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The Penrose transform for $Sp(n, \mathbf{R})$ and singular unitary representations

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Abstract. We give a general definition of the Radon-Penrose transform for a Zuckerman-Vogan derived functor module of a reductive Lie group G, which maps from the Dolbeault cohomology group over a pseudo-Kähler homogeneous manifold into the space of smooth sections of a vector bundle over a Riemannian symmetric space. Furthermore, we formulate a functorial property between two Penrose transforms in the context of the Kobayashi theory of discretely decomposable restrictions of unitary representations.

Based on this general theory, we study the Penrose transform for a family of singular unitary representations of $Sp(n, \mathbf{R})$ in details. We prove that the image of the Penrose transform is exactly the space of global holomorphic solutions of the system of partial differential equations of minor determinant type of odd degree over the bounded symmetric domain of type CI, which is biholomorphic to the Siegel upper half space. This system might be regarded as a generalization of the Gauss-Aomoto-Gelfand hypergeometric differential equations to higher order. We also find a new phenomenon that the kernel of the Penrose transform is non-zero, which we determine explicitly by means of representation theory.

Contents.

§0. Introduction and statement of results.

0.1. In the summer seminar 1994, a general scheme interacting

A) a characterization of singular irreducible infinite dimensional representations by means of differential equations,

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B) a generalization of the Gauss-Aomoto-Gelfand hypergeometric differential equations to higher order,
C) integral geometry, arising from (non-minimal) parabolic subalgebras,
D) invariant theory (prehomogeneous vector spaces, b-functions, Capelli identities)

was posed by T. Kobayashi, especially a suggestion of the usage of (D) for the characterization of the image of integral transforms. In the dissertation 1995 ([**Se**]), a special case of this program was carried out for some real forms of type A_n. It includes:

1) Construction of the Penrose transform $\mathscr{R}$, from the irreducible representations attached to degenerate elliptic orbits (singular unitary representations $A_{\mathfrak{q}}(\lambda)$ in the sense of Zuckerman-Vogan) with $\mathfrak{q}$ a maximal θ-stable parabolic subalgebra.
2) A proof of the bijectivity of the Penrose transform $\mathscr{R}$ onto the space of global solutions to differential equations of determinant type of order k $(1 \leq k \leq n)$. The novelty is a surjectivity in higher dimensional cases, and also a completely new approach in a proof of the injectivity.

From the view point of representation theory, the above result treats the integral transform of the singular representations associated to *open* G-orbits on a generalized flag variety of G_C (namely, $A_{\mathfrak{q}}(\lambda)$).

After [**Se**], another special case of the above program was carried out for some real forms of type A_n by T. Oshima. Namely, parallel to our formulation and using a similar idea in a different setting, he studied the integral transform of the singular representations associated to *closed* G-orbits on a generalized flag variety of G_C (i.e. degenerate principal series representations), for $G = GL(n, \boldsymbol{R})$. The difference between [**Se**] and [**O**] is summarized as:

Sekiguchi [**Se**]		Oshima [**O**]	
$A_{\mathfrak{q}}(\lambda)$	$\Rightarrow$	principal series representations	((A) representation theory),
Penrose transform	$\Rightarrow$	Poisson transform	((C) integral geometry),
b-function	$\Rightarrow$	Capelli identity	((D) invariant theory).

One of the common parts of [**Se**] and [**O**] is (B), the differential equations that are satisfied by the image of the Penrose-Poisson transform. Here is a common complexified setting:

(0.1.1)
$$\begin{array}{ccc} & G_C/(P_C \cap Q_C) & \\ \swarrow & & \searrow \\ G_C/Q_C & & G_C/P_C \end{array}$$

where $G_C = GL(n, \boldsymbol{C})$, P_C and Q_C are maximal parabolic subgroups of G_C.

Recently, T. Tanisaki studied a 'complex version' of the Radon-Penrose transform for sheaves on the above complex double fibration ([**Ta**]).

All of [**Se**], [**O**] and [**Ta**] gave some sufficient conditions for the injectivity of the Radon-Penrose-Poisson transform $\mathscr{R}$ for type A_n, and then proved a characterization of the image of $\mathscr{R}$ by means of differential equations of determinant type.

0.2. On the other hand, very little has been known for the Radon-Penrose transform for reductive groups other than type A_n. The present paper investigates a new example where the Penrose transform is *not* injective for a real group of type C_n, $G = Sp(n, \boldsymbol{R})$. The main aim of the current paper is to give:

1) Explicit description of $\operatorname{Ker}\mathscr{R}$ by means of representation theory.
2) Explicit description of $\operatorname{Image}\mathscr{R}$ by means of differential equations of minor determinant type.

Let us explain our main results. Let

$$G = Sp(n, \boldsymbol{R}), \quad G_{\boldsymbol{C}} = Sp(n, \boldsymbol{C}),$$

$$Q(k) = \text{a maximal parabolic subgroup of } G_{\boldsymbol{C}} \text{ with Levi part } GL(k, \boldsymbol{C}) \times Sp(n-k, \boldsymbol{C}), \quad (1 \le k \le n).$$

We realize $Q(k)$ so that the G-orbit at the origin of the generalized flag variety $G_{\boldsymbol{C}}/Q(k)$ is open (see §2.2). Then we have a generalized Borel embedding:

$$G/L(k) \hookrightarrow G_{\boldsymbol{C}}/Q(k)$$

with

$$L(k) \simeq U(k) \times Sp(n-k, \boldsymbol{R}).$$

Then, our setting will yield the diagram (0.1.1) with $Q_{\boldsymbol{C}} = Q(k)$ and $P_{\boldsymbol{C}} = Q(n)$. In particular, open G-orbits in the diagram (0.1.1) are given by

$$\begin{array}{ccccc} & & Sp(n, \boldsymbol{R})/(U(k) \times U(n-k)) & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ G/L(k) & & & & G/L(n), \end{array}$$

where the fiber of the projection π_2 is the Grassmannian manifold:

$$\pi_2^{-1}\{o\} \simeq Q(n)/(Q(k) \cap Q(n)) \xleftarrow{\sim} U(n)/(U(k) \times U(n-k)) \simeq Gr_k(\boldsymbol{C}^n).$$

For $m, k, n \in \boldsymbol{Z}$ and $1 \le k \le n$, we define a G-equivariant holomorphic line bundle over the non-compact complex manifold $G/L(k)$:

$$\mathscr{L}^n_{m,k} \to G/L(k),$$

associated to the character of the isotropy subgroup $L(k) \simeq U(k) \times Sp(n-k, \boldsymbol{R})$:

$$v_m : L(k) \to \boldsymbol{C}^\times, \quad (a, d) \mapsto (\det a)^m.$$

Then, we have a Fréchet representation of G on the Dolbeault cohomology group of degree j (see [**Wo**])

$$H^j_{\bar{\partial}}(G/L(k), \mathscr{L}^n_{m,k}), \tag{0.2.1}$$

which we shall also write as $H^j_{\bar\partial}\left(G \underset{L(k)}{\times} \boldsymbol{C}_{\nu_m}\right)$ in later sections. We note that if the parameter m is in the weakly fair range (see [**Vo4**]), namely, if

$$m - n + \frac{k-1}{2} \geq 0 \quad \text{(see Proposition 1.5)},$$

then the Dolbeault cohomology group (0.2.1) vanishes for $j \neq k(n-k)$ (see [**Vo2**]). By a careful computation of line bundle parameters, we can define a Penrose transform (see Theorem 2.4 and Proposition 4.1):

$$\mathscr{R} : H^{k(n-k)}_{\bar\partial}(G/L(k), \mathscr{L}^n_{n,k}) \to H^0_{\bar\partial}(G/L(n), \mathscr{L}^n_{k,n}).$$

The 0-th cohomology group $H^0_{\bar\partial}(G/L(n), \mathscr{L}^n_{k,n})$ is nothing but the space of global holomorphic sections $\mathscr{O}(G/L(n), \mathscr{L}^n_{k,n})$ of the holomorphic line bundle $\mathscr{L}^n_{k,n}$ over the Hermitian symmetric space $G/L(n) \simeq Sp(n, \boldsymbol{R})/U(n)$. We trivialize the line bundle $\mathscr{L}^n_{k,n}$ by using the realization of $G/L(n)$ as a bounded symmetric domain of type CI in the sense of É. Cartan:

$$D := \{Z \in M(n, \boldsymbol{C}) : {}^tZ = Z, I_n - Z^*Z \gg 0\}.$$

Then, we identify $\mathscr{O}(G/L(n), \mathscr{L}^n_{k,n})$ with $\mathscr{O}(D)$, on which G acts by multiplier representations (see (4.5.7)). We take a coordinate z_{ij} $(1 \leq i \leq j \leq n)$ of D by putting

$$Z = \begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{1n} & \cdots & z_{nn} \end{pmatrix}, \tag{0.2.2}$$

and use the notation

$$\frac{\partial}{\partial Z} := \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \frac{1}{2}\frac{\partial}{\partial z_{12}} & \cdots & \frac{1}{2}\frac{\partial}{\partial z_{1n}} \\ \frac{1}{2}\frac{\partial}{\partial z_{12}} & \frac{\partial}{\partial z_{22}} & & \\ \vdots & & \ddots & \vdots \\ \frac{1}{2}\frac{\partial}{\partial z_{1n}} & & \cdots & \frac{\partial}{\partial z_{nn}} \end{pmatrix}. \tag{0.2.3}$$

0.3. For a subset $I, J \subset \{1, 2, \ldots, n\}$ with $|I| = |J|$, we introduce a partial differential operator

$$P(I, J) = \det\left(\frac{\partial}{\partial Z}\right)_{i \in I, j \in J}. \tag{0.3.1}$$

For instance,

$$P(\{1,2\},\{1,2\}) = \frac{\partial^2}{\partial z_{11}\partial z_{22}} - \frac{1}{4}\frac{\partial^2}{\partial z_{12}^2},$$

$$P(\{1,2\},\{3,4\}) = \frac{1}{4}\left(\frac{\partial^2}{\partial z_{13}\partial z_{24}} - \frac{\partial^2}{\partial z_{14}\partial z_{23}}\right).$$

Definition 0.3. For $l \in \boldsymbol{N}$, we define the system of differential equations:

$(\mathcal{N}_l)$ $\quad P(I,J)F(Z) = 0$ for any $I, J \subset \{1,2,\dots,n\}$ with $|I| = |J| = l$,

and the space of global solutions:

(0.3.2) $$\mathrm{Sol}(\mathcal{N}_l) = \{F \in \mathcal{O}(D) : F \text{ solves } (\mathcal{N}_l)\}.$$

Note that

(0.3.3) $$\mathrm{Sol}(\mathcal{N}_1) \simeq \boldsymbol{C},$$

(0.3.4) $$\mathrm{Sol}(\mathcal{N}_{n+1}) \simeq \mathcal{O}(D).$$

By the Laplace expansion formula of the determinant of matrices, we have

$$\mathrm{Sol}(\mathcal{N}_1) \subset \mathrm{Sol}(\mathcal{N}_2) \subset \cdots \subset \mathrm{Sol}(\mathcal{N}_n) \subset \mathrm{Sol}(\mathcal{N}_{n+1}).$$

We shall prove in Proposition 1.5 that the G-module

$$W(n,k) := H_{\bar\partial}^{k(n-k)}(G/L(k), \mathcal{L}_{n,k}^n)$$

splits into the direct sum of two irreducible G-modules $W(n,k)_+$ and $W(n,k)_-$, which are characterized by the following properties:

$$W(n,k)_+ \text{ contains a one dimensional } K\text{-type } \det^k,$$

$$W(n,k)_- \text{ contains a } K\text{-type } \textstyle\bigwedge^{2k}(\boldsymbol{C}^n) \otimes \det.$$

Now, our main result asserts:

Main Theorem. *Let $n, k \in \boldsymbol{Z}$ satisfy $1 \le k \le [n/2]$ and $G = Sp(n, \boldsymbol{R})$.*

1) *The Penrose transform*

$$\mathcal{R} : H_{\bar\partial}^{k(n-k)}(Sp(n,\boldsymbol{R})/(U(k) \times Sp(n-k,\boldsymbol{R})), \mathcal{L}_{n,k}^n) \to \mathcal{O}(Sp(n,\boldsymbol{R})/U(n), \mathcal{L}_{k,n}^n)$$

is a non-zero G-intertwining operator.

2) $\mathrm{Ker}\,\mathcal{R} = W(n,k)_-$.

3) $\mathrm{Image}\,\mathcal{R} = \mathrm{Sol}(\mathcal{N}_{2k+1})$.

This paper is organized as follows: We shall give a general definition of the Penrose transform in Theorem 2.4 for a real reductive Lie group G. In order to prove that $\mathcal{R}$ is non-zero, we shall employ a new idea based on the theory of discrete decomposable restriction of Kobayashi [**Ko3**] by finding a larger group G^1 containing G, and formulate the restriction map in the context of the Penrose transform (Theorem 2.6). Then we shall investigate the Penrose transform $\mathscr{R}$ for $G = Sp(n, \boldsymbol{R})$ as a "restriction" of the Penrose transform $\mathscr{R}^1$ for a larger group $G^1 = U(n, n)$. A new feature in our setting is that the Penrose transform $\mathscr{R}$ is not injective. This is related to the fact that the Beilinson-Bernstein correspondence fails for some generalized flag varieties of a reductive group of type C_n (it always holds for type A_n). The proof of the second statement (an explicit characterization of $\mathrm{Ker}\,\mathscr{R}$) will be given in §3 and §4. The proof of the surjectivity (an explicit characterization of $\mathrm{Image}\,\mathscr{R}$) parallels to the case of type A_n (see [**Se**]), where we employed the b-function of prehomogeneous vector spaces for separation of variables. This will be done in §5 and §6.

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§1. Properties of the derived functor $(\mathfrak{g}, K)$-module $W(n,k)$.

1.1. This section prepares a representation theoretic part of our main theorem, especially algebraic properties of singular infinite dimensional representations $W(n,k)_{\pm}$ of $Sp(n, \boldsymbol{R})$. Let $G = Sp(n, \boldsymbol{R})$ be the real symplectic group of rank n. We denote by $\mathfrak{g}_0$ the Lie algebra of G and $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\boldsymbol{R}} \boldsymbol{C}$. Analogous notation will be used for other Lie groups denoted by Roman upper case letters. Take a maximal compact subgroup K of G. Then K is isomorphic to the unitary group $U(n)$. Fix a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{k}_0$, which is also that of $\mathfrak{g}_0$ because $\mathrm{rank}\,\mathfrak{g}_0 = \mathrm{rank}\,\mathfrak{k}_0$. We choose a basis $\{f_j\}$ of $\sqrt{-1}\mathfrak{h}_0^*$ and fix a positive system $\Delta^+(\mathfrak{k}, \mathfrak{h})$ such that

$$\Delta(\mathfrak{g}, \mathfrak{h}) = \{\pm(f_i \pm f_j) : 1 \le i < j \le n\} \cup \{\pm 2f_l : 1 \le l \le n\},$$

$$\Delta^+(\mathfrak{k}, \mathfrak{h}) = \{f_i - f_j : 1 \le i < j \le n\}.$$

We shall identify $\sqrt{-1}\mathfrak{h}_0^*$ with $\boldsymbol{R}^n$ via $\{f_j\}$. Then the weight lattice is identified with $\boldsymbol{Z}^n$. If $\lambda = (\lambda_1, \dots, \lambda_n) \in \boldsymbol{Z}^n$ is dominant with respect to $\Delta^+(\mathfrak{k}, \mathfrak{h})$, namely, if $\lambda_1 \ge \cdots \ge \lambda_n$, we write

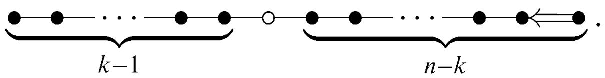
$$F(U(n), \lambda) \equiv F\left(U(n), \sum_{i=1}^{n} \lambda_i f_i\right) \tag{1.1.1}$$

for the finite dimensional irreducible representation of $U(n)$ with highest weight $\lambda = \lambda_1 f_1 + \cdots + \lambda_n f_n$.

