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On endomorphism algebras with small homological dimensions

Dedicated to Idun Reiten on the occasion of her sixtieth birthday

By Mark Kleiner, Andrzej Skowroński and Dan Zacharia

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Abstract. We investigate the endomorphism algebras Γ of finite dimensional modules having the property that every indecomposable finite dimensional Γ-module is of projective dimension at most one or injective dimension at most one. In particular, we describe all matrix algebras $\begin{bmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{bmatrix}$ with this homological property.

0. Introduction.

Throughout the paper by an algebra we mean a finite dimensional K-algebra (associative, with an identity) over a fixed field K. By a module we mean a finite dimensional left module. For an algebra Λ, we denote by $\operatorname{mod}\Lambda$ the category of all (finite dimensional) Λ-modules, by $\operatorname{ind}\Lambda$ the full subcategory of $\operatorname{mod}\Lambda$ consisting of indecomposable modules, and by D the standard duality $\operatorname{Hom}_K(-,K)$ on $\operatorname{mod}\Lambda$. Further, we denote by Γ_Λ the Auslander-Reiten quiver of Λ and by $D\mathrm{Tr}$, $\mathrm{Tr}D$ the Auslander-Reiten translations in $\operatorname{mod}\Lambda$. For a Λ-module M, we denote by $\operatorname{pd}_\Lambda M$ and $\operatorname{id}_\Lambda M$ the projective dimension and the injective dimension of M, respectively. Following [**4**], an algebra Λ is said to be a shod algebra (for **s**mall **ho**mological **d**imension) provided, for each indecomposable Λ-module X, we have $\operatorname{pd}_\Lambda X \leq 1$ or $\operatorname{id}_\Lambda X \leq 1$.

The class of shod algebras contains all tilted, or more generally quasitilted, algebras, and has been recently the object of extensive investigation (see [**5**], [**6**], [**8**], [**10**], [**15**], [**17**], [**22**]). We are interested in the problem of when the endomorphism algebra $\Gamma = \operatorname{End}_\Lambda(M)^{\mathrm{op}}$ of a module M over a shod algebra Λ is again a shod algebra. We prove that it is the case if:

(1) M is a projective module (Section 1)

or

(2) M has no selfextensions and belongs to the additive closure of the maximal predecessor closed subcategory of $\operatorname{ind}\Lambda$ consisting entirely of modules of projective dimension at most one (Section 2).

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As an application, we obtain (in Section 3) a complete description of shod 2×2 lower triangular matrix algebras $\Lambda = \begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ of finite dimensional algebras A over an algebraically closed field. In particular, we show that, if such an algebra Λ is shod, then Λ is tame of linear growth.

1. Endomorphism algebras of projective modules.

Let A be an algebra. For X and Y in $\operatorname{ind} A$, X is said to be a predecessor of Y (respectively, Y is said to be a successor of X) in $\operatorname{ind} A$ if there exists a sequence of nonzero morphisms $X = Z_0 \to Z_1 \to \cdots \to Z_r = Y$, $r \geq 1$, in $\operatorname{ind} A$. Following [**10**], denote by $\mathscr{L}_A$ the family of all indecomposable A-modules M such that $\operatorname{pd}_A X \leq 1$ for every predecessor X of M in $\operatorname{ind} A$, and by $\mathscr{R}_A$ the family of all indecomposable A-modules N such that $\operatorname{id}_A Y \leq 1$ for every successor Y of N in $\operatorname{ind} A$. It has been shown in [**5**, Theorem 2.1] that A is a shod algebra if and only if $\operatorname{ind} A = \mathscr{L}_A \cup \mathscr{R}_A$. We know also that if A is shod then $\operatorname{gl.dim} A \leq 3$ ([**10**, Proposition II.2.1]). We say that A is a strict shod if A is shod with $\operatorname{gl.dim} A = 3$ ([**5**]), and A is quasitilted if A is shod with $\operatorname{gl.dim} A \leq 2$ ([**10**]). Finally, A is called tilted if A is of the form $\operatorname{End}_H(T)^{\mathrm{op}}$, where H is a hereditary algebra and T is a tilting H-module. Recall that an A-module T is called a tilting module if $\operatorname{pd}_A T \leq 1$, $\operatorname{Ext}_A^1(T, T) = 0$, and the number of pairwise non-isomorphic indecomposable direct summands of T equals the rank of the Grothendieck group $K_0(A)$ of A (see [**3**], [**11**]).

Let now Λ be a fixed algebra, P a projective Λ-module, and $\Gamma = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$. Denote by $\operatorname{mod} P$ the full subcategory of $\operatorname{mod} \Lambda$ consisting of all modules X which have a projective presentation $P_1 \to P_0 \to X \to 0$ with P_0 and P_1 in the additive category $\operatorname{add} P$ of P. Then $\operatorname{Hom}_\Lambda(P, -)|_{\operatorname{mod} P} : \operatorname{mod} P \to \operatorname{mod} \Gamma$ is an equivalence of categories with $\operatorname{add} P$ corresponding to the category of projective Γ-modules. For a projective Λ-module Q, we denote by Q^* the projective Λ^{op}-module $\operatorname{Hom}_\Lambda(Q, \Lambda)$. Observe that $\Gamma = \operatorname{End}_\Lambda(P)^{\mathrm{op}} = \operatorname{End}_{\Lambda^{\mathrm{op}}}(P^*)$. We need the following simple lemma (see [**23**]).

Lemma 1.1. *Assume Λ is basic, $1 = e_1 + \cdots + e_n$ for some primitive orthogonal idempotents $e_1, \ldots, e_n$, $P = \Lambda e_2 \oplus \cdots \oplus \Lambda e_n$ and S is the simple module $\Lambda e_1/(\operatorname{rad} \Lambda)e_1$. Then the following hold*

(a) *If $\operatorname{pd}_\Lambda S \leq 1$ then, for every projective Λ-module Q, $\operatorname{Hom}_\Lambda(P, Q)$ is a projective Γ-module.*

(b) *If $\operatorname{id}_\Lambda S \leq 1$ then, for every projective Λ-module Q, $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(P^*, Q^*)$ is a projective Γ^{op}-module.*

Proof. (a) Let $\operatorname{pd}_\Lambda S \leq 1$, Q be a projective Λ-module and $Q = Q' \oplus Q''$ with $Q' \in \operatorname{add} P$ and $Q'' \in \operatorname{add} \Lambda e_1$. Then $\operatorname{Hom}_\Lambda(P, Q) = \operatorname{Hom}_\Lambda(P, Q' \oplus \operatorname{rad} Q'')$ with $Q' \oplus \operatorname{rad} Q'' \in \operatorname{add} P$, and hence $\operatorname{Hom}_\Lambda(P, Q)$ is a projective Γ-module.

(b) Let $\mathrm{id}_\Lambda S \le 1$. Then $D(S) = \mathrm{Hom}_K(S, K) \cong e_1\Lambda/e_1(\mathrm{rad}\,\Lambda)$ is a simple Λ^{op}-module with $\mathrm{pd}_{\Lambda^{\mathrm{op}}} D(S) \le 1$, and the claim follows. □

THEOREM 1.2. *In the above notation the following hold*
(a) *If Λ is shod then Γ is shod.*
(b) *If Λ is quasitilted then Γ is quasitilted.*
(c) *If Λ is tilted then Γ is tilted.*
(d) *If Λ is strict shod then Γ is strict shod or tilted.*

PROOF. Since the projective, injective and global dimensions are preserved by the Morita equivalences we may assume that Λ is basic. Moreover, by induction on the rank of $K_0(\Lambda)$, we may also assume $\Lambda = \Lambda e_1 \oplus \Lambda e_2 \oplus \cdots \oplus \Lambda e_n$, and $P = \Lambda e_2 \oplus \cdots \oplus \Lambda e_n$. Let $S = \Lambda e_1/(\mathrm{rad}\,\Lambda)e_1$.

(a) Assume that Λ is shod. Let X be an indecomposable Γ-module. We shall prove that $\mathrm{pd}_\Gamma X \le 1$ or $\mathrm{id}_\Gamma X \le 1$. We know that $X = \mathrm{Hom}_\Lambda(P, M)$ for some Λ-module M from $\mathrm{mod}\,P$. We have two cases to consider.

Assume first that $\mathrm{id}_\Lambda S \le 1$. Since M is an indecomposable Λ-module, we have $\mathrm{pd}_\Lambda M \le 1$ or $\mathrm{id}_\Lambda M \le 1$. If $\mathrm{pd}_\Lambda M \le 1$ then we have a short exact sequence

$$0 \to P_1 \to P_0 \to M \to 0$$

with $P_0, P_1 \in \mathrm{add}\,P$, and applying the functor $\mathrm{Hom}_\Lambda(P, -)$ we obtain the projective resolution

$$0 \to \mathrm{Hom}_\Lambda(P, P_1) \to \mathrm{Hom}_\Lambda(P, P_0) \to \mathrm{Hom}_\Lambda(P, M) \to 0$$

of X in $\mathrm{mod}\,\Gamma$, and hence $\mathrm{pd}_\Gamma X \le 1$. Assume now $\mathrm{pd}_\Lambda M \ge 2$. Since Λ is shod, we then have $\mathrm{id}_\Lambda M \le 1$, and hence $\mathrm{pd}_{\Lambda^{\mathrm{op}}} D(M) \le 1$. Let

$$0 \to Q_1^* \to Q_0^* \to D(M) \to 0$$

be a minimal projective resolution of $D(M)$ in $\mathrm{mod}\,\Lambda^{\mathrm{op}}$. Applying Lemma 1.1(b) we obtain a (not necessarily minimal) projective resolution

$$0 \to \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P^*, Q_1^*) \to \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P^*, Q_0^*) \to \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P^*, D(M)) \to 0$$

of $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P^*, D(M))$ in $\mathrm{mod}\,\Gamma^{\mathrm{op}}$, and hence $\mathrm{pd}_{\Gamma^{\mathrm{op}}} \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P^*, D(M)) \le 1$. Observe now that we have a canonical isomorphism of Γ-modules

$$D\,\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P^*, D(M)) \cong \mathrm{Hom}_\Lambda(P, M) = X$$

induced by isomorphisms $D(P^* \otimes_\Lambda M) \cong \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P^*, D(M))$ and $P^* \otimes_\Lambda M \cong \mathrm{Hom}_\Lambda(P, M)$. Therefore, we obtain $\mathrm{id}_\Gamma X \le 1$. Note that in fact we have proved the following: if $\mathrm{id}_\Lambda S \le 1$ and $\mathrm{id}_\Lambda M \le 1$ then $\mathrm{id}_\Gamma \mathrm{Hom}_\Lambda(P, M) \le 1$.

Assume now that $\mathrm{id}_\Lambda S \ge 2$. Then $\mathrm{pd}_\Lambda S \le 1$, since Λ is shod. Hence we have $\mathrm{id}_{\Lambda^{\mathrm{op}}} D(S) \leq 1$. Note that $\Gamma \cong \mathrm{End}_{\Lambda^{\mathrm{op}}}(P^*)$. Therefore, we prove as above, that for the indecomposable Γ^{op}-module $Y = D(X)$ we have $\mathrm{pd}_{\Gamma^{\mathrm{op}}} Y \leq 1$ or $\mathrm{id}_{\Gamma}^{\mathrm{op}} Y \leq 1$, and hence $\mathrm{id}_{\Gamma} X \leq 1$ or $\mathrm{pd}_{\Gamma} X \leq 1$. This shows that Γ is shod.

(b) Assume that Λ is quasitilted. The required fact that Γ is quasitilted has been established in [**10**, Proposition II.1.15] as an application of a characterization [**10**, Theorem II.1.14] of quasitilted algebras. Here, we obtain an elementary direct proof. Indeed, due to (a) it remains to show that $\mathrm{gl.dim}\, \Gamma \leq 2$. But this fact follows immediately from Lemma 1.1.

