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Polygonal presentations of semisimple tensor categories

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Abstract. A polygonal description of semisimple tensor categories is presented and the rigidity as well as the associated involutions are analysed in terms of this.

Introduction.

The combinatorial structures behind tensor categories play fundamental roles in recent studies of quantum symmetries ([**1**], [**2**], [**5**], [**11**]). Although it is common to impose the strictness on associativity in tensor categories, which does not lose the information thanks to the coherence theorem, an explicit use of associativity constraints is often convenient in concrete computations.

A direct manipulation of such combinatorial data, however, can easily lose the navigation. When tensor categories are semisimple, we can divide the relevant structure into two parts: the skeleton information of fusion rule (algebra) and the remaining flesh part, which provides a kind of (non-linear) cohomological information.

Viewing this way, semisimple tensor categories can be reconstructed from hom-vector spaces relating three simple objects. These are then pictorially assigned to edges of a triangle and the general hom-sets are expressed in terms of polygons together with triangular decompositions.

This kind of geometrical presentation of tensor categories is useful in actual computations to get perspectives: we shall describe the rigidity as well as the associated involutions (if any exists) and show that the variety of duality isomorphisms comes from choices of "characters" of fusion rules.

1. Polygonal vector spaces.

A monoidal category $(\mathscr{C}, \otimes, I)$ with a, l and r denoting associativity, left-unit and right-unit constraints respectively is called a *tensor category* over a field $\boldsymbol{K}$ if hom-sets are $\boldsymbol{K}$-vector spaces and all the relevant operations are $\boldsymbol{K}$-linear. In what follows, we shall exculisvely deal with tensor categories over the complex number field $\boldsymbol{C}$ and vector spaces are assumed to be $\boldsymbol{C}$-linear unless otherwise stated.

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Given tensor categories $\mathscr{C}$ and $\mathscr{C}'$, a monoidal functor $F : \mathscr{C} \to \mathscr{C}'$ is called an *isomorphism* if (i) F gives an isomorphism of vector spaces

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$$

for any pair (X, Y) of objects in $\mathscr{C}$ and (ii) each object X' in $\mathscr{C}'$ is isomorphic to $F(X)$ for some object X in $\mathscr{C}$. Two tensor categories $\mathscr{C}$ and $\mathscr{C}'$ are said to be *isomorphic* if there is an isomorphism between them.

An object X in a tensor category is *simple* if $\mathrm{End}(X) = \boldsymbol{C}1_X$. A tensor category $\mathscr{C}$ is *semisimple* if the unit object is simple and every object is isomorphic to a direct sum of simple objects.

Given a semisimple tensor category $\mathscr{C}$, we denote by $S = \mathrm{Spec}(\mathscr{C})$ the set of equivalence classes of simple objects in $\mathscr{C}$, which is referred to as the *spectrum* of $\mathscr{C}$. The free module $\boldsymbol{Z}[S]$ generated by the set $\mathrm{Spec}(\mathscr{C})$ has the ring structure defined by

$$[X][Y] = \sum_{[Z] \in S} \dim(\mathrm{Hom}(Z, X \otimes Y))[Z],$$

i.e., the ring $\boldsymbol{Z}[S]$ has a special basis $\mathrm{Spec}(\mathscr{C})$ for which structure constants are non-negative integers.

Rings, furnished with such bases, are referred to as *fusion algebras* although this terminology is usually used in a more restrictive sense.

For $x = [X]$, $y = [Y]$ and $z = [Z]$ in $\mathrm{Spec}(\mathscr{C})$, a non-negative integer N_z^{xy} is defined as the multiplicity of Z-component in the decomposition of $X \otimes Y$, i.e.,

$$N_z^{xy} = \dim \mathrm{Hom}(Z, X \otimes Y).$$

The totality $\{N_z^{xy}\}_{x,y,z \in \mathrm{Spec}}$ is, by definition, the *fusion rule* of $\mathscr{C}$.

Let $\mathscr{C}$ be a semisimple tensor category with the spectrum $S = \mathrm{Spec}(\mathscr{C})$. For the time being, we fix a specific representative containing the unit object I and regard S as a set consisting of simple objects in $\mathscr{C}$. For a finite family $\{X_j\}_{0 \le j \le m}$ of simple objects in S, we set

$$\begin{bmatrix} X_1 \cdots X_m \\ X_0 \end{bmatrix} = \mathrm{Hom}(X_1 \otimes \cdots \otimes X_m, X_0),$$

$$\begin{bmatrix} X_0 \\ X_1 \cdots X_m \end{bmatrix} = \mathrm{Hom}(X_0, X_1 \otimes \cdots \otimes X_m),$$

which are finite dimensional vector spaces with the pairing

$$\begin{bmatrix} X_1 \cdots X_m \\ X_0 \end{bmatrix} \times \begin{bmatrix} X_0 \\ X_1 \cdots X_m \end{bmatrix} \ni f \times g \mapsto \langle f, g \rangle \in \boldsymbol{C}$$

defined by

$$\langle f, g \rangle 1_{X_0} = f \circ g,$$

i.e., we have the natural identification

$$\begin{bmatrix} X_1 \cdots X_m \\ X_0 \end{bmatrix} = \begin{bmatrix} X_0 \\ X_1 \cdots X_m \end{bmatrix}^*.$$

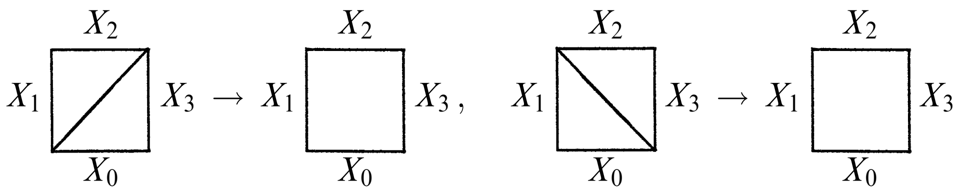
The vector space $\begin{bmatrix} X_1 \cdots X_m \\ X_0 \end{bmatrix}$ is graphically represented by a polygon of $m+1$ vertices with edges labeled by the sequence $\{X_0, X_1, \ldots, X_m\}$ clockwise, where the initial label X_0 plays a special role and we place it at the bottom edge.

Triangular vector spaces $\left\{\begin{bmatrix} XY \\ Z \end{bmatrix}\right\}_{X,Y,Z \in S}$ then form building blocks of polygonal vector spaces such as $\begin{bmatrix} X_1 \cdots X_m \\ X_0 \end{bmatrix}$ in the following sense: We begin with the case $m = 3$ and consider $X_1 \otimes X_2 \otimes X_3$. According to two ways of grouping $(X_1 \otimes X_2) \otimes X_3$ and $X_1 \otimes (X_2 \otimes X_3)$, we have two natural isomorphisms of vector spaces

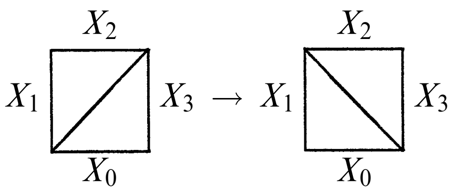
$$\bigoplus_{X_{12} \in S} \begin{bmatrix} X_1 X_2 \\ X_{12} \end{bmatrix} \otimes \begin{bmatrix} X_{12} X_3 \\ X_0 \end{bmatrix} \to \begin{bmatrix} X_1 X_2 X_3 \\ X_0 \end{bmatrix},$$

$$\bigoplus_{X_{23} \in S} \begin{bmatrix} X_2 X_3 \\ X_{23} \end{bmatrix} \otimes \begin{bmatrix} X_1 X_{23} \\ X_0 \end{bmatrix} \to \begin{bmatrix} X_1 X_2 X_3 \\ X_0 \end{bmatrix}.$$

We denote this situation graphically as

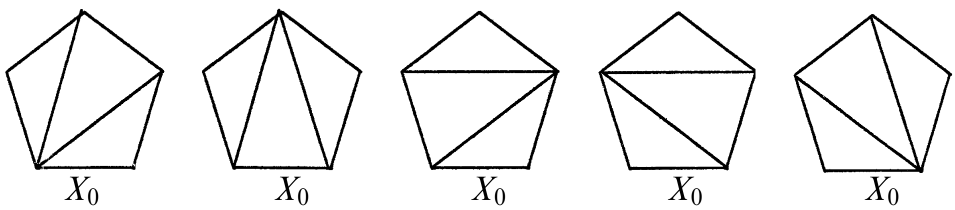


and the composite isomorphism



is referred to as an *associativity transformation*.

For the case $m = 4$, we have five groupings $((X_1X_2)X_3)X_4, (X_1X_2)(X_3X_4), (X_1(X_2X_3))X_4, X_1((X_2X_3)X_4), X_1(X_2(X_3X_4))$ with the associated vector spaces



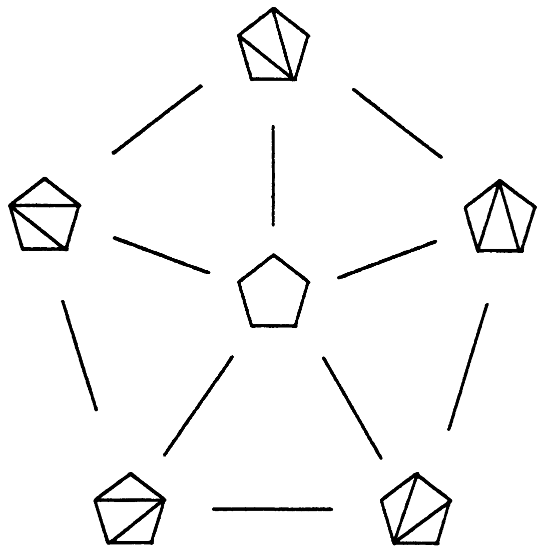
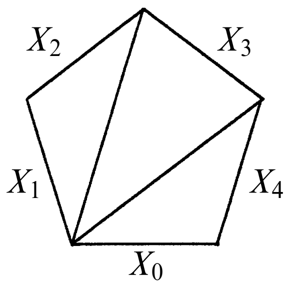


Figure 1.

where



$$= \bigoplus_{X_{12}, X_{123}} \begin{bmatrix} X_1 X_2 \\ X_{12} \end{bmatrix} \otimes \begin{bmatrix} X_{12} X_3 \\ X_{123} \end{bmatrix} \otimes \begin{bmatrix} X_{123} X_4 \\ X_0 \end{bmatrix}$$

and so on.