1.2. For each integer k $(1 \le k \le n)$, we define a θ-stable parabolic subalgebra

$$\mathfrak{q}(k) = \mathfrak{l}(k) + \mathfrak{u}(k)$$

of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{l}(k)$ and that $\mathfrak{l}(k)$ and $\mathfrak{u}(k)$ are $\mathrm{ad}(\mathfrak{h})$-stable subspaces of $\mathfrak{g}$, corresponding to the black dots in the Dynkin diagram below:



Then we have

$$\begin{aligned}\Delta(\mathfrak{l}(k), \mathfrak{h}) &= \{\pm(f_i - f_j) : 1 \le i < j \le k\} \cup \{\pm(f_i \pm f_j) : k+1 \le i < j \le n\} \\ &\quad \cup \{\pm 2f_l : k+1 \le l \le n\}, \\ \Delta(\mathfrak{u}(k), \mathfrak{h}) &= \{f_i + f_j : 1 \le i \le k, i < j \le n\} \\ &\quad \cup \{f_i - f_j : 1 \le i \le k, k+1 \le j \le n\} \cup \{2f_l : 1 \le l \le k\}.\end{aligned}$$

We write $Q(k)$ for the maximal parabolic subgroup of $G_{\boldsymbol{C}} = Sp(n, \boldsymbol{C})$ with Lie algebra $\mathfrak{q}(k)$. Let

$$L(k) := G \cap Q(k) = N_G(\mathfrak{q}(k)) \simeq U(k) \times Sp(n-k, \boldsymbol{R}).$$

Then $\mathfrak{l}(k)$ is a complexification of the Lie algebra of $L(k)$. We put

$$\begin{aligned}\rho(\mathfrak{u}) \equiv \rho(\mathfrak{u}(k)) &:= \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{u}(k), \mathfrak{h})} \alpha \\ &= (\underbrace{n - (k-1)/2, \dots, n - (k-1)/2}_{k}, \underbrace{0, \dots, 0}_{n-k}) \in \mathfrak{h}^*.\end{aligned}$$

By using the notation $\mathbf{1}_k = (1, 1, \dots, 1) \in \boldsymbol{Z}^k$, we shall write

$$\rho(\mathfrak{u}(k)) = (n - (k-1)/2)\mathbf{1}_k \oplus 0\mathbf{1}_{n-k}.$$

1.3. Let us explain the notation in §0.2 in details. We define a character ν_m of $L(k) \simeq U(k) \times Sp(n-k, \boldsymbol{R})$ parametrized by $m \in \boldsymbol{Z}$:

(1.3.1) $$\nu_m \equiv \nu_m^{(k)} : U(k) \times Sp(n-k, \boldsymbol{R}) \to \boldsymbol{C}^\times, \quad (a, d) \mapsto (\det a)^m.$$

The one dimensional representation $(\nu_m^{(k)}, \boldsymbol{C})$ of $L(k)$ will be denoted by $\boldsymbol{C}_m$.

1.4. The inclusion $G \subset G_{\boldsymbol{C}}$ induces an open embedding:

$$G/L(k) \subset G_{\boldsymbol{C}}/Q(k),$$

through which $G/L(k)$ carries a G-invariant complex structure (see §2.2 for a general setting). We define a G-equivariant holomorphic line bundle over $G/L(k)$ by

$$\mathscr{L}_{m,k}^n \equiv G \underset{L(k)}{\times} \boldsymbol{C}_m = Sp(n, \boldsymbol{R}) \underset{U(k) \times Sp(n-k, \boldsymbol{R})}{\times} (\nu_m^{(k)}, \boldsymbol{C}) \to G/L(k).$$

Let $H^j(G/L(k), \mathcal{O}(\mathscr{L}_{m,k}^n))$ be the j-th cohomology group with coefficients in

$\mathcal{O}(\mathcal{L}^n_{m,k})$, the sheaf of germs of holomorphic sections $\mathcal{L}^n_{m,k} \to G/L(k)$. By the Dolbeault lemma, $H^j(G/L(k), \mathcal{O}(\mathcal{L}^n_{m,k}))$ is isomorphic to the Dolbeault cohomology group $H^j_{\bar\partial}(G/L(k), \mathcal{L}^n_{m,k})$ (see §2.3 for a general setting). The latter space is equipped with the Fréchet topology, on which G acts continuously by left translations by a theorem of Schmid-Wong ([**Sch1**], [**Wo**]; see a survey [**Ko2**]). We denote by $H^j_{\bar\partial}(G/L(k), \mathcal{L}^n_{m,k})_K$ the space of K-finite vectors for each j. Then, the vector space $H^j_{\bar\partial}(G/L(k), \mathcal{L}^n_{m,k})_K$ $(j \in \boldsymbol{N})$ is the underlying $(\mathfrak{g}, K)$-module of the Fréchet representation $H^j_{\bar\partial}(G/L(k), \mathcal{L}^n_{m,k})$.

Let $\mathcal{R}^j_{\mathfrak{q}(k)}$ be the j-th derived functor in the sense of Zuckerman-Vogan, which is a covariant functor from the category of metaplectic $(\mathfrak{l}(k), (L(k) \cap K)^\sim)$-modules to that of $(\mathfrak{g}, K)$-modules ([**Vo3**]).

For $m \in \boldsymbol{Z}$, we define a character of $\mathfrak{h}$ by

$$\begin{aligned}\lambda^{(k)}_m &= (m - n + (k-1)/2)\mathbf{1}_k \oplus 0\mathbf{1}_{n-k} \\ &= (\underbrace{m - n + (k-1)/2, \ldots, m - n + (k-1)/2}_{k}, \underbrace{0, \ldots, 0}_{n-k}).\end{aligned}$$

Then $\lambda^{(k)}_m$ lifts to a metaplectic character, denoted by $\boldsymbol{C}_{\lambda^{(k)}_m}$, of the metaplectic cover $\widetilde{L(k)}$.

Lemma 1.4. *We have isomorphisms of $(\mathfrak{g}, K)$-modules for any $j \in \boldsymbol{N}$ and any $k \in \boldsymbol{N}$ such that $1 \le k \le n$:*

$$H^j(G/L(k), \mathcal{O}(\mathcal{L}^n_{m,k}))_K \simeq H^j_{\bar\partial}(G/L(k), \mathcal{L}^n_{m,k})_K \simeq \mathcal{R}^j_{\mathfrak{q}(k)}(\boldsymbol{C}_{\lambda^{(k)}_m}). \tag{1.4.1}$$

Proof. The statement is a special case of the main theorem of [**Wo**]. □

1.5. Here are basic properties of the $(\mathfrak{g}, K)$-modules $\mathcal{R}^j_{\mathfrak{q}(k)}(\boldsymbol{C}_{\lambda^{(k)}_m})$.

Proposition 1.5. *Let $G = Sp(n, \boldsymbol{R})$ and retain the notation as above.*

1) *The $\mathfrak{l}(k)$-module $\lambda^{(k)}_m$ is in the weakly fair range with respect to $\mathfrak{q} \equiv \mathfrak{q}(k)$ in the sense of Vogan* ([**Vo4**]) *if and only if*

$$m - n + \frac{k-1}{2} \ge 0.$$

2) *If $\lambda^{(k)}_m$ is in the weakly fair range, then $\mathcal{R}^j_{\mathfrak{q}(k)}(\boldsymbol{C}_{\lambda^{(k)}_m})$ and $H^j_{\bar\partial}(G/L(k), \mathcal{L}^n_{m,k})$ vanish for $j \ne k(n-k)$. The remaining module $\mathcal{R}^{k(n-k)}_{\mathfrak{q}(k)}(\boldsymbol{C}_{\lambda^{(k)}_m})$ is unitarizable or zero.*

3) *$\mathcal{R}^{k(n-k)}_{\mathfrak{q}(k)}(\boldsymbol{C}_{\lambda^{(k)}_m})$ has a $Z(\mathfrak{g})$-infinitesimal character*

$$\begin{aligned}&\sum_{i=1}^{k}(m - n + k - i)f_i + \sum_{j=k+1}^{n}(n + 1 - j)f_j \\ &\qquad = (m - n + k - 1, m - n + k - 2, \ldots, m - n, n - k, \ldots, 2, 1)\end{aligned}$$

in the Harish-Chandra parametrization

$$\mathrm{Hom}_{\boldsymbol{C}\text{-algebra}}(Z(\mathfrak{g}),\boldsymbol{C})\simeq\mathfrak{h}^*/(\mathfrak{S}_n\ltimes(\boldsymbol{Z}/2\boldsymbol{Z})^n).$$

Here, we use the normalization of the Harish-Chandra parametrization such that the trivial one dimensional representation has the $Z(\mathfrak{g})$-*infinitesimal character*

$$(n,n-1,\dots,2,1).$$

4) *If* $m\geq n$, *then* $\mathscr{R}_{\mathfrak{q}(k)}^{k(n-k)}(\boldsymbol{C}_{\lambda_m^{(k)}})$ *is non-zero and has a unique* K-*type with highest weight*

$$\mu_{\lambda_m^{(k)}}=\lambda_m^{(k)}+\rho(\mathfrak{u})-2\rho(\mathfrak{u}\cap\mathfrak{k})=(m-n+k)\mathbf{1}_k\oplus k\mathbf{1}_{n-k}.$$

5) *If* $m>n$, *then* $\mathscr{R}_{\mathfrak{q}(k)}^{k(n-k)}(\boldsymbol{C}_{\lambda_m^{(k)}})$ *is a non-zero irreducible* $(\mathfrak{g},K)$-*module and* $H_{\bar{\partial}}^{k(n-k)}(G/L(k),\mathscr{L}_{m,k}^n)$ *is a non-zero irreducible Fréchet* G-*module.*

6) *If* $m=n\geq 2k$, *then*

$$W(n,k):=H_{\bar{\partial}}^{k(n-k)}(G/L(k),\mathscr{L}_{n,k}^n)$$

splits into two irreducible components $W(n,k)_+\oplus W(n,k)_-$, *where* $W(n,k)_\pm$ *are irreducible highest weight modules characterized by the following* K-*type formulae*:

$$(W(n,k)_+)_K\simeq\bigoplus_{\substack{x_1\geq\cdots\geq x_{2k}\geq 0\\ x_j\in 2\boldsymbol{N}}}F(U(n),(\underbrace{x_1+k,\dots,x_{2k}+k}_{2k},\underbrace{k,\dots k}_{n-2k})),\tag{1.5.1}$$

$$(W(n,k)_-)_K\simeq\bigoplus_{\substack{x_1\geq\cdots\geq x_{2k}\geq 0\\ x_j\in 2\boldsymbol{N}+1}}F(U(n),(\underbrace{x_1+k,\dots,x_{2k}+k}_{2k},\underbrace{k,\dots k}_{n-2k})).\tag{1.5.2}$$

Proof. (1) is straightforward from the definition of the weakly fair range ([**Vo4**]).

The statements (2) and (3) follow from Vogan ([**Vo2**]). The statement (4) is a consequence of the generalized Blattner formula and its proof ([**Vo1**]).

The statements (5) and (6) do not follow from general theory. In fact, the character $\lambda_m^{(k)}$ is in the good range in the sense of Zuckerman-Vogan ([**Vo3**]) if and only if

$$m-2n+k>0,\tag{1.5.3}$$

where a theorem of Vogan [**Vo3**] guarantees that $\mathscr{R}_{\mathfrak{q}(k)}^{k(n-k)}(\boldsymbol{C}_{\lambda_m^{(k)}})$ is non-zero and irreducible. However, the assumption $m\geq n$ of (5) or (6) is weaker than (1.5.3). Before proving (5) and (6), we prepare the following setting. Let

$$\mathscr{P}(k)=\mathfrak{m}(k)+\mathfrak{n}(k)$$

be a θ-stable parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{m}(k) \subset \mathfrak{l}(k)$ and $\mathfrak{n}(k) \supset \mathfrak{u}(k)$ with

$$\Delta(\mathfrak{m}(k), \mathfrak{h}) = \{\pm(f_i \pm f_j) : k+1 \le i < j \le n\} \cup \{\pm 2f_l : k+1 \le l \le n\},$$

$$\Delta(\mathfrak{n}(k), \mathfrak{h}) = \{f_i \pm f_j : 1 \le i \le k, i < j \le n\} \cup \{2f_l : 1 \le l \le k\}.$$

Then, $\mathscr{P}(k) \subset \mathfrak{q}(k)$, and $\mathfrak{m}(k)$ is a complexified Lie algebra of the Lie group

$$M(k) := N_G(\mathscr{P}(k)) \simeq \boldsymbol{T}^k \times Sp(n-k, \boldsymbol{R}).$$

We set

$$\zeta_m^{(k)} := \sum_{i=1}^{k} (m - n + k - i) f_i.$$

Then, from the cohomological parabolic induction by stages corresponding to $\mathscr{P}(k) \subset \mathfrak{q}(k) \subset \mathfrak{g}$ ([**Vo1**], Proposition 6.3.6; see also [**Ko1**], Lemma 3.2.1 for the metaplectic normalization), we have the following sublemma:

Sublemma 1.6. *We have an isomorphism of* $(\mathfrak{g}, K)$*-modules*:

$$\mathscr{R}_{\mathfrak{q}(k)}^{k(n-k)}(\boldsymbol{C}_{\lambda_m^{(k)}}) \simeq \mathscr{R}_{\mathscr{P}(k)}^{k(2n-k-1)}(\boldsymbol{C}_{\zeta_m^{(k)}}).$$

We note that the moment map for the generalized flag variety of $Sp(n, \boldsymbol{C})$ is not always normal and birational. Then, a standard $\mathscr{D}$-module theory due to Beilinson-Bernstein is not applicable. In fact, the Zuckerman-Vogan derived functor module in the weakly fair range is not always irreducible for type C_n (this is a distinguished feature from type A_n, where irreducibility always holds in the weakly fair range unless it vanishes). A sufficient condition of irreducibility in the weakly fair range for type C_n is obtained by Kobayashi [**Ko1**], Corollary 6.4.1, which is a generalization of [**Vo4**], Theorem 5.11. Then, applying [**Ko1**], Corollary 6.4.1 to our special setting, and using Sublemma 1.6, we conclude that $H_{\bar{\partial}}^{k(n-k)}(G/L(k), \mathscr{L}_{m,k}^n)$ $(n < m \le 2n-k)$ is irreducible as a G-module, proving Proposition 1.5 (5).

Proposition 1.5 (6) follows from Proposition 5.1 in [**Ad**] and from the observation that $W(n,k)_+$ (respectively, $W(n,k)_-$) is obtained by the Howe correspondence to the trivial (respectively, signature) representation of $O(2k)$ in the reductive dual pair $O(2k) \times \widetilde{Sp}(n, \boldsymbol{R})$ in $\widetilde{Sp}(2nk, \boldsymbol{R})$. We note that $W(n,k)_\pm$ are originally defined as the representation of the metaplectic group $Mp(n, \boldsymbol{R}) \equiv \widetilde{Sp}(n, \boldsymbol{R})$, and then are well-defined as modules of $Sp(n, \boldsymbol{R}) \simeq \widetilde{Sp}(n, \boldsymbol{R})/\{\pm 1\}$ because $2k$ is even.

Thus, the proof of Proposition 1.5 is completed. □

1.7. The bounded symmetric domain $G/K \simeq G/L(n) = Sp(n, \boldsymbol{R})/U(n)$ carries a complex structure induced from the generalized flag variety $G_{\boldsymbol{C}}/Q(n)$ (see §1.4).

LEMMA 1.7. *With notation in* (1.1.1), *we have an isomorphism of K-modules*:

$$\mathcal{O}(\mathcal{L}^n_{k,n})_K \simeq \bigoplus_{\substack{b_1 \geq \cdots \geq b_n \geq 0 \\ b_j \in \mathbf{N}}} F(U(n), (2b_1, 2b_2, \ldots, 2b_n) + k\mathbf{1}_n).$$

PROOF. We have isomorphisms as K-modules:

$$\begin{aligned}
\mathcal{O}(\mathcal{L}^n_{k,n})_K &\simeq \mathcal{O}(G/K)_K \otimes \mathbf{C}_k \\
&\simeq \mathrm{Pol}(\bar{\mathfrak{u}}(n)) \otimes \mathbf{C}_k \\
&\simeq S(\mathfrak{u}(n)) \otimes \mathbf{C}_k \\
&\simeq S(F(U(n), 2f_1)) \otimes \mathbf{C}_k \\
&\simeq \bigoplus_{\substack{b_1 \geq \cdots \geq b_n \geq 0 \\ b_j \in \mathbf{N}}} F(U(n), (2b_1, 2b_2, \ldots, 2b_n) + k\mathbf{1}_n).
\end{aligned}$$

Here, $\mathrm{Pol}(V)$ denotes the polynomial ring over V and $S(V)$ denotes the symmetric tensor algebra over V. The last isomorphism is a theorem of Kostant-Schmid ([**Sch2**]). □

§2. The restriction theorem of the Penrose transform.

