove that $w\bar{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g(w^*)$ is in $(M\otimes B(l^2(G)))^K$, then $w\bar{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g(w^*)$ is an $\alpha\otimes\mathrm{Ad}\,\varrho = \bar{\alpha}\sigma\otimes\mathrm{Ad}\,\varrho$ cocycle and

$$\begin{aligned}\bar{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g\circ\theta\circ\sigma_{v(g)}\otimes\mathrm{id}\circ\theta^{-1} &= \bar{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g\circ\mathrm{Ad}\,w^*\circ\sigma_{v(g)}\otimes\mathrm{id}\circ\mathrm{Ad}\,w\\ &= \mathrm{Ad}(\bar{\alpha}\otimes\mathrm{Ad}\,\varrho_g(w^*))\bar{\alpha}_g\sigma_{v(g)}\otimes\mathrm{Ad}\,\varrho_g\circ\mathrm{Ad}\,w\\ &= \mathrm{Ad}\,w^*\circ\mathrm{Ad}(w\bar{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g(w^*))\alpha_g\otimes\mathrm{Ad}\,\varrho_g\circ\mathrm{Ad}\,w\end{aligned}$$

holds and we have the following.

$$\begin{aligned}\alpha\otimes\mathrm{Ad}\,\varrho &= \bar{\alpha}\otimes\mathrm{Ad}\,\varrho\circ\sigma\otimes\mathrm{id}\\ &\sim \bar{\alpha}\otimes\mathrm{Ad}\,\varrho\circ\theta\circ\sigma\otimes\mathrm{id}\circ\theta^{-1}\\ &= \theta\circ\bar{\beta}\otimes\mathrm{Ad}\,\varrho\circ\theta^{-1}\theta\circ\sigma\otimes\mathrm{id}\circ\theta^{-1}\\ &= \theta\circ\bar{\beta}\sigma\otimes\mathrm{Ad}\,\varrho\circ\theta\\ &= \theta\circ\beta\otimes\mathrm{Ad}\,\varrho\circ\theta^{-1}.\end{aligned}$$

Hence α and β are stably conjugate. So we only have to prove that $w\bar{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g(w^*)$ is in $(M\otimes B(l^2(G)))^K$.

It is easy to see $u_{\alpha_g\otimes\mathrm{Ad}\,\varrho_g,\sigma_k\otimes\mathrm{id}} = 1$, so that $u_{\bar{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g,\sigma_k\otimes\mathrm{id}} = u_{\alpha_g\sigma_{v(g)}^{-1}\otimes\mathrm{Ad}\,\varrho_g,\sigma_k\otimes\mathrm{id}} = \kappa_a(k, v(g))$ holds. In the same way, we can see $u_{\bar{\beta}_g\otimes\mathrm{Ad}\,\varrho_g,\sigma_k\otimes\mathrm{id}} = \kappa_a(k, v(g))$. Hence

$$\begin{aligned}u_{\bar{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g,\mathrm{Ad}\,w^*\circ\sigma_k\otimes\mathrm{id}\circ\mathrm{Ad}\,w} &= u_{\theta\circ\bar{\beta}_g\otimes\mathrm{Ad}\,\varrho_g\circ\theta^{-1},\theta\circ\sigma_k\otimes\mathrm{id}\circ\theta^{-1}}\\ &= \theta(u_{\bar{\beta}_g\otimes\mathrm{Ad}\,\varrho_g,\sigma_k\otimes\mathrm{id}})\\ &= \kappa_a(k, v(g))\end{aligned}$$

holds. Take $0 \neq a \in M_n\otimes B(l^2(G))$ such that $\sigma_k\otimes\mathrm{id}(x)a = ax$ holds for every $x \in M\otimes B(l^2(G))$. Then $\bar{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g(a) = \kappa_a(k, v(g))a$ holds by [**26**, Theorem 3.1].