(c) Assume that Λ is a tilted algebra. Then $\Lambda = \mathrm{End}_H(T)^{\mathrm{op}}$ where H is a basic hereditary algebra, T is a tilting H-module, and $T = T_1 \oplus \cdots \oplus T_n$ with $T_1, \dots, T_n$ pairwise nonisomorphic indecomposable Λ-modules such that $\Lambda e_i = \mathrm{Hom}_H(T, T_i)$ for any $i \in \{1, \dots, n\}$. Hence $P = \mathrm{Hom}_H(T, R)$, for the partial tilting H-module $R = T_2 \oplus \cdots \oplus T_n$. It follows from [**7**, Corollary III.6.5] that $\mathrm{End}_H(R)^{\mathrm{op}}$ is a tilted algebra. Invoking now the Brenner-Butler theorem ([**11**]), we conclude that

$$\Gamma = \mathrm{End}_{\Lambda}(P)^{\mathrm{op}} = \mathrm{End}_{\Lambda}(\mathrm{Hom}_H(T, R))^{\mathrm{op}} \cong \mathrm{End}_H(R)^{\mathrm{op}}$$

is a tilted algebra.

(d) Assume that Λ is a strict shod. If $P \in \mathrm{add}\, \mathscr{L}_{\Lambda}$ then it follows from [**17**, Theorem 8.2] that P is a projective module over a tilted factor algebra Λ_l of Λ (called the left tilted algebra of Λ) and then, from (c), $\Gamma = \mathrm{End}_{\Lambda}(P)^{\mathrm{op}} = \mathrm{End}_{\Lambda_l}(P)^{\mathrm{op}}$ is a tilted algebra. Therefore, we may assume that P has at least one indecomposable direct summand, say P_n, from $\mathscr{R}_{\Lambda} \backslash \mathscr{L}_{\Lambda}$. But then it follows from the arguments applied in (a) (in the both cases: $\mathrm{id}_{\Lambda} S \leq 1$ and $\mathrm{pd}_{\Lambda} S \leq 1$) that $\mathrm{Hom}_{\Lambda}(P, P_n)$ is an indecomposable projective Γ-module from $\mathscr{R}_{\Gamma}$. If $\mathrm{gl.dim}\, \Gamma = 3$ then Γ is strict shod, because Γ is shod by (a). Finally, if $\mathrm{gl.dim}\, \Gamma \leq 2$ then Γ is quasitilted with $\mathscr{R}_{\Gamma}$ containing a projective module, and consequently is tilted by [**10**, Corollary II.3.4]. □

The following examples show that we may have $\mathrm{Hom}_{\Lambda}(P, M) \in \mathscr{L}_{\Gamma}$ (respectively, $\mathrm{Hom}_{\Lambda}(P, M) \in \mathscr{R}_{\Gamma}$) for an indecomposable Λ-module M from $(\mathscr{R}_{\Lambda} \backslash \mathscr{L}_{\Lambda}) \cap \mathrm{mod}\, P$ (respectively, from $(\mathscr{L}_{\Lambda} \backslash \mathscr{R}_{\Lambda}) \cap \mathrm{mod}\, P$).

EXAMPLE 1.3. Let Λ be a bound quiver algebra KQ/I, where K is a field, Q is the quiver

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3,$$

and I is the ideal in the path algebra KQ of Q generated by $\alpha\beta$. Then Λ is a tilted algebra of Dynkin type A_3. Denote by S_i the simple Λ-module associated to the vertex i and by P_i the projective cover of S_i in $\mathrm{mod}\, \Lambda$, $1 \leq i \leq 3$. Let $P = P_2 \oplus P_3$ and $\Gamma = \mathrm{End}_{\Lambda}(P)^{\mathrm{op}}$. Clearly, Γ is the path algebra $K\Delta$, where Δ is the full subquiver of Q consisting of the vertices 2 and 3. We have the following minimal projective resolution

$$0 \to P_1 \to P_2 \to P_3 \to S_3 \to 0$$

and consequently $S_3 \in \mathscr{R}_\Lambda \backslash \mathscr{L}_\Lambda$. On the other hand, $\mathrm{Hom}_\Lambda(P, S_3)$ is the simple Γ-module associated to the vertex 3 of Δ, and clearly belongs to $\mathscr{L}_\Gamma = \mathrm{ind}\,\Gamma = \mathscr{R}_\Gamma$, because Γ is hereditary. Similarly, taking $P' = P_1 \oplus P_2$ and $\Gamma' = \mathrm{End}_\Lambda(P')^{\mathrm{op}}$, we conclude that $S_1 \in \mathscr{L}_\Lambda \backslash \mathscr{R}_\Lambda$, because $\mathrm{id}_\Lambda S_1 = 2$, and $\mathrm{Hom}_\Lambda(P', S_1) \in \mathscr{R}_{\Gamma'} = \mathrm{ind}\,\Gamma'$.

2. Endomorphism algebras of modules without selfextensions.

The aim of this section is to prove a generalization of Theorem 1.2 for modules without selfextensions. We need a preliminary fact.

Lemma 2.1. *Let Λ be a connected tilted algebra and M a Λ-module from $\mathrm{add}\,\mathscr{L}_\Lambda$. Moreover, assume that Λ is not a representation-infinite tilted algebra of Euclidean type whose preprojective component is the unique connecting component of Γ_Λ. Then there exists a hereditary algebra H and a tilting H-module T such that $\Lambda = \mathrm{End}_H(T)^{\mathrm{op}}$ and M belongs to the torsion-free part*

$$\mathscr{Y}(T) = \{N \in \mathrm{mod}\,\Lambda \mid \mathrm{Tor}_1^\Lambda(T, N) = 0\}$$

of $\mathrm{mod}\,\Lambda$ *determined by* T.

Proof. Without loss of generality, we may assume that Λ is basic. Then $\Lambda \cong \mathrm{End}_{H'}(T')^{\mathrm{op}}$ for a connected hereditary algebra H', say of type Δ', and a multiplicity-free tilting H'-module T'. Then Γ_Λ admits a connected component $\mathscr{C} = \mathscr{C}_{T'}$ containing a faithful selection of type $(\Delta')^{\mathrm{op}}$, consisting of the images of the indecomposable injective H'-modules via the functor $\mathrm{Hom}_{H'}(T', -)$. Moreover, if Λ is a concealed algebra, we may assume that $\mathscr{C}$ is preinjective. Recall also that if Λ is not concealed then $\mathscr{C}$ is a unique component of Γ_Λ containing a faithful section (see [**7**, Theorem III.7.2]).

We shall prove that then $\mathscr{C}$ admits a faithful section Δ such that all indecomposable direct summands of M are predecessors of Δ in $\mathrm{ind}\,\Lambda$. Assume first that $\mathscr{C}$ contains at least one injective module. Then there exists a (faithful) section Δ in $\mathscr{C}$ whose all sources are injective (see [**17**, Proposition 7.4]). Then for each noninjective indecomposable module X from Δ we have $\mathrm{Hom}_\Lambda(D(\Lambda), D\mathrm{Tr}_\Lambda(\mathrm{Tr}D_\Lambda X)) = \mathrm{Hom}_\Lambda(D(\Lambda), X) \neq 0$, because there is a sectional path in $\mathscr{C}$ (in fact in Δ) from an injective module I to X, and the composition of irreducible morphisms forming a sectional path is nonzero ([**1**, Theorem VII.2.4]). Hence, for such a module X, we have $\mathrm{pd}_\Lambda \mathrm{Tr}DX \geq 2$ (see [**18**, (2.4)]). Observe also that, if an indecomposable Λ-module Y is a successor of a module on Δ but is not from Δ, then Y is a successor of a module $\mathrm{Tr}D_\Lambda X$, where X is an indecomposable module lying on Δ. This shows that $\mathscr{L}_\Lambda$ consists of all predecessors of Δ in $\mathrm{ind}\,\Lambda$. In particular, the indecomposable direct summands of M are predecessors of Δ in $\operatorname{ind}\Lambda$. Finally, assume that $\mathscr{C}$ has no injective modules. Then $\mathscr{C}$ is not preinjective, and, by our assumption on $\mathscr{C} = \mathscr{C}_{T'}$, Λ is not concealed. This implies also that Δ' is a wild quiver. Then invoking the results of [**13**], [**14**] we conclude that the family of all components of Γ_Λ contained entirely in the torsion part $\mathscr{X}(T') = \{N \in \operatorname{mod}\Lambda \mid T' \otimes_\Lambda N = 0\}$ consists of a unique preinjective component $\mathscr{Q}(\Lambda)$ of Γ_Λ and a family $\mathscr{R}(\Lambda)$ of connected components whose stable parts are of the form $\boldsymbol{Z}\boldsymbol{A}_\infty$. Moreover, since $\mathscr{Q}(\Lambda)$ has no faithful section (because Λ is not concealed), the family $\mathscr{R}(\Lambda)$ contains at least one injective module. Applying now [**2**, Proposition 3.1], [**14**, Sections 1 and 2] and [**6**, Lemma 1.5], we conclude that, for every indecomposable Λ-module Z from $\mathscr{Q}(\Lambda)$ or $\mathscr{R}(\Lambda)$, there exists a path in $\operatorname{ind}\Lambda$ of the form $I \to D\mathrm{Tr}_\Lambda X \to Y \to X \to \cdots \to Z$ with I an indecomposable injective Λ-module from $\mathscr{R}(\Lambda)$. In particular, $\operatorname{Hom}_\Lambda(I, D\mathrm{Tr}_\Lambda X) \neq 0$ implies $\operatorname{pd}_\Lambda X \geq 2$, and consequently $Z \notin \mathscr{L}_\Lambda$. Observe also that every indecomposable injective Λ-module lies in $\mathscr{Q}(\Lambda)$ or $\mathscr{R}(\Lambda)$. Therefore, $\mathscr{L}_\Lambda$ consists of all indecomposable modules from $\mathscr{Y}(T')$ and the indecomposable modules from $\mathscr{C}_{T'}$. Then there exists a positive integer m such that $\Delta = (\mathrm{Tr}D_\Lambda)^m(\Delta')^{\mathrm{op}}$ is a faithful section of $\mathscr{C} = \mathscr{C}_{T'}$ and the indecomposable direct summands of M are predecessors of Δ in $\operatorname{ind}\Lambda$. In the both cases, let U be the direct sum of all indecomposable Λ-modules lying on Δ. Then, applying [**21**, Theorem 3] we conclude that U is a tilting Λ-module, $H = \operatorname{End}_\Lambda(U)^{\mathrm{op}}$ is a hereditary algebra of type Δ^{op}, $T = D(U_H)$ is a tilting H-module, $\Lambda = \operatorname{End}_H(T)^{\mathrm{op}}$, $\mathscr{C} = \mathscr{C}_{T'}$ is the connecting component $\mathscr{C}_T$ of Γ_Λ determined by T, and the indecomposable Λ-modules from the torsion-free part $\mathscr{Y}(T)$ of $\operatorname{mod}\Lambda$ determined by T are exactly the predecessors of Δ in $\operatorname{ind}\Lambda$. In particular, M is a module from $\mathscr{Y}(T)$. This finishes the proof. $\square$

Lemma 2.2. *Let Λ be a connected representation-infinite tilted algebra of Euclidean type such that the preprojective component of Γ_Λ is the unique connecting component of Γ_Λ. Then $\mathscr{L}_\Lambda$ consists of all indecomposable preprojective modules and all τ_Λ-periodic modules. Moreover, for every preprojective module M, there exists a hereditary algebra H of Euclidean type and a tilting H-module T such that $\Lambda = \operatorname{End}_H(T)^{\mathrm{op}}$ and M belongs to the torsion-free part $\mathscr{Y}(T) = \{N \in \operatorname{mod}\Lambda \mid \operatorname{Tor}_1^\Lambda(T, N) = 0\}$ determined by T.*