2.1. In this section, we give a definition of the *Penrose transform* (see Theorem 2.4) as an intertwining operator from the Dolbeault cohomology construction of the Zuckerman-Vogan derived functor modules of a real reductive Lie group G to the space of smooth sections of G-equivariant vector bundles over the Riemannian symmetric space G/K. Furthermore, we give a *restriction theorem* of the Penrose transform (Theorem 2.6), from a larger group G^1 to its subgroup G.

2.2. First we recall a Dolbeault cohomology construction of Zuckerman-Vogan derived functor modules ([**Sch1**], [**Wo**]; see also a survey [**Ko2**], §3). Throughout this section, G will be a connected real reductive linear Lie group. We shall assume that G is a real form of a connected complex Lie group $G_{\mathbf{C}}$. Let θ be a Cartan involution of G, and K the corresponding maximal compact subgroup of G. We write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the complexification of the Cartan decomposition of $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ defined by θ. Take a Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$ and extend it to a fundamental Cartan subalgebra

$$\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0 \subset \mathfrak{k}_0 + \mathfrak{p}_0 = \mathfrak{g}_0.$$

We write T for the analytic subgroup of K whose Lie algebra is $\mathfrak{t}_0$. We fix a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$ once for all. We shall write $F(K, \mu)$ for a finite dimensional irreducible representation of K with highest weight $\mu \in \mathfrak{t}^*$. Given a

$\Delta^+(\mathfrak{k}, \mathfrak{t})$-dominant element X of $\sqrt{-1}\mathfrak{t}_0$, we define a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$ as follows: $\mathfrak{l}$, $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ are $\mathrm{ad}(\mathfrak{h})$-stable subspaces of $\mathfrak{g}$ such that the set of weights are given by

$$\Delta(\mathfrak{l}, \mathfrak{h}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) : \alpha(X) = 0\},$$

$$\Delta(\mathfrak{u}, \mathfrak{h}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) : \alpha(X) > 0\},$$

$$\Delta(\bar{\mathfrak{u}}, \mathfrak{h}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) : \alpha(X) < 0\}.$$

Then $\mathfrak{l}$ is a complexified Lie algebra of the reductive Lie group:

$$L = Z_G(X) \equiv \{g \in G : \mathrm{Ad}(g)X = X\}.$$

Let Q be a parabolic subgroup of G_C with Lie algebra $\mathfrak{q}$. In light of $Q \cap G = L$, we have an open embedding (a generalized Borel embedding)

$$G/L \hookrightarrow G_C/Q,$$

from which a G-invariant complex structure on G/L is induced. Likewise, we have a diffeomorphism

$$K/L \cap K \xrightarrow{\sim} K_C/Q \cap K_C.$$

Let S be the complex dimension of the generalized flag variety of $K/L \cap K$.

Let $2\rho(\mathfrak{u}) \in \mathfrak{l}^*$ be the differential of the character of L acting on $\bigwedge^{\mathrm{top}} \mathfrak{u}$, and $2\rho(\mathfrak{u} \cap \mathfrak{k}) \in (\mathfrak{l} \cap \mathfrak{k})^*$ that of $L \cap K$ on $\bigwedge^{\mathrm{top}} (\mathfrak{u} \cap \mathfrak{k})$.

2.3. Suppose $\lambda \in \mathfrak{l}^*$ is a one dimensional representation of the Lie algebra $\mathfrak{l}$. Then $\lambda|_{[\mathfrak{l}, \mathfrak{l}]} \equiv 0$. We define $\mu_\lambda \in (\mathfrak{l} \cap \mathfrak{k})^*$ by

$$\mu_\lambda := \lambda|_{\mathfrak{l} \cap \mathfrak{k}} + \rho(\mathfrak{u})|_{\mathfrak{l} \cap \mathfrak{k}} - 2\rho(\mathfrak{u} \cap \mathfrak{k}).$$

We shall sometimes regard as $\lambda \in \mathfrak{h}^*$ (respectively, $\mu_\lambda \in \mathfrak{t}^*$) by the restriction $\mathfrak{l} \downarrow \mathfrak{h}$ (respectively, $(\mathfrak{l} \cap \mathfrak{k}) \downarrow \mathfrak{t}$) without changing the notation. Assume that the character $\lambda + \rho(\mathfrak{u})$ of the Lie algebra $\mathfrak{l}$ lifts to L. We denote it by $\boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}$. We notice that $\lambda + \rho(\mathfrak{u})$ lifts to L if and only if μ_λ lifts to $L \cap K$. We define a G-equivariant holomorphic line bundle

$$G \times_L \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})} \to G/L \tag{2.3.1}$$

associated to the character $\boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}$ of L. Let $\mathscr{E}^{p,q}(G/L)$ be the space of smooth (p, q) forms on G/L, and $\mathscr{E}\left(G \times_L \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right)$ the space of smooth sections of (2.3.1). We set

$$\mathscr{E}^{p,q}\left(G \times_L \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right) := \mathscr{E}\left(G \times_L \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right) \otimes_{\mathscr{E}(G/L)} \mathscr{E}^{p,q}(G/L).$$

We define the space of $\bar{\partial}$-closed q-forms by

$$Z^{0,q}\Big(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big) := \operatorname{Ker}\Big(\bar{\partial} : \mathscr{E}^{0,q}\Big(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big) \to \mathscr{E}^{0,q+1}\Big(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big)\Big),$$

and the space of $\bar{\partial}$-exact q-forms by

$$B^{0,q}\Big(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big) := \operatorname{Image}\Big(\bar{\partial} : \mathscr{E}^{0,q-1}\Big(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big) \to \mathscr{E}^{0,q}\Big(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big)\Big).$$

The Dolbeault cohomology group $H^q_{\bar{\partial}}\Big(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big)$ is defined as the quotient space of $Z^{0,q}\Big(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big)$ by $B^{0,q}\Big(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big)$. We write the natural quotient map as

$$Z^{0,q}\Big(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big) \to H^q_{\bar{\partial}}\Big(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big), \quad \omega \mapsto [\omega].$$

Likewise, we define the Dolbeault cohomology group $H^q_{\bar{\partial}}\Big(K \underset{L\cap K}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big)$ $(q \in \boldsymbol{N})$ for the holomorphic line bundle $K \underset{L\cap K}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})} \to K/L\cap K$. Then it follows from the Borel-Weil-Bott theorem for compact groups that

$$H^q_{\bar{\partial}}\Big(K \underset{L\cap K}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big) \simeq \begin{cases} 0 & q \neq S \\ F(K,\mu_\lambda) & q = S \end{cases} \tag{2.3.2}$$

provided μ_λ is $\Delta^+(\mathfrak{k},\mathfrak{t})$-dominant. For the simplicity of notation, we shall write U_{μ_λ} for $F(K,\mu_\lambda)$.

2.4. Assume that μ_λ is dominant integral with respect to $\Delta^+(\mathfrak{k},\mathfrak{t})$. Let us define the Penrose transform in this generality. Let

$$i : K/L\cap K \to G/L$$

be the natural embedding map which is K-equivariant and holomorphic. The left action of G on G/L is denoted by

$$l_g : G/L \to G/L, \quad xL \mapsto gxL, \quad \text{for } g \in G.$$

Then the representation of G on $W := H^S_{\bar{\partial}}\Big(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big)$ is defined by

$$\pi(g) : W \to W, \quad [\omega] \mapsto \pi(g)[\omega] := [l^*_{g^{-1}}\omega].$$

The underlying $(\mathfrak{g},K)$-module of (π,W) is isomorphic to $\mathscr{R}^S_{\mathfrak{q}}(\boldsymbol{C}_\lambda)$ in the sense of Zuckerman-Vogan, where we follow the normalization of the ρ-shift in [**Vo3**].

If $\omega \in Z^{0,S}\left(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right)$, then $i^* l_g^* \omega \in \mathscr{E}^{0,S}\left(K \underset{L \cap K}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right)$ is also a $\bar{\partial}$-closed form on $K/L \cap K$, giving rise to a cohomology class

$$[i^* l_g^* \omega] \in H_{\bar{\partial}}^S\left(K \underset{L \cap K}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right) = U_{\mu_\lambda}.$$

Thus, we have defined a map

$$\tilde{\mathscr{R}} : Z^{0,S}\left(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right) \times G \to U_{\mu_\lambda}, \quad (\omega, g) \mapsto [i^* l_g^* \omega]. \tag{2.4.1}$$

If $\omega \in \mathscr{E}^{0,S}\left(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right)$ is a $\bar{\partial}$-exact form on G/L, namely if $\omega = \bar{\partial}\eta$, then

$$i^* l_g^* \omega = i^* l_g^* \bar{\partial}\eta = \bar{\partial} i^* l_g^* \eta \in \mathscr{E}^{0,S}\left(K \underset{L \cap K}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right)$$

is also a $\bar{\partial}$-exact form on $K/L \cap K$. Therefore $\tilde{\mathscr{R}}(\omega, \cdot)$ depends only on the cohomology class $[\omega] \in H_{\bar{\partial}}^S\left(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right)$. Hence the map (2.4.1) is well-defined on the level of cohomology:

$$\tilde{\mathscr{R}} : H_{\bar{\partial}}^S\left(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right) \times G \to U_{\mu_\lambda}, \quad ([\omega], g) \mapsto [i^* l_g^* \omega]. \tag{2.4.2}$$

It follows from definition that the map $\tilde{\mathscr{R}}$ in (2.4.2) satisfies:

$$\tilde{\mathscr{R}}(\pi(g_0)[\omega], g) = \tilde{\mathscr{R}}([l_{g_0^{-1}}^* \omega], g) = [i^* l_g^* l_{g_0^{-1}}^* \omega] = \tilde{\mathscr{R}}([\omega], g_0^{-1} g),$$

$$\tilde{\mathscr{R}}([\omega], gh) = [i^* l_h^* l_g^* \omega] = [l_h^* i^* l_g^* \omega] = h^{-1} \tilde{\mathscr{R}}([\omega], g),$$

for any $g, g_0 \in G$, $h \in K$. These two relations imply that the map $\tilde{\mathscr{R}}$ ((2.4.2)) induces a G-intertwining operator between representations of G:

$$\mathscr{R} : H_{\bar{\partial}}^S\left(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right) \to \mathscr{E}\left(G \underset{K}{\times} U_{\mu_\lambda}\right), \quad [\omega] \mapsto \tilde{\mathscr{R}}([\omega], \cdot). \tag{2.4.3}$$

Hence, we have proved:

Theorem 2.4. *Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra defined by a $\Delta^+(\mathfrak{k}, \mathfrak{t})$-dominant element $X \in \sqrt{-1}\mathfrak{t}_0$. Let $\lambda \in \mathfrak{t}^*$. We assume that*

$$\mu_\lambda|_{\mathfrak{t}} \equiv (\lambda + \rho(\mathfrak{u}) - 2\rho(\mathfrak{u} \cap \mathfrak{k}))|_{\mathfrak{t}}$$

is a $\Delta^+(\mathfrak{k}, \mathfrak{t})$-dominant weight, which lifts to the torus T. Let U_{μ_λ} be an irreducible K-module with highest weight μ_λ. Then, we have a G-intertwining map:

$$\mathscr{R} : H_{\bar{\partial}}^S\left(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right) \to \mathscr{E}\left(G \underset{K}{\times} U_{\mu_\lambda}\right), \quad [\omega] \mapsto \tilde{\mathscr{R}}([\omega], \cdot).$$

We say that $\mathscr{R}$ is the *Penrose transform* for the Dolbeault cohomology construction of a Zuckerman-Vogan derived functor module.

2.5. Next, we consider a restriction theorem of two Penrose transforms corresponding to two reductive groups $G \subset G^1$. Here is the setting that we need:

SETTING 2.5. 1) G^1 is a connected real reductive linear Lie group.
2) G is a closed subgroup of G^1 that is reductive in G^1.
3) K^1 is a maximal compact subgroup of G^1, and $K := K^1 \cap G$ is that of G.
4) $\mathfrak{h}_0^1 = \mathfrak{t}_0^1 + \mathfrak{a}_0^1$ be a fundamental Cartan subalgebra of $\mathfrak{g}_0^1$, and

$$\mathfrak{h}_0 := \mathfrak{h}_0^1 \cap \mathfrak{g}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$$

is that of $\mathfrak{g}_0$.
5) Let $\Delta^+(\mathfrak{k}^1, \mathfrak{t}^1)$ be a positive system compatible with a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$. (Namely, if $\alpha \in \Delta^+(\mathfrak{k}^1, \mathfrak{t}^1)$ and $\alpha|_{\mathfrak{t}} \in \Delta(\mathfrak{k}, \mathfrak{t})$, then $\alpha|_{\mathfrak{t}} \in \Delta^+(\mathfrak{k}, \mathfrak{t})$.)
6) Let $\mathfrak{q}^1 = \mathfrak{l}^1 + \mathfrak{u}^1$ be a θ-stable parabolic subalgebra of $\mathfrak{g}^1 = \overline{\mathfrak{u}^1} + \mathfrak{l}^1 + \mathfrak{u}^1$ defined by a $\Delta^+(\mathfrak{k}^1, \mathfrak{t}^1)$-dominant element $X \in \sqrt{-1}\mathfrak{t}_0^1$.
7) Let $\mathfrak{g} = \bar{\mathfrak{u}} + \mathfrak{l} + \mathfrak{u}$ and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a Levi decomposition of a θ-stable parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{q} := \mathfrak{q}^1 \cap \mathfrak{g}$, and $\mathfrak{l} := \mathfrak{l}^1 \cap \mathfrak{g}$.
8) $\boldsymbol{C}_{\lambda^1+\rho(\mathfrak{u}^1)}$ lifts to a character of L^1 and $\mu_{\lambda^1} := \lambda^1 + \rho(\mathfrak{u}^1) - 2\rho(\mathfrak{u}^1 \cap \mathfrak{k}^1)$ is dominant with respect to $\Delta^+(\mathfrak{k}^1, \mathfrak{t}^1)$.

2.6. Here is a restriction theorem of the Penrose transform.

THEOREM 2.6. *Suppose we are in the Setting* 2.5. *Assume that* $\mathfrak{k} + \mathfrak{l}^1 \supset \mathfrak{k}^1$. *We set*

$$\lambda := \lambda^1|_{\mathfrak{l}} + \rho(\mathfrak{u}^1)|_{\mathfrak{l}} - \rho(\mathfrak{u}) \in \mathfrak{l}^*. \tag{2.6.1}$$

Let $U_{\mu_{\lambda^1}}$ be an irreducible finite dimensional representation of K^1 with highest weight μ_{λ^1}. Then the following statements hold.
1) *The restriction of $U_{\mu_{\lambda^1}}$ to K is still irreducible as a K-module with highest weight $\mu_\lambda := \lambda + \rho(\mathfrak{u}) - 2\rho(\mathfrak{u} \cap \mathfrak{k})$.*
2) *The following diagram commutes*:

$$\begin{array}{ccc} H_{\bar\partial}^S\left(G^1 \underset{L^1}{\times} \boldsymbol{C}_{\lambda^1+\rho(\mathfrak{u}^1)}\right) & \xrightarrow{\mathscr{R}^1} & \mathscr{E}\left(G^1 \underset{K^1}{\times} U_{\mu_{\lambda^1}}\right) \\ \downarrow & & \downarrow \\ H_{\bar\partial}^S\left(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right) & \xrightarrow{\mathscr{R}} & \mathscr{E}\left(G \underset{K}{\times} U_{\mu_\lambda}\right). \end{array} \tag{2.6.2}$$

Here the vertical maps are defined by the restrictions, and the horizontal maps are the Penrose transforms defined in Theorem 2.4.

PROOF. First we note that $\mathfrak{k} + \mathfrak{l}^1 \supset \mathfrak{k}^1$ implies that the embedding $K/L \cap K \hookrightarrow K^1/L^1 \cap K^1$ is bijective.

