J. Math. Soc. Japan
Vol. 54, No. 1, 2002

Perturbation theory for m-accretive operators and generalized complex Ginzburg-Landau equations

By Noboru OKAZAWA* and Tomomi YOKOTA

(Received Oct. 6, 1999)

Abstract. Global existence of unique strong solutions is proved for the generalized complex Ginzburg-Landau equation. The proof is based on a new type perturbation theorem for m-accretive operators in complex Hilbert spaces.

1. Introduction.

Let Ω be a bounded or unbounded domain in $\boldsymbol{R}^N$ with compact C^2-boundary $\partial\Omega$ (including $\boldsymbol{R}^N$ itself). In $L^2(\Omega)$ we consider the initial-boundary value problem for the "generalized" complex Ginzburg-Landau equation:

$$(1.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} + (\lambda + i\alpha)A(x,D)u + (\kappa + i\beta)g(x,|u|^2)u - \gamma u = 0 \quad \text{on } \Omega \times \boldsymbol{R}_+, \\ u = 0 \quad \text{on } \partial\Omega \times \boldsymbol{R}_+, \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{cases}$$

Here u is a complex-valued unknown function, $i = \sqrt{-1}$, $\lambda, \kappa \in \boldsymbol{R}_+$, $\alpha, \beta, \gamma \in \boldsymbol{R}$ are constants, $g \in C^1(\Omega \times (0,\infty); \boldsymbol{R})$, and $A(x,D)$ is the second order elliptic differential operator in divergence form:

$$A(x,D)u := -\sum_{j,k=1}^{N} \frac{\partial}{\partial x_k}\left(a_{jk}(x)\frac{\partial u}{\partial x_j}\right).$$

Problem (1.1), originally derived by Newell and Whitehead [**16**], appears in the mathematical description of spatial pattern formation and of the onset of instabilities in nonequilibrium fluid dynamical systems (see Cross and Hohenberg [**4**]). Problem (1.1) is formally a mixed type model in the sense that it is reduced to a nonlinear Schrödinger equation when $\lambda = \kappa = \gamma = 0$ and to a nonlinear heat equation when $\alpha = \beta = \gamma = 0$.

2000 *Mathematics Subject Classification*. Primary 47H06; Secondary 35Q55, 47H20.
Key Words and Phrases. Perturbation theory, accretive operators, semigroups of nonlinear operators, the complex Ginzburg-Landau equation.
*Partially supported by Grant-in-Aid for Scientific Research (No. 11640185), Ministry of Education, Culture, Sports, Science and Technology, Japan.

When $A(x,D) = -\Delta$ and $g(x,|u|^2) = |u|^{p-1}$ $(p > 1)$, the existence and uniqueness of solutions to (1.1) have been established by many authors using various methods (cf. Bu [**3**], Doering, Gibbon and Levermore [**5**], Levermore and Oliver [**12**], Okazawa and Yokota [**18**], Temam [**20**], Unai and Okazawa [**22**] and Yang [**23**]). The case that $g(x,|u|^2)$ does not depend on x explicitly has been systematically studied in more general situations by Ginibre and Velo [**6**], [**7**], [**8**]. However, there seems to be no work in which the linear and nonlinear terms depend explicitly on the spatial variables.

The purpose of this paper is to prove the global existence of unique strong solutions to (1.1) under the condition that the coefficient $\kappa + i\beta$ of nonlinear term satisfies

$$\frac{|\beta|}{\kappa} \le \frac{\sqrt{1+2\sigma}}{\sigma} \tag{1.2}$$

without any restriction on the dimension $N \ge 1$ and the constant $\sigma > 0$, where σ is an upper bound of the ratio $s(\partial g/\partial s)(x,s)/g(x,s)$ (see (2.2) below); note that $\sigma = (p-1)/2$ and $\sqrt{1+2\sigma}/\sigma = 2\sqrt{p}/(p-1)$ if $g(x,|u|^2) = |u|^{p-1}$ $(p > 1)$. It should be noted that condition (1.2) excludes nonlinear Schrödinger equations.

We regard (1.1) as the initial value problem for abstract evolution equation

$$\frac{du}{dt} + Au = 0, \quad u(0) = u_0, \tag{1.3}$$

in $X := L^2(\Omega)$ by setting

$$A := (\lambda + i\alpha)S + (\kappa + i\beta)B - \gamma \quad \text{with } D(A) := D(S) \cap D(B),$$

where

$$Su := A(x,D)u \quad \text{for } u \in D(S) := H^2(\Omega) \cap H_0^1(\Omega),$$

$$Bu := g(x,|u|^2)u \quad \text{for } u \in D(B) := \{u \in X; g(x,|u|^2)u \in X\}.$$

According to the theory of nonlinear semigroups we have only to show that $A + \gamma$ is m-accretive in X. In a previous paper [**18**] we proved the same result for (1.1) with $A(x,D) = -\Delta$ and $g(x,|u|^2) = |u|^{p-1}$. In this special case the m-accretivity of $A + \gamma$ is a consequence of a perturbation theorem for m-accretive operators prepared in [**18**]. Unfortunately, the perturbation theorem is too simple to be applied to (1.1) itself. So the main task in this paper is to generalize the perturbation theorem in [**18**] to control the contribution of x-dependence in the nonlinear term. Actually, the m-accretivity of $A + \gamma$ under condition (1.2) is guaranteed by the following two inequalities:

$$|\mathrm{Im}(Bu_1 - Bu_2, u_1 - u_2)| \le \frac{\sigma}{\sqrt{1+2\sigma}} \mathrm{Re}(Bu_1 - Bu_2, u_1 - u_2), \tag{1.4}$$

(1.5)

$$|\mathrm{Im}(Su, B_\varepsilon u)| \le \frac{\sigma}{\sqrt{1+2\sigma}} \mathrm{Re}(Su, B_\varepsilon u) + \frac{1+\sigma}{\sqrt{(1+2\sigma)\delta}} \|S^{1/2}u\|(c_1\|B_\varepsilon u\| + c_2\|u\|),$$

where B_ε is the Yosida approximation of B (for δ, c_1 and c_2 see (2.1) and (2.3) below). Evidently, (1.4) implies that $(\kappa + i\beta)B$ is accretive in X under condition (1.2). Therefore it remains to prove the maximality of the accretive operator $(\lambda + i\alpha)S + (\kappa + i\beta)B$. This is achieved by the key inequality (1.5). In this connection, we note that the second term on the right-hand side of (1.5) is absent from the perturbation theorem in [**18**].

This paper is organized as follows. In Section 2 we state our main result on the global existence of unique strong solutions to (1.1) (Theorem 2.2). Section 3 is a review of the nonlinear semigroup theory. In Section 4 we develop the perturbation theory mentioned above. Namely, we consider the m-accretivity of linear combinations of a nonnegative selfadjoint operator and a nonlinear m-accretive operator with complex coefficients. We prove the main result in Section 5 which is largely devoted to the proofs of (1.4) and (1.5) (Lemmas 5.3 and 5.5).

2. Results.

We impose the following conditions on $A(x, D)$ and $g(x, s)$:

(A) $A(x, D)$ is uniformly elliptic in Ω, that is, there is a constant δ $(0 < \delta \le 1)$ such that for $\xi \in \boldsymbol{R}^N$ and $x \in \Omega$,

$$\delta|\xi|^2 \le \sum_{j,k=1}^{N} a_{jk}(x)\xi_j\xi_k \le \delta^{-1}|\xi|^2, \tag{2.1}$$

where $a_{jk} = a_{kj} \in C^1(\bar{\Omega}; \boldsymbol{R}) \cap W^{1,\infty}(\Omega; \boldsymbol{R})$ (cf. Brezis [**1**, Remarque IX.25]).

(B) $g \in C(\Omega \times [0, \infty); \boldsymbol{R}) \cap C^1(\Omega \times (0, \infty); \boldsymbol{R})$ and there are constants $\sigma > 0$ and c_1, $c_2 \ge 0$ such that for all $(x, s) \in \Omega \times (0, \infty)$,

$$0 \le s\frac{\partial g}{\partial s}(x, s) \le \sigma g(x, s), \tag{2.2}$$

$$|\nabla_x g(x, s)| \le c_1 g(x, s) + c_2. \tag{2.3}$$

For example, let $p > 1$. Then $g(x, s) := (|x|^2 + 1)s^{(p-1)/2} + |x|^2$ satisfies (2.2) and (2.3) with respective constants $\sigma = (p-1)/2$ and $c_1 = c_2 = 1$.