Proof. We may assume that Λ is basic. Then $\Lambda \cong \operatorname{End}_{H'}(T')^{\mathrm{op}}$ for a connected hereditary algebra H' of Euclidean type Δ' and a multiplicity-free tilting H'-module T'. It follows from our assumption that the preprojective component $\mathscr{P}(\Lambda)$ of Γ_Λ is the connecting component $\mathscr{C}_{T'}$ of Γ_Λ determined by T' and admits a faithful section of type $(\Delta')^{\mathrm{op}}$. Moreover, Γ_Λ consists of $\mathscr{P}(\Lambda)$, a preinjective component $\mathscr{Q}(\Lambda)$, and an infinite family of coray tubes containing at least one injective module, because $\mathscr{Q}(\Lambda) \neq \mathscr{P}(\Lambda)$ is not a connecting component of Γ_Λ. Then for any indecomposable Λ-module Z from $\mathscr{Q}(\Lambda)$ or a nonstable tube of $\mathscr{T}(\Lambda)$

there exists a path in $\operatorname{ind} \Lambda$ of the form $I \to \cdots \to D\mathrm{Tr}X \to Y \to X \to \cdots \to Z$ with I injective, and hence $Z \notin \mathscr{L}_\Lambda$, because $\operatorname{Hom}_\Lambda(I, D\mathrm{Tr}X) \neq 0$ implies $\operatorname{pd}_\Lambda X \geq 2$. Therefore, $\mathscr{L}_\Lambda$ consists of all modules from $\mathscr{P}(\Lambda)$ and all modules from the stable tubes of $\mathscr{T}(\Lambda)$ (equivalently all indecomposable τ_Λ-periodic modules). Finally, assume that M is a preprojective Λ-module, that is, a direct sum of modules from $\mathscr{P}(\Lambda)$. Since $\mathscr{P}(\Lambda)$ contains all projective Λ-modules but no injective module, there exists a positive integer m such that $\Delta = (\mathrm{Tr}D)^m(\Delta')^{\mathrm{op}}$ is faithful section of $\mathscr{P}(\Lambda)$ and all indecomposable direct summands of M are predecessors of Δ in $\mathscr{P}(\Lambda)$. Let U be the direct sum of all modules lying on Δ. Applying again [**21**, Theorem 3] we conclude that U is a tilting Λ-module, $H = \operatorname{End}_\Lambda(U)^{\mathrm{op}}$ is a hereditary algebra of Euclidean type Δ^{op}, $T = D(U_H)$ is a tilting H-module, $\Lambda = \operatorname{End}_H(T)^{\mathrm{op}}$, $\mathscr{P}(\Lambda)$ is the connecting component $\mathscr{C}_T$ of Γ_Λ determined by T, and the indecomposable modules from the torsion-free part $\mathscr{Y}(T)$ determined by T are exactly the predecessors of Δ in $\operatorname{mod} \Lambda$. In particular, M is a module from $\mathscr{Y}(T)$. $\square$

Proposition 2.3. *Let Λ be a connected tilted algebra, M a Λ-module with $\operatorname{Ext}^1_\Lambda(M, M) = 0$ from $\operatorname{add} \mathscr{L}_\Lambda$ (respectively, $\operatorname{add} \mathscr{R}_\Lambda$), and $\Gamma = \operatorname{End}_\Lambda(M)^{\mathrm{op}}$. Moreover, assume that M is preprojective (respectively, preinjective) if Λ is a representation-infinite tilted algebra of Euclidean type such that the preprojective (respectively, preinjective) component of Γ_Λ is the unique connecting component of Γ_Λ. Then Γ is a tilted algebra.*

Proof. We may assume that $M \in \mathscr{L}_\Lambda$. Applying Lemmas 2.1 and 2.2 we conclude that there exists a hereditary algebra H and a tilting H-module T such that $\Lambda = \operatorname{End}_H(T)^{\mathrm{op}}$ and M belongs to the torsion-free part of $\mathscr{Y}(T)$ of $\operatorname{mod} \Lambda$ determined by T. Moreover, it follows from the Brenner-Butler theorem that $\operatorname{Hom}_H(T, -) : \operatorname{mod} H \to \operatorname{mod} \Lambda$ establishes an equivalence between $\mathscr{T}(T) = \{X \in \operatorname{mod} H \mid \operatorname{Ext}^1_H(T, X) = 0\}$ and $\mathscr{Y}(T) = \{Y \in \operatorname{mod} \Lambda \mid \operatorname{Tor}^\Lambda_1(T, N) = 0\}$. Hence there exists an H-module V in $\mathscr{T}(T)$ such that $M = \operatorname{Hom}_H(T, V)$. Moreover, we have

$$\operatorname{Ext}^1_H(V, V) \cong \operatorname{Ext}^1_\Lambda(\operatorname{Hom}_H(T, V), \operatorname{Hom}_H(T, V)) = \operatorname{Ext}^1_\Lambda(M, M) = 0,$$

and consequently V is a partial tilting H-module, because H is hereditary. Applying now [**7**, Corollary III.6.5] we conclude that $\operatorname{End}_H(V)^{\mathrm{op}}$ is a tilted algebra. Therefore, applying again the Brenner-Butler theorem, we infer that $\operatorname{End}_\Lambda(M)^{\mathrm{op}} \cong \operatorname{End}_H(V)^{\mathrm{op}}$ is a tilted algebra. $\square$

Theorem 2.4. *Let Λ be a connected algebra, M a Λ-module with $\operatorname{Ext}^1_\Lambda(M, M) = 0$ from $\operatorname{add} \mathscr{L}_\Lambda$ (respectively, $\operatorname{add} \mathscr{R}_\Lambda$), and $\Gamma = \operatorname{End}_\Lambda(M)^{\mathrm{op}}$. Then the following hold*

(a) *If Λ is quasitilted then Γ is quasitilted.*
(b) *If Λ is strict shod then Γ is tilted.*
(c) *If Λ is shod then Γ is shod.*

Proof. We may assume that $M \in \operatorname{add} \mathscr{L}_\Lambda$. Then $\operatorname{pd}_\Lambda M \leq 1$, and consequently M is a partial tilting Λ-module. Invoking now [**3**, Lemma 2.1] we conclude that there exists a short exact sequence

$$0 \to \Lambda \to E \to M^d \to 0,$$

where $d = \dim_K \operatorname{Ext}^1_\Lambda(M, \Lambda)$, such that $N = E \oplus M$ is a tilting Λ-module, and, if X is an indecomposable direct summand of E, then $\operatorname{Hom}_\Lambda(X, M) \neq 0$ or X is projective.

(a) Assume Λ is quasitilted. Then $\mathscr{L}_\Lambda$ contains all indecomposable projective Λ-modules ([**10**, Theorem II.1.14]), and consequently N is a tilting Λ-module from $\operatorname{add} \mathscr{L}_\Lambda$. Applying now [**10**, Proposition II.2.4] we conclude that $A = \operatorname{End}_\Lambda(N)^{\mathrm{op}}$ is a quasitilted algebra. Observe now that $\Gamma = \operatorname{End}_A(P)^{\mathrm{op}}$, where P is the projective A-module $\operatorname{Hom}_\Lambda(N, M)$. Therefore, a direct application of Theorem 1.2(b), or [**10**, Proposition II.1.15], gives that Γ is a quasitilted algebra.

(b) Assume that Λ is strict shod. Then it follows from [**17**, Theorem 8.2] that Λ is a (strict) double tilted algebra, and hence Γ_Λ admits a connected component $\mathscr{C}$ with a faithful double section Δ whose left part Δ_l is a disjoint union $\Delta_l = \Delta_l^{(1)} \cup \cdots \cup \Delta_l^{(m)}$ of faithful sections $\Delta_l^{(i)}$ of connecting components of the Auslander-Reiten quivers $\Gamma_{\Lambda_l^{(i)}}$ of the connected parts $\Lambda_l^{(i)}$, $1 \leq i \leq m$, of a tilted factor algebra $\Lambda_l = \Lambda_l^{(1)} \times \cdots \times \Lambda_l^{(m)}$ of Λ, and such that $\mathscr{L}_\Lambda$ consists of all predecessors of Δ_l in $\operatorname{ind} \Lambda_l$. Since M belongs to $\operatorname{add} \mathscr{L}_\Lambda$, we obtain that M is a Λ_l-module and all indecomposable direct summands of M are predecessors of Δ_l in $\operatorname{ind} \Lambda_l$. Let $M = M^{(1)} \oplus \cdots \oplus M^{(m)}$, where $M^{(i)}$ is a $\Lambda_l^{(i)}$-module, for each $1 \leq i \leq m$. Note that each $M^{(i)}$ belongs to $\mathscr{L}_{\Lambda_l^{(i)}}$, $\operatorname{Ext}^1_{\Lambda_l^{(i)}}(M^{(i)}, M^{(i)}) = 0$, and $\Gamma = \operatorname{End}_{\Lambda(M)^{\mathrm{op}}} = \operatorname{End}_{\Lambda_l^{(i)}}(M^{(i)}) \times \cdots \times \operatorname{End}_{\Lambda_l^{(m)}}(M^{(m)})$. Moreover, if $\Lambda_l^{(i)}$ is a representation-infinite tilted algebra of Euclidean type such that the preprojective component of $\Gamma_{\Lambda_l^{(i)}}$ is the unique connecting component of $\Gamma_{\Lambda_l^{(i)}}$, then $M^{(i)}$ is a preprojective $\Lambda_l^{(i)}$, because all its indecomposable direct summands are predecessors of $\Delta_l^{(i)}$ in $\operatorname{ind} \Lambda_l^{(i)}$. Therefore, applying Proposition 2.3, we conclude that $\Gamma = \operatorname{End}_\Lambda(M)^{\mathrm{op}}$ is a tilted algebra.

The statement (c) is a direct consequence of (a) and (b). □

We end this section with an example showing that the additional assumptions in Proposition 2.3, concerning the Euclidean case, are necessary.

Example 2.5. Let K be a field and Λ be the bound quiver algebra KQ/I, where Q is the quiver

$$\begin{tikzcd}
& 2 \arrow[dl, "\alpha"'] & & 6 \arrow[ll, "\beta"'] & \\
1 & & & & 5 \arrow[ul, "\gamma"'] \arrow[dll, "\xi"] \\
& & 4 \arrow[ull, "\sigma"] \arrow[dll, "\eta"] & & \\
3 & & & &
\end{tikzcd}$$

and I is the ideal in the path algebra KQ of Q generated by $\eta\xi$. Then Λ is the one-point coextension $[S(3)]H$ of the hereditary algebra $H = K\Delta$, where Δ is the convex subquiver of Q given by the vertices 1, 2, 4, 5, 6, by the simple module $S(4)$ at the vertex 4, lying in the unique stable tube of rank 2 in Γ_H. Hence Λ is a representation-infinite tilted algebra of Euclidean type $\tilde{A}_5$ and the preprojective component $\mathscr{P}(\Lambda)$ of Γ_Λ is the unique connecting component of Γ_Λ (see [**18**, (4.9)]). Applying Lemma 2.2 we conclude that $\mathscr{L}_\Lambda$ consists of all modules from $\mathscr{P}(\Lambda)$ and all modules from the stable tubes of Γ_Λ, or equivalently, all tubes of Γ_Λ except the coray tube containing the injective module $E(3)$ with socle $S(3)$ and top $S(4)$. Further, Γ_Λ admits a stable tube of rank 3 whose mouth is formed by the simple modules $S(2)$, $S(6)$ and the module X of the form

$$\begin{tikzcd}
& 0 \arrow[dl] & & 0 \arrow[ll] & \\
K & & & & K \arrow[ul] \arrow[dll, "1"] \\
& & K \arrow[ull, "1"] \arrow[dll] & & \\
0 & & & &
\end{tikzcd}$$

and such that $\tau_\Lambda X = S(6)$, $\tau_\Lambda(S(6)) = S(2)$, and $\tau_\Lambda(S(2)) = X$. Consider the Λ-module $M = P(1) \oplus P(2) \oplus P(3) \oplus P(4) \oplus P(5) \oplus X$. Observe that M belongs to $\mathscr{L}_\Lambda$, and hence $\operatorname{pd}_\Lambda M \leq 1$. Moreover, $\operatorname{Ext}^1_\Lambda(M, M) = D\overline{\operatorname{Hom}}_\Lambda(M, \tau_\Lambda M) = D\overline{\operatorname{Hom}}_\Lambda(M, S(6)) = 0$. This implies that M is a tilting Λ-module, and a direct calculation shows that $\Gamma = \operatorname{End}_\Lambda(M)^{\mathrm{op}}$ is the bound quiver algebra KQ'/I', where Q' is the quiver

$$\begin{tikzcd}
& 2 \arrow[dl, "\alpha"'] & & 6 \arrow[dl, "\omega"] \\
1 & & 5 \arrow[ul, "\varrho"'] \arrow[dl, "\xi"] & \\
& 4 \arrow[ul, "\sigma"] \arrow[dl, "\eta"] & & \\
3 & & &
\end{tikzcd}$$

and I' is the ideal in KQ' generated by $\eta\xi$ and $\varrho\omega$. Hence, Γ is obtained from the hereditary algebra $H' = K\Delta'$, where Δ' is the convex subquiver of Q' given by the vertices 1, 2, 4, 5, by the one-point coextension $[S(4)]H$, and next the one-point extension $[S(4)]H'[X']$, with X' of the form

$$\begin{tikzcd}
 & 0 \arrow[dl] & \\
K & & K \arrow[ul] \arrow[dl, "1"] \\
 & K \arrow[ul, "1"'] &
\end{tikzcd}$$

Since $S(4)$ and X' lie in different tubes of rank 2 in $\Gamma_{H'}$, the Auslander-Reiten quiver of Γ admits a coray tube containing the injective module $E(3)$ with $\operatorname{soc} E(3) = S(3)$ and a ray tube containing the projective module $P(6)$ with $\operatorname{rad} P(6) = X'$. Therefore, Γ is a representation-infinite iterated tilted algebra of Euclidean type $\tilde{A}_5$ but is not tilted (see [**18**, (4.9)]). We also note that Γ is a quasitilted algebra of canonical type $(3,3)$, because is a semiregular branch extension of the canonical algebra H' of type $(2,2)$ (see [**15**]). Finally, observe that Λ^{op} is a representation-infinite tilted algebra of Euclidean type $\tilde{A}_5$, the preinjective component $\mathcal{Q}(\Lambda^{\mathrm{op}})$ of $\Gamma_{\Lambda^{\mathrm{op}}}$ is the unique connecting component of $\Gamma_{\Lambda^{\mathrm{op}}}$, $D(M)$ is a cotilting Λ^{op}-module from $\mathscr{R}_{\Lambda^{\mathrm{op}}}$, $\operatorname{Ext}^1_{\Lambda^{\mathrm{op}}}(D(M), D(M)) = 0$, and $\operatorname{End}_{\Lambda^{\mathrm{op}}}(D(M))^{\mathrm{op}} = \Gamma^{\mathrm{op}}$ is iterated tilted of Euclidean type $\tilde{A}_5$ (quasitilted of canonical type $(3,3)$) but is not tilted.