1) Let $\mathrm{pr}: \mathfrak{g}^{1*} \to \mathfrak{g}^*$ be the natural projection with respect to $\mathfrak{g} \hookrightarrow \mathfrak{g}^1$. Likewise, $\mathrm{pr}: (\mathfrak{l}^1 \cap \mathfrak{k}^1)^* \to (\mathfrak{l} \cap \mathfrak{k})^*$ denotes the natural projection with respect to $\mathfrak{l} \cap \mathfrak{k} \hookrightarrow \mathfrak{l}^1 \cap \mathfrak{k}^1$. In light of the biholomorphic map $K/L \cap K \xrightarrow{\sim} K^1/L^1 \cap K^1$, we have

$$\mathrm{pr}(\rho(\mathfrak{u}^1 \cap \mathfrak{k}^1)) = \rho(\mathfrak{u} \cap \mathfrak{k}) \ \ (\in (\mathfrak{l} \cap \mathfrak{k})^*).$$

By the definition of λ ((2.6.1)), we have $\mathrm{pr}(\mu_{\lambda^1}) = \mu_\lambda$. Then μ_λ is also $\Delta^+(\mathfrak{k}, \mathfrak{t})$-dominant. Thus we have a biholomorphic bundle map:

$$\begin{array}{ccc} K \underset{L \cap K}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})} & \xrightarrow{\sim} & K^1 \underset{L^1 \cap K^1}{\times} \boldsymbol{C}_{\lambda^1+\rho(\mathfrak{u}^1)} \\ \downarrow & & \downarrow \\ K/L \cap K & \xrightarrow{\sim} & K^1/L^1 \cap K^1. \end{array}$$

Passing to the cohomology groups, we have a K-module isomorphism

$$H^S_{\bar\partial}\Big(K \underset{L \cap K}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big) \xleftarrow{\sim} H^S_{\bar\partial}\Big(K^1 \underset{L^1 \cap K^1}{\times} \boldsymbol{C}_{\lambda^1+\rho(\mathfrak{u}^1)}\Big). \tag{2.6.3}$$

Thus, we have $F(K, \mu_\lambda) \xleftarrow{\sim} F(K^1, \mu_{\lambda^1})$ by the Borel-Weil-Bott theorem, proving the first statement.

2) First, we observe from Setting 2.5(7) the following commutative diagram of complex manifolds and holomorphic maps:

$$\begin{array}{ccc} G/K & \hookrightarrow & G_C/Q \\ \cap & & \cap \\ G^1/K^1 & \hookrightarrow & G^1_C/Q^1. \end{array}$$

From the construction of $\tilde{\mathscr{R}}$ and $\tilde{\mathscr{R}}^1$, the following diagram commutes:

$$\begin{array}{ccc} \tilde{\mathscr{R}}^1 : H^S_{\bar\partial}\Big(G^1 \underset{L^1}{\times} \boldsymbol{C}_{\lambda^1+\rho(\mathfrak{u}^1)}\Big) \times G^1 & \longrightarrow & H^S_{\bar\partial}\Big(K^1 \underset{L^1 \cap K^1}{\times} \boldsymbol{C}_{\lambda^1+\rho(\mathfrak{u}^1)}\Big) \\ \cup & \nearrow & \\ H^S_{\bar\partial}\Big(G^1 \underset{L^1}{\times} \boldsymbol{C}_{\lambda^1+\rho(\mathfrak{u}^1)}\Big) \times G & & \Big\downarrow \text{Restriction} \\ \text{Restriction} \Big\downarrow & & \\ \tilde{\mathscr{R}} : H^S_{\bar\partial}\Big(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big) \times G & \longrightarrow & H^S_{\bar\partial}\Big(K \underset{L \cap K}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\Big). \end{array}$$

Then, passing from (2.4.2) to (2.4.3) for both G^1 and G, we have the following commutative diagram.

$$
\begin{array}{ccc}
H^S_{\bar{\partial}}\left(G^1 \underset{L^1}{\times} \boldsymbol{C}_{\lambda^1+\rho(\mathfrak{u}^1)}\right) & \xrightarrow{\mathscr{R}^1} & \mathscr{E}\left(G^1 \underset{K^1}{\times} H^S_{\bar{\partial}}\left(K^1 \underset{L^1\cap K^1}{\times} \boldsymbol{C}_{\lambda^1+\rho(\mathfrak{u}^1)}\right)\right) \\
 & & \big\downarrow \text{Restriction} \\
\text{Restriction} \Big\downarrow & & \mathscr{E}\left(G \underset{K}{\times} H^S_{\bar{\partial}}\left(K^1 \underset{L^1\cap K^1}{\times} \boldsymbol{C}_{\lambda^1+\rho(\mathfrak{u}^1)}\right)\right) \\
 & & \big\downarrow \simeq \\
H^S_{\bar{\partial}}\left(G \underset{L}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right) & \xrightarrow{\mathscr{R}} & \mathscr{E}\left(G \underset{K}{\times} H^S_{\bar{\partial}}\left(K \underset{L\cap K}{\times} \boldsymbol{C}_{\lambda+\rho(\mathfrak{u})}\right)\right).
\end{array}
$$

Then, Theorem 2.6 (2) follows from the above vertical isomorphism, which is induced from (2.6.3). □

2.7. In §4.4, we shall apply Theorem 2.6 to study the Penrose transform of $G = Sp(n, \boldsymbol{R})$ by putting $G^1 = U(n, n)$. Here is a sketch of the setting:

Example 2.7 (see §4.3 for details). Let $(G^1, G) = (U(n, n), Sp(n, \boldsymbol{R}))$, $\mathfrak{q}^1 \supset \mathfrak{q}$ be θ-stable parabolic subalgebras, and

$$(L^1, L) = (U(k) \times U(n-k, n), U(k) \times Sp(n-k, \boldsymbol{R})),$$

$$\lambda^1 = \frac{k}{2}\mathbf{1}_{2n}, \qquad \rho(\mathfrak{u}^1) = \frac{2n-k}{2}\mathbf{1}_k \oplus \left(-\frac{k}{2}\right)\mathbf{1}_{2n-k} \in (\mathfrak{h}^1)^*,$$

$$\lambda = \frac{k-1}{2}\mathbf{1}_k \oplus 0\mathbf{1}_{n-k}, \qquad \rho(\mathfrak{u}) = \left(n - \frac{k-1}{2}\right)\mathbf{1}_k \oplus 0\mathbf{1}_{n-k} \in \mathfrak{h}^*.$$

Then, the Penrose transform for $Sp(n, \boldsymbol{R})$ is obtained as a restriction of the Penrose transform for $U(n, n)$ (see (2.6.2)).

§3. Kernel of the Penrose transform.

3.1. From this section, we return to our setting $G = Sp(n, \boldsymbol{R})$. Let $1 \le k \le n$. We apply Theorem 2.4 with $L \simeq U(k) \times Sp(n-k, \boldsymbol{R})$, $\lambda = ((k-1)/2)\mathbf{1}_k + 0\mathbf{1}_{n-k}$ and $U_{\mu_\lambda} \simeq \boldsymbol{C}_k$. This section studies the kernel of the Penrose transform $\mathscr{R}$: $W(n, k) \to \mathscr{E}\left(G \underset{K}{\times} \boldsymbol{C}_k\right)$. We recall from §1.5 that

$$W(n, k) = W(n, k)_+ \oplus W(n, k)_-$$

is an irreducible decomposition as G-modules. We shall prove that the kernel of $\mathscr{R}$ is roughly half, namely $W(n, k)_-$. In this section, we shall prove:

Proposition 3.1. $W(n, k)_- \subset \operatorname{Ker} \mathscr{R}$.

Then the equality $\operatorname{Ker} \mathscr{R} = W(n, k)_-$ follows immediately from the fact that $\mathscr{R}|_{W(n,k)_+}$ is injective (see Proposition 4.1).

3.2. We recall that $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0 = \mathfrak{sp}(n, \boldsymbol{R})$. We take a maximal abelian subspace $\mathfrak{a}_0$ in $\mathfrak{p}_0$ and write A for the analytic subgroup with Lie algebra $\mathfrak{a}_0$. Then the centralizer of $\mathfrak{a}_0$ in $G = Sp(n, \boldsymbol{R})$ is the direct product group $M \times A$, where $M \simeq \{\mathrm{diag}(\underbrace{\pm 1, \pm 1, \ldots, \pm 1}_{n})\}$. Given $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in (\boldsymbol{Z}/2\boldsymbol{Z})^n \simeq \{0, 1\}^n$, we define a character of M by

$$\varepsilon : M \to \{\pm 1\}, \quad (a_1, \ldots, a_n) \mapsto \prod_{j=1}^{n} a_j^{\varepsilon_j}.$$

We set $l(\varepsilon) = \sum_{j=1}^{n} \varepsilon_j \in \{0, 1, \ldots, n\}$, where we take representatives of $\varepsilon_1, \ldots, \varepsilon_n \in \boldsymbol{Z}/2\boldsymbol{Z}$ in $\{0, 1\}$.

LEMMA 3.3. *Let $k \in \boldsymbol{Z}$, and $0 \le j \le n$. Then $j = 0$, if*

$$\mathrm{Hom}_K\Big(F(K, (\underbrace{k+1, \ldots, k+1}_{j}, \underbrace{k, \ldots, k}_{n-j})), \mathscr{E}\Big(G \underset{K}{\times} \boldsymbol{C}_k\Big)\Big) \neq 0.$$

PROOF. Let $f \in \mathscr{E}\Big(G \underset{K}{\times} \boldsymbol{C}_k\Big)$ be a non-zero image of $F(K, (k+1, \ldots, k+1, k, \ldots, k))$. In view of the Cartan decomposition, f is determined by its restriction to A. As M centralizes A, we have

$$(m \cdot f)(a) = f(m^{-1} a m m^{-1}) = f(a m^{-1}) = \nu_k^{(n)}(m) f(a),$$

where $\nu_k^{(n)} : K \to \boldsymbol{C}^\times$ is given by $a \mapsto (\det a)^k$. This implies that

(3.3.1) $$\mathrm{Hom}_M(F(K, (k+1, \ldots, k+1, k, \ldots, k)), \nu_k^{(n)}|_M) \neq 0.$$

We set $\varepsilon \in \hat{M} \simeq (\boldsymbol{Z}/2\boldsymbol{Z})^n$ by

$$\varepsilon := \nu_k^{(n)}|_M = \begin{cases} (1, \ldots, 1) & \text{if } k \equiv 1 \mod 2, \\ (0, \ldots, 0) & \text{if } k \equiv 0 \mod 2. \end{cases}$$

In view of the branching law $K \downarrow M$

$$F(K, (1, \ldots, 1, 0, \ldots, 0))|_M \simeq \bigwedge\nolimits^j (\boldsymbol{C}^n)|_M \simeq \bigoplus_{\substack{\delta \in (\boldsymbol{Z}/2\boldsymbol{Z})^n \\ l(\delta) = j}} \delta,$$

(3.3.1) leads us to

$$\bigoplus_{\substack{\delta \in (\boldsymbol{Z}/2\boldsymbol{Z})^n \\ l(\delta) = j}} \mathrm{Hom}_M(\delta + \varepsilon, \varepsilon) \neq 0.$$

$\mathrm{Hom}_M(\delta + \varepsilon, \varepsilon) \neq 0$ only if $\delta = 0$ in $(\boldsymbol{Z}/2\boldsymbol{Z})^n$. Then $j = l(\delta) = 0$. Thus we have proved the lemma. □

3.4. Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. The Penrose transform $\mathscr{R} : W(n,k) \to \mathscr{E}\left(G \underset{K}{\times} \boldsymbol{C}_k\right)$ is a G-intertwining operator. As $W(n,k)_-$ contains a K-type $F(K,(k+1,\dots,k+1,k,\dots,k))$, this K-type must be contained in the kernel of $\mathscr{R}|_{W(n,k)_-}$ by Lemma 3.3. Since $W(n,k)_-$ is irreducible, $\mathscr{R}$ must be the zero map on $W(n,k)_-$. □

§4. Application of restriction theorem.

4.1. The purpose of §4 is to prove the following:

Proposition 4.1. *$\mathscr{R}|_{W(n,k)_+} : W(n,k)_+ \to \mathscr{E}\left(G \underset{K}{\times} \boldsymbol{C}_k\right)$ is injective. Furthermore,*

$$\mathrm{Image}\,\mathscr{R} \subset \mathscr{O}\left(G \underset{K}{\times} \boldsymbol{C}_k\right).$$

The main idea is to use the restriction theorem (see Theorem 2.6) of two Penrose transforms associated to a pair of reductive groups $G \subset G^1$.

4.2. In order to apply Theorem 2.6 to our special setting $(G^1, G) = (U(n,n), Sp(n,\boldsymbol{R}))$, first let us give an explicit matrix realization of the embedding $Sp(n,\boldsymbol{R}) \subset U(n,n)$ and the corresponding complexification $Sp(n,\boldsymbol{C}) \subset GL(2n,\boldsymbol{C})$. The point here is that $GL(2n,\boldsymbol{R})$ is realized in $GL(2n,\boldsymbol{C})$ in a different way from a standard one. Let $G^1_{\boldsymbol{C}} = GL(2n,\boldsymbol{C})$ and we put elements of $G^1_{\boldsymbol{C}}$ by

$$C_n := \begin{pmatrix} I_n & \sqrt{-1}I_n \\ I_n & -\sqrt{-1}I_n \end{pmatrix}, \quad I_{n,n} := \begin{pmatrix} I_n & O \\ O & -I_n \end{pmatrix},$$

$$Y_n := C_n\overline{C_n^{-1}} = \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix}, \quad J_n := Y_n I_{n,n} = \begin{pmatrix} O & -I_n \\ I_n & O \end{pmatrix}.$$

We define two involutions σ and τ of $GL(2n,\boldsymbol{C})$ by

$$\sigma(g) := Y_n \bar{g} Y_n,$$

$$\tau(g) := I_{n,n}(g^*)^{-1}I_{n,n} = I_{n,n}\overline{{}^t g^{-1}}I_{n,n}.$$

Then σ commutes with τ, and we have another involution $\sigma\tau$ given by

$$\sigma\tau(g) = J_n {}^t g^{-1} J_n^{-1}.$$

Lemma 4.2. 1) *The Lie algebras of the fixed point groups $(G^1_{\boldsymbol{C}})^{\sigma\tau}$, $(G^1_{\boldsymbol{C}})^{\sigma}$, $(G^1_{\boldsymbol{C}})^{\tau}$, and $(G^1_{\boldsymbol{C}})^{\sigma,\tau} := (G^1_{\boldsymbol{C}})^{\sigma} \cap (G^1_{\boldsymbol{C}})^{\tau}$ are given by*

$$\mathfrak{gl}(2n, \boldsymbol{C})^{\sigma\tau} = \left\{ \begin{pmatrix} A & B \\ C & -{}^t\!A \end{pmatrix} \in \mathfrak{gl}(2n, \boldsymbol{C}) : \begin{array}{c} A, B, C \in M(n, \boldsymbol{C}), \\ {}^tB = B, {}^tC = C \end{array} \right\} \simeq \mathfrak{sp}(n, \boldsymbol{C}),$$

$$\mathfrak{gl}(2n, \boldsymbol{C})^{\sigma} = \left\{ \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \in \mathfrak{gl}(2n, \boldsymbol{C}) : A, B \in M(n, \boldsymbol{C}) \right\} \simeq \mathfrak{gl}(2n, \boldsymbol{R}),$$

$$\mathfrak{gl}(2n, \boldsymbol{C})^{\tau} = \left\{ \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \in \mathfrak{gl}(2n, \boldsymbol{C}) : \begin{array}{c} A^* = -A, \\ D^* = -D \end{array} \right\} \simeq \mathfrak{u}(n, n),$$

$$\mathfrak{gl}(2n, \boldsymbol{C})^{\sigma,\tau} = \left\{ \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \in \mathfrak{gl}(2n, \boldsymbol{C}) : \begin{array}{c} A^* = -A, \\ {}^tB = B \end{array} \right\} \simeq \mathfrak{sp}(n, \boldsymbol{R}).$$

2) *We have the following isomorphisms of Lie groups*:

$$GL(2n, \boldsymbol{C})^{\sigma\tau} \simeq Sp(n, \boldsymbol{C}),$$

$$GL(2n, \boldsymbol{C})^{\sigma} = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in GL(2n, \boldsymbol{C}) : a, b \in M(n, \boldsymbol{C}) \right\} \overset{\sim}{\leftarrow} GL(2n, \boldsymbol{R}),$$

$$GL(2n, \boldsymbol{C})^{\tau} \simeq U(n, n),$$

$$GL(2n, \boldsymbol{C})^{\sigma,\tau} = \left\{ \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} : {}^tac = {}^tca, a^*a - c^*c = I_n \right\} \simeq Sp(n, \boldsymbol{R}).$$

The isomorphism $GL(2n, \boldsymbol{R}) \overset{\sim}{\to} GL(2n, \boldsymbol{C})^{\sigma}$ *in* (2) *is given by* $x \mapsto C_n x C_n^{-1}$.