Before stating our result we give a definition of strong solutions to (1.1).

Definition 2.1. Let X and A be as defined in Section 1. Then the global *strong solution* to (1.1) is defined as an X-valued function $u(t) := u(x, t)$ with the following properties:

(a) $u(t) \in D(A)$ $\forall t \geq 0$ and $Au(\cdot) \in L^\infty(0,T;X)$ $\forall T > 0$.
(b) $u(\cdot)$ is Lipschitz continuous on $[0,T]$: $u(\cdot) \in C^{0,1}([0,T];X)$ $\forall T > 0$.
(c) The strong derivative $u'(t)$ exists for a.a. $t \geq 0$ and is bounded in X on $[0,T]$: $u(\cdot) \in W^{1,\infty}(0,T;X)$ $\forall T > 0$.
(d) $u(\cdot)$ satisfies the equation in (1.3) a.e. on $[0,\infty)$ as well as the initial condition.

We now state our main result in this paper.

Theorem 2.2. *Let $\lambda > 0$, $\kappa > 0$, and $\kappa^{-1}|\beta| \leq \sqrt{1+2\sigma}/\sigma$. Then for any initial value $u_0 \in D(A)$ there exists a unique global strong solution $u(t) := u(x,t)$ to* (1.1) *in X such that*

$$u(\cdot) \in L^\infty(0,T;H^2(\Omega)) \cap L^\infty(0,T;D(B)) \cap C^{0,1/2}([0,T];H_0^1(\Omega)) \quad \forall T > 0 \tag{2.4}$$

with the estimates

$$\|u(t)\| \leq e^{\gamma t}\|u_0\|, \tag{2.5}$$

$$\|u(t) - v(t)\| \leq e^{\gamma t}\|u_0 - v_0\|, \tag{2.6}$$

$$\|\nabla u(t) - \nabla v(t)\|^2 \leq c(u_0,v_0)e^{2\gamma t}\|u_0 - v_0\|, \tag{2.7}$$

where $v(t)$ is a solution to (1.1) *with initial value $v_0 \in D(A)$ and $c(u_0,v_0)$ is given by*

$$c(u_0,v_0) := (\delta K)^{-1}[L(\|Au_0\| + \|Av_0\|) + (L|\gamma| + \sqrt{cK})(\|u_0\| + \|v_0\|)].$$

Here K, L and c depend on $\lambda + i\alpha$, $\kappa + i\beta$ and the constants appearing in condition (B).

Remark. 1) It is remarkable that condition (1.2) is free from the fact that A and g depend explicitly on x.

2) Let $N \leq 3$. Then it seems that the solution is of class C^1; this may be shown by regarding (1.1) as a semilinear evolution equation (cf. [**3**], [**5**], [**12**], [**21**], [**23**]).

3) Let $N\sigma \leq 2$. Then it is desirable to show that (1.1) has unique global solutions with no restriction on the coefficients $\lambda + i\alpha$ and $\kappa + i\beta$ (for mild solution cf. [**6**], [7], [**8**] and for C^1-solution cf. [**3**], [**5**], [**12**], [**20**], [**23**]).

Corollary 2.3. *In* Theorem 2.2 *assume further that $c_1 = 0$ in condition* (2.3). *Then*

$$\delta\|\nabla u(t)\|^2 + \varepsilon\|u(t)\|^2 \leq (\delta^{-1}\|\nabla u_0\|^2 + \varepsilon\|u_0\|^2)\exp\left[\left(2\gamma + \frac{(1+\sigma)\kappa}{\sigma\sqrt{\delta\varepsilon}}c_2\right)t\right] \tag{2.8}$$

for $\varepsilon > 0$. In particular, if $c_1 = c_2 = 0$, then one can take $\varepsilon = 0$ in (2.8):

$$\|\nabla u(t)\| \leq \delta^{-1}e^{\gamma t}\|\nabla u_0\|.$$

Note that if g is independent of x: $g(x,s) = g(s)$, then $c_1 = c_2 = 0$.

3. Preliminaries.

In this section we briefly review the abstract Cauchy problem with m-accretive operator and its relation to the theory of nonlinear semigroups.

Let X be a complex Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

An operator A with domain $D(A)$ and range $R(A)$ in X is said to be *accretive* (or monotone) if $\mathrm{Re}(Au_1 - Au_2, u_1 - u_2) \geq 0$ for $u_1, u_2 \in D(A)$. If, in addition, $R(1 + A) = X$, then we say that A is *m-accretive* (or maximal monotone) in X.

The following is fundamental (see Kōmura and Konishi [**11**], Miyadera [**14**, pp. 145–148] and Showalter [**19**, Theorem IV.4.1]; cf. also [**18**, Lemma 2.1]).

Theorem 3.1. *Let A be an operator in X and $\gamma \in \mathbf{R}$. If $A + \gamma$ is m-accretive in X, then for every $u_0 \in D(A)$ there exists a unique strong solution $u(t)$ to the initial value problem*

$$\frac{du}{dt} + Au = 0, \quad u(0) = u_0, \tag{3.1}$$

in the following sense:

(a) $u(t) \in D(A)$ *and* $\|Au(t)\| \leq e^{\gamma t}\|Au_0\|$ *for all* $t \geq 0$.

(b) $\|u(t) - u(s)\| \leq e^{\gamma_+(t+s)}\|Au_0\| \cdot |t - s|,\ t, s \geq 0,$ *where* $\gamma_+ := \max\{0, \gamma\}$.

(c) du/dt *exists a.e. on* $[0, \infty)$, *with* $\|(du/dt)(t)\| \leq e^{\gamma t}\|Au_0\|$ *(a.e.)*.

(d) $u(\cdot)$ *satisfies the equation in* (3.1) *a.e. on* $[0, \infty)$ *as well as the initial condition.*

We can define the solution operator $U(t) : D(A) \to D(A)$ by $U(t)u_0 := u(t)$, $t \geq 0$, where $u(\cdot)$ is a unique solution to (3.1) in the sense of Theorem 3.1 (a)–(d). Denoting the continuous extension again by $U(t)$, we obtain a one-parameter family $\{U(t); t \geq 0\}$ on $\overline{D(A)}$ (the closure of $D(A)$ in X) which satisfies

(a) $U(0) = 1,\ U(t + s) = U(t)U(s),\ t, s \geq 0,$

(b) $U(t)v \to v\ (t \downarrow 0),\ v \in \overline{D(A)},$

(c) $\|U(t)v_1 - U(t)v_2\| \leq e^{\gamma t}\|v_1 - v_2\|,\ v_1, v_2 \in \overline{D(A)},\ t \geq 0.$

In this paper the family $\{U(t); t \geq 0\}$ is called a *semigroup of type γ on $\overline{D(A)}$ generated by $-A$.*

4. Perturbations of m-accretive operators.

The following is an essence in our perturbation theory for m-accretive operators.

Proposition 4.1. *Let $v_0, v_1, v_2 \in X$ and $\lambda + i\alpha$, $\kappa + i\beta \in \mathbf{C}$ with $\lambda > 0$, $\kappa > 0$. Assume that there are constants $k_1 > 0$, $a \geq 0$, $b \geq 0$ and $c \geq 0$ such that*

$$(4.1)\qquad |\mathrm{Im}(v_1, v_2)| \le k_1 \,\mathrm{Re}(v_1, v_2) + a\|v_1\|^2 + b\|v_2\|^2 + c\|v_0\|^2, \quad \textit{and}$$

$$(4.2)\qquad K := k_1 \frac{\lambda}{\kappa} - b\frac{\lambda^2 + \alpha^2}{\kappa^2 + \beta^2} - a > 0.$$

If $\kappa^{-1}|\beta| \le k_1^{-1}$, *then*

$$(4.3)\qquad K\|v_1\| \le L\|(\lambda + i\alpha)v_1 + (\kappa + i\beta)v_2\| + \sqrt{cK}\|v_0\|,$$

where

$$(4.4)\qquad L := \frac{k_1}{\kappa} + 2b\frac{\sqrt{\lambda^2 + \alpha^2}}{\kappa^2 + \beta^2} + \frac{\sqrt{bK}}{\sqrt{\kappa^2 + \beta^2}}.$$