Since $\mathrm{Ad}\, w^* \circ \sigma_k \otimes \mathrm{id} \circ \mathrm{Ad}\, w(x) w^* a w = w^* a w x$, we also have $\bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(w^* a w) = \kappa_a(k, v(g)) w^* a w$. From these two equalities, we get $\bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(w^*) a \bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(w) = w^* a w$. Hence $w \bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(w^*)$ satisfies the condition in Proposition 2.5, and $w \bar{\alpha}_g \otimes \mathrm{Ad}\, \varrho_g(w^*)$ is in the fixed point algebra $(M \otimes B(l^2(G)))^K$. □

Corollary 2.6. *If G is a finite group in Theorem 2.4, then α and β are cocycle conjugate if and only if $H_\alpha = H_\beta$, $\Lambda(\alpha) = \Lambda(\beta)$ and $v_\alpha = v_\beta$ hold.*

Proof. On one hand, $\alpha \otimes \mathrm{Ad}\, \varrho$ and $\beta \otimes \mathrm{Ad}\, \varrho$ are conjugate by Theorem 2.4. On the other hand, in the same way as in the proof of [**16**, Lemma 6.5], we can prove that α is cocycle conjugate to $\alpha \otimes m$, where m is an outer action of G/H on the injective type II_1 factor R_0 and we regard m as an action of G in the natural way. Hence α and β are cocycle conjugate since m and $m \otimes \mathrm{Ad}\, \varrho$ are cocycle conjugate. □

In the rest of this section, we treat examples which satisfy the assumption in Theorem 2.4. The first example is taken from [**20**, Theorem 4.1].

Example 2.7. We consider the case $G = \boldsymbol{Z}$. Take $\alpha \in \mathrm{Aut}(M, N)$. Let p be the strongly outer period of α. Set $\sigma := \alpha^p$. Then v_α is given by $v_\alpha(pm) = [\sigma^m]$. Let n be the outer period of σ. Here assume $(p, n) = 1$. Then we can find $k, l \in \boldsymbol{Z}$ such that $pk + nl = 1$. Set $v(g) := [\sigma^{gk}]$. Then we have $v(p) = [\sigma^{pk}] = [\sigma^{-nl+1}] = [\sigma]$, and hence v_α can be extended to a homomorphism from $\boldsymbol{Z}$.

Example 2.8. Assume that G is of the form $G = H_\alpha \rtimes K$. For $(h, k) \in G = H_\alpha \rtimes K$, define $v(h, k) := v_\alpha(h)$. Then by using the fact $v_\alpha(knk^{-1}) = v_\alpha(n)$, we get

$$\begin{aligned} v((h_1, k_1)(h_2, k_2)) &= v(h_1 k_1 h_2 k_1^{-1}, k_1 k_2) \\ &= v_\alpha(h_1 k_1 h_2 k_1^{-1}) \\ &= v_\alpha(h_1) v_\alpha(k_1 h_2 k_1^{-1}) \\ &= v(h_1, k_1) v(h_2, k_2). \end{aligned}$$

Hence we can extend v_α to a homomorphism from G, and we can apply the main theorem.

A. On liftings of the relative χ group.

In [**26**] and this paper, we fixed a lifting of $\chi_a(M, N)$ to $\mathrm{Aut}(M, N)$ for the definition of the characteristic invariants and classification of group actions on subfactors. However, in general, a different choice of a lifting produces a 2-

cocycle, and this 2-cocycle may change the characteristic invariants of actions. Hence the definition of characteristic invariants depends on the choice of a lifting. In this appendix, we show that we can choose a unique lifting up to cocycle perturbation by using the algebraic κ invariant.

Take a lifting σ. Then the algebraic κ invariant $\kappa_a(h,k)$ is defined as $\kappa_a(h,k) := u^*_{\sigma_k,\sigma_h}$. To specify σ, we denote this κ_a by $\kappa^\sigma_a(h,k)$.

Fix σ and take another lifting $\tilde{\sigma}$. Then we can find a unitary $u_h \in U(N)$ with $\operatorname{Ad} u_h\sigma_h = \tilde{\sigma}_h$. Since $\tilde{\sigma}$ is a lifting, there exists a 2-cocycle $\mu(h,k) \in Z^2(\chi_a(M,N),\boldsymbol{T})$ with $u_h\sigma_h(u_k) = \mu(h,k)u_{hk}$. To compare κ^σ_a and $\kappa^{\tilde{\sigma}}_a$, we compute $\kappa^{\tilde{\sigma}}_a$. Then

$$\begin{aligned}
\overline{\kappa^{\tilde{\sigma}}_a(h,k)} &= u_{\tilde{\sigma}_k,\tilde{\sigma}_h} \\
&= u_{\operatorname{Ad} u_k\sigma_k,\operatorname{Ad} u_h\sigma_h} \\
&= u_{\operatorname{Ad} u_k\sigma_k,\operatorname{Ad} u_h} \operatorname{Ad} u_h(u_{\operatorname{Ad} u_k\sigma_k,\sigma_h}) \\
&= u_k\sigma_k(u_h)u_k^*u_h^* \operatorname{Ad} u_h(\operatorname{Ad} u_k(u_{\sigma_k,\sigma_h})u_{\operatorname{Ad} u_k,\sigma_h}) \\
&= u_k\sigma_k(u_h)u_k^*u_{\sigma_k,\sigma_h}u_k\sigma_h(u_k^*)u_h^* \\
&= \overline{\kappa^\sigma_a(h,k)\mu(h,k)}\mu(k,h)
\end{aligned}$$

holds, so we get $\kappa^{\tilde{\sigma}}_a(h,k) = \kappa^\sigma_a(h,k)\mu(h,k)\overline{\mu(k,h)}$.