4.3. Suppose that we are in the setting of Lemma 4.2. We shall explain the details of Example 2.7 in order to apply Theorem 2.6.

We fix $k \in \{1, 2, \ldots, n\}$. We define a maximal parabolic subgroup $Q^1 \equiv Q^1(k)$ of $G^1_{\boldsymbol{C}}$ by

(4.3.1) $\quad Q^1(k) := \{g = (g_{ij}) \in GL(2n, \boldsymbol{C}) : g_{ij} = 0 \ \ (k+1 \le i \le 2n, 1 \le j \le k)\}.$

The Lie algebra $\mathfrak{q}^1 \equiv \mathfrak{q}^1(k)$ of the Lie group $Q^1 \equiv Q^1(k)$ is given by

(4.3.2) $\quad \mathfrak{q}^1(k) = \{X = (x_{ij}) \in M(2n, \boldsymbol{C}) : x_{ij} = 0 \ \ (k+1 \le i \le 2n, 1 \le j \le k)\}.$

Then $\mathfrak{q}^1(k) = \mathfrak{l}^1(k) + \mathfrak{u}^1(k)$ is a Levi decomposition, where

$$\mathfrak{l}^1 \equiv \mathfrak{l}^1(k) := \{X = (x_{ij}) \in M(2n, \boldsymbol{C}) : x_{ij} = 0 \ \ (j \le k < i \ \text{ or } \ i \le k < j)\},$$

$$\mathfrak{u}^1 \equiv \mathfrak{u}^1(k) := \{X = (x_{ij}) \in M(2n, \boldsymbol{C}) : x_{ij} = 0 \ \ (k < i \ \text{ or } \ j \le k)\}.$$

Next, we define a subalgebra of $\mathfrak{g}$ by

$$\mathfrak{q} \equiv \mathfrak{q}(k) := \mathfrak{q}^1(k) \cap \mathfrak{g}.$$

In view of the matrix realization $\mathfrak{g} = \mathfrak{gl}(2n, \boldsymbol{C})^{\sigma\tau}$, we have

$$\mathfrak{q} = \left\{ \left(\begin{array}{cc|cc} A_1 & A_2 & \multicolumn{2}{c}{\multirow{2}{*}{B}} \\ O & A_4 & & \\ \hline O & O & -{}^tA_1 & O \\ O & C_4 & -{}^tA_2 & -{}^tA_4 \end{array} \right) : \begin{array}{l} A_1 \in M(k, \boldsymbol{C}), A_2 \in M(k, n-k, \boldsymbol{C}), \\ A_4, C_4 \in M(n-k, \boldsymbol{C}), \\ {}^tC_4 = C_4, \\ {}^tB = B \in M(n, \boldsymbol{C}) \end{array} \right\}.$$

We define a Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ by

$$\mathfrak{l} = \mathfrak{l}^1 \cap \mathfrak{g} = \left\{ \left(\begin{array}{c|c|c|c} A_1 & O & O & O \\ \hline O & A_4 & O & B_4 \\ \hline O & O & -{}^tA_1 & O \\ \hline O & C_4 & O & -{}^tA_4 \end{array} \right) : \begin{array}{l} A_1 \in M(k, \boldsymbol{C}), \\ A_4 \in M(n-k, \boldsymbol{C}), \\ {}^tB_4 = B_4, {}^tC_4 = C_4 \in M(n-k, \boldsymbol{C}) \end{array} \right\},$$

$$\mathfrak{u} = \left\{ \left(\begin{array}{c|c|c|c} O & A_2 & B_1 & B_2 \\ \hline O & O & {}^tB_2 & O \\ \hline O & O & O & O \\ \hline O & O & -{}^tA_2 & O \end{array} \right) : \begin{array}{l} A_2, B_2 \in M(k, n-k, \boldsymbol{C}), \\ {}^tB_1 = B_1 \in M(k, \boldsymbol{C}) \end{array} \right\}.$$

Then $\mathfrak{l} = \mathfrak{l}^1 \cap \mathfrak{q}$. We note that $\mathfrak{u} \not\subset \mathfrak{u}^1$. Note that $(\mathfrak{l}^1, \mathfrak{l})$ are complexified Lie algebras of $(L^1, L) = (U(k) \times U(n-k, n), U(k) \times Sp(n-k, \boldsymbol{R}))$.

Take a fundamental Cartan subalgebra $\mathfrak{h}^1 = \sum_{i=1}^{2n} \boldsymbol{C} E_{ii}$ of $\mathfrak{g}^1$ where E_{ij} is the matrix unit. We put $\mathfrak{h} := \mathfrak{h}^1 \cap \mathfrak{g}$.

Take maximal compact subgroups $K \simeq U(n)$ and $K^1 \simeq U(n) \times U(n)$ in the natural matrix realization. The corresponding embedding $K \subset K^1$ is given by

(4.3.3) $$U(n) \hookrightarrow U(n) \times U(n), \quad g \mapsto (g, {}^tg^{-1}).$$

We take a standard base $\{e_1, \ldots, e_{2n}\}$ for $(\mathfrak{h}^1)^*$ and $\{f_1, \ldots, f_n\}$ for $\mathfrak{h}^*$ such that

$$\Delta(\mathfrak{g}^1, \mathfrak{h}^1) = \{\pm(e_i - e_j) : 1 \leq i < j \leq 2n\},$$
$$\Delta(\mathfrak{g}, \mathfrak{h}) = \{\pm(f_i \pm f_j) : 1 \leq i < j \leq n\} \cup \{\pm 2f_l : 1 \leq l \leq n\}.$$

Here is a list of explicit computations of $\rho(\mathfrak{u}^1)$, $\rho(\mathfrak{u}^1 \cap \mathfrak{k}^1)$, etc.:

$$\rho(\mathfrak{u}^1) = \left(\frac{2n-k}{2}\right) \sum_{i=1}^{k} e_i + \left(-\frac{k}{2}\right) \sum_{i=k+1}^{2n} e_i = \frac{2n-k}{2} \mathbf{1}_k \oplus \left(-\frac{k}{2}\right) \mathbf{1}_{2n-k},$$

$$\rho(\mathfrak{u}^1 \cap \mathfrak{k}^1) = \left(\frac{n-k}{2}\right) \sum_{i=1}^{k} e_i + \left(-\frac{k}{2}\right) \sum_{i=k+1}^{n} e_i = \frac{n-k}{2} \mathbf{1}_k \oplus \left(-\frac{k}{2}\right) \mathbf{1}_{n-k} \oplus 0 \mathbf{1}_n,$$

$$\rho(\mathfrak{u}^1) - 2\rho(\mathfrak{u}^1 \cap \mathfrak{k}^1) = \frac{k}{2}\sum_{i=1}^{n} e_i + \left(-\frac{k}{2}\right)\sum_{i=n+1}^{2n} e_i \qquad = \frac{k}{2}\mathbf{1}_n \oplus \left(-\frac{k}{2}\right)\mathbf{1}_n,$$

$$\rho(\mathfrak{u}) = \left(n - \frac{k-1}{2}\right)\sum_{i=1}^{k} f_i \qquad = \frac{2n-k+1}{2}\mathbf{1}_k \oplus 0\mathbf{1}_{n-k},$$

$$\rho(\mathfrak{u} \cap \mathfrak{k}) = \left(\frac{n-k}{2}\right)\sum_{i=1}^{k} f_i + \left(-\frac{k}{2}\right)\sum_{i=k+1}^{n} f_i = \frac{n-k}{2}\mathbf{1}_k \oplus \left(-\frac{k}{2}\right)\mathbf{1}_{n-k},$$

$$\rho(\mathfrak{u}) - 2\rho(\mathfrak{u} \cap \mathfrak{k}) = \left(\frac{k+1}{2}\right)\sum_{i=1}^{k} f_i + k\sum_{i=k+1}^{n} f_i \qquad = \frac{k+1}{2}\mathbf{1}_k \oplus k\mathbf{1}_{n-k}.$$

The projection $(\mathfrak{h}^1)^* \to \mathfrak{h}^*$ dual to the embedding $\mathfrak{h} = \mathfrak{h}^1 \cap \mathfrak{g} \hookrightarrow \mathfrak{h}^1$ is given by

$$(\mathfrak{h}^1)^* \to \mathfrak{h}^*, \quad \sum_{i=1}^{2n} \lambda_i e_i \mapsto \sum_{j=1}^{n} (\lambda_j - \lambda_{n+j}) f_j.$$

4.4. We define a character of $\mathfrak{l}^1$ such that its restriction to $\mathfrak{h}^1$ is given by

$$\lambda^1 = \frac{k}{2}\sum_{i=1}^{2n} e_i.$$

Then we have

$$\lambda^1 + \rho(\mathfrak{u}^1) = n\sum_{i=1}^{k} e_i, \quad \mu_{\lambda^1} = \lambda^1 + \rho(\mathfrak{u}^1) - 2\rho(\mathfrak{u}^1 \cap \mathfrak{k}^1) = k\sum_{i=1}^{n} e_i.$$

In particular, $\lambda^1 + \rho(\mathfrak{u}^1)$ lifts to a character of L^1. Next, let us define a character of $\mathfrak{l}$ by

$$\lambda := \frac{k-1}{2}\sum_{i=1}^{k} f_i,$$

then we have

$$\lambda + \rho(\mathfrak{u}) = n\sum_{i=1}^{k} f_i, \quad \mu_\lambda = \lambda + \rho(\mathfrak{u}) - 2\rho(\mathfrak{u} \cap \mathfrak{k}) = k\sum_{i=1}^{n} f_i.$$

In particular, the condition (2.6.1) of Theorem 2.6 is satisfied:

$$\begin{cases} (\lambda^1 + \rho(\mathfrak{u}^1))|_{\mathfrak{l}} = \lambda + \rho(\mathfrak{u}) \\ \mu_{\lambda^1}|_{\mathfrak{l}} = \mu_\lambda. \end{cases}$$

Furthermore, we have a biholomorphic map:

$$K/(L(k)\cap K) = U(n)/(U(k)\times U(n-k))$$
$$\xrightarrow{\sim} K^1/(L^1(k)\cap K^1) = (U(n)\times U(n))/(U(k)\times U(n-k)\times U(n))$$

between flag varieties of complex dimension $k(n-k)$. Since $\mu_{\lambda^1} = k\sum_{i=1}^n e_i$, $U_{\mu_{\lambda^1}}$ is the one dimensional representation of $K^1 = U(n)\times U(n)$ given by

$$\nu_k^{1(n)} : U(n)\times U(n) \to \boldsymbol{C}^\times, \quad (a,d)\mapsto (\det a)^k,$$

and its restriction to $K \simeq U(n)$ is an irreducible (one dimensional) representation of K with highest weight $\mu_\lambda = k\sum_{i=1}^n f_i$ by Theorem 2.6 (1), which is isomorphic to $\nu_k^{(n)}$ (see notation (1.3.1)). We define a character of $L^1(k) = U(k)\times U(n-k,n)$ by

$$\nu_n^{1(k)} : U(k)\times U(n-k,n) \to \boldsymbol{C}^\times, \quad (a,d)\mapsto (\det a)^n.$$

By Theorem 2.6 (2), we have:

Lemma 4.4. *The following diagram commutes*:

$$\begin{array}{ccc}
V(n,k) := H_{\bar\partial}^{k(n-k)}\left(G^1 \underset{L^1(k)}{\times} \boldsymbol{C}_{\nu_n^{1(k)}}\right) & \xrightarrow{\mathscr{R}^1} & \mathscr{E}\left(G^1 \underset{K^1}{\times} \boldsymbol{C}_{\nu_k^{1(n)}}\right) \\
\text{Restriction}\big\downarrow & & \big\downarrow\text{Restriction} \\
W(n,k) = H_{\bar\partial}^{k(n-k)}\left(G \underset{L(k)}{\times} \boldsymbol{C}_{\nu_n^{(k)}}\right) & \xrightarrow{\mathscr{R}} & \mathscr{E}\left(G \underset{K}{\times} \boldsymbol{C}_{\nu_k^{(n)}}\right).
\end{array}$$

4.5. We recall a realization of a Hermitian symmetric space

$$G^1/K^1 = U(n,n)/(U(n)\times U(n))$$

as a classical bounded symmetric domain in $\boldsymbol{C}^{n^2} \simeq M(n,\boldsymbol{C})$. Let

$$\mathrm{Symm}(n,\boldsymbol{C}) := \{Z \in M(n,\boldsymbol{C}) : {}^tZ = Z\}.$$

We define unipotent subgroups of $G_{\boldsymbol{C}} \subset G_{\boldsymbol{C}}^1$ by

$$\overline{U} := \left\{\begin{pmatrix} I_n & O \\ Z & I_n\end{pmatrix} : Z \in \mathrm{Symm}(n,\boldsymbol{C})\right\} \subset \overline{U^1} := \left\{\begin{pmatrix} I_n & O \\ Z & I_n\end{pmatrix} : Z \in M(n,\boldsymbol{C})\right\}.$$

We consider open Bruhat cells induced from $G_{\boldsymbol{C}}^1 \supset \overline{U^1}Q^1(n)$ and $G_{\boldsymbol{C}} \supset \overline{U}Q(n)$:

$$\begin{array}{ccccc}
G^1/K^1 & \underset{\text{open}}{\overset{j_1^1}{\subset}} & G_{\boldsymbol{C}}^1/Q^1(n) & \underset{\text{open dense}}{\overset{j_2^1}{\hookleftarrow}} & \overline{U^1} \simeq M(n,\boldsymbol{C}) \\
\cup & & \cup & & \cup \\
G/K & \underset{\text{open}}{\overset{j_1}{\subset}} & G_{\boldsymbol{C}}/Q(n) & \underset{\text{open dense}}{\overset{j_2}{\hookleftarrow}} & \overline{U} \simeq \mathrm{Symm}(n,\boldsymbol{C}).
\end{array}$$

Here j_1 and j_1^1 are Borel embeddings. The key point in the above diagram is $j_1(G/K) \subset j_2(\overline{U})$ and $j_1^1(G^1/K^1) \subset j_2^1(\overline{U^1})$, so that $(j_2)^{-1}j_1$ (respectively, $(j_2^1)^{-1}j_1^1$) gives a global coordinate of G/K (respectively, G^1/K^1). We put

$$D := (j_2)^{-1}j_1(G/K) \subset D^1 := (j_2^1)^{-1}j_1^1(G^1/K^1) \subset M(n, \boldsymbol{C}).$$

Then D and D^1 are precisely given by classical bounded symmetric domains

$$D = \{Z \in \mathrm{Symm}(n, \boldsymbol{C}) : I_n - Z^*Z \gg 0\}$$

$$\subset D^1 = \{Z \in M(n, \boldsymbol{C}) : I_n - Z^*Z \gg 0\}.$$

We note that if $n = 1$, then $(G^1, G) = (U(1,1), Sp(1, \boldsymbol{R}))$ and

$$D = D^1 \simeq \{z \in \boldsymbol{C} : |z| < 1\} \quad \text{(the Poincaré disk)}.$$

Hereafter, we shall identify G^1/K^1 with D^1 and trivialize vector bundles over G^1/K^1 by using a global coordinate on $D^1 \subset M(n, \boldsymbol{C})$. With the identification $G^1/K^1 \simeq D^1$, the action of G^1 on D^1 is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Z = (c + dZ)(a + bZ)^{-1} \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n, n), \quad Z \in D^1,$$

because we have

$$\text{(4.5.1)} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} I_n & O \\ Z & I_n \end{pmatrix} = \begin{pmatrix} I_n & O \\ (c + dZ)(a + bZ)^{-1} & I_n \end{pmatrix}\begin{pmatrix} a + bZ & b \\ O & d - (c + dZ)(a + bZ)^{-1}b \end{pmatrix},$$

corresponding to the Bruhat cell $\overline{U^1}Q(n) \subset G_{\boldsymbol{C}}^1$.