Proof. Suppose that $\kappa^{-1}|\beta| \le k_1^{-1}$. Then it follows from (4.1) that

$$\begin{aligned}\mathrm{Re}(v_1, (\kappa + i\beta)v_2) &\ge \kappa\,\mathrm{Re}(v_1, v_2) - |\beta| \cdot |\mathrm{Im}(v_1, v_2)|\\ &\ge (k_1^{-1}\kappa - |\beta|)|\mathrm{Im}(v_1, v_2)| - k_1^{-1}\kappa(a\|v_1\|^2 + b\|v_2\|^2 + c\|v_0\|^2)\\ &\ge -k_1^{-1}\kappa(a\|v_1\|^2 + b\|v_2\|^2 + c\|v_0\|^2).\end{aligned}$$

Setting $N(v_1, v_2) := \|(\lambda + i\alpha)v_1 + (\kappa + i\beta)v_2\|$, we see that

$$\begin{aligned}\lambda\|v_1\|^2 &= \mathrm{Re}(v_1, (\lambda + i\alpha)v_1)\\ &= \mathrm{Re}(v_1, (\lambda + i\alpha)v_1 + (\kappa + i\beta)v_2) - \mathrm{Re}(v_1, (\kappa + i\beta)v_2)\\ &\le N(v_1, v_2)\|v_1\| + k_1^{-1}\kappa(a\|v_1\|^2 + b\|v_2\|^2 + c\|v_0\|^2).\end{aligned}$$

Since $\|v_2\| \le s^{-1}[N(v_1, v_2) + r\|v_1\|]$ for $r := \sqrt{\lambda^2 + \alpha^2}$ and $s := \sqrt{\kappa^2 + \beta^2}$, it follows that

$$K\|v_1\|^2 \le (k_1\kappa^{-1} + 2brs^{-2})N(v_1, v_2)\|v_1\| + bs^{-2}N(v_1, v_2)^2 + c\|v_0\|^2,$$

where K is given by (4.2). Solving this inequality, we obtain the assertion. □

Corollary 4.2. *Let* $v_0, v_1, v_2 \in X$. *Assume that* $(v_1, v_0) \ge 0$ *and there are constants* $k_1, k_2 > 0$ *and* $c_1, c_2 \ge 0$ *such that*

$$(4.5)\qquad |\mathrm{Im}(v_1, v_2)| \le k_1 \,\mathrm{Re}(v_1, v_2) + k_2(v_1, v_0)^{1/2}(c_1\|v_2\| + c_2\|v_0\|).$$

Then for every pair of $\lambda + i\alpha,\ \kappa + i\beta \in \boldsymbol{C}$ *with* $\lambda > 0$, $\kappa > 0$ *the assumption of* Proposition 4.1 *is satisfied.*

Proof. Let $\varepsilon > 0$. Then (4.1) is derived from (4.5). In fact, for example, we can take

$$(4.6)\qquad a := \frac{k_2}{4}(c_1 + c_2)\varepsilon, \quad b := \frac{k_2}{2}c_1\varepsilon,$$

and $c := (k_2/4)(c_1\varepsilon^{-3} + 2c_2\varepsilon^{-1} + c_2\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, (4.2) is also satisfied for any pair of $\lambda + i\alpha,\ \kappa + i\beta \in \boldsymbol{C}$ with $\lambda > 0,\ \kappa > 0$. Thus we obtain (4.3) under the condition $\kappa^{-1}|\beta| \le k_1^{-1}$. □

In applications we encounter (4.5) rather than (4.1). But K and L in (4.3) are easily determined with the coefficients in (4.1) rather than those in (4.5).

An operator B in X is said to be *m-sectorial of type* $S(k)$ if B is m-accretive and *sectorial of type* $S(k)$: for $u_1, u_2 \in D(B)$,

$$|\mathrm{Im}(Bu_1 - Bu_2, u_1 - u_2)| \le k\,\mathrm{Re}(Bu_1 - Bu_2, u_1 - u_2) \tag{4.7}$$

(cf. Goldstein [**9**, Definition 1.5.8]). Let $\kappa + i\beta \in \boldsymbol{C}$ with $\kappa > 0$ and $\kappa^{-1}|\beta| \le k^{-1}$. Then (4.7) implies that $(\kappa + i\beta)B$ is accretive in X:

$$\mathrm{Re}((\kappa + i\beta)(Bu_1 - Bu_2), u_1 - u_2) \ge (k^{-1}\kappa - |\beta|)|\mathrm{Im}(Bu_1 - Bu_2, u_1 - u_2)| \ge 0.$$

Let $\{B_\varepsilon; \varepsilon > 0\}$ be the Yosida approximation of B:

$$B_\varepsilon := \varepsilon^{-1}(1 - J_\varepsilon) = BJ_\varepsilon,$$

where $J_\varepsilon := (1 + \varepsilon B)^{-1}$, $\varepsilon > 0$. Then $(\kappa + i\beta)B_\varepsilon = (\kappa + i\beta)BJ_\varepsilon$ is also accretive in X:

$$\begin{aligned}&\mathrm{Re}((\kappa + i\beta)(B_\varepsilon v_1 - B_\varepsilon v_2), v_1 - v_2)\\ &\qquad = \varepsilon\kappa\|B_\varepsilon v_1 - B_\varepsilon v_2\|^2 + \mathrm{Re}((\kappa + i\beta)(B(J_\varepsilon v_1) - B(J_\varepsilon v_2)), J_\varepsilon v_1 - J_\varepsilon v_2) \ge 0.\end{aligned}$$

Next let S be a nonnegative selfadjoint operator in X, and $\lambda + i\alpha \in \boldsymbol{C}$ with $\lambda \ge 0$. Since $(\lambda + i\alpha)S$ is m-accretive in X, it follows that $(\lambda + i\alpha)S + (\kappa + i\beta)B_\varepsilon$ is also m-accretive in X (see e.g. [**19**, Lemma IV.2.1]). Therefore for $f \in X$ and $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in D(S)$ of the equation

$$(\lambda + i\alpha)Su_\varepsilon + (\kappa + i\beta)B_\varepsilon u_\varepsilon + u_\varepsilon = f. \tag{4.8}$$

Now we can state a criterion for the m-accretivity of $(\lambda + i\alpha)S + (\kappa + i\beta)B$.

Lemma 4.3. *Let S be a nonnegative selfadjoint operator in X. Let B be a nonlinear m-sectorial operator of type $S(k_1)$ in X. Let $\lambda + i\alpha,\ \kappa + i\beta \in \boldsymbol{C}$ with $\lambda \ge 0,\ \kappa > 0$ and $\kappa^{-1}|\beta| \le k_1^{-1}$.*

Then $(\lambda + i\alpha)S + (\kappa + i\beta)B$ is m-accretive in X if and only if for every $f \in X$, $\|B_\varepsilon u_\varepsilon\|$ is bounded as $\varepsilon \downarrow 0$, where u_ε is a unique solution of (4.8).

This lemma is essentially proved by Brezis, Crandall and Pazy [**2**, Theorem 2.1] in which $\lambda = \kappa = 1$ and $\alpha = \beta = 0$. In the proof of the "only if" part we obtain

$$\|B_\varepsilon u_\varepsilon\| \le \kappa^{-1}\sqrt{\kappa^2+\beta^2}\|Bu\|,$$

where u is a unique solution of the equation $(\lambda + i\alpha)Su + (\kappa + i\beta)Bu + u = f$.

Now we can state and prove a perturbation theorem for m-accretive operators which extends [**18**, Theorem 2.5] and applies to problem (1.1).

Theorem 4.4. *Let S be a nonnegative selfadjoint operator in X. Let B be a nonlinear m-sectorial operator of type $S(k_1)$ in X. Assume that $D(S) \cap D(B) \neq \emptyset$ and there are constants $k_2 > 0$ and $c_j \ge 0$ $(j = 1, 2)$ such that for $u \in D(S)$ and $\varepsilon > 0$,*

$$|\mathrm{Im}(Su, B_\varepsilon u)| \le k_1 \mathrm{Re}(Su, B_\varepsilon u) + k_2\|S^{1/2}u\|(c_1\|B_\varepsilon u\| + c_2\|u\|). \tag{4.9}$$

Let $\lambda + i\alpha$, $\kappa + i\beta \in \boldsymbol{C}$ with $\lambda > 0$, $\kappa > 0$ and $\kappa^{-1}|\beta| \le k_1^{-1}$. Then for $\gamma \in \boldsymbol{R}$,

$$A + \gamma := (\lambda + i\alpha)S + (\kappa + i\beta)B, \quad D(A) := D(S) \cap D(B),$$

is m-accretive in X and hence for $u_0 \in D(A)$ and $t \ge 0$,

$$K\|SU(t)u_0\| \le Le^{\gamma t}\|Au_0\| + (L|\gamma| + \sqrt{cK})\|U(t)u_0\|, \tag{4.10}$$

where $\{U(t); t \ge 0\}$ is the semigroup of type γ on $\overline{D(A)}$ generated by $-A$ and K, L are the constants defined by (4.2), (4.4) *and* (4.6).