If we apply associativity transformations to squares inside triangulated pentagons, then we have the commutative diagram in Figure 1, i.e., associativity transformations satisfy the *pentagonal relation*.

Generalizing these, groupings in $X_1 \cdots X_m$ are parametrized by triangular decompositions of an $(m+1)$-polygon, which in turn give rise to triangular decompositions of the vector space $\begin{bmatrix} X_1 \cdots X_m \\ X_0 \end{bmatrix}$ and composed associativity transformations provide the same result as long as the initial and final vector spaces are the same (the coherence for associativity transformations).

Triangular vector spaces of the form $\begin{bmatrix} IY \\ X \end{bmatrix}$ and $\begin{bmatrix} YI \\ X \end{bmatrix}$ are non-trivial only for $X = Y$ and, if this is the case, they admit distinguished vectors $l_X \in \begin{bmatrix} IX \\ X \end{bmatrix}$ and $r_X \in \begin{bmatrix} XI \\ X \end{bmatrix}$ given by the unit constraints.

The associativity transformation

$$T : \begin{bmatrix} (XI)Y \\ Z \end{bmatrix} \to \begin{bmatrix} X(IY) \\ Z \end{bmatrix}$$

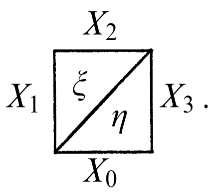
then takes an especially simple form:

$$T(r_X \otimes \xi) = l_X \otimes \xi \quad \text{for } \xi \in \begin{bmatrix} XY \\ Z \end{bmatrix}.$$

To facilitate the visual notation further, given a triangular decomposition of a polygon and a family of vectors in the accompanied triangular vector spaces we denote its tensor product by putting the vectors inside the belonging triangles: for example, in the triangular decomposition

$$\begin{bmatrix} X_1X_2X_3 \\ X_0 \end{bmatrix} = \bigoplus_{X_{12} \in S} \begin{bmatrix} X_1X_2 \\ X_{12} \end{bmatrix} \otimes \begin{bmatrix} X_{12}X_3 \\ X_0 \end{bmatrix}$$

with $\xi \in \mathrm{Hom}(X_1X_2, X_{12})$ and $\eta \in \mathrm{Hom}(X_{12}X_3, X_0)$, the associated vector $\eta(\xi \otimes 1_{X_3})$ is denoted by



We now discuss the reverse process of the construction according to [**13**]. Suppose that we are given a set S with a distinguished element 1 and a family of finite-dimensional vector spaces $\begin{bmatrix} x_1x_2 \\ x_0 \end{bmatrix}$ indexed by triplets (x_0, x_1, x_2) in the set S (called triangular vector spaces) and a family of isomorphisms

$$T^{x_1x_2x_3}_{x_0} : \bigoplus_{x_{12} \in S} \begin{bmatrix} x_1x_2 \\ x_{12} \end{bmatrix} \otimes \begin{bmatrix} x_{12}x_3 \\ x_0 \end{bmatrix} \to \bigoplus_{x_{23} \in S} \begin{bmatrix} x_2x_3 \\ x_{23} \end{bmatrix} \otimes \begin{bmatrix} x_1x_{23} \\ x_0 \end{bmatrix}$$

indexed by quadruplets (x_0, x_1, x_2, x_3) in the set S (called associativity transformations) which satisfies the pentagonal identity. Furthermore, we assume that there are distinguished vectors $l_x \in \begin{bmatrix} 1x \\ x \end{bmatrix}$, $r_x \in \begin{bmatrix} x1 \\ x \end{bmatrix}$ such that

$$\begin{bmatrix} 1x \\ y \end{bmatrix} = \begin{cases} \boldsymbol{C} l_x & \text{if } x = y, \\ \{0\} & \text{otherwise,} \end{cases} \qquad \begin{bmatrix} x1 \\ y \end{bmatrix} = \begin{cases} \boldsymbol{C} r_x & \text{if } x = y, \\ \{0\} & \text{otherwise,} \end{cases}$$

and

$$T(r_x \otimes \sigma) = l_y \otimes \sigma, \quad \sigma \in \begin{bmatrix} xy \\ s \end{bmatrix}$$

for the associativity transformation $T : \begin{bmatrix} (x1)y \\ s \end{bmatrix} \to \begin{bmatrix} x(1y) \\ s \end{bmatrix}$.

The totality of these data is referred to as a *monoidal system*. Given a monoidal system

$$\left\{ \begin{bmatrix} xy \\ z \end{bmatrix}, T_s^{xyz}, l_s, r_s \right\}_{s,x,y,z \in S},$$

we define the *dual system* by

$$\left\{ \begin{bmatrix} xy \\ z \end{bmatrix}^*, \bar{T}_s^{xyz}, l_s^*, r_s^* \right\}_{s,x,y,z \in S},$$

where $\begin{bmatrix} xy \\ z \end{bmatrix}^*$ is the dual vector space of $\begin{bmatrix} xy \\ z \end{bmatrix}$,

$$l_s^* \in \begin{bmatrix} 1s \\ s \end{bmatrix}^*, \quad r_s^* \in \begin{bmatrix} s1 \\ s \end{bmatrix}^*$$

are specified by $\langle l_s, l_s^* \rangle = 1 = \langle r_s, r_s^* \rangle$ and

$$\bar{T} = \bar{T}_s^{xyz} : \bigoplus_{t \in S} \begin{bmatrix} xy \\ t \end{bmatrix}^* \otimes \begin{bmatrix} tz \\ s \end{bmatrix}^* \to \bigoplus_{t \in S} \begin{bmatrix} yz \\ t \end{bmatrix}^* \otimes \begin{bmatrix} xt \\ s \end{bmatrix}^*$$

is defined to be the transposed inverse of $T = T_s^{xyz}$.

Given a monoidal system, consider a family $X = \{X(s)\}_{s \in S}$ of finite-dimensional $\boldsymbol{C}$-vector spaces with $X(s) = \{0\}$ except for finitely many $s \in S$. Let $Y = \{Y(s)\}_{s \in S}$ be another such family. Thinking of these as objects and defining hom-sets by

$$\operatorname{Hom}(X, Y) = \bigoplus_{s \in S} \operatorname{Hom}(X(s), Y(s))$$

with the pointwise composition, we obtain a (semisimple) category $\mathscr{C}(S)$.

Using the assumed triangular vector spaces, the tensor product operation in $\mathscr{C}(S)$ is introduced by

$$(X \otimes Y)(s) = \bigoplus_{x,y \in S} X(x) \otimes Y(y) \otimes \begin{bmatrix} xy \\ s \end{bmatrix}^*$$

for objects X, Y in $\mathscr{C}(S)$ and

$$(f \otimes g)(s) = \bigoplus_{x,y \in S} f(x) \otimes g(y) \otimes 1 :$$

$$\bigoplus_{x,y \in S} X(x) \otimes Y(y) \otimes \begin{bmatrix} xy \\ s \end{bmatrix}^* \to \bigoplus_{x,y \in S} X'(x) \otimes Y'(y) \otimes \begin{bmatrix} xy \\ s \end{bmatrix}^*$$

for $f \in \operatorname{Hom}(X, X')$ and $g \in \operatorname{Hom}(Y, Y')$.

The unit object I in $\mathscr{C}(S)$ is defined to be

$$I(s) = \begin{cases} \boldsymbol{C} & \text{if } s = 1, \\ \{0\} & \text{otherwise.} \end{cases}$$

Associativity and unit constraints are defined by

$$a_{X,Y,Z}(s) = \bigoplus_{x,y,z} 1_{X(x)} \otimes 1_{Y(y)} \otimes 1_{Z(z)} \otimes \bar{T}_s^{x,y,z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

with

$$((X \otimes Y) \otimes Z)(s) = \bigoplus_{x,y,z,t} X(x) \otimes Y(y) \otimes Z(z) \otimes \begin{bmatrix} xy \\ t \end{bmatrix}^* \otimes \begin{bmatrix} tz \\ s \end{bmatrix}^*,$$

$$(X \otimes (Y \otimes Z))(s) = \bigoplus_{x,y,z,t} X(x) \otimes Y(y) \otimes Z(z) \otimes \begin{bmatrix} yz \\ t \end{bmatrix}^* \otimes \begin{bmatrix} xt \\ s \end{bmatrix}^*,$$

and

$$l_X(x) : X(x) \otimes \begin{bmatrix} 1x \\ x \end{bmatrix}^* \ni \xi \otimes l_x^* \mapsto \xi \in X(x),$$

$$r_X(x) : X(x) \otimes \begin{bmatrix} x1 \\ x \end{bmatrix}^* \ni \xi \otimes r_x^* \mapsto \xi \in X(x)$$

with

$$(I \otimes X)(x) = X(x) \otimes \begin{bmatrix} 1x \\ x \end{bmatrix}^*, \quad (X \otimes I)(x) = X(x) \otimes \begin{bmatrix} x1 \\ x \end{bmatrix}^*.$$

It is immediate to see that these in fact satisfy the axioms of tensor category: So far we have defined a semisimple tensor category $\mathscr{C}(S, T)$.

If we identify an element $x \in S$ with the object X in $\mathscr{C}(S, T)$ defined by

$$X(s) = \begin{cases} \boldsymbol{C} & \text{if } s = x, \\ \{0\} & \text{otherwise,} \end{cases}$$

then the vector space $\mathrm{Hom}(x \otimes y, z)$ is naturally identified with the triangular vector space $\begin{bmatrix} xy \\ z \end{bmatrix}$ and we recover the associativity transformation T from the associativity constraint for $\mathscr{C}(S, T)$.