3. Triangular matrix algebras.

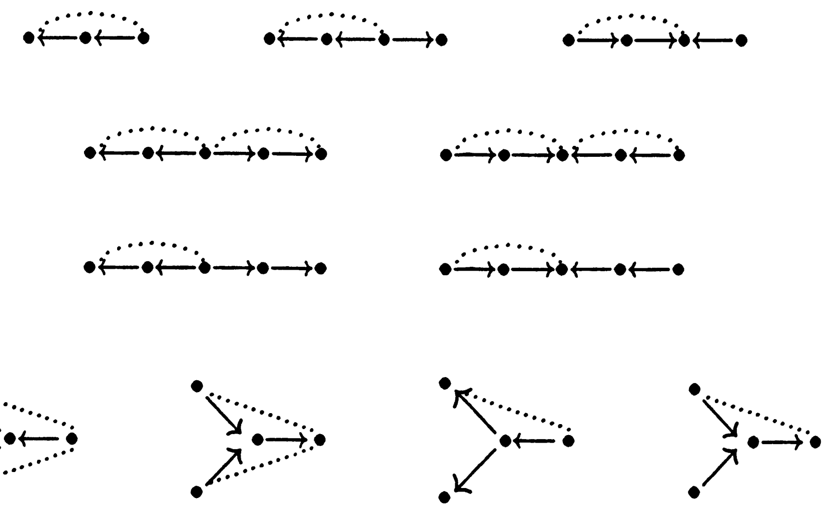
Throughout this section K will be an algebraically closed field and A a fixed basic connected (finite dimensional) algebra over K. We denote by Λ the algebra $\begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ of 2×2 lower triangular matrices over A. It is well known that $\operatorname{mod} \Lambda$ is equivalent to the category whose objects are morphisms $f : X \to Y$ in $\operatorname{mod} A$ and morphisms are pairs of morphisms in $\operatorname{mod} A$ making the obvious squares commutative. The modules over the algebras $\Lambda = \begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ have been the object of studies during the last 20 years. We refer to [**12**] and [**16**] for a complete description of all representation-finite and tame algebras of the form $\begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ and further references.

Here we are interested in a complete description of algebras A such that the algebra $\Lambda = \begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ is shod. It is known that $\operatorname{gl.dim} \Lambda = 1 + \operatorname{gl.dim} A$ (see [**1**, Proposition III.2.6]). Hence, if Λ is quasitilted (respectively, strict shod) then $\operatorname{gl.dim} A \leq 1$ (respectively, $\operatorname{gl.dim} A = 2$). Recall also that A can be presented as an algebra $A = KQ/I$, where $Q = Q_A$ is the Gabriel quiver of A and I is an admissible ideal in the path algebra KQ of Q. Moreover, $A = KQ/I$ is hereditary if and only if $I = 0$ and Q has no oriented cycles. The following description of all quasitilted 2×2 lower triangular algebras has been established in [**9**, Theorem 3.1].

Theorem 3.1. *The algebra* $\Lambda = \begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ *is quasitilted if and only if* $A = KQ$ *for* Q *one of the Dynkin quivers of type* $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{A}_3$, $\boldsymbol{A}_4$, $\boldsymbol{D}_4$ (*any orientation*) *or* $\boldsymbol{A}_5$ (*orientation different from* $\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet$).

The following main result of this section extends the above theorem to a complete description of all shod algebras of the form $\begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$.

Theorem 3.2. *The algebra* $\Lambda = \begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ *is a strict shod if and only if* $A \cong KQ/I$, *where* (Q, I) *is one of the following bound quivers*



where $\bullet \to \bullet \to \bullet$ *means that the composition of these arrows is a generator of the ideal* I.

The proof of this theorem will be a combination of several facts established below. We would like first to state a direct consequence of Theorems 3.1 and 3.2, and the main results of [**12**] and [**16**]. Recall that an algebra Γ is called tame if, for any dimension d, there is a finite number of Γ-$K[X]$-bimodules M_i which are finitely generated and free as right $K[X]$-modules, and satisfy the following condition: all but a finite number of isomorphism classes of indecomposable Γ-modules of dimension d are of the form $M_i \otimes K[X]/(X - \lambda)$ for some $\lambda \in K$ and for some i. Denote by $\mu_\Gamma(d)$ the least number of bimodules M_i satisfying the above condition for d. Then Γ is said to be of linear growth if there is a natural number m such that $\mu_\Gamma(d) \leq md$ for all $d \geq 1$ (see [**20**] for more details). It follows also from the validity of the second Brauer-Thrall conjecture that $\mu_\Gamma(d) = 0$ for all $d \geq 1$ if and only if Γ is representation-finite (the number of isomorphism classes of indecomposable Γ-modules is finite).

COROLLARY 3.3. *For* $\Lambda = \begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ *the following hold*:
(1) *If* Λ *is shod then* Λ *is of linear growth.*
(2) *If* Λ *is strict shod then* Λ *is representation-finite.*

We start our proofs with the following

PROPOSITION 3.4. *Assume that* $\Lambda = \begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ *is a strict shod. Then* A *is representation-finite and tilted.*

PROOF. Observe that $A = \operatorname{End}(P)^{\mathrm{op}}$, where P is the projective Λ-module $\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} = \Lambda \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Moreover, $\operatorname{gl.dim} \Lambda = 3$ implies $\operatorname{gl.dim} A = 2$. Hence, applying Theorem 1.2(d) we conclude that A is tilted. It has been proved in [**9**, Proposition 3.3] that if H is a hereditary algebra and there exists an indecomposable H-module X with $D\mathrm{Tr}_H^4 X \neq 0$ then there exists an indecomposable module Z over $\begin{bmatrix} H & 0 \\ H & H \end{bmatrix}$ of both projective and injective dimension 2. A simple analysis of arguments used there shows that the same holds for algebras of global dimension at most 2. Since $\operatorname{gl.dim} A = 2$ and Λ is shod, we obtain that $D\mathrm{Tr}_A^4 M = 0$ for every indecomposable A-module M. This implies that every $D\mathrm{Tr}$-orbit in the Auslander-Reiten quiver Γ_A of A consists of at most 4 indecomposable modules and contains a projective module. Therefore A is representation-finite. □

From now on we may assume that A is representation-finite. Moreover, it follows from [**4**] that A has a presentation $A = KQ/I$ where the ideal I is generated by paths or differences of paths (having common sources and targets) in Q. Let $Q = (Q_0, Q_1)$, where Q_0 is the set of vertices of Q and Q_1 is the set of arrows of Q. Then the quiver $\Delta = (\Delta_0, \Delta_1)$ of $\Lambda = \begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ can be described as follows: $\Delta_0 = \{i, i^* \mid i \in Q_0\}$ and $\Delta_1 = \{\alpha, \alpha^* \mid \alpha \in Q_1\} \cup \{\gamma_i : i^* \to i \mid i \in Q_0\}$. Denote by J the ideal in the path algebra KQ of Δ generated by the elements:

(1) $\alpha_1 \cdots \alpha_r$, $\alpha_1^* \cdots \alpha_r^*$, for all paths $\alpha_1 \cdots \alpha_r \in I$,
(2) $\alpha_1 \cdots \alpha_s - \beta_1 \cdots \beta_t$, $\alpha_1^* \cdots \alpha_s^* - \beta_1^* \cdots \beta_t^*$ for all differences $\alpha_1 \cdots \alpha_s - \beta_1 \cdots \beta_t \in I$,
(3) $\gamma_j \alpha^* - \alpha \gamma_i$ for all arrows $i \xrightarrow{\alpha} j$ from Q_1.

Then $\Lambda \cong K\Delta/J$ (see [**19**]). We also note that if A is tilted then Q, and hence Δ, has no oriented cycles. Further, there exists a canonical choice of primitive orthogonal idempotents e_i, e_i^*, $i \in Q_0$, of Λ such that

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \sum_{i \in Q_0} e_i \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \sum_{i \in Q_0} e_i^*.$$

For a multiplicity-free projective A-module $P = Ae_{i_1} \oplus \cdots \oplus Ae_{i_r}$, with $i_1, \ldots, i_r$ pairwise different elements of Q_0, we denote by $\bar{P}$ the multiplicity-free projective Λ-module $\Lambda e_{i_1} \oplus \cdots \oplus \Lambda e_{i_r} \oplus \Lambda e_{i_1}^* \oplus \cdots \oplus \Lambda e_{i_r}^*$. Moreover, for a vertex a of Q_0, we put $P(a) = \Lambda e_a$, $P(a^*) = \Lambda e_a^*$, $E(a) = D(e_a \Lambda)$ and $E(a^*) = D(e_a^* \Lambda)$.

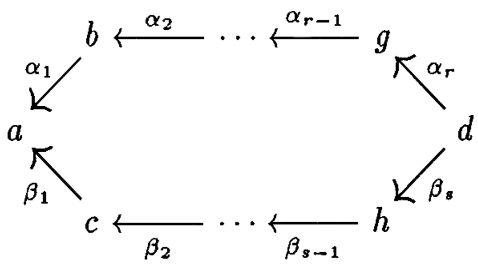
Lemma 3.5. *Let P be a multiplicity-free projective A-module, $B = \mathrm{End}_A(P)^{\mathrm{op}}$, and $\Gamma = \begin{bmatrix} B & 0 \\ B & B \end{bmatrix}$. Then $\Gamma \cong \mathrm{End}_A(\bar{P})^{\mathrm{op}}$.*

Proof. Obvious. $\square$

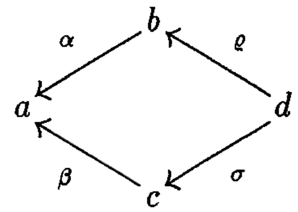
Recall that $A = KQ/I$ is called a monomial algebra provided I is generated by paths.