Proof. First we note that (4.9) is nothing but (4.5) with $v_1 := Su$, $v_2 := B_\varepsilon u$ and $v_0 = u$. Thus we can determine the constants K and L such that for $u \in D(S)$,

$$K\|Su\| \le L\|(\lambda + i\alpha)Su + (\kappa + i\beta)B_\varepsilon u\| + \sqrt{cK}\|u\|. \tag{4.11}$$

Now let u_ε be a unique solution of (4.8). Then we see from (4.11) that

$$K\|Su_\varepsilon\| \le L\|f - u_\varepsilon\| + \sqrt{cK}\|u_\varepsilon\|.$$

Since $D(S) \cap D(B) \neq \emptyset$, we can conclude that $\{\|u_\varepsilon\|\}$ is bounded as $\varepsilon \downarrow 0$ (see [**19**, Lemma IV.2.2]) and so are $\{\|Su_\varepsilon\|\}$ and $\{\|B_\varepsilon u_\varepsilon\|\}$, too. Therefore Lemma 4.3 yields that $A + \gamma = (\lambda + i\alpha)S + (\kappa + i\beta)B$ is m-accretive in X.

Finally we prove (4.10). Setting $u = U(t)u_0 \in D(S) \cap D(B)$ in (4.11) and noting that $B_\varepsilon U(t)u_0 \to BU(t)u_0$ $(\varepsilon \downarrow 0)$, we obtain

$$K\|SU(t)u_0\| \le L\|AU(t)u_0\| + (L|\gamma| + \sqrt{cK})\|U(t)u_0\|.$$

Thus (4.10) follows from Theorem 3.1 (a). □

Here is an information on invariant sets for $U(t)$ (cf. [**18**, Corollary 2.6]).

Corollary 4.5. *In* Theorem 4.4 *assume further that* $B0 = 0$, $c_1 = 0$ *in* (4.9) *and $D(A)$ is dense in $D(S^{1/2})$ (that is, $D(A)$ is a core for $S^{1/2}$). Then $U(t)$ leaves $D(S^{1/2})$ invariant and for $v \in D(S^{1/2})$, $t \ge 0$ and $\varepsilon > 0$,*

$$\|(\varepsilon+S)^{1/2}U(t)v\| \le \|(\varepsilon+S)^{1/2}v\| \exp\left[\left(\gamma+\frac{k_2\kappa}{2k_1\sqrt{\varepsilon}}c_2\right)t\right]. \tag{4.12}$$

In particular, if $c_1 = c_2 = 0$ *in* (4.9), *then one can take* $\varepsilon = 0$ *in* (4.12).

PROOF. It suffices to prove (4.12) for the elements in $D(A)$. Let $v \in D(A)$ and $\varepsilon > 0$. Setting $u(s) := U(s)v$, we see that $(d/ds)\|(\varepsilon+S)^{1/2}u(s)\|^2 = 2\,\mathrm{Re}(u'(s), (\varepsilon+S)u(s))$ a.e. on $[0,\infty)$. Integrating this equality on $[0,t]$, we have

$$\|(\varepsilon+S)^{1/2}u(t)\|^2 = \|(\varepsilon+S)^{1/2}v\|^2 - 2\int_0^t \mathrm{Re}(Au(s), (\varepsilon+S)u(s))\,ds.$$

It follows from (4.9) with $c_1 = 0$ that

$$\|(\varepsilon+S)^{1/2}u(t)\|^2 \le \|(\varepsilon+S)^{1/2}v\|^2 + 2\gamma\int_0^t \|(\varepsilon+S)^{1/2}u(s)\|^2\,ds + \frac{2c_2k_2\kappa}{k_1}\int_0^t \|S^{1/2}u(s)\|\cdot\|u(s)\|\,ds.$$

Noting that $2\sqrt{\varepsilon}\|S^{1/2}u(s)\|\cdot\|u(s)\| \le \|(\varepsilon+S)^{1/2}u(s)\|^2$, we have

$$\|(\varepsilon+S)^{1/2}u(t)\|^2 \le \|(\varepsilon+S)^{1/2}v\|^2 + \left(2\gamma+\frac{c_2k_2\kappa}{k_1\sqrt{\varepsilon}}\right)\int_0^t \|(\varepsilon+S)^{1/2}u(s)\|^2\,ds.$$

According to the Gronwall inequality we can obtain (4.12) for $v \in D(A)$. □

5. Proof of the main theorem.

Throughout this section we assume that conditions (A) and (B) introduced in Section 2 are satisfied. As stated in Section 1, we define two operators S, B in $X := L^2(\Omega)$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$:

$$Su := -\sum_{j,k=1}^{N}\frac{\partial}{\partial x_k}\left(a_{jk}(x)\frac{\partial u}{\partial x_j}\right) \quad \text{for} \quad u \in D(S) := H^2(\Omega)\cap H_0^1(\Omega),$$

$$Bu := g(x,|u|^2)u \quad \text{for} \quad u \in D(B) := \{u \in X; g(x,|u|^2)u \in X\}.$$

To apply the results in Section 4, we shall show that the operators S, B satisfy the assumption of Theorem 4.4. It is well-known (see e.g. Mizohata [**15**, Section 3.16]) that S is a nonnegative selfadjoint operator in X, satisfying

$$\|u\|_{H^2(\Omega)} \le c(\|Su\| + \|u\|) \quad \forall u \in D(S). \tag{5.1}$$

We start with the meaning of inequality (2.2).

Lemma 5.1. *Let* $h \in C([0,\infty); \boldsymbol{R}) \cap C^1((0,\infty); \boldsymbol{R})$ *and assume that*

$$0 \le s\frac{dh}{ds} \le \sigma h(s) \quad \forall s > 0 \tag{5.2}$$

for some constant $\sigma > 0$. *Then the function* $s \mapsto s[h(s)]^2$ *is nondecreasing and for* $t, s > 0$,

$$ts|h(t) - h(s)|^2 \le \frac{\sigma^2}{1+2\sigma}(t-s)\{t[h(t)]^2 - s[h(s)]^2\}. \tag{5.3}$$

Proof. We modify the proof of Liskevich and Perelmuter [**13**, Lemma 2.2]. First we note that

$$r^2\left(\frac{dh}{dr}\right)^2 \le \frac{\sigma^2}{1+2\sigma}\frac{d}{dr}\{r[h(r)]^2\} \quad \forall r > 0.$$

In fact, by (5.2) we have

$$[h(r)]^2 + 2rh(r)\frac{dh}{dr} - \frac{1+2\sigma}{\sigma^2}r^2\left(\frac{dh}{dr}\right)^2 = \left(h(r) - \frac{1}{\sigma}r\frac{dh}{dr}\right)\left(h(r) + \frac{1+2\sigma}{\sigma}r\frac{dh}{dr}\right) \ge 0.$$

Therefore the Cauchy-Schwarz inequality yields that for $t, s > 0$,

$$\begin{aligned} |h(t) - h(s)|^2 &= \left(\int_s^t r\frac{dh}{dr}r^{-1}\,dr\right)^2 \\ &\le \int_s^t r^2\left(\frac{dh}{dr}\right)^2 dr \int_s^t r^{-2}\,dr \\ &\le \frac{\sigma^2}{1+2\sigma}(t[h(t)]^2 - s[h(s)]^2)(s^{-1} - t^{-1}). \end{aligned}$$

This is equivalent to (5.3). □

For $h \in C([0,\infty); \boldsymbol{R}) \cap C^1((0,\infty); \boldsymbol{R})$ we define the operator $H : \boldsymbol{C} \to \boldsymbol{C}$ by

$$Hz := zh(|z|^2), \quad z \in \boldsymbol{C}. \tag{5.4}$$

Lemma 5.2. *Let* h *and* H *be as defined above. Then under condition* (5.2), H *is sectorial of type* $S(\sigma/\sqrt{1+2\sigma})$ *in* $\boldsymbol{C}$: *for* $z_1, z_2 \in \boldsymbol{C}$,

$$|\mathrm{Im}(\overline{z_1} - \overline{z_2})(Hz_1 - Hz_2)| \le \frac{\sigma}{\sqrt{1+2\sigma}}\mathrm{Re}(\overline{z_1} - \overline{z_2})(Hz_1 - Hz_2). \tag{5.5}$$