Two monoidal systems (S, T), (S', T') with the unit elements 1 and $1'$ are said to be *equivalent* if we can find a bijection $\phi : S \to S'$ such that $\phi(1) = 1'$ and a family of isomorphisms $\phi_z^{xy} : \begin{bmatrix} xy \\ z \end{bmatrix} \to \begin{bmatrix} \phi(x)\phi(y) \\ \phi(z) \end{bmatrix}$ which makes the diagram in Fig. 2 commutative and satisfies

$$\phi_x^{1x}(l_x) = l_{\phi(x)}, \quad \phi_x^{x1}(r_x) = r_{\phi(x)}.$$

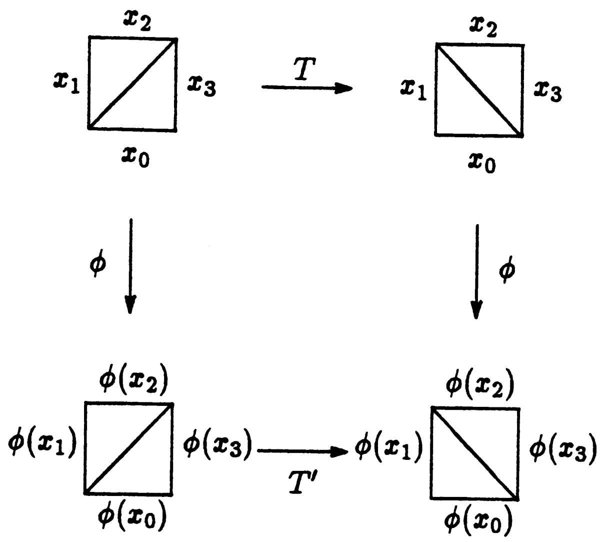


Figure 2.

An equivalence $\{\phi_z^{xy}\}$ is called a *gauge transformation* if it is associated to the identity map of the index set S.

Summarizing these, we have the following.

Proposition 1.1 (Reconstruction). *Two tensor categories $\mathscr{C}(S, T)$ and $\mathscr{C}(S', T')$ are isomorphic if and only if (S, T) and (S', T') are equivalent.*

Example 1.2. Semisimple tensor categories with the fusion rule given by a countable group G are parametrized by elements in the third cohomology group $H^3(G)$ up to gauge transformations.

Proof. In fact, non-trivial triangular vector spaces are 1-dimensional and given by $\begin{bmatrix} g, h \\ gh \end{bmatrix}$, $g, h \in H$. With a choice of associative bases $0 \neq [g, h] \in \begin{bmatrix} g, h \\ gh \end{bmatrix}$ such that $[1, g] = l_g$ and $[g, 1] = r_g$, associativity transformations are specified as

$$T : [g_1, g_2] \otimes [g_1 g_2, g_3] \mapsto c(g_1, g_2, g_3)[g_2 \otimes g_3] \otimes [g_1, g_2 g_3],$$

where $c(g_1, g_2, g_3)$ is a three cocycle of G.

The pentagonal identity and the unit constraint condition for T take the form

$$c(g_2, g_3, g_4) c(g_1 g_2, g_3, g_4)^{-1} c(g_1, g_2 g_3, g_4) c(g_1, g_2, g_3 g_4)^{-1} c(g_1, g_2, g_3) = 1$$

and $c(g, 1, h) = 1$ respectively.

The Kelley's result on unit constraint conditions (see [7]) is then reduced to $c(1, g, h) = c(g, h, 1) = 1$, which is a direct consequence of the cocycle condition if we take $g_2 = 1$ or $g_3 = 1$.

If we denote by $\mathscr{C}(G,c)$ the semisimple tensor category associated to a normalized 3 cocycle c on a group G, then it is immediate to see that $\mathscr{C}(G,c) \cong \mathscr{C}(G',c')$ iff there is a group isomorphism $\phi : G \to G'$ such that c and $c' \circ \phi$ are cohomologous in $H^3(G)$. □

2. Rigidity.

Recall that an object X in a tensor category $\mathscr{C}$ is said to be rigid if we can find an object X^* and a pair of morphisms $\varepsilon : X \otimes X^* \to I$, $\delta : I \to X^* \otimes X$ such that the composite morphisms (the associativity isomorphisms being omitted by coherence theorem)

$$X \xrightarrow{1\otimes\delta} X \otimes X^* \otimes X \xrightarrow{\varepsilon\otimes 1} X, \quad X^* \xrightarrow{\delta\otimes 1} X^* \otimes X \otimes X^* \xrightarrow{1\otimes\varepsilon} X^*$$

are identities. The object X^* is unique up to isomorphisms and is referred to as a *dual object* of X. The tensor category $\mathscr{C}$ is *rigid* if every object is rigid and isomorphic to a dual of another object.

Lemma 2.1 (cf. [**6**]). *For a simple object X in a rigid (semisimple) tensor category, its dual object X^* is again simple and X itself is a dual of X^*.*

Proof. If X^* is not simple, $X^* \cong Y^* \oplus Z^*$ by the rigidity of $\mathscr{C}$ and then $X \cong Y \oplus Z$ by the uniqueness of (pre)dual objects. Thus X^* is simple if X is so. By Frobenius reciprocity and the semisimplicity of $\mathscr{C}$, we have

$$\mathrm{Hom}(X, X^{**}) \cong \mathrm{Hom}(X^* \otimes X, I) \cong \mathrm{Hom}(I, X^* \otimes X) \cong \mathrm{End}(X),$$

whence there is a non-trivial morphism $X \to X^{**}$. Since X^{**} is simple as a dual of the simple object X^*, they are actually isomorphic. □

As a consequence of the above lemma, an involution is defined on the spectrum set S of a rigid tensor category by $[X]^* = [X^*]$, which satisfies the duality relation

$$N_1^{xy} = \begin{cases} 1 & \text{if } x = y^*, \\ 0 & \text{otherwise.} \end{cases}$$

The fusion algebra $\boldsymbol{C}[S]$ is a *-algebra by extending the involution on the set S (see [**12**], [**4**], [**14**] for more information on fusion algebras in the present context).

Assume that in the tensor category $\mathscr{C}(S,T)$ the fusion set S is furnished with an involution $*$ satisfying the duality relation. For an object X in $\mathscr{C}(S,T)$ and a morphism $f : X \to Y$, we then define the object X^* by

$$X^*(s) = (X(s^*))^* \ (= \text{the dual vector space of } X(s^*))$$

and the morphism ${}^t f : Y^* \to X^*$ by

$$({}^t f)(s) = {}^t f(s^*) : Y(s^*)^* \to X(s^*)^*,$$

where ${}^t f(s^*)$ denotes the transposed map of $f(s^*)$.

The operation X^*, ${}^t f$ gives a contravariant functor from $\mathscr{C}(S,T)$ into itself.

LEMMA 2.2 (Local Rigidity). *The semisimple tensor category $\mathscr{C}(S,T)$ is rigid if and only if S admits an involution $*$ satisfying $N_1^{x^*y} = \delta_{x,y}$ and*

$$\langle \delta \otimes r_s^*, T(\varepsilon \otimes l_s)\rangle = \langle \delta \otimes l_{s^*}^*, T^{-1}(\varepsilon \otimes r_{s^*})\rangle \neq 0$$

for $0 \neq \varepsilon \in \begin{bmatrix} ss^* \\ 1 \end{bmatrix}$, $0 \neq \delta \in \begin{bmatrix} s^*s \\ 1 \end{bmatrix}^*$ *with* $s \in S$.

PROOF. For $\varepsilon \in \begin{bmatrix} xy \\ 1 \end{bmatrix}$ and $\delta \in \begin{bmatrix} yx \\ 1 \end{bmatrix}^*$, we have

$$(\varepsilon \otimes 1)a^{-1}(1 \otimes \delta) = \langle \delta \otimes r_x^*, T(\varepsilon \otimes l_x)\rangle 1_x,$$

$$(1 \otimes \varepsilon)a(\delta \otimes 1) = \langle \delta \otimes l_y^*, T^{-1}(\varepsilon \otimes r_y)\rangle 1_y$$

with obvious identifications on unit constraints. The condition in question is then equivalent to the rigidity of simple objects in $\mathscr{C}(S,T)$, which in turn implies the rigidity for arbitrary objects: For each $s \in S$, choose $\varepsilon_s \in \begin{bmatrix} ss^* \\ 1 \end{bmatrix}$ and $\delta_s \in \begin{bmatrix} s^*s \\ 1 \end{bmatrix}^*$ so that

$$\langle \delta_s \otimes r_s^*, T(\varepsilon_s \otimes l_s)\rangle = \langle \delta_s \otimes l_{s^*}^*, T^{-1}(\varepsilon_s \otimes r_{s^*})\rangle = 1.$$

For any object X in $\mathscr{C}(S,T)$, we then define morphisms $\varepsilon_X : X \otimes X^* \to I$, $\delta_X : I \to X^* \otimes X$ by

$$\bigoplus_{s \in S} X(s) \otimes X(s)^* \otimes \begin{bmatrix} ss^* \\ 1 \end{bmatrix}^* \ni \bigoplus_{s \in S} \xi_s \otimes \eta_s^* \otimes \sigma_s \mapsto \sum_s \langle \xi_s, \eta_s^*\rangle \langle \varepsilon_s, \sigma_s\rangle \in \boldsymbol{C} = I(1),$$

$$I(1) = \boldsymbol{C} \ni 1 \mapsto \bigoplus_s \delta_{X(s)} \otimes \delta_s \in \bigoplus_{s \in S} X(s)^* \otimes X(s) \otimes \begin{bmatrix} s^*s \\ 1 \end{bmatrix}^*.$$

Here

$$\delta_{X(s)} = \sum_j \xi_j^* \otimes \xi_j$$

with $\{\xi_j\}$ a basis in $X(s)$ and $\{\xi_j^*\}$ its dual basis.

It is straightforward to check that these in fact give a rigidity pairing between X and X^*. $\square$

EXAMPLE 2.3. The tensor category $\mathscr{C}(G,c)$ associated to a 3 cocycle c of a group G is rigid.

Proof. The fusion rule set S is the group G itself and the involution is given by $g^* = g^{-1}$ for $g \in G$. Let $\varepsilon = [g, g^{-1}] \in \begin{bmatrix} g, g^{-1} \\ 1 \end{bmatrix}$ and $\delta = [g^{-1}, g]^* \in \begin{bmatrix} g^{-1}, g \\ 1 \end{bmatrix}^*$ in Lemma 2.2. Then

$$\langle \delta \otimes r_g^*, T(\varepsilon \otimes l_g) \rangle = c(g, g^{-1}, g), \quad \langle \delta \otimes l_{g^{-1}}^*, T^{-1}(\varepsilon \otimes r_{g^{-1}}) \rangle = c(g^{-1}, g, g^{-1})^{-1}$$

shows that the rigidity is equivalent to

$$c(g, g^{-1}, g) = c(g^{-1}, g, g^{-1})^{-1},$$

whence it follows from the cocycle condition of c if we consider the condition $\delta c(g, g^{-1}, g, g^{-1}) = 1$. $\square$

Let $\mathscr{C}(S, T)$ be a semisimple tensor category described by a monoidal system (S, T) with S a fusion rule set and T a system of associativity transformations on S. Assume that the tensor category $\mathscr{C}(S, T)$ is rigid, i.e., S admits an involution $*$ satisfying the duality relation and the condition in Lemma 2.2. We then choose $\varepsilon_s \in \begin{bmatrix} ss^* \\ 1 \end{bmatrix}$, $\delta_s \in \begin{bmatrix} s^*s \\ 1 \end{bmatrix}^*$ and extend it to rigidity pairings $\varepsilon_X : X \otimes X^* \to I$, $\delta_X : I \to X^* \otimes X$ as described in the proof of Lemma 2.2.