Lemma 3.6. *Assume A is strict shod. Then A is a monomial algebra.*

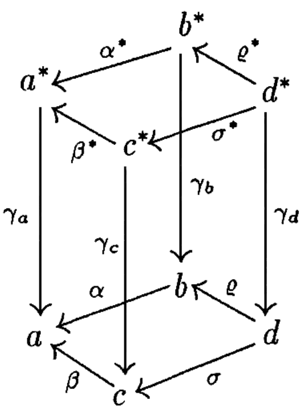
Proof. Suppose $A = KQ/I$ is not a monomial algebra. Since A is representation-finite it follows from the above discussion that Q contains a subquiver



such that $\alpha_1 \cdots \alpha_r - \beta_1 \cdots \beta_s \in I$ but $\alpha_1 \cdots \alpha_r \notin I$, $\beta_1 \cdots \beta_s \notin I$. Take the projective A-module $P = Ae_a \oplus Ae_b \oplus Ae_c \oplus Ae_d$ and $B = \mathrm{End}_A(P)$. Then $B = KQ'/I'$ where Q' is the quiver



and I' is generated by $\alpha\varrho - \beta\sigma$. Then $\Gamma = \begin{bmatrix} B & 0 \\ B & B \end{bmatrix} = K\Delta'/J'$ where Δ' is the quiver



and J' is generated by $\alpha\varrho - \beta\sigma$, $\alpha^*\varrho^* - \beta^*\sigma^*$, $\gamma_a\alpha^* - \alpha\gamma_b$, $\gamma_a\beta^* - \beta\gamma_c$, $\gamma_b\varrho^* - \varrho\gamma_d$, $\gamma_c\sigma^* - \sigma\gamma_d$. Observe that Γ admits a unique indecomposable projective-injective Γ-module $P(d^*) = E(a)$. Further, $M = \operatorname{rad} P(d^*)/\operatorname{soc} P(d^*)$ is an indecomposable Γ-module. Moreover, M has a minimal projective resolution

$$0 \to P(a) \to P(a^*) \oplus P(b) \oplus P(c) \to P(b^*) \oplus P(c^*) \oplus P(d) \to M \to 0,$$

and a minimal injective resolution

$$0 \to M \to E(a^*) \oplus E(b) \oplus E(c) \to E(b^*) \oplus E(c^*) \oplus E(d) \to E(d^*) \to 0,$$

in $\operatorname{mod} \Gamma$. Hence $\operatorname{pd}_\Gamma M = 2$ and $\operatorname{id}_\Gamma M = 2$. On the other hand, it follows from Theorem 2.1(d) and Lemma 3.5 that $\Gamma = \operatorname{End}_\Lambda(\bar{P})^{\mathrm{op}}$ is a shod, a contradiction. Therefore, Λ is a monomial algebra. $\square$

LEMMA 3.7. *Assume Λ is a strict shod and the bound quiver (Q, I) of Λ contains a full subquiver Q' of Dynkin type $\boldsymbol{A}_5$ or $\boldsymbol{D}_4$. Then Q' contains a path belong to I.*

PROOF. Suppose that (Q, I) contains a subquiver Q' of type $\boldsymbol{A}_5$ or $\boldsymbol{D}_4$, which has no subpath belonging to I. Let P be the direct sum of indecomposable projective Λ-modules corresponding to the vertices of Q' and $B = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$. Then $B \cong KQ'$ and $\Gamma = \begin{bmatrix} B & 0 \\ B & B \end{bmatrix} \cong \operatorname{End}_\Lambda(\bar{P})^{\mathrm{op}}$ for the corresponding projective Λ-module $\bar{P}$. It has been shown in [**9**] that either there exists an indecomposable Γ-module M with $\operatorname{pd}_\Gamma M = 2$ and $\operatorname{id}_\Gamma M = 2$, if Q' is the quiver $\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet$, or Γ is a quasitilted but not tilted algebra, in the remaining cases. On the other hand, it follows from Theorem 1.2(d) that Γ is either strict shod or tilted. Since $\operatorname{gl.dim} B = 1$ implies $\operatorname{gl.dim} \Gamma = 2$, we have a contradiction. This finishes the proof. $\square$

LEMMA 3.8. *Assume Λ is strict shod. Then the bound quiver (Q, I) of Λ does not contain a full bound subquiver (Q', I') of one of the forms*

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3 \xrightarrow{\gamma} 4 \xleftarrow{\sigma} 5 \quad \text{or} \quad 1 \xrightarrow{\beta} 2 \xrightarrow{\alpha} 3 \xleftarrow{\gamma} 4 \xrightarrow{\sigma} 5$$

with I' generated by $\alpha\beta$.

PROOF. Suppose that (Q, I) contains a full bound subquiver (Q', I') of one of the above forms and $B = KQ'/I'$. By duality we may assume that (Q', I') is the left quiver. Clearly, $B = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$, where P is the direct sum of the indecomposable projective Λ-modules corresponding to the vertices of Q', and $\Gamma = \begin{bmatrix} B & 0 \\ B & B \end{bmatrix} \cong \operatorname{End}_\Lambda(\bar{P})^{\mathrm{op}}$. Moreover, $\Gamma \cong K\Delta'/J'$ where Δ' is the quiver

$$\begin{tikzcd}
1^* \arrow[d, "\gamma_1"] & 2^* \arrow[l, "\alpha^*"'] \arrow[d, "\gamma_2"] & 3^* \arrow[l, "\beta^*"'] \arrow[r, "\varrho^*"] \arrow[d, "\gamma_3"] & 4^* \arrow[d, "\gamma_4"] & 5^* \arrow[l, "\sigma^*"'] \arrow[d, "\gamma_5"] \\
1 & 2 \arrow[l, "\alpha"] & 3 \arrow[l, "\beta"] \arrow[r, "\varrho"'] & 4 & 5 \arrow[l, "\sigma"]
\end{tikzcd}$$

and I' is generated by $\alpha\beta$, $\alpha^*\beta^*$, $\gamma_1\alpha^* - \alpha\gamma_3^*$, $\gamma_2\beta^* - \beta\gamma_3$, $\gamma_4\varrho^* - \varrho\gamma_3$ and $\gamma_4\sigma^* - \sigma\gamma_5$. Consider the indecomposable Γ-module (representation of (Δ', J'))

$$M : \begin{tikzcd}
0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[r] \arrow[d] & 0 \arrow[d] & 0 \arrow[l] \arrow[d] \\
0 & 0 \arrow[l] & K \arrow[l] \arrow[r, "1"'] & K & 0 \arrow[l]
\end{tikzcd}$$

Then the minimal projective and injective resolutions of M in $\operatorname{mod} \Gamma$ are of the forms

$$0 \to P(1) \to P(2) \to P(3) \to M \to 0,$$

$$0 \to M \to E(4) \to E(4^*) \oplus E(5) \to E(5^*) \to 0,$$

and hence $\operatorname{pd}_\Gamma M = 2$ and $\operatorname{id}_\Gamma(M) = 2$. This contradicts Theorem 1.2, because, by Lemma 3.4, $\Gamma = \operatorname{End}_A(\bar{P})$ is a shod algebra. $\square$

LEMMA 3.9. *Assume A is strict shod. Then (Q, I) does not contain a full bound subquiver (Q', I') where Q' is the quiver*

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\sigma} 4$$

and $I' \neq 0$.

PROOF. Suppose (Q, I) contains a full bound subquiver (Q', I') of the above form and $I' \neq 0$, and $B = KQ'/I$. Then $B = \operatorname{End}_A(P)^{\mathrm{op}}$, for the corresponding projective A-module P, and $\Gamma = \begin{bmatrix} B & 0 \\ B & B \end{bmatrix} \cong \operatorname{End}_\Lambda(\bar{P})^{\mathrm{op}}$, for the corresponding projective Λ-module P. Moreover, $\Gamma = K\Delta'/J'$, where Δ' is the quiver

$$\begin{tikzcd}
1^* \arrow[d, "\gamma_1"] & 2^* \arrow[l, "\alpha^*"'] \arrow[d, "\gamma_2"] & 3^* \arrow[l, "\beta^*"'] \arrow[r, "\sigma^*"] \arrow[d, "\gamma_3"] & 4^* \arrow[d, "\gamma_4"] \\
1 & 2 \arrow[l, "\alpha"] & 3 \arrow[l, "\beta"] & 4 \arrow[l, "\sigma"]
\end{tikzcd}$$

Observe that I' is generated only by one path. Indeed, if it is not the case, then $\alpha\beta, \beta\sigma \in I'$, and then $\operatorname{gl.dim} A = 3$, a contradiction, because Γ shod implies $\operatorname{gl.dim} A \leq 2$.

Assume now that I' is generated by $\alpha\beta\sigma$. Then J' is generated by the elements $\alpha\beta\sigma$, $\alpha^*\beta^*\sigma^*$, $\gamma_1\alpha^* - \alpha\gamma_2$, $\gamma_2\beta^* - \beta\gamma_3$, $\gamma_3\sigma^* - \sigma\gamma_4$. Consider the indecomposable Γ-module (representation of (Q', I'))

$$M : \begin{array}{ccccccc} K & \xleftarrow{(0,1)} & K^2 & \xleftarrow{\binom{1}{0}} & K & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle (1,1)} & & \downarrow{\scriptstyle \binom{0}{1}} & & \downarrow \\ 0 & \longleftarrow & K & \xleftarrow[(1,1)]{} & K^2 & \xrightarrow[(1,0)]{} & K \end{array}$$

Then the minimal projective and injective resolutions of M in $\operatorname{mod}\Gamma$ are of the forms

$$0 \to P(1) \to P(1^*) \oplus P(2) \to P(3^*) \oplus P(2^*) \oplus P(4) \to M \to 0$$

$$\begin{aligned} 0 \to M \to\ & E(1^*) \oplus E(2) \oplus E(3) \\ & \to E(3^*) \oplus E(3^*) \oplus E(4) \oplus E(4) \to E(4^*) \oplus E(4^*) \to 0, \end{aligned}$$

and consequently $\operatorname{pd}_\Gamma M = 2$ and $\operatorname{id}_\Gamma M = 2$, a contradiction since Γ is a shod.

Assume now that I' is generated by a path of length 2. Without loss of generality, we may assume that I' is generated by $\alpha\beta$. Then J' is generated by $\alpha\beta$, $\alpha^*\beta^*$, $\gamma_1\alpha^* - \alpha\gamma_2$, $\gamma_2\beta^* - \beta\gamma_3$, and $\gamma_3\sigma^* - \sigma\gamma_4$. Consider the simple Γ-module $S(3)$ given by the vertex 3. Then the minimal projective and injective resolutions of $S(3)$ in $\operatorname{mod}\Gamma$ are of the forms

$$0 \to P(1) \to P(2) \to P(3) \to S(3) \to 0,$$

$$0 \to S(3) \to E(3) \to E(3^*) \oplus E(4) \to E(4^*) \to 0,$$

and hence $\operatorname{pd}_\Gamma S(3) = 2$ and $\operatorname{id}_\Gamma S(3) = 2$, again a contradiction since Γ is shod. $\square$

COROLLARY 3.10. *Assume Λ is strict shod. Then Q does not contain a full subquiver Q' of the form*

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\varrho} 4 \xleftarrow{\sigma} 5.$$

PROOF. Let (Q', I') be the full bound subquiver of (Q, I) given by Q', and $B = KQ'/I'$. Applying Lemmas 3.7 and 3.9, we may assume that I' is generated by $\alpha\beta\varrho\sigma$. Consider the projective Λ-module $P = P(1) \oplus P(2) \oplus P(3) \oplus P(5)$ and $C = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$. Then $C = KQ''/I''$ where Q'' is the quiver

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\omega} 5$$

and I'' is generated by $\alpha\beta\omega$. Since $\begin{bmatrix} C & 0 \\ C & C \end{bmatrix} \cong \operatorname{End}_\Lambda(\bar{P})$ is a shod algebra we obtain a contradiction with Lemma 3.9. $\square$

Lemma 3.11. *Assume Λ is strict shod. Then (Q, I) does not contain a full bound subquiver (Q', I') with Q' of the form*

$$1 \overset{\xi}{\leftarrow} 2 \overset{\beta}{\to} 3 \overset{\alpha}{\to} 4 \overset{\sigma}{\leftarrow} 5$$

and I' generated by $\alpha\beta$.