Proof. Let $w(z_1, z_2) := (\overline{z_1} - \overline{z_2})(Hz_1 - Hz_2)$ and $\theta := \arg(\overline{z_1}z_2)$. Then we have

$$\begin{aligned}\operatorname{Re} w(z_1,z_2) &= |z_1|^2 h(|z_1|^2) + |z_2|^2 h(|z_2|^2) - [h(|z_1|^2) + h(|z_2|^2)] \operatorname{Re}(\overline{z_1} z_2) \\ &\geq |z_1|^2 h(|z_1|^2) + |z_2|^2 h(|z_2|^2) - [h(|z_1|^2) + h(|z_2|^2)] |z_1| \cdot |z_2| \cdot |\cos\theta| \\ &\geq (|z_1| - |z_2|)[|z_1| h(|z_1|^2) - |z_2| h(|z_2|^2)], \\ \operatorname{Im} w(z_1,z_2) &= [h(|z_1|^2) - h(|z_2|^2)] \operatorname{Im}(\overline{z_1} z_2) \\ &= [h(|z_1|^2) - h(|z_2|^2)] |z_1| \cdot |z_2| \sin\theta.\end{aligned}$$

We see from (5.2) that the function $s \mapsto sh(s^2)$ is nondecreasing which implies that $\operatorname{Re} w(z_1,z_2) \geq 0$. Setting $t := |z_1|^2$ and $s := |z_2|^2$, we have

$$\frac{|\operatorname{Im} w(z_1,z_2)|}{\operatorname{Re} w(z_1,z_2)} \leq \frac{|h(t) - h(s)|\sqrt{ts}|\sin\theta|}{th(t) + sh(s) - [h(t) + h(s)]\sqrt{ts}|\cos\theta|}.$$

Noting that

$$\frac{|\sin\theta|}{p - q|\cos\theta|} \leq \frac{1}{\sqrt{p^2 - q^2}} \quad (0 \leq q \leq p),$$

we obtain

$$\frac{|\operatorname{Im} w(z_1,z_2)|}{\operatorname{Re} w(z_1,z_2)} \leq \frac{|h(t) - h(s)|\sqrt{ts}}{\sqrt{(t-s)\{t[h(t)]^2 - s[h(s)]^2\}}}.$$

Therefore (5.5) follows from (5.3). □

Here we note that H is m-accretive on $\boldsymbol{C}$. The question is reduced to the real-space case. In fact, the equation $z + Hz = f$ in $\boldsymbol{C}$ is equivalent to

$$|z| + |z|h(|z|^2) = |f|, \quad \arg z = \arg f.$$

Obviously, the mapping $s \mapsto s + sh(s^2)$ is a bijection of $[0,\infty)$.

As a consequence of Lemma 5.2 we can obtain

Lemma 5.3. *Let* $(Bu)(x) = g(x, |u(x)|^2)u(x)$ *for* $u \in D(B)$. *Then* B *is* m-*sectorial of type* $S(\sigma/\sqrt{1+2\sigma})$ *in* X: *for* $u_1, u_2 \in D(B)$,

$$\text{(5.6)} \qquad |\operatorname{Im}(Bu_1 - Bu_2, u_1 - u_2)| \leq \frac{\sigma}{\sqrt{1+2\sigma}} \operatorname{Re}(Bu_1 - Bu_2, u_1 - u_2).$$

Proof. Let $z \in \boldsymbol{C}$ and $x \in \Omega$. Replace $h(|z|^2)$ in (5.4) with $g(x,|z|^2)$. Then we have (5.5) with $Hz_j = g(x,|z_j|^2)z_j$ $(j = 1,2)$. Setting $z_j = u_j(x)$ $(j = 1,2)$ and integrating the inequality over Ω, we obtain (5.6).

Next we show that B is m-accretive in X. Let $f \in X$ and $\varepsilon > 0$. Since the operator H is m-accretive on $\boldsymbol{C}$ (as noted above), we see that for almost all $x \in \Omega$ the equation

$$z + \varepsilon g(x, |z|^2) z = f(x) \tag{5.7}$$

in $\boldsymbol{C}$ has a unique solution $z = u_\varepsilon(x)$ such that

$$|u_\varepsilon(x)| + \varepsilon g(x, |u_\varepsilon(x)|^2)|u_\varepsilon(x)| \le |f(x)|, \tag{5.8}$$

$$|u_\varepsilon(x) - \tilde{u}_\varepsilon(x)| \le |f(x) - \tilde{f}(x)|, \tag{5.9}$$

where $\tilde{u}_\varepsilon(x)$ is a unique solution of (5.7) with f replaced with $\tilde{f}$. Using approximation by simple functions, we see from (5.9) that u_ε is measurable on Ω. Therefore $u_\varepsilon \in D(B)$ and we obtain $R(1 + \varepsilon B) = X$. □

Lemma 5.4. *$C^1(\Omega)$ is invariant under $(1 + \varepsilon B)^{-1}$ for every $\varepsilon > 0$. More precisely, put $u_\varepsilon(x) := (1 + \varepsilon B)^{-1} f(x)$ for $f \in C^1(\Omega)$ and $\varepsilon > 0$. Then $u_\varepsilon \in C^1(\Omega)$ and*

$$\nabla u_\varepsilon = \frac{1}{1 + \varepsilon g(x, |u_\varepsilon|^2)} \nabla f - \frac{2\varepsilon}{\mathrm{Jac}} \frac{\partial g}{\partial s}(x, |u_\varepsilon|^2) u_\varepsilon \operatorname{Re}(\overline{u_\varepsilon} \nabla f) \tag{5.10}$$
$$- \frac{\varepsilon}{\mathrm{Jac}} \{1 + \varepsilon g(x, |u_\varepsilon|^2)\} \nabla_x g(x, |u_\varepsilon|^2) u_\varepsilon,$$

where

$$\mathrm{Jac} = \{1 + \varepsilon g(x, |u_\varepsilon|^2)\} \left\{ 1 + \varepsilon g(x, |u_\varepsilon|^2) + 2\varepsilon \frac{\partial g}{\partial s}(x, |u_\varepsilon|^2) |u_\varepsilon|^2 \right\}. \tag{5.11}$$

In particular, $W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ $(1 \le p < \infty)$ is invariant under $(1 + \varepsilon B)^{-1}$ for $\varepsilon > 0$.

Proof. Given $\varepsilon > 0$ let $u_\varepsilon(x) = v_\varepsilon(x) + i w_\varepsilon(x) = (1 + \varepsilon B)^{-1} f(x)$ for $f = f_1 + i f_2 \in C^1(\Omega)$. Then it follows from (5.7) that

$$u_\varepsilon(x) + \varepsilon g(x, |u_\varepsilon(x)|^2) u_\varepsilon(x) = f(x). \tag{5.12}$$

To show that $u_\varepsilon \in C^1(\Omega)$ we set

$$U_\varepsilon(x) := {}^t(v_\varepsilon(x), w_\varepsilon(x)), \quad F(x) := {}^t(f_1(x), f_2(x)). \tag{5.13}$$

To apply the inverse function theorem, (5.12) is usually regarded as

$$\Phi(x, U_\varepsilon(x)) = (x, F(x))$$

and $(x, U_\varepsilon(x)) = \Phi^{-1}(x, F(x))$ if Φ is proved to be C^1-bijection.