The following is immediate from definitions.

Lemma 2.4. *The contravariant functor $(X, f) \mapsto (X^*, {}^t f)$ is compatible with the rigidity pairing $\{(\varepsilon_X, \delta_X)\}$:*

$$\varepsilon_Y(f \otimes 1) = \varepsilon_X(1 \otimes {}^t f) \quad \textit{or equivalently} \quad (1 \otimes f)\delta_X = ({}^t f \otimes 1)\delta_Y$$

for $f : X \to Y$.

By the uniqueness of rigidity pairings, we can define isomorphisms $c_{X,Y} : Y^* \otimes X^* \to (X \otimes Y)^*$ by the commutativity of the diagram

$$\begin{tikzcd}
X \otimes Y \otimes Y^* \otimes X^* \arrow[r, "1 \otimes \varepsilon_Y"] \arrow[d, "1 \otimes c_{X,Y}"'] & X \otimes X^* \arrow[d, "\varepsilon_X"] \\
X \otimes Y \otimes (X \otimes Y)^* \arrow[r, "\varepsilon_{XY}"'] & I
\end{tikzcd},$$

where parentheses are omitted thanks to the coherence theorem.

By the above lemma, the family $\{c_{X,Y}\}$ is natural in X, Y and hence it is determined by isomorphisms $c_{x,y}$ for x, $y \in S$ as

$$c_{X,Y}(s) = \bigoplus_{x,y \in S} 1 \otimes c_{x,y}(s) : (Y^* \otimes X^*)(s) \to (X \otimes Y)^*(s)$$

with

$$(Y^* \otimes X^*)(s) = \bigoplus_{x,y \in S} Y(y)^* \otimes X(x)^* \otimes \begin{bmatrix} y^*x^* \\ s \end{bmatrix}^*,$$

$$(X \otimes Y)^*(s) = \bigoplus_{x,y \in S} Y(y)^* \otimes X(x)^* \otimes \begin{bmatrix} xy \\ s^* \end{bmatrix}.$$

Here the isomorphism

$$c_{x,y}(s) : \begin{bmatrix} y^*x^* \\ s \end{bmatrix}^* \to \begin{bmatrix} xy \\ s^* \end{bmatrix}$$

is specified in the follwoing way: Starting with a vector $\alpha \otimes \beta \otimes \sigma$ in the vector space

$$((xy)(y^*x^*))(1) = \bigoplus_{s \in S} \begin{bmatrix} xy \\ s^* \end{bmatrix}^* \otimes \begin{bmatrix} y^*x^* \\ s \end{bmatrix}^* \otimes \begin{bmatrix} s^*s \\ 1 \end{bmatrix}^*,$$

the evaluation by the morphism

$$(xy)(y^*x^*) \xrightarrow{1 \otimes c_{x,y}} (xy)(xy)^* \xrightarrow{\varepsilon_{xy}} 1$$

gives

$$\langle \alpha, c_{x,y}(s)\beta \rangle \langle \varepsilon_{s^*}, \sigma_s \rangle,$$

whereas the evaluation by the morphism (see Fig. 3)

$$(xy)(y^*x^*) \xrightarrow{a} x(y(y^*x^*)) \xrightarrow{1 \otimes a^{-1}} x((yy^*)x^*) \xrightarrow{1 \otimes (\varepsilon_y \otimes 1)} x(1x^*)$$
$$\xrightarrow{1 \otimes l} xx^* \xrightarrow{\varepsilon_x} 1$$

gives the expression

$$\langle (1 \otimes T^{-1})(T \otimes 1)(\varepsilon_y \otimes l_{x^*} \otimes \varepsilon_x), \alpha \otimes \beta \otimes \sigma \rangle.$$

Equating these, we get an explicit formula which determines $c_{x,y}(s)$:

$$\langle \alpha, c_{x,y}(s)\beta \rangle = \langle (1 \otimes T^{-1})(T \otimes 1)(\varepsilon_y \otimes l_{x^*} \otimes \varepsilon_x), \alpha \otimes \beta \otimes \varepsilon_{s^*}^* \rangle,$$

where $\varepsilon_{s^*}^* \in \mathrm{Hom}(1, s^*s)$ is specified by $\langle \varepsilon_{s^*}, \varepsilon_{s^*}^* \rangle = 1$.

Figure 3.

If we change $\{\varepsilon_s\}_{s\in S}$ into $\{\varepsilon'_s = \phi(s^*)^{-1}\varepsilon_s\}_{s\in S}$ with $\phi(s^*) \in \boldsymbol{C}^\times$ and $c_{x,y}$ into $c'_{x,y}$, then $c'_{x,y}$ satisfies

$$\langle \alpha, c'_{x,y}(s)\beta\rangle \phi(s)^{-1} = \phi(y^*)^{-1}\phi(x^*)^{-1}\langle (1\otimes T^{-1})(T\otimes 1)(\varepsilon_y \otimes l_{x^*}\otimes \varepsilon_x), \alpha\otimes\beta\otimes\varepsilon^*_{s^*}\rangle$$

and the comparison with the equation for $c_{x,y}$ yields

$$c'_{x,y}(s) = \frac{\phi(s)}{\phi(x^*)\phi(y^*)}c_{x,y}(s).$$

If we define a natural family of isomorphisms $\{\phi_X : X^* \to X^*\}$ by

$$\phi_X : \bigoplus_{s\in S}\xi(s) \mapsto \bigoplus_{s\in S}\phi(s)\xi(s),$$

then it intertwines between $c_{X,Y}$ and $c'_{X,Y}$:

$$\begin{array}{ccc} Y^*\otimes X^* & \xrightarrow{c_{X,Y}} & (X\otimes Y)^* \\ {\scriptstyle \phi_Y\otimes\phi_X}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi_{X\otimes Y}} \\ Y^*\otimes X^* & \xrightarrow[c'_{X,Y}]{} & (X\otimes Y)^* \end{array}.$$

In other words, the monoidal functor $(X^*, {}^tf, c_{X,Y})$ is unique up to natural equivalences.

For an object X in $\mathscr{C}(S,T)$, we defined $\delta_{X^*} : I \to X^{**}\otimes X^*$, whereas we have

$$I \xrightarrow{{}^t\varepsilon_X} (X\otimes X^*)^* \xrightarrow{c^{-1}_{X,X^*}} X^{**}\otimes X^*.$$

Lemma 2.5. *For an object X in $\mathscr{C}(S,T)$,*

$${}^t\varepsilon_X = c_{X,X^*}\circ\delta_{X^*} : I \to (X\otimes X^*)^*.$$

Proof. If we regard ${}^t\varepsilon_X$ as an element in $(X\otimes X^*)^*(1) = (X\otimes X^*)(1)^*$, then

$${}^t\varepsilon_X = \bigoplus_{x\in S}\delta_{X(x)^*}\otimes\varepsilon_x \in \bigoplus_{x\in S}X(x)\otimes X(x)^*\otimes\begin{bmatrix} xx^* \\ 1\end{bmatrix},$$

where the second dual of (finite-dimensional) vector spaces are identified with the original ones. Similarly δ_{X^*} is identified with an element in $(X^{**}\otimes X^*)(1)$ by

$$\delta_{X^*} = \bigoplus_{x\in S}\delta_{X(x)^*}\otimes\delta_{x^*} \in \bigoplus_{x\in S}X(x)\otimes X(x)^*\otimes\begin{bmatrix} xx^* \\ 1\end{bmatrix}^*$$

and then $c_{X,X^*}\circ\delta_{X^*}$ takes the form

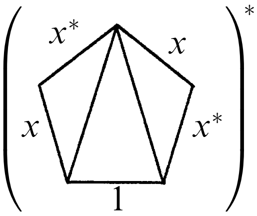
$$\bigoplus_{x\in S}\delta_{X(x)^*}\otimes c_{x,x^*}(\delta_{x^*}) \in \bigoplus_{x\in S}X(x)\otimes X(x)^*\otimes\begin{bmatrix} xx^* \\ 1\end{bmatrix}.$$

Thus we need to show $\varepsilon_x = c_{x,x^*}(\delta_{x^*})$ in $\begin{bmatrix} xx^* \\ 1 \end{bmatrix}$. According to the morphism

$$(x \otimes x^*) \otimes (x \otimes x^*) \xrightarrow{1 \otimes c_{x,x^*}} (x \otimes x^*) \otimes (x \otimes x^*)^* \xrightarrow{\varepsilon_{xx^*}} I,$$

the vector

$$\delta_{x^*} \otimes \delta_{x^*} \otimes r_1^* \in \begin{bmatrix} xx^* \\ 1 \end{bmatrix}^* \otimes \begin{bmatrix} xx^* \\ 1 \end{bmatrix}^* \otimes \begin{bmatrix} 11 \\ 1 \end{bmatrix}^* \subset$$



is mapped by $1 \otimes c_{x,x^*} \otimes 1$ to the vector

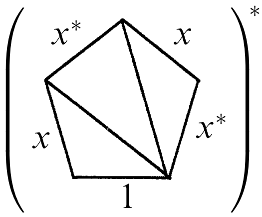
$$\delta_{x^*} \otimes c_{x,x^*}(\delta_{x^*}) \otimes r_1^* \in \begin{bmatrix} xx^* \\ 1 \end{bmatrix}^* \otimes \begin{bmatrix} xx^* \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 11 \\ 1 \end{bmatrix}^*$$

and then evaluated by ε_{xx^*}, resulting in the scalar

$$\langle \delta_{x^*}, c_{x,x^*}(\delta_{x^*}) \rangle.$$

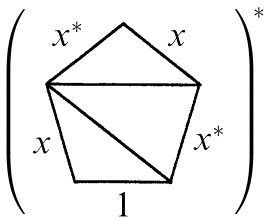
On the other hand, the vector $\delta_{x^*} \otimes \delta_{x^*} \otimes r_1^*$ is transformed by an associativity transformation into $\delta_{x^*} \otimes \overline{T}(\delta_{x^*} \otimes r_1^*)$, which is equal to

$$\delta_{x^*} \otimes r_{x^*}^* \otimes \delta_{x^*} \in \begin{bmatrix} xx^* \\ 1 \end{bmatrix}^* \otimes \begin{bmatrix} x^*1 \\ x^* \end{bmatrix}^* \otimes \begin{bmatrix} xx^* \\ 1 \end{bmatrix}^* \subset$$



by Kelly's theorem ([**7**]) and then again by an associativity transformation into

$$\overline{T}^{-1}(\delta_{x^*} \otimes r_x^*) \otimes \delta_{x^*} \in \begin{bmatrix} x^*x \\ 1 \end{bmatrix}^* \otimes \begin{bmatrix} 1x^* \\ x^* \end{bmatrix}^* \otimes \begin{bmatrix} xx^* \\ 1 \end{bmatrix}^* \subset$$



.