Proof. Suppose (Q, I) contains a full bound subquiver (Q', I') of the above form and $B = KQ'/I'$. Then $B = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$, where P is the direct sum of the indecomposable projective Λ-modules corresponding to the vertices of Q', and $\Gamma = \begin{bmatrix} B & 0 \\ B & B \end{bmatrix} \cong \operatorname{End}_\Lambda(\bar{P})^{\mathrm{op}}$. Moreover, $\Gamma \cong K\Delta'/J'$, where Δ' is the quiver

$$\begin{tikzcd} 1^* \arrow[d, "\gamma_1"] & 2^* \arrow[l, "\xi^*"'] \arrow[r, "\beta^*"] \arrow[d, "\gamma_2"] & 3^* \arrow[r, "\alpha^*"] \arrow[d, "\gamma_3"] & 4^* \arrow[d, "\gamma_4"] & 5^* \arrow[l, "\sigma^*"'] \arrow[d, "\gamma_5"] \\ 1 & 2 \arrow[l, "\xi"] \arrow[r, "\beta"'] & 3 \arrow[r, "\alpha"'] & 4 & 5 \arrow[l, "\sigma"] \end{tikzcd}$$

and I' is generated by $\alpha\beta$, $\alpha^*\beta^*$, $\gamma_1\xi^* - \xi\gamma_2$, $\gamma_3\beta^* - \beta\gamma_2$, $\gamma_4\alpha^* - \alpha\gamma_3$, and $\gamma_4\sigma^* - \sigma\gamma_5$. Consider the indecomposable Γ-module

$$M: \begin{tikzcd} 0 \arrow[d] & 0 \arrow[l] \arrow[r] \arrow[d] & K \arrow[r, "1"] \arrow[d, "1"] & K \arrow[d] & K \arrow[l, "1"'] \arrow[d] \\ K & K \arrow[l, "1"] \arrow[r, "1"'] & K \arrow[r] & 0 & 0 \arrow[l] \end{tikzcd}$$

Then the minimal projective and injective resolutions of M in $\operatorname{mod}\Gamma$ are of the forms

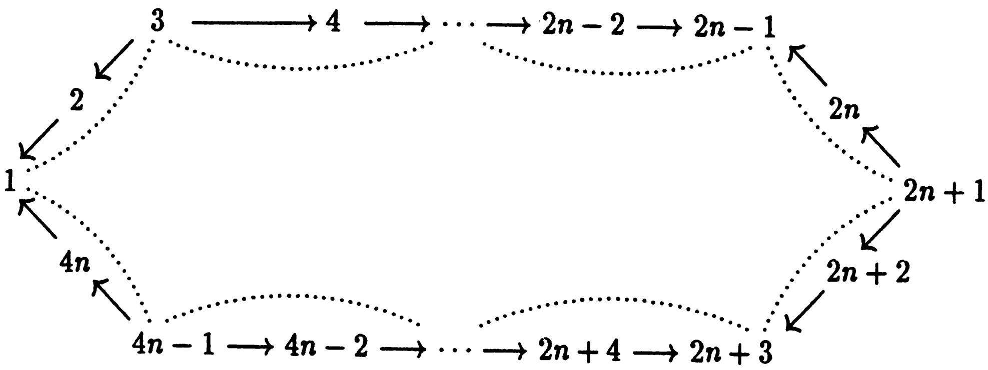
$$0 \to P(4) \to P(3^*) \oplus P(4^*) \oplus P(5) \to P(2) \oplus P(3^*) \oplus P(5^*) \to M \to 0,$$

$$0 \to M \to E(1) \oplus E(3) \oplus E(4^*)$$
$$\to E(1^*) \oplus E(2) \oplus E(3^*) \to E(2^*) \to 0,$$

and hence $\operatorname{pd}_\Gamma M = 2$ and $\operatorname{id}_\Gamma M = 2$. This contradicts Theorem 1.2, because, by Lemma 3.4, $\Gamma = \operatorname{End}_\Lambda(\bar{P})^{\mathrm{op}}$ is a shod algebra. □

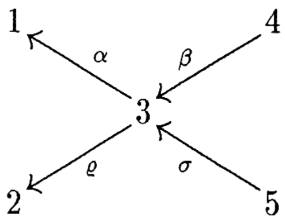
Lemma 3.12. *Assume Λ is strict shod. Then Q is a tree.*

Proof. Suppose that the quiver Q of Λ contains a cycle. Since $\Lambda = KQ/I$ is representation-finite, such a cycle contains at least one subpath from I. We know also that Q has no oriented cycles. Invoking now our assumption on Λ and the properties of (Q, I) established above, we conclude that there exists a multiplicity-free projective Λ-module P such that $B = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$ is isomorphic to the bound quiver algebra KQ'/I' of the bound quiver (Q', I') of the form

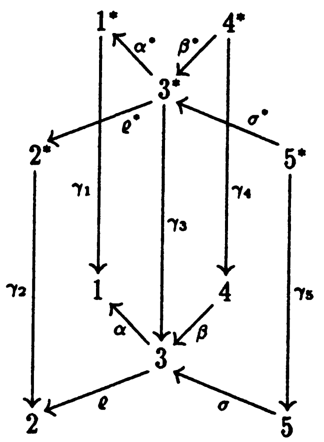


where $\bullet \to \bullet \to \bullet$ means that the composition of these two arrows belongs to I'. But this contradicts Lemma 3.11. $\square$

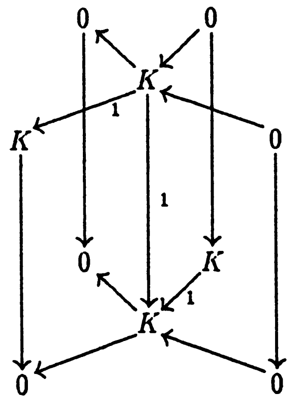
Lemma 3.13. *Assume Λ is strict shod. Then (Q, I) does not contain a full bound subquiver (Q', I') with Q' of the form*



Proof. Suppose (Q, I) contains a full bound subquiver (Q', I') of the above form, and let $B = KQ'/I'$, $\Gamma = \begin{bmatrix} B & 0 \\ B & B \end{bmatrix} = K\Delta'/J'$. We know that $B = \mathrm{End}_\Lambda(P)^{\mathrm{op}}$ and $\Gamma = \mathrm{End}_\Lambda(\bar{P})^{\mathrm{op}}$, for the corresponding projective modules P in $\mathrm{mod}\,\Lambda$ and $\bar{P}$ in $\mathrm{mod}\,\Lambda$, and in particular Γ is shod. Applying Lemma 3.7, we may assume that (Q', I') does not contain a full bound subquiver (Q'', I''), where Q'' is a Dynkin quiver of type $\boldsymbol{D}_4$ and $I'' = 0$. Hence I' is generated by at least two paths (of length 2). The quiver Δ' of Γ is of the form



Consider the indecomposable Γ-module M of the form



Without loss of generality we may assume that $\alpha\beta \in I'$ and $\varrho\sigma \in I'$. Then M has a minimal projective resolution in $\operatorname{mod}\Gamma$ of the form

$$0 \to P(1) \to P(1^*) \oplus P(3) \oplus N \to P(3^*) \oplus P(4) \to M \to 0$$

where $N = 0$, if $\varrho\beta \in I'$, and $N = P(2)$ if $\varrho\beta \notin I'$. Similarly, we conclude that the minimal injective resolution of M in $\operatorname{mod}\Gamma$ is of the form

$$0 \to M \to E(2^*) \oplus E(3) \to E(3^*) \oplus E(5) \oplus R \to E(5^*) \to 0,$$

where $R = 0$, if $\varrho\beta \in I'$, and $R = E(4^*)$ if $\varrho\beta \notin I'$. Therefore, we have always $\operatorname{pd}_\Gamma M = 2$ and $\operatorname{id}_\Gamma M = 2$, a contradiction because Γ is a shod. □

Lemma 3.14. *Assume Λ is strict shod. Then (Q, I) does not contain a full bound subquiver (Q', I') of one of the forms*

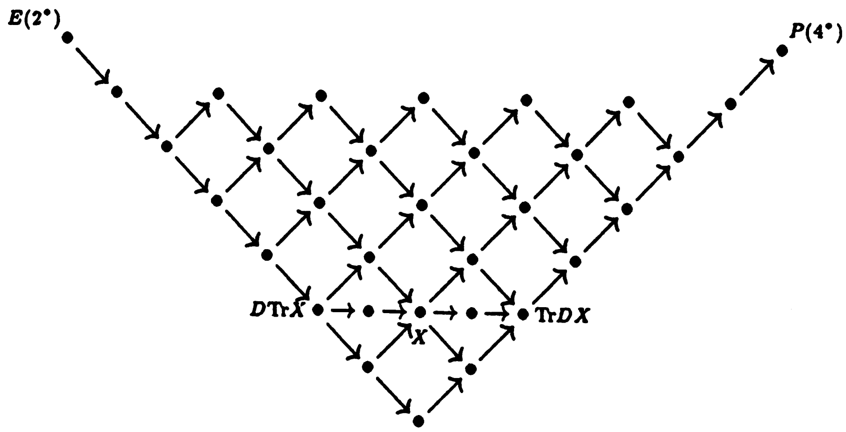
$$\begin{tikzcd} 1 & & \\ & 3 \arrow[ul, "\alpha"'] \arrow[dl, "\sigma"] & 4 \arrow[l, "\beta"'] \arrow[r, "\varrho"] & 5 \\ 2 & & \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[dr, "\beta"] & & \\ & 3 \arrow[r, "\alpha"] & 4 & 5 \arrow[l, "\varrho"'] \\ 2 \arrow[ur, "\sigma"'] & & \end{tikzcd}$$

with I' generated by $\alpha\beta$.

Proof. Suppose, by duality, that (Q, I') contains a full bound subquiver (Q', I') of the left form, and let $B = KQ'/I'$, $\Gamma = \begin{bmatrix} B & 0 \\ B & B \end{bmatrix} = K\Delta'/J'$. Then Δ' is of the form

$$\begin{tikzcd} 1^* \arrow[dd, "\gamma_1"'] & & 3^* \arrow[ll, "\alpha^*"'] \arrow[dl, "\sigma^*"] \arrow[dd, "\gamma_3"] & 4^* \arrow[l, "\beta^*"'] \arrow[r, "\varrho^*"] \arrow[dd, "\gamma_4"] & 5^* \arrow[dd, "\gamma_5"] \\ & 2^* \arrow[dd, "\gamma_2"'] & & & \\ 1 & & 3 \arrow[ll, "\alpha"'] \arrow[dl, "\sigma"] & 4 \arrow[l, "\beta"] \arrow[r, "\varrho"'] & 5 \\ & 2 & & & \end{tikzcd}$$

and J' is generated by $\alpha\beta$, $\alpha^*\beta^*$, $\gamma_1\alpha^* - \alpha\gamma_3$, $\gamma_2\sigma^* - \sigma\gamma_3$, $\gamma_3\beta^* - \beta\gamma_4$, $\gamma_5\varrho^* - \varrho\gamma_4$. It follows from [**12**] and [**16**] that Γ is representation-finite, and hence its Auslander-Reiten quiver consists of a finite preprojective (and preinjective) translation quiver. A direct but tedious calculation shows that it contains a full translation subquiver of the form



where X is the indecomposable Γ-module with the dimension-vector

$$\dim X = \begin{array}{cccc} 2 & 4 & 0 & 1 \\ & 3 & & \\ 0 & 5 & 4 & 3 \\ & 2 & & \end{array}$$

Since the composition of irreducible morphisms between modules forming a sectional path is nonzero ([**1**, Theorem VII.2.4]), we have $\mathrm{Hom}_\Gamma(E(1), D\mathrm{Tr}X) \neq 0$ and $\mathrm{Hom}_\Gamma(\mathrm{Tr}DX, P(4^*)) \neq 0$, and consequently $\mathrm{pd}_\Gamma X \geq 2$ and $\mathrm{id}_\Gamma X \geq 2$. This leads to a contradiction because $\Gamma = \mathrm{End}_\Lambda(\bar{P})^{\mathrm{op}}$ for a projective Λ-module $\bar{P}$, and so Γ is shod by Theorem 1.2. $\square$

LEMMA 3.15. *Assume Λ is strict shod. Then (Q, I) does not contain a full bound subquiver (Q', I') of one of the forms*

$$\begin{tikzcd} 1 \arrow[r, "\sigma"] & 2 & 3 \arrow[l, "\alpha"'] \arrow[d, "\varrho"] & 4 \arrow[l, "\beta"'] \\ & & 5 & \end{tikzcd} \qquad \begin{tikzcd} 1 & 2 \arrow[l, "\sigma"'] \arrow[r, "\beta"] & 3 \arrow[r, "\alpha"] & 4 \\ & & 5 \arrow[u, "\varrho"'] & \end{tikzcd}$$

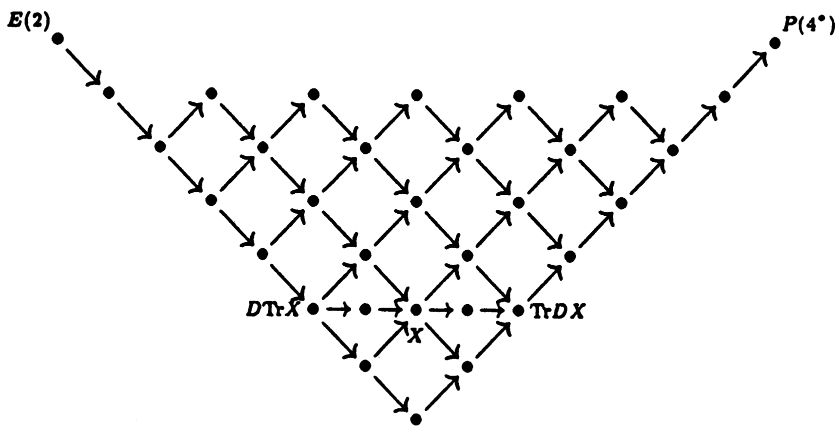
with I' generated by $\alpha\beta$.