To be more precise, for $x = {}^t(x_1, \ldots, x_N) \in \Omega$ and $\xi = {}^t(\xi_1, \xi_2) \in \boldsymbol{R}^2$ we set

$$G(x,\xi) := {}^t(G_1(x,\xi), G_2(x,\xi)) := \xi + \varepsilon g(x,|\xi|^2)\xi,$$
$$\Phi(x,\xi) := (x, G(x,\xi))$$
$$= {}^t(x_1,\dots,x_N,\xi_1+\varepsilon g(x,|\xi|^2)\xi_1,\xi_2+\varepsilon g(x,|\xi|^2)\xi_2).$$

Here it is worth noticing that

$$(\xi-\eta)\cdot(g(x,|\xi|^2)\xi - g(x,|\eta|^2)\eta) = \mathrm{Re}(\bar{z}-\bar{w})(g(x,|z|^2)z - g(x,|w|^2)w),$$

where $z := \xi_1 + i\xi_2$ and $w := \eta_1 + i\eta_2$. Therefore we see from Lemma 5.2 that $g(x,|\xi|^2)\xi$ is accretive (or nondecreasing) on $\boldsymbol{R}^2$ with respect to ξ so that both $G(x,\cdot):\boldsymbol{R}^2\to\boldsymbol{R}^2$ and $\Phi:\Omega\times\boldsymbol{R}^2\to\Omega\times\boldsymbol{R}^2$ are bijections. Moreover we can show that Φ is a C^1-bijection. In fact, it follows from condition (B) that Φ is of class C^1 and its Jacobian matrix is given as follows: for $(x,\xi)\in\Omega\times\boldsymbol{R}^2$,

$$D\Phi(x,\xi) := \begin{pmatrix} \left(\dfrac{\partial x_j}{\partial x_k}\right)_{j,k} & \left(\dfrac{\partial x_j}{\partial \xi_m}\right)_{j,m} \\ \left(\dfrac{\partial G_l}{\partial x_k}\right)_{l,k} & \left(\dfrac{\partial G_l}{\partial \xi_m}\right)_{l,m} \end{pmatrix} = \begin{pmatrix} I_{N\times N} & 0_{N\times 2} \\ \left(\dfrac{\partial G_l}{\partial x_k}\right)_{l,k} & \left(\dfrac{\partial G_l}{\partial \xi_m}\right)_{l,m} \end{pmatrix},$$
$$j,k=1,\dots,N,\quad l,m=1,2,$$

where $I_{N\times N}$ is the $N\times N$ unit matrix and $0_{N\times 2}$ is the $N\times 2$ zero matrix. Denoting by $\partial_\xi G(x,\xi)$ the matrix $(\partial G_l/\partial\xi_m)_{l,m}$, we have

$$\partial_\xi G(x,\xi) = \begin{pmatrix} 1+\varepsilon g(x,|\xi|^2) + 2\varepsilon\dfrac{\partial g}{\partial s}(x,|\xi|^2)\xi_1^2 & 2\varepsilon\dfrac{\partial g}{\partial s}(x,|\xi|^2)\xi_1\xi_2 \\ 2\varepsilon\dfrac{\partial g}{\partial s}(x,|\xi|^2)\xi_1\xi_2 & 1+\varepsilon g(x,|\xi|^2) + 2\varepsilon\dfrac{\partial g}{\partial s}(x,|\xi|^2)\xi_2^2 \end{pmatrix}$$

and hence $\mathrm{Jac}(x,\xi) := \det D\Phi(x,\xi) = \det\partial_\xi G(x,\xi)\ge 1$. Therefore we can conclude by the inverse function theorem that Φ^{-1} is also of class C^1 so that $u_\varepsilon\in C^1(\Omega)$.

Next we prove (5.10). It follows from (5.12) and (5.13) that $G(x,U_\varepsilon(x)) = F(x)$. Differentiating both sides of this equality with respect to x_k, we have

$$\partial_\xi G(x,U_\varepsilon(x))\frac{\partial}{\partial x_k}U_\varepsilon(x) = \frac{\partial}{\partial x_k}F(x) - \varepsilon\frac{\partial g}{\partial x_k}(x,|u_\varepsilon|^2)U_\varepsilon(x).$$

Solving this linear system of equations with respect to $\partial v_\varepsilon/\partial x_k$ and $\partial w_\varepsilon/\partial x_k$, we obtain

$$\frac{\partial v_\varepsilon}{\partial x_k} = \frac{1}{\mathrm{Jac}}\left\{1+\varepsilon g(x,|u_\varepsilon|^2) + 2\varepsilon\frac{\partial g}{\partial s}(x,|u_\varepsilon|^2)w_\varepsilon^2\right\}\frac{\partial f_1}{\partial x_k}$$
$$-\frac{2\varepsilon}{\mathrm{Jac}}\frac{\partial g}{\partial s}(x,|u_\varepsilon|^2)v_\varepsilon w_\varepsilon\frac{\partial f_2}{\partial x_k} - \frac{\varepsilon}{\mathrm{Jac}}\{1+\varepsilon g(x,|u_\varepsilon|^2)\}\frac{\partial g}{\partial x_k}(x,|u_\varepsilon|^2)v_\varepsilon.$$

Using the equality $w_\varepsilon^2 = |u_\varepsilon|^2 - v_\varepsilon^2$ and noting that $\mathrm{Jac} = \det \partial_\xi G(x, U_\varepsilon(x))$ coincides with (5.11), we have

$$\frac{\partial v_\varepsilon}{\partial x_k} = \frac{1}{1+\varepsilon g(x,|u_\varepsilon|^2)} \frac{\partial f_1}{\partial x_k} - \frac{2\varepsilon}{\mathrm{Jac}} \frac{\partial g}{\partial s}(x,|u_\varepsilon|^2) v_\varepsilon \operatorname{Re}\left(\overline{u_\varepsilon} \frac{\partial f}{\partial x_k}\right) - \frac{\varepsilon}{\mathrm{Jac}} \{1+\varepsilon g(x,|u_\varepsilon|^2)\} \frac{\partial g}{\partial x_k}(x,|u_\varepsilon|^2) v_\varepsilon. \tag{5.14}$$

Since $\partial w_\varepsilon / \partial x_k$ is given by (5.14) with v_ε and f_1 replaced with w_ε and f_2, respectively, we see that $\nabla u_\varepsilon = \nabla v_\varepsilon + i\nabla w_\varepsilon$ can be written as (5.10). Furthermore, it follows from (5.10), (2.3) and (5.8) that $|u_\varepsilon(x)| \le |f(x)|$ and

$$\begin{aligned} |\nabla u_\varepsilon(x)| &\le 2|\nabla f(x)| + \varepsilon|\nabla_x g(x,|u_\varepsilon(x)|^2)| \cdot |u_\varepsilon(x)| \\ &\le 2|\nabla f(x)| + c_1 \varepsilon g(x,|u_\varepsilon(x)|^2)|u_\varepsilon(x)| + c_2\varepsilon|u_\varepsilon(x)| \\ &\le 2|\nabla f(x)| + (c_1 + c_2\varepsilon)|f(x)|. \end{aligned}$$

This proves that $u_\varepsilon \in W_0^{1,p}(\Omega) \cap C^1(\bar{\Omega})$ if $f \in W_0^{1,p}(\Omega) \cap C^1(\bar{\Omega})$. □

Now we want to prove the key inequality which generalizes [**18**, Lemma 3.2].

LEMMA 5.5. *For $u \in D(S)$ and $\varepsilon > 0$,*

$$|\operatorname{Im}(Su, B_\varepsilon u)| \le \frac{\sigma}{\sqrt{1+2\sigma}} \operatorname{Re}(Su, B_\varepsilon u) + \frac{1+\sigma}{\sqrt{(1+2\sigma)\delta}} \|S^{1/2}u\|(c_1\|B_\varepsilon u\| + c_2\|u\|), \tag{5.15}$$

where δ, σ, c_1 and c_2 are the constants in (2.1)–(2.3).

PROOF. Put $D_0 := H^2(\Omega) \cap H_0^1(\Omega) \cap C^1(\bar{\Omega})$. Then it follows from the elliptic regularity and Morrey's theorem that

$$\begin{aligned} C_0(\Omega) &\subset (1+S)(H^2(\Omega) \cap H_0^1(\Omega) \cap C^{1,\alpha}(\bar{\Omega})) \quad (0 < \alpha < 1) \\ &\subset (1+S)D_0 \end{aligned}$$

(see e.g. Brezis [**1**, p. 198]). This implies that $(1+S)D_0$ is dense in X and hence D_0 is a core for S (see Kato [**10**, Problem III.5.19]). Therefore it suffices to prove (5.15) for the elements in D_0. Let $f \in D_0$ and set $u_\varepsilon := (1+\varepsilon B)^{-1} f$. Then it follows from Lemma 5.4 (with $p = 2$) that $B_\varepsilon f = \varepsilon^{-1}(f - u_\varepsilon) \in H_0^1(\Omega) \cap C^1(\bar{\Omega})$ and $(\partial/\partial x_k) B_\varepsilon f$ is given by

$$\frac{1}{\varepsilon}\left(\overline{\frac{\partial f}{\partial x_k}-\frac{\partial u_\varepsilon}{\partial x_k}}\right)=\frac{g(x,|u_\varepsilon|^2)}{1+\varepsilon g(x,|u_\varepsilon|^2)}\frac{\overline{\partial f}}{\partial x_k}+\frac{2}{\mathrm{Jac}}\frac{\partial g}{\partial s}(x,|u_\varepsilon|^2)\overline{u_\varepsilon}\,\mathrm{Re}\left(\overline{u_\varepsilon}\frac{\partial f}{\partial x_k}\right)$$
$$+\frac{1}{\mathrm{Jac}}\{1+\varepsilon g(x,|u_\varepsilon|^2)\}\frac{\partial g}{\partial x_k}(x,|u_\varepsilon|^2)\overline{u_\varepsilon}.$$