The last vector is evaluated according to the morphism

$$(x \otimes (x^* \otimes x)) \otimes x^* \xrightarrow{(1 \otimes \varepsilon_{x^*}) \otimes 1} (x \otimes 1) \otimes x^* \xrightarrow{r_x \otimes 1} x \otimes x^* \xrightarrow{\varepsilon_x} I$$

into the scalar

$$\langle \varepsilon_{x^*} \otimes l_{x^*}, \overline{T}^{-1}(\delta_{x^*} \otimes r_{x^*}^*) \rangle \langle \varepsilon_x, \delta_{x^*} \rangle = \langle \varepsilon_x, \delta_{x^*} \rangle.$$

(In the last line, we used the local rigidity.)

Comparing these, we have $\varepsilon_x = c_{x,x^*}(\delta_{x^*})$, proving the assertion. $\square$

3. Duality.

In this section, we assume that the antimonoidal functor $f \mapsto {}^t f$ is supplemented to an involution by duality isomorphisms $\{d_X : X \to X^{**}\}_{X \in \mathrm{Object}(\mathscr{C})}$, i.e., the family $\{d_X\}$ is natural in X and multiplicative in the sense that the following diagrams commute,

$$\begin{array}{ccc} X & \xrightarrow{d_X} & X^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle {}^{tt}f} \\ Y & \xrightarrow[d_Y]{} & Y^{**} \end{array}, \qquad \begin{array}{ccc} X \otimes Y & \xrightarrow{d \otimes d} & X^{**} \otimes Y^{**} \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle c} \\ (X \otimes Y)^{**} & \xrightarrow[{}^t c]{} & (Y^* \otimes X^*)^* \end{array},$$

and ${}^t d_X = d_{X^*}^{-1} : X^{***} \to X^*$ (see [**16**] for more information on this).

By the naturality in X, d takes the form

$$X(s) \ni \xi \mapsto D_s \xi^{**} \in X(s)^{**} = X^{**}(s)$$

for $s \in S$ with $D_s \in \boldsymbol{C}^\times$, where $\xi \mapsto \xi^{**}$ denotes the natural identification $X(s) = X(s)^{**}$. The multiplicativity of d is then reduced to the commutativity of

$$\begin{array}{ccc} \begin{bmatrix} xy \\ s \end{bmatrix}^* & \xrightarrow{D_x D_y} & \begin{bmatrix} xy \\ s \end{bmatrix}^* \\ {\scriptstyle D_s}\downarrow & & \downarrow{\scriptstyle c_{y^*,x^*}(s)} \\ \begin{bmatrix} xy \\ s \end{bmatrix} & \xrightarrow[{}^t c_{x,y}(s)]{} & \begin{bmatrix} y^* x^* \\ s^* \end{bmatrix} \end{array}$$

(note that ${}^t c_{x,y}(s) = {}^t(c_{x,y}(s^*)))$, i.e.,

$$D_s^t c_{x,y}(s) = D_x D_y c_{y^*,x^*}(s) : \begin{bmatrix} xy \\ s \end{bmatrix}^* \to \begin{bmatrix} y^* x^* \\ s^* \end{bmatrix}.$$

If $\{D_s'\}_{s \in S}$ gives a duality d' for the antimultiplicativity $\{c_{x,y}'\}$, then it satisfies

$$D_s'^t c_{x,y}'(s) = D_x' D_y' c_{y^*,x^*}'(s),$$

which is equivalent to

$$\frac{D_s'}{D_s} \frac{\phi(s^*)}{\phi(x^*)\phi(y^*)} = \frac{D_x' D_y'}{D_x D_y} \frac{\phi(s)}{\phi(x)\phi(y)}.$$

As a solution, we may take

$$D_s' = \frac{\phi(s)}{\phi(s^*)} D_s.$$

In other words, there is a natural correspondance between duality isomorphisms for different antimultiplicativities and we may restrict ourselves to the case $(X^*, {}^tf, c) = (X^{*'}, {}^{t'}f, c')$ for the studies of possibility of rigidity-compatible involutions: the information on involutions is then stacked up to the choice of duality isomorphisms.

REMARK. Since the vector space $\begin{bmatrix} xx^* \\ 1 \end{bmatrix}$ is 1-dimensional, the linear map

$$C_{x,x^*}(1) : \begin{bmatrix} xx^* \\ 1 \end{bmatrix} \to \begin{bmatrix} xx^* \\ 1 \end{bmatrix}^*$$

and its transposed map ${}^tC_{x,x^*}(1)$ coincides. Thus we have

$$D_1 = D_x D_{x^*}$$

for any $x \in S$. In particular, we see that

$$D_1 = 1, \quad D_x D_{x^*} = 1.$$

The condition ${}^td_X = d_{X^*}^{-1}$ is a consequence of the multiplicativity.

Let d and d' be dualities based on a common antimultiplicative functor $(X^*, {}^tf, c)$. If we define a family of isomorphisms $\{\phi_X : X \to X\}$ by $d'_X = d_X \circ \phi_X$, then it is natural and multiplicative: $f\phi_X = \phi_Y f$ for $f : X \to Y$ and $\phi_{X\otimes Y} = \phi_X \otimes \phi_Y$. By the naturality, such a family is determined by scalars $\{\phi(s)\}_{s\in S}$, $\phi_s \in \boldsymbol{C}^\times$, defined by $\phi_s = \phi(s)1_s$ and the multiplicativity is reduced to the condition

$$\phi(s) = \phi(x)\phi(y) \quad \text{if} \quad \begin{bmatrix} xy \\ s \end{bmatrix} \neq \{0\}.$$

In particular, it satisfies

$$\phi(1) = 1, \quad \phi(s^*) = \phi(s)^{-1}$$

and gives a character of the fusion rule set S. (This is different from the notion of character of the fusion algebra $\boldsymbol{C}[S]$.)

Conversely, given a character $\{\phi(s)\}_{s\in S}$ of S, it induces a natural and multiplicative family of isomorphisms by

$$\phi_X(x) : X(x) \ni \xi \mapsto \phi(x)\xi \in X(x).$$

By point-wise multiplication, the set $G(S)$ of characters of S becomes a commutative group and choices of dualities for the rigidity-compatible antimultiplicative functor $(X^*, {}^tf, c)$ are parametrized by elements in $G(S)$. Since the antimultiplicativity $\{c_{X,Y}\}$ is uniquely determined by the choice $\{\varepsilon_X\}$, the isomorphism classes of the structure $(\{\varepsilon_X\}, \{d_X\})$ is parametrized by the set $G(S)$ as well.

We now proceed into the description of Frobenius duality ([**15**], [**16**], cf. also [**3**]), which is a family $\{\varepsilon_X : X \otimes X^* \to I\}$ together with an involution $(X^*, {}^t f, c, d)$ satisfying

(i) (Multiplicativity)

$$\begin{array}{ccccc} (X \otimes (Y \otimes Y^*)) \otimes X^* & \xrightarrow{(1\otimes \varepsilon_Y)\otimes 1} & (X \otimes I) \otimes X^* & \xrightarrow{r_X \otimes 1} & X \otimes X^* \\ \downarrow & & & & \downarrow \varepsilon_X \\ (X \otimes Y) \otimes (Y^* \otimes X^*) & \xrightarrow[1\otimes c_{X,Y}]{} & (X \otimes Y) \otimes (X \otimes Y)^* & \xrightarrow[\varepsilon_{X\otimes Y}]{} & I \end{array} .$$

(ii) (Naturality)

$$\begin{array}{ccc} X \otimes Y^* & \xrightarrow{f\otimes 1} & Y \otimes Y^* \\ {\scriptstyle 1\otimes {}^t f}\downarrow & & \downarrow \varepsilon_Y \\ X \otimes X^* & \xrightarrow[\varepsilon_X]{} & I \end{array} .$$

(iii) (Faithfulness) The map $\mathrm{End}(X) \ni f \mapsto \varepsilon_X(f \otimes 1)$ is faithful.

(iv) (Neutrality) If we define left and right dimensions (denoted by $\mathrm{ldim}(X)$ and $\mathrm{rdim}(X)$ respectively) of an object X by the following composites

$$I \xrightarrow{{}^t\varepsilon_X} (X \otimes X^*)^* \xrightarrow{c^{-1}_{X,X^*}} X^{**} \otimes X^* \xrightarrow{d_X^{-1}\otimes 1} X \otimes X^* \xrightarrow{\varepsilon_X} I,$$

$$I \xrightarrow{{}^t\varepsilon_{X^*}} (X^* \otimes X^{**})^* \xrightarrow{c^{-1}_{X^*,X^{**}}} X^{***} \otimes X^{**} \xrightarrow{d_{X^*}^{-1}\otimes 1} X^* \otimes X^{**} \xrightarrow{\varepsilon_{X^*}} I,$$

then they coincide (the common scalar is called the dimension of X and denoted by $\dim(X)$).

Starting with a choice of rigidity pairings $\left\{\varepsilon_s \in \begin{bmatrix} ss^* \\ 1 \end{bmatrix}\right\}$, we enlarge it to the family $\{\varepsilon_X\}$ and define an antimultiplicativity $\{c_{X,Y}\}$ as discussed before. Then the first three properties, multiplicativity, naturality and faithfulness, are satisfied by just giving a duality $\{d_X : X \to X^{**}\}$ for $(X^*, {}^t f, c)$. Thus whether it gives a Frobenius duality depends on the validity of neutrality.