Let χ_1, χ_2 and χ be characters of $U(n)$. In view of the embedding $K \subset K^1$ (see (4.3.3)), the following diagram commutes:

$$\begin{array}{rcl} (\chi_1, \chi_2) : K^1 = U(n) \times U(n) & \longrightarrow & \boldsymbol{C}^\times \\ \cup & & \| \\ \chi : K = U(n) & \longrightarrow & \boldsymbol{C}^\times \end{array}$$

if $\chi = \chi_1 \cdot (\chi_2)^{-1}$. We extend (χ_1, χ_2) to a holomorphic character

$$(\chi_1, \chi_2) : Q^1(n) \to \boldsymbol{C}^\times,$$

by letting the unipotent radical act trivially. Likewise, χ to a holomorphic character $Q(n) \to \boldsymbol{C}^\times$. Thus, if $\chi = \chi_1 \cdot (\chi_2)^{-1}$, we have G-equivariant line bundles:

$$
\begin{array}{ccc}
G \underset{K}{\times} \boldsymbol{C}_{\chi} & \hookrightarrow & G^1 \underset{K^1}{\times} \boldsymbol{C}_{(\chi_1,\chi_2)} \\
\downarrow & & \downarrow \\
G/K & \hookrightarrow & G^1/K^1.
\end{array} \tag{4.5.2}
$$

Let us trivialize the above bundles by using global coordinates of $\bar{U} \subset \overline{U^1}$. In view of the bundle maps:

$$
\begin{array}{ccccccc}
\bar{U} \times \boldsymbol{C} & \subset & G_{\boldsymbol{C}} \underset{Q(n)}{\times} \boldsymbol{C}_{\chi} & \longrightarrow & G^1_{\boldsymbol{C}} \underset{Q^1(n)}{\times} \boldsymbol{C}_{(\chi_1,\chi_2)} & \supset & \overline{U^1} \times \boldsymbol{C} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bar{U} & \subset & G_{\boldsymbol{C}}/Q(n) & \longrightarrow & G^1_{\boldsymbol{C}}/Q^1(n) & \supset & \overline{U^1},
\end{array}
$$

we have homeomorphisms of Fréchet spaces:

$$
\mathscr{E}\Big(G^1 \underset{K^1}{\times} \boldsymbol{C}_{(\chi_1,\chi_2)}\Big) \simeq C^{\infty}(D^1), \quad \mathscr{E}\Big(G \underset{K}{\times} \boldsymbol{C}_{\chi}\Big) \simeq C^{\infty}(D). \tag{4.5.3}
$$

Through the isomorphism, one can define the representation π^1 of $G^1 = U(n,n)$ on $C^{\infty}(D^1)$. By using (4.5.1), this representation is of the form:

$$
\begin{aligned}
&(\pi^1(g)F)(Z) \\
&\quad = \chi_1^{-1}(a+bZ)\chi_2^{-1}(d-(c+dZ)(a+bZ)^{-1}b)F((c+dZ)(a+bZ)^{-1}) \\
&\quad \text{if } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^1 = U(n,n) \quad \text{and} \quad Z \in D^1.
\end{aligned} \tag{4.5.4}
$$

Likewise, the representation π of $G = Sp(n,\boldsymbol{R})$ on $C^{\infty}(D)$ is defined to be:

$$
\begin{aligned}
&(\pi(g)F)(Z) = \chi^{-1}(a+\bar{c}Z)F((c+\bar{a}Z)(a+\bar{c}Z)^{-1}) \\
&\text{if } g^{-1} = \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} \in G = Sp(n,\boldsymbol{R}) \quad \text{and} \quad Z \in D.
\end{aligned} \tag{4.5.5}
$$

LEMMA 4.5. *If $\chi = \chi_1 \cdot (\chi_2)^{-1}$, then the restriction map $C^{\infty}(D^1) \to C^{\infty}(D)$ is a G-intertwining operator.*

PROOF. Lemma follows immediately from the G-equivariance of the bundle map (4.5.2) and from our definition of π and π^1. Lemma also follows from a direct computation by using the formulae (4.5.4), (4.5.5) and

$$
\bar{a} - (c+\bar{a}Z)(a+\bar{c}Z)^{-1}\bar{c} = (c^*)^{-1}(a+\bar{c}Z)^{-1}\bar{c} \tag{4.5.6}
$$

for any $a, c \in M(n,\boldsymbol{C})$ satisfying ${}^t ac = {}^t ca$, and $a^*a - c^*c = I_n$. □

In particular, we fix $k \in \{1, \dots, n\}$ and let $\chi = \nu_k^{(n)}$, and $(\chi_1, \chi_2) = \nu_k^{1(n)}$, then the corresponding representation π of G on $C^\infty(D)$ (π^1 of G^1 on $C^\infty(D^1)$, respectively) is denoted by

$$(4.5.7) \qquad (\tilde{\pi}_{n,k}(g)F)(Z) = (\det(a + \bar{c}Z))^{-k} F((c + \bar{a}Z)(a + \bar{c}Z)^{-1})$$

$$\text{if } g^{-1} = \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} \in G, \quad \text{and} \quad Z \in D,$$

$$(\tilde{\pi}^1_{n,k}(g)F)(Z) = (\det(a + bZ))^{-k} F((c + dZ)(a + bZ)^{-1})$$

$$\text{if } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^1, \quad \text{and} \quad Z \in D^1.$$

Then, Lemma 4.5 in this special case means that the restriction map $(\tilde{\pi}^1_{n,k}, C^\infty(D^1)) \to (\tilde{\pi}_{n,k}, C^\infty(D))$ is G-equivariant.

Similarly, we have isomorphisms:

$$\mathcal{O}\Big(G^1 \underset{K^1}{\times} \boldsymbol{C}_{\nu_k^{1(n)}}\Big) \simeq \mathcal{O}(D^1) \quad \text{and} \quad \mathcal{O}\Big(G \underset{K}{\times} \boldsymbol{C}_{\nu_k^{(n)}}\Big) \simeq \mathcal{O}(D),$$

through which $\mathcal{O}(D)$ is a subrepresentation of $(G, \tilde{\pi}_{n,k}, C^\infty(D))$, and $\mathcal{O}(D^1)$ is that of $(G^1, \tilde{\pi}^1_{n,k}, C^\infty(D^1))$. Then the restriction $\mathcal{O}(D^1) \to \mathcal{O}(D)$ also intertwines between the restriction $\tilde{\pi}^1_{n,k}|_G$ and $\tilde{\pi}_{n,k}$.

4.6. Now, we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. We recall $V(n,k) = H_{\bar{\partial}}^{k(n-k)}\Big(G^1 \underset{L^1(k)}{\times} \boldsymbol{C}_{(n,0)}\Big)$ is an irreducible infinite dimensional representation of $U(n,n)$ (see [**Se**], Fact 2.1 (1)). Here $\boldsymbol{C}_{(n,0)}$ denotes the one dimensional representation $\nu_n^{1(k)}$ of $L^1(k)$ (see §4.4). We write $V(n,k)_0$ ($\subset V(n,k)$) for the representation space the minimal K-type of $V(n,k)$, which is given by $\det^k \boxtimes \mathbf{1} \equiv \nu_k^{1(n)}$ ([**Se**], Fact 2.1 (2)). Then

$$\mathcal{R}^1(V(n,k)_0) = \boldsymbol{C}\mathbf{1}$$

in

$$\mathcal{O}(D^1) \subset C^\infty(D^1) \simeq \mathcal{E}\Big(G^1 \underset{K^1}{\times} \boldsymbol{C}_{(k,0)}\Big)$$

by [**Se**], Lemma 5.1. In view of the commutative diagram:

$$\begin{array}{ccccccc}
V(n,k) & = H_{\bar{\partial}}^{k(n-k)}\Big(G^1 \underset{L^1(k)}{\times} \boldsymbol{C}_{(n,0)}\Big) & \xrightarrow[\mathcal{R}^1]{} & \mathcal{E}\Big(G^1 \underset{K^1}{\times} \boldsymbol{C}_{(k,0)}\Big) & \simeq & C^\infty(D^1) \\
\downarrow & \downarrow \text{Rest} & & \downarrow & & \downarrow \text{Rest} \\
W(n,k) & = H_{\bar{\partial}}^{k(n-k)}\Big(G \underset{L(k)}{\times} \boldsymbol{C}_n\Big) & \xrightarrow[\mathcal{R}]{} & \mathcal{E}\Big(G \underset{K}{\times} \boldsymbol{C}_k\Big) & \simeq & C^\infty(D),
\end{array}$$

$$(4.6.1) \qquad \mathcal{R} \circ \mathrm{Rest}(V(n,k)_0) = \mathrm{Rest} \circ \mathcal{R}^1(V(n,k)_0).$$

Because the restriction of the constant function **1** on D^1 to a submanifold D does not vanish, the right side of (4.6.1) contains a non-zero holomorphic function. This implies that Image $\mathscr{R}$ is non-zero. Since $\mathscr{R}$ vanishes on $W(n,k)_-$ (by Proposition 3.1) and since $W(n,k)_+$ is irreducible (by Proposition 1.5), $\mathscr{R}$ must be injective on $W(n,k)_+$. Also, because $\mathcal{O}(D) \simeq \mathcal{O}\left(G \underset{K}{\times} \boldsymbol{C}_k\right)$ is a subrepresentation of $C^\infty(D) \simeq \mathscr{E}\left(G \underset{K}{\times} \boldsymbol{C}_k\right)$, and because $\mathscr{R}(W(n,k)_+)$ is irreducible, we conclude $\mathscr{R}(W(n,k)_+) \subset \mathcal{O}\left(G \underset{K}{\times} \boldsymbol{C}_k\right)$. This completes the proof of Proposition 4.1. $\square$

§5. Differential equations $(\mathscr{N}_{2k+1})$.

5.1. The purpose of this section is to prove that the image of the Penrose transform satisfies the differential equations $(\mathscr{N}_{2k+1})$ (Proposition 5.1). The surjectivity (difficult part of Main Theorem) will be proved in Proposition 6.1.

PROPOSITION 5.1. *With notation of Main Theorem in* §0, *we have*

$$\mathscr{R}W(n,k)_+ \subset \mathrm{Sol}(\mathscr{N}_{2k+1}) \quad (1 \le 2k \le n).$$

If $2k = n$, we note that $\mathrm{Sol}(\mathscr{N}_{n+1}) = \mathcal{O}\left(Sp(n,\boldsymbol{R}) \underset{U(n)}{\times} \boldsymbol{C}_k\right)$, namely, there is no differential equation (see (0.3.4)).

5.2. For $A = (a_{ij})_{1 \le i,j \le n} \in M(n,\boldsymbol{C})$ and for $1 \le l \le n$, the l-th principal minor of A is denoted by

$$\det_l(A) := \det(a_{ij})_{1 \le i,j \le l}.$$

Let $\mathrm{Pol}^m(V)$ be the complex vector space of polynomials on a vector space V of homogeneous degree m. Then, the direct product group $GL(n,\boldsymbol{C}) \times GL(n,\boldsymbol{C})$ acts on $\mathrm{Pol}^m(M(n,\boldsymbol{C}))$ by

$$f(X) \mapsto f({}^t g_1 X g_2) \quad (g_1, g_2 \in GL(n,\boldsymbol{C}), f \in \mathrm{Pol}^m(M(n,\boldsymbol{C}))).$$

For $I, J \subset \{1,2,\ldots,n\}$ such that $|I| = |J| = l$, we define a polynomial on $M(n,\boldsymbol{C})$ of homogeneous degree l by

$$\det_{IJ}(A) := \det(a_{ij})_{i \in I, j \in J}, \quad A = (a_{ij})_{1 \le i,j \le n}.$$

In particular, we have

$$\det_l(A) = \det_{\{1,2,\ldots,l\}\{1,2,\ldots,l\}}(A).$$

The polynomial $\det_{IJ}$ is also a polynomial on $\mathrm{Symm}(n,\boldsymbol{C})$ by the restriction, denoted by the same $\det_{IJ}$. We note that $\det_{IJ} = \det_{JI}$ as polynomials on $\mathrm{Symm}(n,\boldsymbol{C})$.

5.3. We define two subspaces of $\mathrm{Pol}^m(M(n, \boldsymbol{C}))$ by

$$\widetilde{V_m} = \boldsymbol{C}\text{-span}\langle \det_m(g_1 X g_2) : g_1, g_2 \in GL(n, \boldsymbol{C})\rangle,$$

$$\widetilde{W_m} = \boldsymbol{C}\text{-span}\langle \det_{IJ}(X) : |I| = |J| = m\rangle.$$

LEMMA 5.3. *For* $1 \le m \le n$, *we have*

$$\widetilde{V_m} = \widetilde{W_m},$$

on which $GL(n, \boldsymbol{C}) \times GL(n, \boldsymbol{C})$ *acts irreducibly as*

$$F(GL(n, \boldsymbol{C}), (\underbrace{1, \dots, 1}_{m}, \underbrace{0, \dots, 0}_{n-m})) \boxtimes F(GL(n, \boldsymbol{C}), (\underbrace{1, \dots, 1}_{m}, \underbrace{0, \dots, 0}_{n-m})).$$

PROOF. By the elementary transformation of matrices, we have $\widetilde{W_m} \subset \widetilde{V_m}$. Let $g_1, g_2 \in GL(n, \boldsymbol{C})$ be upper triangular matrices with 1 in the diagonal entries. Then $\det({}^t g_1 X g_2) = \det(X)$. This means that $\det_m(X)$ is annihilated by $\mathfrak{n} \times \mathfrak{n}$, where $\mathfrak{n}$ is the nilpotent Lie algebra consisting of strictly upper triangular matrices. Since $\widetilde{V_m}$ is a $GL(n, \boldsymbol{C}) \times GL(n, \boldsymbol{C})$-span of a highest weight vector $\det_m(X)$, $\widetilde{V_m}$ is irreducible and isomorphic to

$$F(GL(n, \boldsymbol{C}), (\underbrace{1, \dots, 1}_{m}, \underbrace{0, \dots, 0}_{n-m})) \boxtimes F(GL(n, \boldsymbol{C}), (\underbrace{1, \dots, 1}_{m}, \underbrace{0, \dots, 0}_{n-m}))$$

as a $GL(n, \boldsymbol{C}) \times GL(n, \boldsymbol{C})$-module. Therefore,

$$\dim \widetilde{V_m} = \left(\dim \textstyle\bigwedge^m (\boldsymbol{C}^n)\right)^2 = \binom{n}{m}^2.$$

On the other hand, $\{\det_{IJ}(X)\}$ are linearly independent. Thus

$$\dim \widetilde{W_m} = \binom{n}{m}^2.$$

Now, we have proved $\dim \widetilde{W_m} = \dim \widetilde{V_m}$. As $\widetilde{W_m} \subset \widetilde{V_m}$, we have $\widetilde{W_m} = \widetilde{V_m}$. □

5.4. We define two subspaces of $\mathrm{Pol}^m(\mathrm{Symm}(n, \boldsymbol{C}))$ by

$$V_m = \boldsymbol{C}\text{-span}\langle \det_m({}^t g X g) : g \in GL(n, \boldsymbol{C})\rangle,$$

$$W_m = \boldsymbol{C}\text{-span}\langle \det_{IJ}(X) : |I| = |J| = m\rangle,$$

where $X \in \mathrm{Symm}(n, \boldsymbol{C})$ stands for a variable of polynomials.