The integration by parts gives $(Sf,B_\varepsilon f)=I_1(f)+2I_2(f)+I_3(f)$, where

$$I_1(f):=\int_\Omega\frac{g(x,|u_\varepsilon|^2)}{1+\varepsilon g(x,|u_\varepsilon|^2)}\sum_{j,k=1}^N a_{jk}\frac{\partial f}{\partial x_j}\frac{\overline{\partial f}}{\partial x_k}\,dx,$$

$$I_2(f):=\int_\Omega\frac{1}{\mathrm{Jac}}\frac{\partial g}{\partial s}(x,|u_\varepsilon|^2)\sum_{j,k=1}^N a_{jk}\left(\overline{u_\varepsilon}\frac{\partial f}{\partial x_j}\right)\mathrm{Re}\left(\overline{u_\varepsilon}\frac{\partial f}{\partial x_k}\right)dx,$$

$$I_3(f):=\int_\Omega\frac{1}{\mathrm{Jac}}\{1+\varepsilon g(x,|u_\varepsilon|^2)\}\sum_{j,k=1}^N a_{jk}\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_k}(x,|u_\varepsilon|^2)\overline{u_\varepsilon}\,dx.$$

Modifying the proof in [**17**], we shall show that

$$|\mathrm{Im}(Sf,B_\varepsilon f)-\mathrm{Im}\,I_3(f)|\le\frac{\sigma}{\sqrt{1+2\sigma}}\{\mathrm{Re}(Sf,B_\varepsilon f)-\mathrm{Re}\,I_3(f)\}. \tag{5.16}$$

First we see from the symmetry of $(a_{jk}(x))$ and ellipticity (2.1) that $I_1(f)$ and $\mathrm{Re}\,I_2(f)$ are nonnegative; note that

$$\mathrm{Re}\,I_2(f)=\int_\Omega\frac{1}{\mathrm{Jac}}\frac{\partial g}{\partial s}(x,|u_\varepsilon|^2)Q(\mathrm{Re}(\overline{u_\varepsilon}\nabla f))\,dx, \tag{5.17}$$

where we have set $Q(h):=\sum_{j,k=1}^N a_{jk}(x)h_j(x)\overline{h_k(x)}$ for $h\in L^2(\Omega;\boldsymbol{C}^N)$. Thus we have

$$\mathrm{Re}(Sf,B_\varepsilon f)-\mathrm{Re}\,I_3(f)=I_1(f)+2\,\mathrm{Re}\,I_2(f)\ge 0, \tag{5.18}$$

$$\mathrm{Im}(Sf,B_\varepsilon f)-\mathrm{Im}\,I_3(f)=2\,\mathrm{Im}\,I_2(f). \tag{5.19}$$

Applying the Cauchy-Schwarz inequality to the sum in the integrand and then to the integral $I_2(f)$, we see from (5.17) that

$$|I_2(f)|^2\le\int_\Omega\frac{1}{\mathrm{Jac}}\frac{\partial g}{\partial s}(x,|u_\varepsilon|^2)Q(\overline{u_\varepsilon}\nabla f)\,dx\cdot\mathrm{Re}\,I_2(f).$$

Since $Q(\overline{u_\varepsilon}\nabla f)=|u_\varepsilon|^2Q(\nabla f)$ and $\mathrm{Jac}\ge 1+\varepsilon g(x,|u_\varepsilon|^2)$, it follows from (2.2) that

$$|I_2(f)|^2\le\sigma\int_\Omega\frac{1}{\mathrm{Jac}}g(x,|u_\varepsilon|^2)Q(\nabla f)\,dx\cdot\mathrm{Re}\,I_2(f)$$
$$\le\sigma I_1(f)\,\mathrm{Re}\,I_2(f).$$

This enables us to estimate $|\mathrm{Im}\,I_2(f)|^2=|I_2(f)|^2-|\mathrm{Re}\,I_2(f)|^2$. In fact, we see that

$$|\mathrm{Im}\, I_2(f)|^2 \le \sigma I_1(f)\, \mathrm{Re}\, I_2(f) - |\mathrm{Re}\, I_2(f)|^2.$$

Since $I_1(f)$ is given by (5.18), we obtain

$$|\mathrm{Im}\, I_2(f)|^2 \le \sigma\{\mathrm{Re}(Sf, B_\varepsilon f) - \mathrm{Re}\, I_3(f)\}\, \mathrm{Re}\, I_2(f) - (1+2\sigma)|\mathrm{Re}\, I_2(f)|^2.$$

Applying the geometric-arithmetic mean inequality $2ab \le a^2 + b^2$ with

$$a := \frac{\sigma}{2\sqrt{1+2\sigma}}\{\mathrm{Re}(Sf, B_\varepsilon f) - \mathrm{Re}\, I_3(f)\}, \quad b := \sqrt{1+2\sigma}\, \mathrm{Re}\, I_2(f)$$

to the first term on the right-hand side, we have

$$|\mathrm{Im}\, I_2(f)|^2 \le \frac{\sigma^2}{4(1+2\sigma)}\{\mathrm{Re}(Sf, B_\varepsilon f) - \mathrm{Re}\, I_3(f)\}^2.$$

In view of (5.19) this is equivalent to (5.16). It follows from (5.16) that

$$\begin{aligned} |\mathrm{Im}(Sf, B_\varepsilon f)| &\le \frac{\sigma}{\sqrt{1+2\sigma}}\, \mathrm{Re}(Sf, B_\varepsilon f) + \frac{\sigma}{\sqrt{1+2\sigma}}|\mathrm{Re}\, I_3(f)| + |\mathrm{Im}\, I_3(f)| \\ &\le \frac{\sigma}{\sqrt{1+2\sigma}}\, \mathrm{Re}(Sf, B_\varepsilon f) + \frac{1+\sigma}{\sqrt{1+2\sigma}}|I_3(f)|. \end{aligned}$$

This proves (5.15) because $|I_3(f)| \le \delta^{-1/2}\|S^{1/2}f\|(c_1\|B_\varepsilon f\| + c_2\|f\|)$. In fact, we have

$$\begin{aligned} |I_3(f)| &\le \int_\Omega |u_\varepsilon| \left| \sum_{j,k=1}^N a_{jk} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k}(x, |u_\varepsilon|^2) \right| dx \\ &\le \int_\Omega |u_\varepsilon| [Q(\nabla f)]^{1/2} [Q(\nabla_x g(x, |u_\varepsilon|^2))]^{1/2}\, dx. \end{aligned}$$

Since $B_\varepsilon f = Bu_\varepsilon$, we see from (2.1), (2.3) and (5.8) that

$$\begin{aligned} |u_\varepsilon|[Q(\nabla_x g(x, |u_\varepsilon|^2))]^{1/2} &\le \delta^{-1/2}|u_\varepsilon| \cdot |\nabla_x g(x, |u_\varepsilon|^2)| \\ &\le \delta^{-1/2}(c_1 g(x, |u_\varepsilon|^2)|u_\varepsilon| + c_2|u_\varepsilon|) \\ &\le \delta^{-1/2}(c_1|(B_\varepsilon f)(x)| + c_2|f(x)|). \end{aligned}$$

Noting that $D(S^{1/2}) = H_0^1(\Omega)$ and

$$\|S^{1/2}f\|^2 = \int_\Omega Q(\nabla f)\, dx \quad \forall f \in H_0^1(\Omega), \tag{5.20}$$

we can obtain (5.15). □

We are now in a position to prove Theorem 2.2 and Corollary 2.3.

Proof of Theorem 2.2. Define S and B as in Section 1. Then Lemmas 5.3 and 5.5 show that the assumption of Theorem 4.4 is satisfied, with

$$k_1 = \frac{\sigma}{\sqrt{1+2\sigma}}, \quad k_2 = \frac{1+\sigma}{\sqrt{(1+2\sigma)\delta}}.$$

Let $\lambda > 0$, $\kappa > 0$ and $\kappa^{-1}|\beta| \le \sqrt{1+2\sigma}/\sigma$. Then it follows from Theorem 4.4 that

$$A + \gamma = (\lambda + i\alpha)S + (\kappa + i\beta)B$$

is an m-accretive operator with domain $D(A)$ dense in X. Denoting by $\{U(t)\}$ the semigroup of type γ on X generated by $-A$, we see that if $u_0 \in D(A)$ then $u(t) := U(t)u_0$ is a unique solution to (3.1). This implies that (1.1) admits a unique global strong solution $u(x,t)$ in the sense of Definition 2.1.