LEMMA 3.1. *A duality family $\{D_s\}_{s\in S}$ for a local rigidity family $\{\varepsilon_s, \delta_s\}_{s\in S}$ with $\varepsilon_s \in \begin{bmatrix} ss^* \\ 1 \end{bmatrix}$, $\delta_s \in \begin{bmatrix} s^*s \\ 1 \end{bmatrix}^*$ gives a Frobenius duality if and only if*

$$D_s^2 = \frac{\langle \delta_{s^*}, \varepsilon_s \rangle}{\langle \delta_s, \varepsilon_{s^*} \rangle}.$$

PROOF. By the naturality, it is enough to check the condition for simple X. In that case, by Lemma 2.5, the composite morphisms in the neutrality are reduced to

$$1 \xrightarrow{\delta_{s^*}} s^{**}s^* \xrightarrow{d_s^{-1}} ss^* \xrightarrow{\varepsilon_s} 1, \quad 1 \xrightarrow{\delta_s} s^*s \xrightarrow{1\otimes d_s} s^*s^{**} \xrightarrow{\varepsilon_{s^*}} 1$$

and the equality of these is

$$D_s^{-1}\langle\delta_{s^*}, \varepsilon_s\rangle = D_s\langle\delta_s, \varepsilon_{s^*}\rangle. \qquad \square$$

As a conclusion of our discussions, we have the following.

THEOREM 3.2. *Isomorphism classes of Frobenius dualities in the tensor category $\mathscr{C}(S, T)$ is, if it exists, parametrized by characters of S taking values in $\{\pm 1\}$.*

EXAMPLE 3.3. Let G be a (discrete) group and c be a normalized cyclic 3-cocycle (see Appendix A). The associated tensor category $\mathscr{C}(G, c)$ is reflexive and reflexivity is parametrized by characters of G. For a cyclic cocycle c, $d = \{D_g\}_{g\in G}$ itself is a character (generally, the parameter space is a principal homogeneous space of the character group of G) and the dimension function is given by

$$\mathrm{ldim}(g) = \frac{c(g, g^{-1}, g)}{D_g}.$$

Moreover, a Frobenius duality is defined by choosing $D_g = c(g, g^{-1}, g)$ so that $\dim(g) = 1$ for $g \in G$.

PROOF. Let $\varepsilon_g = [g, g^{-1}] \in \begin{bmatrix} gg^* \\ 1 \end{bmatrix}$ with the accompanied $\delta_g \in \begin{bmatrix} g^*g \\ 1 \end{bmatrix}^*$ given by

$$\delta_g = c(g, g^{-1}, g)^{-1}[g, g^{-1}][g^{-1}, g]^*.$$

Given $g,\ h \in G$, let

$$c_{g,h}(h^{-1}g^{-1}) : \begin{bmatrix} h^*g^* \\ h^{-1}g^{-1} \end{bmatrix}^* \ni [h^{-1}, g^{-1}]^* \mapsto \mu[g, h] \in \begin{bmatrix} g, h \\ gh \end{bmatrix}$$

be the non-trivial part of $c_{g,h}$.

If we start with the vector

$$[h, h^{-1}]^* \otimes [g, 1]^* \otimes [g, g^{-1}] \in \begin{bmatrix} (g(hh^{-1}))g^{-1} \\ 1 \end{bmatrix}^*,$$

its evaluation by $\varepsilon_g \circ \varepsilon_h$ is equal to 1, whereas repetitions of associativity transformations show that it corresponds to the vector

$$c(g, h, h^{-1})c(gh, h^{-1}, g^{-1})^{-1}[g, h]^* \otimes [h^{-1}, g^{-1}]^* \otimes [gh, h^{-1}g^{-1}]^* \in \begin{bmatrix} (gh)(h^*g^*) \\ 1 \end{bmatrix}^*$$

in the pentagonal vector space. Now the evaluation by $\varepsilon_{gh} \circ c_{g,h}$ gives the result

$$\mu c(g,h,h^{-1})c(gh,h^{-1},g^{-1})^{-1}.$$

Therefore we have

$$\mu = \frac{c(gh,h^{-1},g^{-1})}{c(g,h,h^{-1})}.$$

Thus the equation for d takes the form

$$D_{gh}\frac{c(gh,h^{-1},g^{-1})}{c(g,h,h^{-1})} = D_g D_h \frac{c(h^{-1}g^{-1},g,h)}{c(h^{-1},g^{-1},g)}.$$

To solve this equation, we assume the cyclic symmetry on the cocycle c. Then

$$c(gh,h^{-1},g^{-1}) = c(h^{-1}g^{-1},g,h) = 1, \quad c(g,h,h^{-1}) = c(h^{-1},g^{-1},g),$$

which reduces the equation to

$$D_{gh} = D_g D_h, \quad g,h \in G,$$

i.e., $\{D_g\}_{g\in G}$ is a character of G.

The left dimension is calculated by

$$\dim(g)1_I = \varepsilon_g(d_g^{-1} \otimes 1)\delta_{g^*} = c(g^{-1},g,g^{-1})^{-1}D_g^{-1}. \qquad \square$$

Remark. For a normalized cyclic 3-cocycle c, we have

$$c(gh,(gh)^{-1},gh) = c(g,g^{-1},g)c(h,h^{-1},h).$$

Proof. From the cocycle relation $\delta c(g,h,h^{-1}g^{-1},gh) = 1$, we have

$$c(gh,h^{-1}g^{-1},gh) = c(h,h^{-1}g^{-1},gh)c(g,g^{-1},gh)c(g,h,h^{-1}g^{-1}),$$

which, combined with the cyclicity of c ($g_1g_2g_3g_4 = 1$ implies $c(g_1,g_2,g_3) = c(g_2,g_3,g_4)^{-1}$) of the form

$$c(g,h,h^{-1}g^{-1}) = 1, \quad c(h,h^{-1}g^{-1},gh) = c(h^{-1},h,h^{-1}g^{-1})^{-1},$$

produces

$$c(gh,h^{-1}g^{-1},gh) = \frac{c(g,g^{-1},gh)}{c(h^{-1},h,h^{-1}g^{-1})}.$$

By the cocycle relations $\delta c(g,g^{-1},g,h) = 1 = \delta c(h^{-1},h,h^{-1},g^{-1})$, we have

$$c(g,g^{-1},gh) = c(g,g^{-1},g)c(g^{-1},g,h),$$

$$c(h^{-1},h,h^{-1}g^{-1}) = c(h^{-1},h,h^{-1})c(h,h^{-1},g^{-1}),$$

whence

$$c(gh, h^{-1}g^{-1}, gh) = \frac{c(g, g^{-1}, g)}{c(h^{-1}, h, h^{-1})} \frac{c(g^{-1}, g, h)}{c(h, h^{-1}, g^{-1})}.$$

Finally, we apply the cyclicity of c in the form

$$c(g^{-1}, g, h) = c(h^{-1}, g^{-1}, g)^{-1} = c(h, h^{-1}, g^{-1}). \qquad \square$$

EXAMPLE 3.4. Let G be a finite abelian group. Given a symmetric non-degenerate bicharacter $\sigma : G \times G \to \boldsymbol{T}$ and a real number τ satisfying $\tau^2 = |G|^{-1}$, we can define a tensor category $\mathscr{C}(\sigma, \tau)$ such that $S = G \sqcup \{m\}$ with the fusion rule $am = m = ma$ and $m^2 = \sum_{a \in G} a$ other than the group operation among elements in G (see [**13**]).

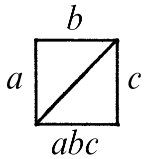
Then the tensor category $\mathscr{C}(\sigma, \tau)$ is reflexive and there are two choices of reflexivity, both of which give rise to Frobenius dualities. More precisely, we have $D_a = 1$ for $a \in G$, $D_m = \pm 1$ and the dimension function is given by

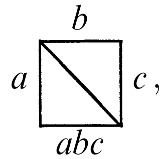
$$\dim(a) = 1, \quad \dim(m) = \frac{1}{\tau D_m} \in \{\pm |G|^{1/2}\}.$$

PROOF. Recall that non-trivial triangular vector spaces are given by

$$\begin{bmatrix} a, b \\ ab \end{bmatrix} = \boldsymbol{C}[a, b], \quad \begin{bmatrix} a, m \\ m \end{bmatrix} = \boldsymbol{C}[a, m], \quad \begin{bmatrix} m, a \\ m \end{bmatrix} = \boldsymbol{C}[m, a], \quad \begin{bmatrix} m, m \\ a \end{bmatrix} = \boldsymbol{C}[a]$$

and associativity transformations are given by



$\ni [a, b] \otimes [ab, c] \mapsto [b, c] \otimes [a, bc] \in$


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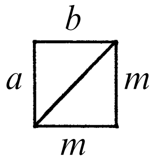
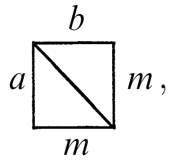


$\ni [a, b] \otimes [ab, m] \mapsto [b, m] \otimes [a, m] \in$


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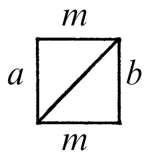
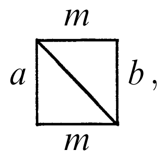


$\ni [a, m] \otimes [m, b] \mapsto \sigma(a, b)[m, b] \otimes [a, m] \in$


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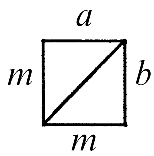
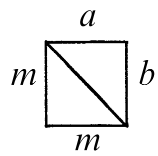