LEMMA 5.4. *For* $1 \le m \le n$, *we have*

$$V_m = W_m,$$

on which $GL(n, \boldsymbol{C})$ *acts as an irreducible representation*

$$F(GL(n, \boldsymbol{C}), (\underbrace{2, \dots, 2}_{m}, \underbrace{0, \dots, 0}_{n-m})).$$

Here, $GL(n, \boldsymbol{C})$ *acts on* $\mathrm{Pol}^m(\mathrm{Symm}(n, \boldsymbol{C}))$ *by*

$$f(X) \mapsto f({}^t gXg) \quad \text{for } g \in GL(n, \boldsymbol{C}),\ f \in \mathrm{Pol}^m(\mathrm{Symm}(n, \boldsymbol{C})).$$

Proof. V_m is a $GL(n, \boldsymbol{C})$-span of a highest weight vector $\det_m(X) \in \mathrm{Pol}^m(\mathrm{Symm}(n, \boldsymbol{C}))$, and therefore we have an isomorphism of $GL(n, \boldsymbol{C})$-modules:

$$V_m \simeq F(GL(n, \boldsymbol{C}), (\underbrace{2, \dots, 2}_{m}, \underbrace{0, \dots, 0}_{n-m})).$$

Let $\widetilde{V_m'} := \boldsymbol{C}\text{-span}\langle \det_m({}^t gXg) : g \in GL(n, \boldsymbol{C}) \rangle \subset \mathrm{Pol}^m(M(n, \boldsymbol{C}))$. We write

$$q : \mathrm{Pol}^m(M(n, \boldsymbol{C})) \to \mathrm{Pol}^m(\mathrm{Symm}(n, \boldsymbol{C}))$$

for the restriction map. If we consider the diagonal action of $GL(n, \boldsymbol{C})$ $(\subset GL(n, \boldsymbol{C}) \times GL(n, \boldsymbol{C}))$ on $\mathrm{Pol}^m(M(n, \boldsymbol{C}))$, then q respects the action of $GL(n, \boldsymbol{C})$. Clearly,

$$W_m = q(\widetilde{W_m}) \quad \text{and} \quad V_m = q(\widetilde{V_m'}).$$

By Lemma 5.3, we have $\widetilde{W_m} = \widetilde{V_m} \supset \widetilde{V_m'}$, and therefore we have proved

$$W_m \supset V_m.$$

Let us prove the opposite inclusion $W_m \subset V_m$. We shall prove $\det_{IJ}(X) \in V_m$ for any I, J by using the differential action of $GL(n, \boldsymbol{C})$ on V_m.

Let $E_{ab} \in M(n, \boldsymbol{C})$ be the matrix unit $(1 \le a, b \le n)$. Then we have

$${}^t(I_n + sE_{ab})X(I_n + sE_{ab}) = X + s(A + {}^t A) + O(s^2),$$

where $A = \sum_{j=1}^n x_{aj} E_{bj}$ and $X = (x_{ij})_{1 \le i, j \le n} \in \mathrm{Symm}(n, \boldsymbol{C})$.

Suppose $1 \le b \le m$ and $m + 1 \le a \le n$. We set

$$I := \{1, \dots, m\} \quad \text{and} \quad J := (I \backslash \{b\}) \cup \{a\}.$$

Then we have

$$\begin{aligned} &\left.\frac{d}{ds}\right|_{s=0} \det_{II}({}^t(I_n + sE_{ab})X(I_n + sE_{ab})) \\ &= \left.\frac{d}{ds}\right|_{s=0} \det_{II}(X + s(A + {}^t A) + O(s^2)) \\ &= 2 \det_{IJ}(X) \\ &= 2 \det_{JI}(X). \end{aligned}$$

This implies that $\det_{IJ} \in V_m$ because $\det_{II} \in V_m$ and because V_m is a representation space of $GL(n, \boldsymbol{C})$. The iteration of this procedure shows that $W_m \subset V_m$. Hence we have proved $V_m = W_m$. □

5.5. We recall on matrix realization of $Sp(n, \boldsymbol{R})$ (Lemma 4.2):

$$Sp(n, \boldsymbol{R}) = \left\{ \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} : {}^t ac = {}^t ca, a^* a - c^* c = I_n \right\}.$$

If $g = \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}$, then $g^{-1} = \begin{pmatrix} {}^t\bar{a} & -{}^t\bar{c} \\ -{}^tc & {}^ta \end{pmatrix} \in Sp(n, \boldsymbol{R})$. Thus we have

$${}^t({}^t\bar{a})(-{}^tc) = {}^t(-{}^tc){}^t\bar{a},$$

namely, $a{}^t\bar{c} = \bar{c}{}^ta$.

Here is a fine structure of the $U(n)$-module $\mathrm{Pol}(\mathrm{Symm}(n, \boldsymbol{C})) = \bigoplus_{r=0}^{\infty} \mathrm{Pol}^r(\mathrm{Symm}(n, \boldsymbol{C}))$ (cf. Lemma 1.7).

LEMMA 5.5. *For $r_j \in \boldsymbol{Z}$ $(1 \le j \le n)$, $r_1 \ge \cdots \ge r_n$, we set*

$$\sigma := F(U(n), (r_1, \dots, r_n)) \in \widehat{U(n)}.$$

Let $r \in \boldsymbol{N}$. Then the following three conditions on $r, r_1, \dots, r_n$ are equivalent:

i) $\mathrm{Hom}_{U(n)}(\sigma, \mathrm{Pol}^r(\mathrm{Symm}(n, \boldsymbol{C}))) \neq 0$.

ii) $\mathrm{Hom}_{U(n)}(\sigma, \mathrm{Pol}^r(\mathrm{Symm}(n, \boldsymbol{C}))) \simeq \boldsymbol{C}$.

iii) $r_j \in 2\boldsymbol{Z}$ $(1 \le j \le n)$, $r_n \ge 0$ *and* $r_1 + \cdots + r_n = 2r$.

We write V_σ as the σ-isotopic subspace of $\mathrm{Pol}^r(\mathrm{Symm}(n, \boldsymbol{C}))$.

We set

$$\beta_\sigma(s) := \prod_{h=1}^{n} \prod_{i=1}^{r_h/2} \left(s - i + \frac{h+1}{2} \right). \tag{5.5.1}$$

5.6. Then it follows from [**Sh**], Theorem 4.3 that:

LEMMA 5.6. *If $s \in \boldsymbol{C}$ satisfies $\beta_\sigma(s) = 0$ then*

$$\xi\left(\frac{\partial}{\partial Z}\right) \det(cZ + d)^s = 0 \quad \textit{for any } \xi \in V_\sigma$$

for any $Z \in \mathrm{Symm}(n, \boldsymbol{C})$, and $c, d \in M(n, \boldsymbol{C})$ such that $c{}^td = d{}^tc$, $\mathrm{rank}(c, d) = n$. Here, for $\xi \in \mathrm{Pol}^r(\mathrm{Symm}(n, \boldsymbol{C}))$, we write $\xi(\partial/\partial Z)$ for the differential operator of degree r on $\mathrm{Symm}(n, \boldsymbol{C})$ defined by using the notation (0.2.2), (0.2.3).

5.7. Let us consider a special case of Lemma 5.6 by putting

$$\sigma := F(U(n), (\underbrace{2, \dots, 2}_{l}, \underbrace{0, \dots, 0}_{n-l})).$$

As $r_1 = \cdots = r_l = 2$, the formula (5.5.1) is given by

$$\beta_\sigma(s) = s\left(s+\frac{1}{2}\right)\cdots\left(s+\frac{l-1}{2}\right).$$

Thus, $\beta_\sigma(s) = 0$ if and only if $-2s \in \boldsymbol{N}$ and $-2s \le l-1$. On the other hand,

$$V_\sigma \simeq V_l = W_l$$

by Lemma 5.4. Since $\det_{IJ}(\partial/\partial Z) = P(I,J)$ (see Definition (0.3.1)) we have:

Proposition 5.7. *Suppose $2k, l \in \boldsymbol{N}$ satisfy $2k \le l-1$. Then for any $I, J \subset \{1, 2, \ldots, n\}$ such that $|I| = |J| = l$, we have*

$$P(I,J)\det(cZ+d)^{-k} = 0$$

for any $c, d \in M(n, \boldsymbol{C})$ such that $c^t d = d^t c$, $\operatorname{rank}(c,d) = n$ and $Z \in \operatorname{Symm}(n, \boldsymbol{C})$.

5.8. Now we complete the proof of Proposition 5.1. We have shown in §4 that

$$\mathscr{R}(W(n,k)_+) \text{ contains } \mathbf{1} \in \mathcal{O}(D) \simeq \mathcal{O}\left(G \underset{K}{\times} \boldsymbol{C}_k\right).$$

Since $\mathscr{R} : W(n,k)_+ \to \mathcal{O}\left(G \underset{K}{\times} \boldsymbol{C}_k\right)$ is a G-homomorphism, and since $W(n,k)_+$ is an irreducible G-module, we conclude:

Lemma 5.8. *$\mathscr{R}(W(n,k)_+)$ contains a subspace*

$$\boldsymbol{C}\text{-}span\langle \tilde{\pi}_{n,k}(g)\mathbf{1} : g \in Sp(n, \boldsymbol{R})\rangle$$

as a dense set in the Fréchet topology on $\mathcal{O}(D)$.

On the other hand, if $g^{-1} = \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} \in Sp(n, \boldsymbol{R})$, and if $Z \in D \subset \operatorname{Symm}(n, \boldsymbol{C})$, we have from the definition (4.5.7)

$$(\tilde{\pi}_{n,k}(g)\mathbf{1})(Z) = (\det(a + \bar{c}Z))^{-k}.$$

Since $\bar{c}^t a = a^t \bar{c}$ and $\operatorname{rank}(\bar{c}, a) = n$, we have

$$P(I,J)(\tilde{\pi}_{n,k}(g)\mathbf{1}) = 0 \quad \text{if } |I| = |J| = 2k+1$$

by Proposition 5.7. Here $P(I,J)$ annihilates $\boldsymbol{C}\text{-span}\langle \tilde{\pi}_{n,k}(g)\mathbf{1} : g \in Sp(n, \boldsymbol{R})\rangle$.
Combining with Lemma 5.8, we have proved:

$$\mathscr{R}(W(n,k)_+) \subset \operatorname{Sol}(\mathscr{N}_{2k+1}).$$

Hence Proposition 5.1 is proved. □

§6. Surjectivity of the Penrose transform $\mathscr{R}$.

6.1. We have already proved that
1) $\mathscr{R}(W(n,k)_+) \subset \mathrm{Sol}(\mathscr{N}_{2k+1})$ (see Proposition 5.1),
2) $\mathscr{R}|_{W(n,k)_+} : W(n,k)_+ \to \mathrm{Sol}(\mathscr{N}_{2k+1})$ is injective (see Proposition 4.1).
The aim of this section is to prove the surjectivity:

Proposition 6.1. $\mathscr{R}(W(n,k)_+) = \mathrm{Sol}(\mathscr{N}_{2k+1})$.

Then the proof of the Main Theorem (see §0) will be finished. In view of (1) and (2), the proof of Proposition 6.1 will be completed if we show that both $W(n,k)_+$ and $\mathrm{Sol}(\mathscr{N}_{2k+1})$ have the same K-types, because $W(n,k)_+$ is a maximal globalization.

6.2. We recall from (4.5.7) that the representation $\tilde{\pi}_{n,k}$ of $G = Sp(n,\boldsymbol{R})$ on $\mathscr{O}\left(G \underset{K}{\times} \boldsymbol{C}_k\right)$ is given by

$$(\tilde{\pi}_{n,k}(g)F)(Z) = (\det(a+\bar{c}Z))^{-k} F((c+\bar{a}Z)(a+\bar{c}Z)^{-1})$$

$$\text{for } g^{-1} = \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} \in G = Sp(n,\boldsymbol{R}), \quad \text{and} \quad Z \in D.$$

In particular, if $g^{-1} = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \in K \simeq U(n)$, then

$$(\tilde{\pi}_{n,k}(g)F)(Z) = (\det a)^{-k} F(\bar{a}Za^{-1}), \quad \text{for } Z \in D.$$

If we consider the double covering $Mp(n,\boldsymbol{R})$ of $Sp(n,\boldsymbol{R})$, then k is allowed to be a half integer. We shall obtain the K-type formula for $\mathrm{Sol}(\mathscr{N}_{2k+1})$ for $k \in \{1/2, 1, 3/2, 2, \dots, (n-1)/2, n/2\}$. (But, we need only the case where $k \in \boldsymbol{Z}$. Note that $\mathrm{Sol}(\mathscr{N}_{2k+1}) = \mathscr{O}\left(G \underset{K}{\times} \boldsymbol{C}_k\right)$ if n is even and $k = n/2$.) In the light of

$$^t(I_n + sE_{ab})Z(I_n + sE_{ab}) = Z + s(E_{ba}Z + ZE_{ab}) + O(s^2),$$

the differential action $d\tilde{\pi}_{n,k}$ of $\mathfrak{sp}(n,\boldsymbol{C})$ on $\mathscr{O}(D) \simeq \mathscr{O}\left(G \underset{K}{\times} \boldsymbol{C}_k\right)$ is given by

$$(6.2.1) \qquad (d\tilde{\pi}_{n,k}(E_{ab})F)(Z) = \sum_{i=1}^{n} (1+\delta_{ib}) z_{ia} \frac{\partial}{\partial z_{ib}} F(Z) + k\delta_{ab} F(Z)$$

for $1 \le a, b \le n$, and $F \in \mathscr{O}(D)$ $\left(\simeq \mathscr{O}\left(G \underset{K}{\times} \boldsymbol{C}_k\right)\right)$.

Remark 6.2. Here, we have used the global coordinate of $\mathrm{Symm}(n,\boldsymbol{C})$ by z_{ij} $(1 \le i \le j \le n)$ for $Z = (z_{ij})_{1 \le i \le j \le n} \in \mathrm{Symm}(n,\boldsymbol{C})$ (see (0.2.2)). It is convenient to allow z_{ji} (resp. $\partial/\partial z_{ji}$) to denote z_{ij} (resp. $\partial/\partial z_{ij}$) $(1 \le i \le j \le n)$.

6.3. We define a nilpotent Lie subalgebra $\mathfrak{n}(\mathfrak{k})$ of $\mathfrak{k} \simeq \mathfrak{gl}(n, \boldsymbol{C})$ by

$$\mathfrak{n}(\mathfrak{k}) := \{X = (x_{ij})_{1 \le i, j \le n} \in \mathfrak{gl}(n, \boldsymbol{C}) : x_{ij} = 0 \text{ if } i \ge j\} \subset \mathfrak{gl}(n, \boldsymbol{C}).$$

For any open set Ω in $\mathrm{Symm}(n, \boldsymbol{C})$, we define

(6.3.1)
$$\mathcal{O}(\Omega)^{\mathfrak{n}(\mathfrak{k})} := \left\{ F \in \mathcal{O}(\Omega) : \sum_{i=1}^{n} (1 + \delta_{ib}) z_{ia} \frac{\partial}{\partial z_{ib}} F(Z) = 0 \text{ for } 1 \le a < b \le n \right\}.$$

Then, it follow from (6.2.1) that

$$\mathcal{O}(D)^{\mathfrak{n}(\mathfrak{k})} = \{F \in \mathcal{O}(D) : d\tilde{\pi}_{n,k}(X) F(Z) = 0 \text{ for any } X \in \mathfrak{n}(\mathfrak{k})\}.$$

Here we note that the right side is in fact independent of k.

We define a Borel subgroup $B(K_{\boldsymbol{C}})$, its unipotent radical $N(K_{\boldsymbol{C}})$ and a Cartan subgroup $H(K_{\boldsymbol{C}})$ of $K_{\boldsymbol{C}} = GL(n, \boldsymbol{C})$ by

$$B(K_{\boldsymbol{C}}) = \{(g_{ij}) \in GL(n, \boldsymbol{C}) : g_{ij} = 0 \ (1 \le j < i \le n)\} = N(K_{\boldsymbol{C}}) H(K_{\boldsymbol{C}}),$$

$$N(K_{\boldsymbol{C}}) = \{(g_{ij}) \in GL(n, \boldsymbol{C}) : g_{ij} = 0 \ (1 \le j < i \le n) \text{ and } g_{ll} = 1 \ (1 \le l \le n)\},$$

$$H(K_{\boldsymbol{C}}) = \{(g_{ij}) \in GL(n, \boldsymbol{C}) : g_{ij} = 0 \ (i \ne j)\}.$$

The Borel subgroup $B(K_{\boldsymbol{C}})$ acts on $\mathrm{Symm}(n, \boldsymbol{C})$ by

$$Z \mapsto {}^t g^{-1} Z g^{-1}, \quad Z \in \mathrm{Symm}(n, \boldsymbol{C}), \quad g \in B(K_{\boldsymbol{C}}).$$

There is a unique open $B(K_{\boldsymbol{C}})$-orbit, denoted by $\mathrm{Symm}(n, \boldsymbol{C})^{\mathrm{reg}}$, in $\mathrm{Symm}(n, \boldsymbol{C})$. Thus, $(B(K_{\boldsymbol{C}}), \mathrm{Symm}(n, \boldsymbol{C}))$ forms a prehomogeneous vector space.