PROOF. Suppose, by duality, that (Q, I) contains a full bound subquiver

(Q', I') of the left form, and let $B = KQ'/I$, $\Gamma = \begin{bmatrix} B & 0 \\ B & B \end{bmatrix} = K\Delta'/J'$. Then Δ' is of the form

$$\begin{tikzcd}
1^* \arrow[r, "\sigma^*"] \arrow[dd, "\gamma_1"'] & 2^* \arrow[dd, "\gamma_2"'] & & 3^* \arrow[ll, "\alpha^*"'] \arrow[dl, "\varrho^*"] \arrow[dd, "\gamma_3"] & 4^* \arrow[l, "\beta^*"'] \arrow[dd, "\gamma_4"] \\
& & 5^* \arrow[dd, "\gamma_5"] & & \\
1 \arrow[r, "\sigma"'] & 2 & & 3 \arrow[ll, "\alpha"] \arrow[dl, "\varrho"] & 4 \arrow[l, "\beta"] \\
& & 5 & &
\end{tikzcd}$$

and J' is generated by $\alpha\beta$, $\alpha^*\beta^*$, $\sigma\gamma_1 - \gamma_2\sigma^*$, $\alpha\gamma_3 - \gamma_2\alpha^*$, $\beta\gamma_4 - \gamma_3\beta^*$, $\varrho\gamma_3 - \gamma_5\varrho^*$. It follows from [**12**] and [**16**] that Γ is a representation-finite algebra and its Auslander-Reiten quiver is a finite preprojective (and preinjective) translation quiver. A direct but tedious calculation shows that it contains a full translation subquiver of the form



Hence, as in the previous lemma, we conclude that $\mathrm{Hom}_\Gamma(D(\Gamma), D\mathrm{Tr}_\Gamma X) \neq 0$, $\mathrm{Hom}_\Gamma(\mathrm{Tr}D_\Gamma X, \Gamma) \neq 0$, and hence $\mathrm{pd}_\Gamma X \geq 2$ and $\mathrm{id}_\Gamma X \geq 2$. Since $\Gamma = \mathrm{End}_\Lambda(\bar{P})$ for a projective Λ-module $\bar{P}$, this contradicts Theorem 1.2, because Λ is shod. □

LEMMA 3.16. *Assume Λ is strict shod. Then (Q, I) does not contain a full bound subquiver (Q', I') of one of the forms*

$$\begin{tikzcd}
1 & & \\
& 3 \arrow[ul, "\alpha"'] \arrow[dl, "\sigma"] & 4 \arrow[l, "\beta"'] \arrow[r, "\varrho"] & 5 \\
2 & &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
1 \arrow[dr, "\eta"] & & \\
& 3 \arrow[r, "\omega"] & 4 & 5 \arrow[l, "\nu"'] \\
2 \arrow[ur, "\xi"'] & &
\end{tikzcd}$$

with I' generated by $\alpha\beta$ and $\sigma\beta$, or $\omega\eta$ and $\omega\xi$, respectively.

PROOF. Suppose, by duality, that (Q, I) contains a full bound subquiver (Q', I') of the left form, and let $B = KQ'/I'$, $\Gamma = \begin{bmatrix} B & 0 \\ B & B \end{bmatrix} = K\Delta'/J'$. Then Δ' is of the form

$$\begin{tikzcd}
& 1^* \arrow[ddd, "\gamma_1"'] & & & \\
& & 3^* \arrow[ul, "\alpha^*"'] \arrow[dll, "\sigma^*"] \arrow[ddd, "\gamma_3"] & 4^* \arrow[l, "\beta^*"'] \arrow[r, "\varrho^*"] \arrow[ddd, "\gamma_4"] & 5^* \arrow[ddd, "\gamma_5"] \\
2^* \arrow[ddd, "\gamma_2"'] & & & & \\
& 1 & & & \\
& & 3 \arrow[ul, "\alpha"] \arrow[dll, "\sigma"] & 4 \arrow[l, "\beta"] \arrow[r, "\varrho"'] & 5 \\
2 & & & &
\end{tikzcd}$$

and J' is generated by $\alpha\beta$, $\sigma\beta$, $\alpha^*\beta^*$, $\sigma^*\beta^*$, $\gamma_1\alpha^* - \alpha\gamma_3$, $\gamma_2\sigma^* - \sigma\gamma_3$, $\gamma_3\beta^* - \beta\gamma_4$, $\gamma_5\varrho^* - \varrho\gamma_4$. Consider the indecomposable Γ-module

$$M: \quad \begin{tikzcd}
& K \arrow[ddd] & & & \\
& & K \arrow[ul, "1"'] \arrow[dll] \arrow[ddd, "1"] & 0 \arrow[l] \arrow[r] \arrow[ddd] & 0 \arrow[ddd] \\
0 \arrow[ddd] & & & & \\
& 0 & & & \\
& & K \arrow[ul] \arrow[dll] & K \arrow[l, "1"] \arrow[r, "1"'] & K \\
0 & & & &
\end{tikzcd}$$

Then M has the following minimal projective and injective resolutions in $\operatorname{mod} \Gamma$

$$0 \to P(2) \to P(2^*) \oplus P(3) \to P(3^*) \oplus P(4) \to M \to 0,$$

$$0 \to M \to E(1^*) \oplus E(3) \oplus E(5) \to E(3^*) \oplus E(4) \to E(4^*) \to 0,$$

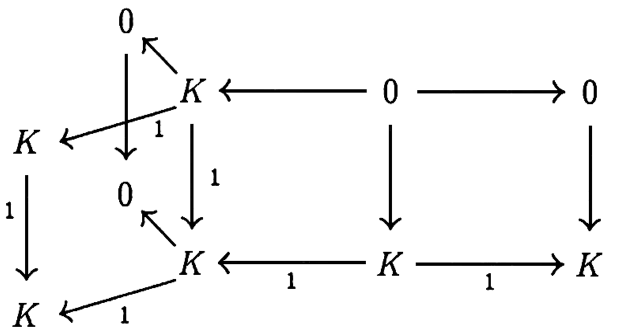
and hence $\operatorname{pd}_\Gamma M = 2$ and $\operatorname{id}_\Gamma M = 2$. This again contradicts the fact that $\Gamma = \operatorname{End}_\Lambda(\bar{P})^{\mathrm{op}}$, for some projective Λ-module $\bar{P}$, is a shod algebra. $\square$

LEMMA 3.17. *Assume Λ is strict shod. Then (Q, I) does not contain a full bound subquiver (Q', I') of one of the forms*

$$\begin{tikzcd}
1 & & & \\
& 3 \arrow[ul, "\alpha"'] \arrow[dl, "\sigma"] & 4 \arrow[l, "\beta"'] \arrow[r, "\varrho"] & 5 \\
2 & & &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
1 \arrow[dr, "\eta"] & & & \\
& 3 \arrow[r, "\omega"] & 4 & 5 \arrow[l, "\nu"'] \\
2 \arrow[ur, "\xi"'] & & &
\end{tikzcd}$$

with I' generated by $\alpha\beta$, or $\omega\eta$, respectively.

Proof. Suppose (by duality) that (Q, I) contains a full bound subquiver (Q', I') of the left form, and let $B = KQ'/I'$, $\Gamma = \begin{bmatrix} B & 0 \\ B & B \end{bmatrix} = K\Delta'/J'$. Then $\Gamma = K\Delta'/J'$, where Δ is the quiver described in the proof of Lemma 3.16 and J' is generated by $\alpha\beta$, $\alpha^*\beta^*$, $\gamma_1\alpha^* - \alpha\gamma_3$, $\gamma_2\sigma^* - \sigma\gamma_3$, $\gamma_3\beta^* - \beta\gamma_4$, and $\gamma_5\varrho^* - \varrho\gamma_4$. Consider the indecomposable Γ-module



Then a direct checking shows that M has the minimal projective and minimal injective resolutions in $\operatorname{mod}\Gamma$ of the forms

$$0 \to P(1) \to P(1^*) \oplus P(3) \to P(3^*) \oplus P(4) \to M \to 0,$$

$$0 \to M \to E(2) \oplus E(5) \to E(4) \oplus E(5^*) \to E(4^*) \to 0,$$

and hence $\operatorname{pd}_\Gamma M = 2$ and $\operatorname{ind}_\Gamma M = 2$. This contradicts again the fact that Γ is shod, as an algebra of the form $\operatorname{End}_\Lambda(\bar{P})$ for the corresponding projective Λ-module $\bar{P}$. □

We may summarize our considerations above as follows: if $\Lambda = \begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ is strict shod then $A = KQ/I$ for a bound quiver (Q, I) listed in Theorem 3.2. In order to proof the sufficiency part of Theorem 3.2 it is enough to show, thanks to Lemma 3.4, that if (Q, I) is maximal bound quiver listed in Theorem 3.2 and $A = KQ/I$ then $\Lambda = \begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ is strict shod. Hence, since the opposite algebra of a strict shod algebra is also strict shod, we have only four cases to consider.

Lemma 3.18. *Let $A = KQ/I$, where Q is of the form*

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3 \xrightarrow{\sigma} 4 \xrightarrow{\varrho} 5$$

and I is generated by $\alpha\beta$ and $\varrho\sigma$. Then Λ is strict shod.

Proof. Since $\operatorname{gl.dim} A = 2$, we have $\operatorname{gl.dim} \Lambda = 3$, and then it remains to show that Λ is shod. We know that $\Lambda = K\Delta/J$, where Δ is the quiver

$$\begin{tikzcd}
1^* \arrow[d, "\gamma_1"] & 2^* \arrow[l, "\alpha^*"'] \arrow[d, "\gamma_2"] & 3^* \arrow[l, "\beta^*"'] \arrow[r, "\sigma^*"] \arrow[d, "\gamma_3"] & 4^* \arrow[r, "\varrho^*"] \arrow[d, "\gamma_4"] & 5^* \arrow[d, "\gamma_5"] \\
1 & 2 \arrow[l, "\alpha"] & 3 \arrow[l, "\beta"] \arrow[r, "\sigma"'] & 4 \arrow[r, "\varrho"'] & 5
\end{tikzcd}$$

and J is generated by $\alpha\beta$, $\alpha^*\beta^*$, $\varrho\sigma$, $\varrho^*\sigma^*$, $\gamma_1\alpha^* - \alpha\gamma_2$, $\gamma_2\beta^* - \beta\gamma_3$, $\gamma_4\sigma^* - \sigma\gamma_3$, and $\gamma_5\varrho^* - \varrho\gamma_4$. Then a direct calculation shows that Λ is a representation-finite algebra and Γ_Λ is of the form

where $P(2^*) = E(1)$ and $P(4^*) = E(5)$. Observe that for each indecomposable Λ-module M we have

$$\mathrm{Hom}_\Lambda(D(\Lambda), D\mathrm{Tr}_\Lambda M) = 0 \quad \text{or} \quad \mathrm{Hom}_\Lambda(\mathrm{Tr}D_\Lambda M, \Lambda) = 0,$$

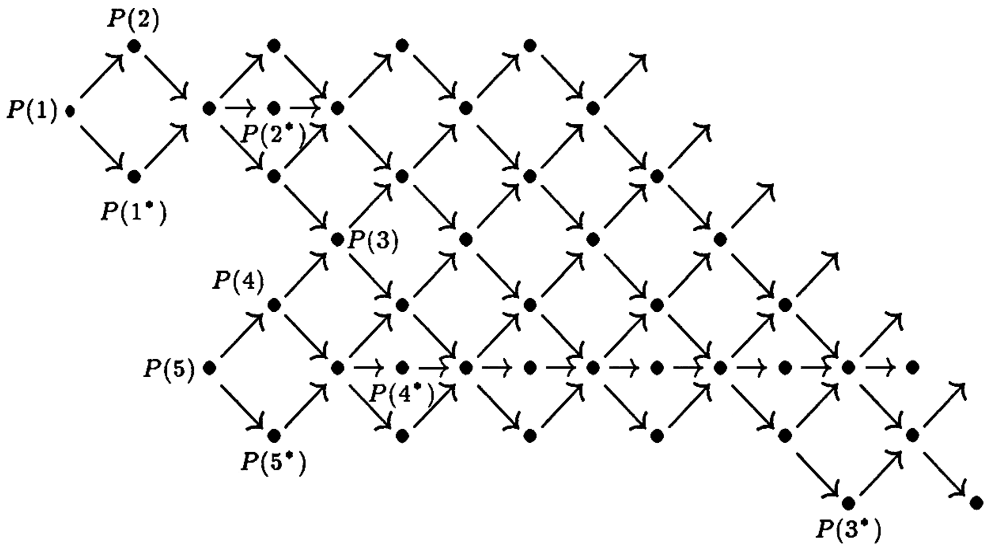
and consequently $\mathrm{pd}_\Lambda M \leq 1$ or $\mathrm{id}_\Lambda M \leq 1$ (see [**15**, (2.4)]). Therefore Λ is shod. □

Lemma 3.19. *Let $A = KQ/I$, where Q is of the form*

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3 \xrightarrow{\sigma} 4 \xrightarrow{\varrho} 5$$

and I is generated by $\alpha\beta$. Then A is strict shod.