$\ni [m, a] \otimes [m, b] \mapsto [a, b] \otimes [m, ab] \in$


,

$$\begin{array}{ccc} & m & \\ a & \boxed{\diagup} & m \\ & b & \end{array} \ni [a,m] \otimes [b] \mapsto [a^{-1}b] \otimes [a, a^{-1}b] \in \begin{array}{ccc} & m & \\ a & \boxed{\diagdown} & m \\ & b & \end{array},$$

$$\begin{array}{ccc} & a & \\ m & \boxed{\diagup} & m \\ & b & \end{array} \ni [m,a] \otimes [b] \mapsto \sigma(a,b)[a,m] \otimes [b] \in \begin{array}{ccc} & a & \\ m & \boxed{\diagdown} & m \\ & b & \end{array},$$

$$\begin{array}{ccc} & m & \\ m & \boxed{\diagup} & a \\ & b & \end{array} \ni [ba^{-1}] \otimes [ba^{-1}, a] \mapsto [m,a] \otimes [b] \in \begin{array}{ccc} & m & \\ m & \boxed{\diagdown} & a \\ & b & \end{array},$$

$$\begin{array}{ccc} & m & \\ m & \boxed{\diagup} & m \\ & m & \end{array} \ni [b] \otimes [b,m] \mapsto \sum_{a \in G} \frac{\tau}{\sigma(a,b)} [a] \otimes [m,a] \in \begin{array}{ccc} & m & \\ m & \boxed{\diagdown} & m \\ & m & \end{array}.$$

Note that, if we denote by $\{[a]^*\}$ the dual basis of $\{[a]\}$ and so on, then we have

$$\left(\begin{array}{ccc} & m & \\ a & \boxed{\diagup} & b \\ & m & \end{array}\right)^* \ni [a,m]^* \otimes [m,b]^* \mapsto \langle a,b \rangle^{-1} [m,b]^* \otimes [a,m]^* \in \left(\begin{array}{ccc} & m & \\ a & \boxed{\diagdown} & b \\ & m & \end{array}\right)^*,$$

$$\left(\begin{array}{ccc} & a & \\ m & \boxed{\diagup} & m \\ & b & \end{array}\right)^* \ni [m,a]^* \otimes [b]^* \mapsto \langle a,b \rangle^{-1} [a,m]^* \otimes [b]^* \in \left(\begin{array}{ccc} & a & \\ m & \boxed{\diagdown} & m \\ & b & \end{array}\right)^*,$$

$$\left(\begin{array}{ccc} & m & \\ m & \boxed{\diagup} & m \\ & m & \end{array}\right)^* \ni [b]^* \otimes [b,m]^* \mapsto \sum_{a} \tau \langle a,b \rangle [a]^* \otimes [m,a]^* \in \left(\begin{array}{ccc} & m & \\ m & \boxed{\diagdown} & m \\ & m & \end{array}\right)^*.$$

As seen in [**13**], the tensor category $\mathscr{C}(\sigma, \tau)$ is rigid. With the choice of pairings

$$\varepsilon_a = [a, a^{-1}] \in \begin{bmatrix} aa^* \\ 1 \end{bmatrix}, \quad \varepsilon_m = [1] \in \begin{bmatrix} mm \\ 1 \end{bmatrix},$$

the associated copairings are given by

$$\delta_a = [a^{-1}, a]^* \in \begin{bmatrix} a^*a \\ 1 \end{bmatrix}^*, \quad \delta_m = \tau^{-1}[1]^* \in \begin{bmatrix} mm \\ 1 \end{bmatrix}^*.$$

Based on these data, we can calculate the maps $c_{x,y}(s)$:

$$c_{a,b}(b^{-1}a^{-1}) : \begin{bmatrix} b^*a^* \\ b^{-1}a^{-1} \end{bmatrix}^* \ni [b^{-1}, a^{-1}]^* \mapsto [a,b] \in \begin{bmatrix} a,b \\ ab \end{bmatrix},$$

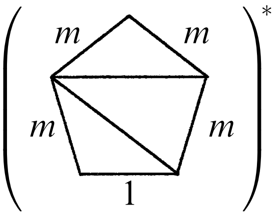
$$c_{a,m}(m) : \begin{bmatrix} ma^* \\ m \end{bmatrix} \ni [m, a^{-1}]^* \mapsto [a,m] \in \begin{bmatrix} am \\ m \end{bmatrix},$$

$$c_{m,a}(m) : \begin{bmatrix} a^*m \\ m \end{bmatrix}^* \ni [a^{-1}, m]^* \mapsto [m,a] \in \begin{bmatrix} ma \\ m \end{bmatrix},$$

$$c_{m,m}(a^*) : \begin{bmatrix} mm \\ a^* \end{bmatrix} \ni [a^{-1}]^* \mapsto \tau[a] \in \begin{bmatrix} mm \\ a \end{bmatrix}.$$

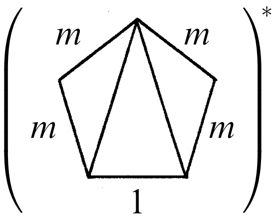
In fact, if we start with the vector

$$[b]^* \otimes [b,m]^* \otimes [1]^* \in$$



the evaluation by $\varepsilon_m(1 \otimes \varepsilon_m \otimes 1)$ gives $\delta_{b,1}$, whereas the vector is changed by associativity transformations into

$$\tau \sum_{a \in G} \sigma(a,b)[a]^* \otimes [a^{-1}]^* \otimes [a^{-1}, a]^* \in$$



If we define $\mu(a) \in \boldsymbol{C}$ by $c_{m,m}(a^*)([a^{-1}]^*) = \mu(a)[a]$, then the last vector goes to

$$\tau \sum_{a \in G} \mu(a)\sigma(a,b)[a]^* \otimes [a] \otimes [a^{-1}, a]^* \in \bigoplus_{a \in G} \begin{bmatrix} mm \\ a \end{bmatrix}^* \otimes \begin{bmatrix} mm \\ a \end{bmatrix} \otimes \begin{bmatrix} a^*a \\ 1 \end{bmatrix}$$

and its evaluation turns out to be

$$\tau \sum_{a \in G} \mu(a)\sigma(a,b).$$

Comparing this with the Kronecker delta $\delta_{b,1}$, we obtain

$$\mu(a) = \frac{1}{\tau|G|} = \tau.$$

Similarly for other $c_{x,y}(s^*)$'s.

It is now immediate to write down the equations for $\{D_s\}$:

$$D_{ab} = D_a D_b, \quad D_a D_m = D_m, \quad D_a = D_m^2$$

with the solutions given by

$$D_a \equiv 1, \quad D_m \in \{\pm 1\}.$$

The left dimension is then calculated by

$$\dim(a) = \varepsilon_a (d_a^{-1} \otimes 1) \delta_{a^*} = 1,$$

$$\dim(m) = \varepsilon_m (d_m^{-1} \otimes 1) \delta_{m^*} = \frac{1}{\tau D_m},$$

which is automatically *-invariant and hence the involution in consideration gives rise to a Frobenius duality. □

4. Positivity.

Now we shall restrict ourselves to tensor categories possessing positivity, i.e., C^*-tensor categories.

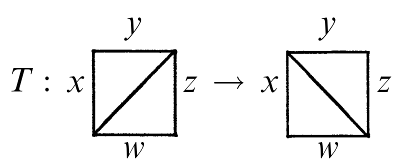
A category $\mathscr{C}$ is a C^*-category if each $\mathrm{Hom}(X, Y)$ is a Banach space with a conjugate-linear involution $\mathrm{Hom}(X, Y) \ni f \mapsto f^* \in \mathrm{Hom}(Y, X)$ such that $\|f^* f\| = \|f\|^2$.

A tensor category $\mathscr{C}$ is, by definition, a C^*-tensor category if it is a C^*-category at the same time in such a way that all the monoidal structures respect the *-operation (the unit and associativity constarints are then unitaries). A monoidal functor F between C^*-tensor categories is called a C^*-monoidal functor if it preserves the *-operation: $F(f)^* = F(f^*)$ for $f : X \to Y$ and the multiplicativity $m_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ is unitary.

Given a semisimple C^*-tensor category $\mathscr{C}$ with the spetrum set S represented by simple objects, each triangular vector space $\begin{bmatrix} xy \\ z \end{bmatrix}$ is a finite-dimensional Hilbert space with the inner product defined by

$$(\xi|\eta) 1_z = \eta \xi^*, \quad \xi, \eta \in [xy].$$

The elements (of unit constarints) $l_x \in \begin{bmatrix} 1x \\ x \end{bmatrix}$ and $r_x \in \begin{bmatrix} x1 \\ x \end{bmatrix}$ are then unit vectors. Moreover, the associativity transformations



are unitaries.

A monoidal system satisfying these conditions is referred to as a C^*-*monoidal system*.

Given a C^*-monoidal system (S, T), we can reconstruct the C^*-tensor category $\mathscr{C}$: an object in $\mathscr{C}$ is a family $X = \{X(s)\}_{s \in S}$ of finite-dimensional Hilbert spaces with $X(s) = \{0\}$ for all but finitely many $s \in S$. Hom-sets are then defined by

$$\mathrm{Hom}(X, Y) = \bigoplus_{s \in S} \mathrm{Hom}(X(s), Y(s)),$$

which is a vector space of linear maps between Hilbert spaces $\bigoplus_s X(s)$ and $\bigoplus_s Y(s)$, whence it admit the norm as well as the *-operation in the obvious manner.

It is now immediate to check that the unit and associativity constraints defined before are unitaries.

In the C^*-tensor category $\mathscr{C}(S, T)$, the operations $X \mapsto X^*$ and $f \mapsto {}^t f$ are defined exactly as in §2. It is then immediate to check the relation $({}^t f)^* = {}^t(f^*)$ for a morphism $f : X \to Y$.

We can apply the discussions on rigidity and (Frobenius) duality to C^*-tensor categories as well. For C^*-tensor cateogries, however, it is natural to require the positivity in Frobenius duality: a Frobenius duality $\{\varepsilon_X : X \otimes X^* \to I\}$ with an involution $(X^*, {}^t f, c, d)$ is *positive* if $(d_X^{-1} \otimes 1) c_{X, X^*}^{-1} \, {}^t \varepsilon_X = \varepsilon_X^*$ (I^* being identified with I).