It remains to prove (2.4)–(2.7). From a property of the semigroup of type γ we obtain (2.5) and (2.6); note that $0 \in D(A)$ and $A0 = 0$. Next, (4.10) together with (2.5) yields that for all $t \ge 0$,

$$K\|Su(t)\| \le [L\|Au_0\| + (L|\gamma| + \sqrt{cK})\|u_0\|]e^{\gamma t} \tag{5.21}$$

which implies by (5.1) that $u(\cdot) \in L^\infty(0,T;H^2(\Omega))$ for any $T > 0$. By virtue of Theorem 3.1 (a) we see also that $u(\cdot) \in L^\infty(0,T;D(B))$. These prove the first two assertions of (2.4). It follows from (5.20), the symmetry of (a_{jk}) and ellipticity (2.1) that

$$\delta\|\nabla u\|^2 \le \|S^{1/2}u\|^2 \le \delta^{-1}\|\nabla u\|^2 \quad \forall u \in D(S). \tag{5.22}$$

The first inequality in (5.22) together with the Cauchy-Schwarz inequality implies that

$$\delta\|\nabla u - \nabla v\|^2 \le (\|Su\| + \|Sv\|)\|u - v\| \quad \forall u, v \in D(S). \tag{5.23}$$

Therefore (2.7) follows from (2.6) and (5.21). To show that $u(\cdot) \in C^{0,1/2}([0,T];H_0^1(\Omega))$ let $t, s \in [0,T]$. Then by Theorem 3.1 (b) we have

$$\|u(t) - u(s)\| \le e^{2\gamma+T}\|Au_0\| \cdot |t - s|.$$

Using (5.21) and (5.23) (with u, v replaced with $u(t)$, $u(s)$), we have

$$\delta K\|\nabla u(t) - \nabla u(s)\|^2 \le 2[L\|Au_0\| + (L|\gamma| + \sqrt{cK})\|u_0\|]\|Au_0\|e^{3\gamma+t}|t - s|.$$

Thus we obtain the remaining part of (2.4). □

Proof of Corollary 2.3. We see from (5.22) that

$$\delta\|\nabla u\|^2 + \varepsilon\|u\|^2 \le \|(\varepsilon + S)^{1/2}u\|^2 \le \delta^{-1}\|\nabla u\|^2 + \varepsilon\|u\|^2 \quad \forall u \in D(S).$$

Since $k_2/k_1 = (1+\sigma)/(\sigma\sqrt{\delta})$, (2.8) follows from (4.12). □

References

[1] H. Brezis, Analyse Fonctionnelle, Théorie et Applications, Masson, Paris, 1983.

[2] H. Brezis, M. Crandall and A. Pazy, Perturbations of nonlinear maximal monotone sets in Banach spaces, Comm. Pure Appl. Math., **23** (1970), 123–144.

[3] C. Bu, On the Cauchy problem for the $1+2$ complex Ginzburg-Landau equation, J. Austral. Math. Soc. Ser. B, **36** (1994), 313–324.
[4] M. C. Cross and P. C. Hohenberg, Pattern formation outside of equilibrium, Rev. Modern Phys., **65** (1993), 851–1089.
[5] C. R. Doering, J. D. Gibbon and C. D. Levermore, Weak and strong solutions of the complex Ginzburg-Landau equation, Phys. D, **71** (1994), 285–318.
[6] J. Ginibre and G. Velo, The Cauchy problem in local spaces for the complex Ginzburg-Landau equation. I. Compactness methods, Phys. D, **95** (1996), 191–228.
[7] J. Ginibre and G. Velo, The Cauchy problem in local spaces for the complex Ginzburg-Landau equation. II. Contraction methods, Comm. Math. Phys., **187** (1997), 45–79.
[8] J. Ginibre and G. Velo, The Cauchy problem in local spaces for the complex Ginzburg-Landau equation, Differential Equations, Asymptotic Analysis, and Mathematical Physics (Potsdam, 1996), 138–152, Mathematical Research, vol. 100, Akademie Verlag, Berlin, 1997.
[9] J. A. Goldstein, Semigroups of Linear Operators and Applications, Oxford University Press, New York, 1985.
[10] T. Kato, Perturbation Theory for Linear Operators, Grundlehren math. Wissenschaften, vol. 132, Springer-Verlag, Berlin and New York, 1966; 2nd ed., 1976.
[11] Y. Kōmura and Y. Konishi, Nonlinear Evolution Equations, Iwanami-Shoten, Tokyo, 1977. (Japanese)
[12] C. D. Levermore and M. Oliver, The complex Ginzburg-Landau equation as a model problem, Dynamical Systems and Probabilistic Methods in Partial Differential Equations (Berkeley, 1994), 141–190, Lectures in Appl. Math., vol. 31, Amer. Math. Soc., Providence, RI, 1996.
[13] V. A. Liskevich and M. A. Perelmuter, Analyticity of submarkovian semigroups, Proc. Amer. Math. Soc., **123** (1995), 1097–1104.
[14] I. Miyadera, Nonlinear Semigroups, Transl. Math. Monogr., vol. 109, Amer. Math. Soc., Providence, RI, 1992.
[15] S. Mizohata, The Theory of Partial Differential Equations, Cambridge University Press, London, 1973.
[16] A. C. Newell and J. A. Whitehead, Finite bandwidth, finite amplitude convection, J. Fluid Mech., **38** (1969), 279–303.
[17] N. Okazawa, Sectorialness of second order elliptic operators in divergence form, Proc. Amer. Math. Soc., **113** (1991), 701–706.
[18] N. Okazawa and T. Yokota, Monotonicity method applied to the complex Ginzburg-Landau and related equations, J. Math. Anal. Appl., to appear.
[19] R. E. Showalter, Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, Math. Surveys Monogr., vol. 49, Amer. Math. Soc., Providence, RI, 1997.
[20] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Applied Math. Sci., vol. 68, Springer-Verlag, Berlin and New York, 1988; 2nd ed., 1997.
[21] A. Unai, A remark on time-dependent Ginzburg-Landau equations, SUT J. Math., **33** (1997), 115–119.
[22] A. Unai and N. Okazawa, Perturbations of nonlinear m-sectorial operators and time-dependent Ginzburg-Landau equations, Dynamical Systems and Differential Equations, vol. II (Springfield, MO, 1996), Discrete Contin. Dynam. Systems 1998, Added Volume II, 259–266.
[23] Y. Yang, On the Ginzburg-Landau wave equation, Bull. London Math. Soc., **22** (1990), 167–170.

Noboru OKAZAWA and Tomomi YOKOTA

Department of Mathematics
Science University of Tokyo
Shinjuku-ku, Tokyo 162-8601, Japan
E-mail: okazawa@ma.kagu.sut.ac.jp
yokota@minserver.ma.kagu.sut.ac.jp

Notes added in proof. 1. The proofs of Lemmas 5.4 and 5.5 require a little more care. In the proof of Lemma 5.4 we have tacitly assumed that $g \in C^1(\Omega \times [0, \infty); \boldsymbol{R})$ instead of that $g \in C^1(\Omega \times [0, \infty); \boldsymbol{R})$. Nevertheless we can prove Lemma 5.5 by this "weak" form of Lemma 5.4. In fact, put $g_\nu(x, s) := g(x, s + \nu)$ for $\nu > 0$. Then g_ν belongs to $C^1(\Omega \times [0, \infty); \boldsymbol{R})$ and satisfies (2.2) and (2.3). Thus the weak form of Lemma 5.4 is meaningful for $(1 + \varepsilon B^\nu)^{-1}$, where $B^\nu u := g_\nu(x, |u|^2)u$. Consequently, we can obtain Lemma 5.5 with B replaced with B^ν. To conclude (5.15) it suffices to note that $(B^\nu)_\varepsilon u \to B_\varepsilon u (\nu \downarrow 0)$ in X.

2. After the submission of the paper, the authors could prove smoothing effect on the solutions to (1.1). To this end the operators S and B should be cast into the language of subdifferential operators. For details see Discrete Contin. Dynam. Systems 2001, Added Volume, 280–288.