6.4. We define

$$W := \{\mathrm{diag}(y_1, \ldots, y_n) : y_1, \ldots, y_n \in \boldsymbol{C}^\times\}.$$

As we shall see in the following Lemma 6.4, $W \subset \mathrm{Symm}(n, \boldsymbol{C})^{\mathrm{reg}}$ and

$$\mathrm{Symm}(n, \boldsymbol{C})^{\mathrm{reg}} = \{Z \in \mathrm{Symm}(n, \boldsymbol{C}) : \det_l(Z) \ne 0 \ (1 \le l \le n)\}.$$

Lemma 6.4. *We consider the following*:

(6.4.1) $$\varphi : N(K_{\boldsymbol{C}}) \times W \to \mathrm{Symm}(n, \boldsymbol{C}), \quad (g, y) \mapsto {}^t g y g.$$

1) *The image of* φ *is* $\mathrm{Symm}(n, \boldsymbol{C})^{\mathrm{reg}}$.
2) (*inversion formula of* φ) *If* $Z = \varphi(g, y)$, *then*

(6.4.2) $$y_l = \frac{\det_l Z}{\det_{l-1} Z} \quad (1 \le l \le n),$$

(6.4.3) $$g_{ij} = \frac{\det_i(Z \tau_{ij})}{\det_i Z} \quad (1 \le i < j \le n).$$

Here, $g=(g_{ij})_{1\le i,j\le n}\in N(K_{\boldsymbol{C}})$ *and* $y=\mathrm{diag}(y_1,\dots,y_n)\in W$*. We write* $\det_{-1} Z=1$ *and* $\tau_{ij}\in GL(n,\boldsymbol{R})$ *stands for the permutation matrix, so that*

$$\det_i(Z\tau_{ij})=\det\begin{pmatrix} z_{11} & \cdots & z_{1,i-1} & z_{1j}\\ \vdots & & \vdots & \vdots\\ z_{i1} & \cdots & z_{i,i-1} & z_{ij}\end{pmatrix}.$$

PROOF. (1) is deduced from (2). Let us prove (2). Fix $1\le i<j\le n$. We divide $n\times n$-matrices into $(i,n-i)$ block and put

$$Z=\begin{pmatrix} Z_1 & Z_2\\ Z_3 & Z_4\end{pmatrix}\in M(n,\boldsymbol{C})\quad \text{with } \det_i Z=\det Z_1\neq 0,$$

$$A=\begin{pmatrix} A_1 & O\\ A_3 & A_4\end{pmatrix}\in GL(n,\boldsymbol{C})\quad \text{(lower triangular matrix)},$$

$$B=\begin{pmatrix} B_1 & B_2\\ O & B_4\end{pmatrix}\in GL(n,\boldsymbol{C})\quad \text{(strictly upper triangular matrix)}.$$

If $Z=AB$, then we have

$$A_1B_1=Z_1,\quad A_1B_2=Z_2,\quad A_3B_1=Z_3,\quad A_3B_2+A_4B_4=Z_4.$$

In particular,

$$Z_1(B_1^{-1}B_2)=(A_1B_1)(B_1^{-1}B_2)=Z_2.$$

As $B_1\in GL(i,\boldsymbol{C})$ is a strictly upper triangular matrix with diagonal entry 1, we have:

$$B_{ij}=(B_2)_{i,j-i}=(B_1^{-1}B_2)_{i,j-i}.$$

Thus, the Cramer's formula leads to (6.4.3). Other statements are easy. □

It follows from Lemma 6.4 that the restriction map

$$\mathrm{Rest}:\mathcal{O}(\mathrm{Symm}(n,\boldsymbol{C})^{\mathrm{reg}})\to\mathcal{O}(W)$$

is injective when restricted to

$$\mathcal{O}(\mathrm{Symm}(n,\boldsymbol{C})^{\mathrm{reg}})^{\mathfrak{n}(\mathfrak{k})}\to\mathcal{O}(W).$$

6.5. We define a differential operator on W by

$$Q:=\prod_{j=1}^{n}\left(y_j\frac{\partial}{\partial y_j}+\frac{1}{2}(n+2-j)\right)\frac{1}{y_j}. \tag{6.5.1}$$

The following Lemma can be proved similarly as in [**Se**], §4.

Lemma 6.5. *The following diagram commutes*:

$$\begin{array}{ccc} \mathcal{O}(\mathrm{Symm}(n, \boldsymbol{C})^{\mathrm{reg}})^{\mathfrak{n}(\mathfrak{k})} & \xrightarrow{\mathrm{Rest}} & \mathcal{O}(W) \\ \Big\downarrow \det\left(\frac{\partial}{\partial Z}\right) & & \Big\downarrow Q \\ \mathcal{O}(\mathrm{Symm}(n, \boldsymbol{C})^{\mathrm{reg}})^{\mathfrak{n}(\mathfrak{k})} & \xrightarrow{\mathrm{Rest}} & \mathcal{O}(W). \end{array}$$

6.6. Then, we have a simple Lemma:

Lemma 6.6. *Let* $\mathcal{O}_0$ *denote the germ of* $\mathcal{O}(\boldsymbol{C}^n)$ *at* $0 \in \boldsymbol{C}^n$. *Then,*

$$\{h \in \mathcal{O}_0 : Qh = 0 \text{ in } V \cap (\boldsymbol{C}^\times)^n \text{ for some open set } 0 \in V \subset \boldsymbol{C}^n\}$$
$$= \left\{h \in \mathcal{O}_0 : \frac{\partial h}{\partial y_n} = 0\right\}.$$

Proof. We shall write $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$. Let

$$h(y) = \sum_{\alpha \in \boldsymbol{N}^n} b_\alpha y^\alpha$$

be the Taylor expansion of $h(y)$ at 0. Then

$$Qh(y) = \sum_{\alpha \in \boldsymbol{N}^n} b_\alpha \left(\alpha_1 + \frac{n-1}{2}\right)\left(\alpha_2 + \frac{n-2}{2}\right) \cdots \left(\alpha_n + \frac{n-n}{2}\right) y^{\alpha - (1, \dots, 1)}.$$

Therefore, $Qh = 0$ if and only if $b_\alpha = 0$ for all $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $\alpha_n \neq 0$. Namely, the Taylor expansion of $h(y)$ is of the form:

$$h(y) = \sum_{\alpha_1, \dots, \alpha_{n-1} \in \boldsymbol{N}} b_{(\alpha_1, \dots, \alpha_{n-1}, 0)} y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}}.$$

This condition is equivalent to $(\partial h / \partial y_n)(y) = 0$. □

6.7. Let Ω be an open neighbourhood of O in $\mathrm{Symm}(n, \boldsymbol{C})$ such that $W \cap \Omega$ is connected.

Lemma 6.7. *If* $f \in \mathcal{O}(\Omega)^{\mathfrak{n}(\mathfrak{k})}$ *satisfies* $\det(\partial/\partial Z) f = 0$, *then* $\partial f / \partial z_{nn}|_{W \cap \Omega} = 0$.
Furthermore, we assume $N(K_{\boldsymbol{C}}) \cdot (W \cap \Omega) \supset \Omega$ *and* $f(n \cdot Z) = f(Z)$ *for any* $n \in N(K_{\boldsymbol{C}})$ *and* $Z \in \Omega$ *such that* $n \cdot Z \in \Omega$. *Then* $d\tilde{\pi}_{n,k}(E_{nn}) f = kf$ *in* Ω.

We recall that $n \cdot Z = {}^t n^{-1} Z n^{-1}$.

Proof. We set $h(y_1, \dots, y_n) := f(\mathrm{diag}(y_1, \dots, y_n)) \in \mathcal{O}(W \cap \Omega)$. By Lemma

6.5, $\det(\partial/\partial Z)f = 0$ implies $Qh = 0$ and then $\partial h/\partial y_n = 0$ on $W \cap \Omega$ by Lemma 6.6. By using the coordinate map φ (see (6.4.1)), we have

$$\frac{\partial f}{\partial z_{nn}} = \sum_{1 \le i < j \le n} \frac{\partial g_{ij}}{\partial z_{nn}} \frac{\partial f}{\partial g_{ij}} + \sum_{l=1}^{n} \frac{\partial y_l}{\partial z_{nn}} \frac{\partial h}{\partial y_l}.$$

As f is $\mathfrak{n}(\mathfrak{k})$-invariant, $\partial f/\partial g_{ij} \equiv 0$. By (6.4.2), $\partial y_l/\partial z_{nn} \equiv 0$ except for $l = n$. Hence we have

$$\left.\frac{\partial f}{\partial z_{nn}}\right|_{W \cap \Omega} \equiv 0.$$

In view of $z_{ij} = 0$ $(i \ne j)$ on W, we have for $Z \in W$

$$\begin{aligned} d\tilde{\pi}_{n,k}(E_{nn})f(Z)|_W &= \sum_{i=1}^{n} (1 + \delta_{in}) z_{in} \frac{\partial}{\partial z_{in}} f(Z) + k\delta_{nn} f(Z) \\ &= \left(2z_{nn} \frac{\partial}{\partial z_{nn}} + k\right) f(Z) = kf(Z). \end{aligned}$$

Now, let $Z \in \Omega$. We take $n \in N(K_{\boldsymbol{C}})$ such that $n^{-1} \cdot Z = {}^t nZn \in W \cap \Omega$. Then

$$\begin{aligned} (d\tilde{\pi}_{n,k}(E_{nn})f)(Z) &= \left.\frac{d}{dt}\right|_{t=0} f(e^{-tE_{nn}} \cdot Z) = \left.\frac{d}{dt}\right|_{t=0} f(e^{-tE_{nn}} n e^{tE_{nn}} e^{-tE_{nn}} n^{-1} \cdot Z) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(e^{-tE_{nn}} n^{-1} \cdot Z) = (d\tilde{\pi}_{n,k}(E_{nn})f)(n^{-1} \cdot Z) \\ &= kf(n^{-1} \cdot Z) \\ &= kf(Z). \end{aligned}$$

Thus, $d\tilde{\pi}_{n,k}(E_{nn})f = kf$ in Ω. □

Lemma 6.8. *Let* $k \in \{1/2, 1, \ldots, (n-1)/2, n/2\}$. *If* $f \in \mathcal{O}(\Omega)^{\mathfrak{n}(\mathfrak{k})} \cap \mathrm{Sol}(\mathcal{N}_{2k+1})$, *then*

$$d\tilde{\pi}_{n,k}(E_{aa})f = kf \quad (2k+1 \le a \le n).$$

Proof. First we prove Lemma in the case $2k+1 = n$. Let $I_n \in U \subset N(K_{\boldsymbol{C}})$ and $O \in V \subset D$ be (small) connected open neighbourhoods, respectively. We put an open set of $\mathrm{Symm}(n, \boldsymbol{C})$ by

$$\Omega = U \cdot V = \{{}^t nZn : n \in U, Z \in V\}.$$

If U and V are small enough, we may and do assume $(O \in)$ $\Omega \subset D$. It follows from

$$d\tilde{\pi}_{n,k}(X)f = 0 \quad \text{for } X \in \mathfrak{n}(\mathfrak{k})$$

that

$$f(n \cdot Z) = f(Z) \quad \text{for any } n \in U \ (\subset N(K_{\boldsymbol{C}})) \quad \text{and} \quad Z \in V \ (\subset D).$$

Then, by the previous Lemma 6.7, we have

$$d\tilde{\pi}_{n,k}(E_{nn})f = kf.$$

Next, we consider the case $2k+1 \leq n$. We fix $I = \{i_1, \ldots, i_{2k+1}\} \subset \{1, 2, \ldots, n\}$ such that $|I| = 2k+1$. In place of $\mathrm{Symm}(n, \boldsymbol{C})$, we consider $Z^{(I)} := (z_{ij})_{i,j \in I} \in \mathrm{Symm}(2k+1, \boldsymbol{C})$ and apply the above argument. Then we have $d\tilde{\pi}_{n,k}(E_{i_{2k+1}, i_{2k+1}})f = kf$. The possible values of i_{2k+1} are $2k+1, 2k+2, \ldots, n$ under the condition $i_1 < \cdots < i_{2k+1}$ and $I = \{i_1, \ldots, i_{2k+1}\} \subset \{1, 2, \ldots, n\}$. Therefore we have

$$d\tilde{\pi}_{n,k}(E_{aa})f = kf \quad \text{for any } a \text{ with } 2k+1 \leq a \leq n.$$

Hence we have proved the lemma. □

6.9. Let $\tilde{U}(n)$ be the double covering group of $U(n)$. Here is a K-type formula of $\mathrm{Sol}(\mathcal{N}_{2k+1})$:

Proposition 6.9. *Let* $k = 1/2, 1, \ldots, (n-1)/2, n/2$.

$$\mathrm{Sol}(\mathcal{N}_{2k+1})_K \subset \bigoplus_{\substack{a_1 \geq \cdots \geq a_{2k} \geq 0 \\ a_j \in \boldsymbol{N}}} F(\tilde{U}(n), (\underbrace{2a_1 + k, \ldots, 2a_{2k} + k}_{2k}, \underbrace{k, \ldots, k}_{n-2k})). \tag{6.9.1}$$

Remark. (6.9.1) is an equality. This will be proved in the next subsection for $k \in \boldsymbol{Z}$.

Proof. First, we recall a multiplicity free K-type formula (Lemma 1.7):

$$\mathcal{O}\Big(G \underset{K}{\times} \boldsymbol{C}_k\Big)_K \simeq \bigoplus_{\substack{a_1 \geq \cdots \geq a_n \geq 0 \\ a_j \in \boldsymbol{N}}} F(\tilde{U}(n), (2a_1, 2a_2, \ldots, 2a_n) + k\mathbf{1}_n).$$

Let $f \in \mathrm{Sol}(\mathcal{N}_{2k+1})_K$ be a K-highest weight vector with respect to $\mathfrak{n}(\mathfrak{k}) \subset \mathfrak{k}$. Let $\sum_{j=1}^n (2a_j + k) f_j \in \mathfrak{t}^*$ be its highest weight. It follows from Lemma 6.8 that

$$d\tilde{\pi}_{n,k}(E_{jj})f = kf \quad \text{for any } j \ (2k+1 \leq j \leq n).$$

The left side equals $(2a_j + k)f$. Therefore, we have

$$2a_j + k = k \quad \text{for any } j \ (2k+1 \leq j \leq n),$$

that is $a_j = 0$ $(2k+1 \leq j \leq n)$. Thus, we have proved Proposition. □

6.10. Now we complete the proof of Proposition 6.1. Let $k = 1, 2, \ldots, [n/2]$. Combining Proposition 6.9 with (1.5.1), any K-type occurring in $(\mathrm{Sol}(\mathcal{N}_{2k+1}))_K$ appears in $(W(n,k)_+)_K$ with multiplicity one. As $\mathcal{R} : W(n,k)_+ \to \mathrm{Sol}(\mathcal{N}_{2k+1})$ is an injective G-homomorphism, we have proved an isomorphism on the level of $(\mathfrak{g}, K)$-modules:

$$\mathscr{R}((W(n,k)_+)_K) = \mathrm{Sol}(\mathcal{N}_{2k+1})_K.$$

Then, the $(\mathfrak{g}, K)$-isomorphism $\mathscr{R} : (W(n,k)_+)_K \xrightarrow{\sim} \mathrm{Sol}(\mathcal{N}_{2k+1})_K$ induces the G-isomorphism $\mathscr{R} : W(n,k)_+ \xrightarrow{\sim} \mathrm{Sol}(\mathcal{N}_{2k+1})$, because $H_{\bar{\partial}}^{k(n-k)}\left(G \underset{L}{\times} \boldsymbol{C}_n\right)$ is a maximal globalization in the sense of Schmid ([**Sch3**]). □

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