We proved in [**16**] the existence and the uniqueness of positive Frobenius duality for a rigid C^*-tensor category with simple unit object. The key notion there is the balancedness of rigidity pairs: a rigidity pair $\{\varepsilon : X \otimes X^* \to I, \delta : I \to X^* \otimes X\}$ is said to be balanced if

$$\varepsilon(a \otimes 1)\varepsilon^* = \delta^*(1 \otimes a)\delta \quad \text{for any } a \in \mathrm{End}(X).$$

By Lemma 2.5, the positivity is equivalent in the present context to requiring

$$D_s^{-1} \delta_{s^*} = \varepsilon_s^*, \quad s \in S.$$

For $s \neq s^*$, we can choose balanced pairings ε_s, ε_{s^*} so that $\delta_{s^*} = \varepsilon_s^*$, which forces D_s to be 1 by positivity. For $s = s^*$, let $\varepsilon \in \begin{pmatrix} ss \\ 1 \end{pmatrix}$ and $\delta \in \begin{pmatrix} ss \\ 1 \end{pmatrix}^*$ be a balanced rigidity pair. Since both of ε^* and δ are non-trivial vectors in the 1-dimensional vector space $\begin{pmatrix} ss \\ 1 \end{pmatrix}^*$, they are proportional

$$\delta = \lambda \varepsilon^*,$$

where $\lambda \in \boldsymbol{C}$ satisfies $|\lambda|$ by $\|\delta\| = \|\varepsilon\|$ (the pair (ε, δ) being balanced). Since the balanced pair (ε, δ) is unique up to the phase choice $(e^{i\theta}\varepsilon, e^{-i\theta}\delta)$ $(\theta \in \boldsymbol{R})$, the phase factor λ does not depend on the choice of balanced pairs and is characterisrtic of

$s = s^*$. Thus it must coincides with the duality factor D_s for a positive duality. On the other hand, D_s satisfies $D_s^2 = 1$ for $s = s^*$; the phase factor is either $+1$ or -1.

Definition 4.1. Let S be the spectrum (fusion rule set) of a rigid C^*-tensor category with simple unit object. A self-dual element $s = s^*$ in S is said to be *real* or *pseudoreal* according to $D_s = 1$ or $D_s = -1$, namely $\lambda > 0$ or $\lambda < 0$, where $\lambda \in \boldsymbol{R}^\times$ is given by

$$\delta = \lambda\varepsilon^*, \quad \varepsilon \in \begin{bmatrix} ss \\ 1 \end{bmatrix}, \quad \delta \in \begin{bmatrix} ss \\ 1 \end{bmatrix}^*$$

with (ε, δ) a rigidity pair for s.

Now we can describe a positive Frobenius duality in terms of polygonal presentations. Given a rigid C^*-monoidal system, a family $\left\{\varepsilon_s \in \begin{bmatrix} ss^* \\ 1 \end{bmatrix}\right\}_{s \in S}$ is called a *balanced system* if $(\varepsilon_s, \varepsilon_{s^*}^*)$ is a rigidity pair for $s \neq \pm s^*$ and $(\varepsilon_s, \pm\varepsilon_s^*)$ is a rigidity pair for $s = \pm s^*$, where $s = \pm s^*$ means that s is real or pseudoreal according to the signature.

Theorem 4.2. *In a rigid C^*-monoidal system, we can always find a balanced system of duality pairings. Given a balanced system $\{\varepsilon_s\}_{s \in S}$, its canonical extension $\{\varepsilon_X\}$ provides a positive Frobenius duality together with the involution $(X^*, {}^t\!f, c_{X,Y}, d_X)$, where the duality isomorphism $\{d_X : X \to X^{**}\}$ is specified by*

$$D_s = \begin{cases} -1 & \text{if } s \in S \text{ is pseudoreal}, \\ 1 & \text{otherwise}. \end{cases}$$

Appendix A. Group Cohomology. In the text, we have occasionally used the cyclic normalization of cocycles in group cohomology, which would be a known fact but we have failed in finding literatures; we shall give an account here for completeness.

Let G be a discrete group and

$$\cdots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \to \boldsymbol{Z}$$

be a projective resolution of the trivial G-module $\boldsymbol{Z}$. For an abelian group A with a G-action, the cohomology groups are defined by

$$H^n(G, A) = H^n(\mathrm{Hom}_G(C_\cdot, A)),$$

which is independent of the choice of projective resolutions.

Commonly used is the standard resolution, where

$$C_n = \bigoplus_{g_0, g_1, \dots, g_n \in G} \boldsymbol{Z}(g_0, g_1, \dots, g_n)$$

is a free $\boldsymbol{Z}$-module with the G-action defined by

$$g(g_0, g_1, \dots, g_n) = (gg_0, gg_1, \dots, gg_n).$$

The differential and a chain contraction s (satisfying $\partial s + s\partial = id$) are given by

$$\partial(g_0, g_1, \dots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \dots, \hat{g}_i, \dots, g_n),$$

$$s(g_0, g_1, \dots, g_n) = (1, g_0, g_1, \dots, g_n).$$

As a $\boldsymbol{Z}(G)$-basis (the so-called bar basis), we can choose

$$|g_1|g_2|\cdots|g_n| = (1, g_1, g_1g_2, \dots, g_1g_2\cdots g_n)$$

$((g_0, g_1, \dots, g_n) = g_0|g_0^{-1}g_1|g_1^{-1}g_2|\cdots|g_{n-1}^{-1}g_n|)$. Note that

$$\partial(|g_1|g_2|\cdots|g_n|) = g_1|g_2|\cdots|g_n| + \sum_{i=1}^{n-1} (-1)^i |g_1|\cdots|g_ig_{i+1}|\cdots|g_n|$$

$$+ (-1)^n |g_1|\cdots|g_{n-1}|$$

and

$$s(g|g_1|\cdots|g_n|) = |g|g_1|\cdots|g_n|.$$

For $g_0,\ g_1, \dots, g_n \in G$, define the wedge product by

$$g_0 \wedge g_1 \wedge \cdots \wedge g_n = P_{n+1}(g_0, g_1, \dots, g_n) = \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} \varepsilon(\sigma)(g_{\sigma(0)}, g_{\sigma(1)}, \dots, g_{\sigma(n)}),$$

which is an element in $\boldsymbol{Q} \otimes_{\boldsymbol{Z}} C_n$. These, when parametrized by unordered $n+1$-tuple $\{g_0, g_1, \dots, g_n\}$, form a $\boldsymbol{Z}$-basis of the image D_n of C_n under the projection P_{n+1} in $\boldsymbol{Q} \otimes_{\boldsymbol{Z}} C_n$ and hence

$$D_n = \sum \boldsymbol{Z} g_0 \wedge g_1 \wedge \cdots \wedge g_n$$

is a free $\boldsymbol{Z}(G)$-submodule of $\boldsymbol{Q} \otimes_{\boldsymbol{Z}} C_n$. Define a $\boldsymbol{Z}(G)$-linear map $d : D_n \to D_{n-1}$ by $d = P_n\partial$. Then by the formula

$$d(g_0 \wedge g_1 \wedge \cdots \wedge g_n) = \sum_{i=0}^{n} (-1)^i g_0 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_n$$

we know that $d^2 = 0$.

In fact,

$$P\left(\sum_{\sigma} \varepsilon(\sigma)\partial(g_{\sigma(0)}, \dots, g_{\sigma(n)})\right)$$

$$= \sum_{i=0}^{n} \sum_{\sigma} \varepsilon(\sigma)(-1)^{i} g_{\sigma(0)} \wedge \cdots \wedge \hat{g}_{\sigma(i)} \wedge \cdots \wedge g_{\sigma(n)}$$

$$= \sum_{i,j} \sum_{\sigma(i)=j} \varepsilon(\sigma)(-1)^{i} g_{\sigma(0)} \wedge \cdots \wedge \hat{g}_{\sigma(i)=j} \wedge \cdots \wedge g_{\sigma(n)}$$

(letting $\sigma = \tau \circ (i, j)$)

$$= \sum_{i,j} \sum_{\tau(j)=j} \varepsilon(\tau)(-1)^{i+1}$$

$$g_{\tau(0)} \wedge \cdots \wedge g_{\tau(i-1)} \wedge \hat{g}_{\tau(j)=j} \wedge g_{\tau(i+1)} \wedge \cdots \wedge g_{\tau(j-1)} \wedge g_{\tau(i)} \wedge g_{\tau(j+1)} \wedge \cdots \wedge g_{\tau(n)}$$

$$= \sum_{i,j} \sum_{\tau(j)=j} \varepsilon(\tau)(-1)^{i+1}(-1)^{j-1-i} g_{\tau(0)} \wedge \cdots \wedge g_{\tau(j-1)} \wedge \hat{g}_{\tau(j)} \wedge g_{\tau((j+1)} \wedge \cdots \wedge g_{\tau(n)}$$

$$= \sum_{i,j} \sum_{\tau(j)=j} (-1)^{j} g_0 \wedge \cdots \wedge \hat{g}_j \wedge \cdots \wedge g_n$$

$$= (n+1)! \sum_{j=0}^{n} (-1)^{j} g_0 \wedge \cdots \wedge \hat{g}_j \wedge \cdots \wedge g_n.$$

The chain contraction c for the differential is then defined by

$$c(g_0 \wedge \cdots \wedge g_n) = P_{n+2}s(g_0 \wedge \cdots \wedge g_n) = 1 \wedge g_0 \wedge \cdots \wedge g_n.$$

Thus we get another free resolution (D, d) (the wedge resolution) of the trivial G-module $\boldsymbol{Z}$.

Since the $\boldsymbol{Z}(G)$-linear projections $\{P_{n+1} : C_n \to D_n\}$ give a chain homomorphism $(C, \partial) \to (D, d)$ (see the explicit formula for d), the induced inclusion of chain complexes

$$\begin{array}{ccc}
\mathrm{Hom}_G(D_n, A) & \xrightarrow{d^*} & \mathrm{Hom}_G(D_{n+1}, A) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_G(C_n, A) & \xrightarrow[\partial^*]{} & \mathrm{Hom}_G(C_{n+1}, A)
\end{array}$$

induces the isomorphisms.

As a consequence, to represent cohomology classes, we can choose cocycles, say $F \in \mathrm{Hom}_G(C_n, A)$, satisfying

$$F(g_{\sigma(0)}, \dots, g_{\sigma(n)}) = \varepsilon(\sigma)F(g_0, \dots, g_n)$$

for $\sigma \in S_{n+1}$. In the case of lower n, we can explicitly write down the conditions: With

$$f(g_1, \dots, g_n) = F(1, g_1, g_1g_2, \dots, g_1g_2\cdots g_n),$$

we have

$$f(g,h) = f(gh, h^{-1}) = -gf(g^{-1}, gh),$$

$$f(g_1, g_2, g_3) = -g_1f(g_2, g_3, g_4)$$

if $g_1g_2g_3g_4 = 1$, and so on.

Remark. When G acts on A trivially, we can deduce

$$-f(g,h) = f(h^{-1}, g^{-1}), \quad f(g_1, g_2, g_3) = -f(g_2, g_3, g_4)$$

from the above conditions.

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