Therefore if $\kappa^{\tilde{\sigma}}_a = \kappa^\sigma_a$, then we must have $\mu(h,k) = \mu(k,h)$. By [**30**, Proposition 3.2], μ is a coboundary, so we can choose u_h as a σ-cocycle. Hence we have shown the following proposition.

Proposition A.1. *Let σ and $\tilde{\sigma}$ be liftings of $\chi_a(M,N)$ to* $\operatorname{Aut}(M,N)$. *If $\kappa^\sigma_a = \kappa^{\tilde{\sigma}}_a$ holds, then $\tilde{\sigma}$ is a cocycle perturbation of σ.*

By Proposition A.1, we can find a unique lifting σ up to cocycle perturbation once we fix the algebraic κ invariant.

In the next proposition, we do not assume $\operatorname{Ker}\Phi = \operatorname{Aut}(M,N)$. Every $\theta \in \operatorname{Aut}(M,N)$ induces an automorphism $\chi_a(\theta)$ of $\chi_a(M,N)$ by $\chi_a(\theta)([\sigma]) := [\theta\circ\sigma\circ\theta^{-1}]$.

Proposition A.2. *Let σ be a lifting of $\chi_a(M,N)$ to* $\operatorname{Aut}(M,N)$. *Assume that $\kappa^\sigma_a(h,k) = \kappa^\sigma_a(\chi_a(\theta)(h),\chi_a(\theta)(k))$ holds for every $\theta \in \operatorname{Aut}(M,N)$. Then there exists a $\sigma_{\chi_a(\theta)(\cdot)}$-cocycle w_h such that $\theta\circ\sigma_h\circ\theta^{-1} = \operatorname{Ad} w_h\sigma_{\chi^\sigma_a(\theta)(h)}$ holds.*

Proof. Take a unitary w_h with $\operatorname{Ad} w_h\sigma_{\chi_a(\theta)(h)} = \theta\circ\sigma_h\circ\theta^{-1}$. Then there exists a 2-cocycle $\mu(h,k)$ satisfying $w_h\sigma_{\chi_a(\theta)(h)}(w_k) = \mu(h,k)w_{hk}$. On one hand, we have $u^*_{\theta\circ\sigma_h\circ\theta^{-1},\theta\circ\sigma_k\circ\theta^{-1}} = \theta(u_{\sigma_h,\sigma_k})^* = \kappa^\sigma_a(k,h)$. On the other hand, we have

$$u^*_{\theta\circ\sigma_h\circ\theta^{-1},\theta\circ\sigma_k\circ\theta^{-1}} = u^*_{\mathrm{Ad}\,w_h\sigma_{\chi_a(\theta)(h)},\mathrm{Ad}\,w_k\sigma_{\chi_a(\theta)(k)}}$$
$$= \kappa_a^\sigma(\chi_a(\theta)(k),\chi_a(\theta)(h))\mu(k,h)\overline{\mu(h,k)}.$$

By the assumption on κ_a^σ, we can choose w_h as a cocycle as in the proof of Proposition A.1. □

The assumption on κ_a in the above proposition is satisfied when either κ_a is trivial or $\chi_a(M,N)$ is cyclic. The former is trivial so that we will see the latter. It is easy to see $\kappa_a^\sigma(h,h) = \kappa_a^\sigma(\chi_a(\theta)(h),\chi_a(\theta)(h))$ holds from the above computation. If $\chi_a(M,N)$ is cyclic and g is a generator of $\chi_a(M,N)$, then

$$\kappa_a^\sigma(g^m,g^n) = \kappa_a^\sigma(g,g)^{mn}$$
$$= \kappa_a^\sigma(\chi_a(\theta)(g),\chi_a(\theta)(g))^{mn}$$
$$= \kappa_a^\sigma(\chi_a(\theta)(g^m),\chi_a(\theta)(g^n))$$

holds.

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