Proof. Since $\mathrm{gl.dim}\, A = 3$, it is enough to show that A is shod. We have $\Lambda = K\Delta/J$ where Δ is the quiver described in the proof of the previous lemma and J is generated by $\alpha\beta$, $\alpha^*\beta^*$, $\gamma_1\alpha^* - \alpha\gamma_2$, $\gamma_2\beta^* - \beta\gamma_3$, $\gamma_4\sigma^* - \sigma\gamma_3$, and $\gamma_5\varrho^* - \varrho\gamma_4$. It follows from [**12**] and [**16**] that Λ is a representation-finite algebra having a directed Auslander-Reiten quiver. A direct calculation shows that Γ_Λ has a full translation subquiver of the form



$(P(2^*) = E(1))$ containing all indecomposable projective Λ-modules. Then we conclude that for every indecomposable Λ-module M we have

$$\mathrm{Hom}_\Lambda(D(\Lambda), D\mathrm{Tr}_\Lambda M) = 0 \quad \text{or} \quad \mathrm{Hom}_\Lambda(\mathrm{Tr}D_\Lambda M, \Lambda) = 0,$$

and hence $\mathrm{pd}_\Lambda M \leq 1$ or $\mathrm{id}_\Lambda M \leq 1$. Therefore Λ is shod. □

Lemma 3.20. *Let $A = KQ/I$, where Q is the quiver*

$$\begin{tikzcd}
1 & & \\
& 3 \arrow[ul, "\alpha"'] \arrow[dl, "\sigma"] & 4 \arrow[l, "\beta"'] \\
2 & &
\end{tikzcd}$$

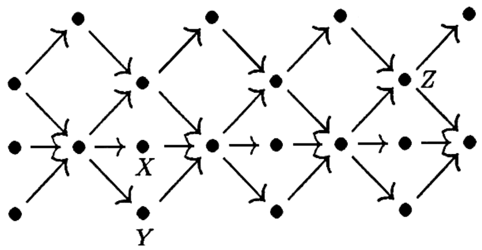
and I is generated by $\alpha\beta$ and $\sigma\beta$. Then Λ is strict shod.

Proof. Since $\mathrm{gl.dim}\,\Lambda = 3$, we have to show that Λ is shod. We have $\Lambda = K\Delta/J$, where Δ is of the form

$$\begin{tikzcd}
& 1^* \arrow[ddd, "\gamma_1"'] & & \\
& & 3^* \arrow[ul, "\alpha^*"'] \arrow[dll, "\sigma^*"] \arrow[ddd, "\gamma_3"] & 4^* \arrow[l, "\beta^*"'] \arrow[ddd, "\gamma_4"] \\
2^* \arrow[ddd, "\gamma_2"'] & & & \\
& 1 & & \\
& & 3 \arrow[ul, "\alpha"] \arrow[dll, "\sigma"] & 4 \arrow[l, "\beta"] \\
2 & & &
\end{tikzcd}$$

and J is generated by $\alpha\beta$, $\alpha^*\beta^*$, $\sigma\beta$, $\sigma^*\beta^*$, $\gamma_1\alpha^* - \alpha\gamma_3$, $\gamma_2\sigma^* - \sigma\gamma_3$, $\gamma_3\beta^* - \beta\gamma_4$. Let $B = K\Omega$ be the path algebra of the subquiver Ω of Δ given by the vertices 1^*, 2^*, 3^*, 3 and 4. Then Λ can be obtained from B by a one-point extension $B[Z]$

of B by the indecomposable B-module $Z = \operatorname{rad} P(4^*)$ and next two one-point coextensions $[X][Y]B[Z]$ of $B[Z]$ by two indecomposable B-modules (hence $B[Z]$-modules) $X = E(1)/\operatorname{soc} E(1)$ and $Y = E(2)/\operatorname{soc} E(2)$. A simple calculation shows that the Auslander-Reiten quiver Γ_B of B is of the form



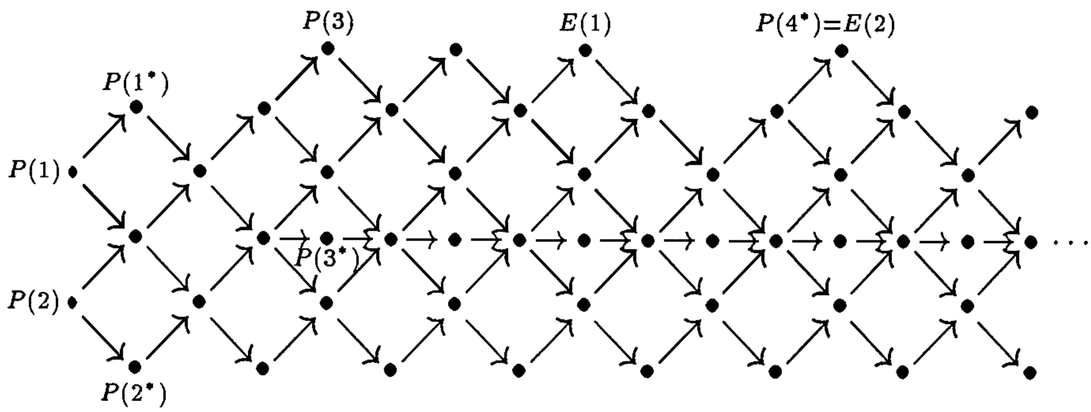
Moreover, we know by [**12**] and [**16**] that Λ is representation-finite and has a directed (finite) Auslander-Reiten quiver Γ_Λ. Then it follows that for every indecomposable Λ-module M we have $\operatorname{Hom}_\Lambda(D(\Lambda), D\mathrm{Tr}_\Lambda M) = 0$ or $\operatorname{Hom}_\Lambda(D\mathrm{Tr}D_\Lambda M, \Lambda) = 0$, and so $\operatorname{pd}_\Lambda M \leq 1$ or $\operatorname{id}_\Lambda M \leq 1$. Therefore Λ is shod. $\square$

LEMMA 3.21. *Let $A = KQ/I$, where Q is the quiver*

$$\begin{tikzcd} 1 & & \\ & 3 \arrow[ul, "\alpha"'] \arrow[dl, "\sigma"] & 4 \arrow[l, "\beta"'] \\ 2 & & \end{tikzcd}$$

and I is generated by $\alpha\beta$. Then Λ is strict shod.

PROOF. Since $\operatorname{gl.dim} \Lambda = 3$, it is enough to show that Λ is shod. We have $\Lambda = K\Delta/J$, where Δ is the quiver described in the proof of Lemma 3.20 and J is generated by $\alpha\beta$, $\alpha^*\beta^*$, $\gamma_1\alpha^* - \alpha\gamma_3$, $\gamma_2\sigma^* - \sigma\gamma_3$ and $\gamma_3\beta^* - \beta\gamma_4$. It follows from [**12**] and [**16**] and Λ is representation-finite and has a directed Auslander-Reiten quiver. A direct calculation shows that Γ_Λ has a full translation subquiver of the form



containing all indecomposable projective Λ-modules. Then we easily deduce that each indecomposable Λ-module M satisfies

$$\mathrm{Hom}_\Lambda(D(\Lambda), D\mathrm{Tr}_\Lambda M) = 0 \quad \text{or} \quad \mathrm{Hom}_\Lambda(\mathrm{Tr}D_\Lambda M, \Lambda) = 0,$$

or equivalently, $\mathrm{pd}_\Lambda M \leq 1$ or $\mathrm{id}_\Lambda M \leq 1$. Therefore Λ is a shod algebra. $\square$

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References

[1] M. Auslander, I. Reiten and S. O. Smalø, Representation theory of artin algebras, Cambridge Stud. Adv. Math., **36**, Cambridge Univ. Press, 1995.

[2] D. Baer, Wild hereditary artin algebras and linear methods, Manuscripta Math., **55** (1986), 69–82.

[3] K. Bongartz, Tilted algebras, In Representations of Algebras, Lecture Notes in Math., vol. 903, Springer, Berlin, 1981, pp. 27–38.

[4] O. Bretscher and P. Gabriel, The standard form of the representation-finite algebra, Bull. Soc. Math. France, **111** (1983), 21–40.

[5] F. U. Coelho and M. Lanzilotta, Algebras with small homological dimension, Manuscripta Math., **100** (1999), 1–11.

[6] F. U. Coelho and A. Skowroński, On the Auslander-Reiten components for quasitilted algebras, Fund. Math., **149** (1996), 67–82.

[7] D. Happel, Triangulated categories in the representation theory of finite dimensional algebras, London Math. Soc. Lecture Note Ser., **119**, Cambridge Univ. Press, 1988.

[8] D. Happel, A characterization of hereditary categories with tilting object, Preprint, 2000.

[9] D. Happel, I. Reiten and S. O. Smalø, Quasitilted algebras, In Finite Dimensional Algebras and Related Topics, NATO ASI Series, Series C, vol. 424, Kluwer Academic, Dordrecht, 1994, pp. 163–181.

[10] D. Happel, I. Reiten and S. O. Smalø, Tilting in abelian categories and quasitilted algebras, Mem. Amer. Math. Soc., **575** (1996).

[11] D. Happel and C. M. Ringel, Tilted algebras, Trans. Amer. Math. Soc., **274** (1982), 399–443.

[12] K. Igusa, M. I. Platzeck, G. Todorov and D. Zacharia, Auslander algebras of finite representation type, Comm. Algebra, **15** (1987), 377–424.

[13] O. Kerner, Tilting wild algebras, J. London Math. Soc., **29** (1989), 29–47.

[14] O. Kerner, Stable components of wild tilted algebras, J. Algebra, **142** (1991), 37–57.

[15] H. Lenzing and A. Skowroński, Quasitilted algebras of canonical type, Colloq. Math., **71** (1996), 161–181.

[16] Z. Leszczyński and A. Skowroński, Tame triangular matrix algebras, Colloq. Math., **86** (2000), 259–303.

[17] I. Reiten and A. Skowroński, Characterizations of algebras with small homological dimensions, Preprint, 2000.

[18] C. M. Ringel, Tame algebras and integral quadratic forms, Lecture Notes in Math., **1099** (1984).

[19] A. Skowroński, On tame triangular matrix algebras, Bull. Polon. Math. Soc., **34** (1986), 517–523.

[20] A. Skowroński, Algebras of polynomial growth, In Topics in Algebras, vol. 26, Part 1, Banach Center Publications, PWN, Warszawa, 1990, pp. 535–568.
[21] A. Skowroński, Generalized standard Auslander-Reiten components without oriented cycles, Osaka J. Math., **30** (1993), 515–527.
[22] A. Skowroński, Tame quasi-tilted algebras, J. Algebra, **203** (1998), 470–490.
[23] D. Zacharia, On the Cartan matrix of an artin algebra of global dimension two, J. Algebra, **82** (1983), 353–357.

Mark KLEINER
Department of Mathematics
Syracuse University
Syracuse, New York 13244
USA

Andrzej SKOWROŃSKI
Faculty of Mathematics and Informatics
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń
Poland

Dan ZACHARIA
Department of Mathematics
Syracuse University
Syracuse, New York 13244
USA