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Topology of compact self-dual manifolds whose twistor space is of positive algebraic dimension

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Abstract. The topology of a compact self-dual manifold whose twistor space has positive algebraic dimension is studied. When the algebraic dimension equals three, it is known by Campana [**4**] that the original self-dual manifold is homeomorphic to a connected sum of copies of a complex projecitve plane. In the remaining cases where the algebraic dimension is equal to two or one, we similarly determine the topology of the self-dual manifold except in a certain exceptional case where the algebraic dimension equals one.

1. Introduction and statement of results.

Let M be a compact connected oriented 4-manifold with self-dual structure $[g]$. Let Z be the associated twistor space, which is a compact connected complex manifold of complex dimension three. Campana [**4**] has shown that if Z is Moishezon, then M is homeomorphic to a connected sum $m\boldsymbol{CP}^2$ of m copies of complex projective plane $\boldsymbol{CP}^2$, where m is the second betti number $b_2(M)$ of M. As a generalization we study in this note the topology of M when the associated twistor space Z has a positive algebraic dimension, namely we consider the cases where the algebraic dimension $a(Z) = 2$ or 1 and show the two theorems below. (Note that the Moishezon case corresponds to the case $a(Z) = 3$.) We also note that under our assumptions Ville [**34**] had obtained the estimate $b_1 \leq 4$ for the first betti number $b_1 = b(Z)$.

The first theorem deals with the case of algebraic dimension two.

Theorem 1.1. *Let $(M, [g])$ be a compact connected self-dual 4-manifold and Z the associated twistor space as above. Suppose that the algebraic dimension $a(Z)$ of Z equals two. Then one of the following is true*:

1) *M is homeomorphic to $m\boldsymbol{CP}^2$, where $m = b_2(M)$,*

or

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2) *there exists a finite unramified covering* $\hat{M}$ *of* M *which is homeomorphic to* $(S^1 \times S^3)\sharp m\boldsymbol{CP}^2$, *where* $m = b_2(M)$, *and* $\sharp$ *denotes the smooth connected sum.*

Before stating the result in the case $a(Z) = 1$ we recall some definitions and related known results. We call the self-dual manifold $(M, [g])$ *of positive* (resp. *zero*) *type* if the conformal class $[g]$ contains a Riemannian metric of constant positive (resp. zero) scalar curvature s. Typical examples of self-dual manifolds of zero type are provided by Kähler surfaces of zero scalar curvature with reversed orientation. There exists a strong relation between the type of a self-dual manifold and the algebraic dimension of the associated twistor space. In fact by the results of Poon [**28**], Gauduchon and Pontecorvo [**25**, 3.5, 3.3, 4.3] we know the following:

Lemma 1.2. *Let* $(M, [g])$ *and* Z *be as above.*

1) *If* $a(Z) \geq 2$, $(M, [g])$ *is of positive type, and*

2) *If* $a(Z) = 1$, $(M, [g])$ *is of positive type or of zero type. If it is of zero type, then one of the following is true*;

a) $(M, [g])$ *is flat,*

b) M *is a* $K3$ *surface with reversed orientation and* $[g]$ *is the classs of a Calabi-Yau Kähler metric, and*

c) *a finite* (*nontrivial*) *unramified Galois covering of* $(M, [g])$ *is isomorphic to one of the self-dual manifolds in* b), *and in this case the fundamental group of* M *is isomorphic to either* $\boldsymbol{Z}/2\boldsymbol{Z}$ *or* $\boldsymbol{Z}/2\boldsymbol{Z} \oplus \boldsymbol{Z}/2\boldsymbol{Z}$.

Remark. 1) Here, $(M, [g])$ being flat in a) above means that $[g]$ contains a flat metric. In this case the fundamental group of M is considered as a 4-dimensional crystallographic group, and the topological classification of manifolds is essentially known.

2) In case c) of the above lemma the fundamental group of M is isomorphic either to $\boldsymbol{Z}/2\boldsymbol{Z}$ or to $\boldsymbol{Z}/2\boldsymbol{Z} \oplus \boldsymbol{Z}/2\boldsymbol{Z}$. The result on the fundamental group of M in c) follows from [**14**, Theorem 1].

In any case, when $a(Z) = 1$ and $(M, [g])$ is of zero type, up to unramified coverings the topology of M is essentially known. So, for our purpose of studying the topology of self-dual manifolds we restrict ourselves to the case where $(M, [g])$ is a self-dual manifold of positive type. In this case our result is stated as follows.

Theorem 1.3. *Let* $(M, [g])$ *be a compact connected self-dual* 4-*manifold and* Z *the associated twistor space as above. Suppose that the algebraic dimension* $a(Z)$ *of* Z *equals one and* $(M, [g])$ *is of positive type. Then the first betti number* $b_1(M)$ *of* M *is either one or zero. Moreover, either of the statements* 1) *and* 2) *of Theorem* 1.1 *holds true except possibly in the following case*: *a general fiber of the* *algebraic reduction is a compact analytic surface of class VII which contains no global spherical shells and with positive second betti number.*

Remark. 1) It is expected that the exceptional case mentioned in the above theorem does not occur. In fact, it is conjectured that there exist no compact analytic surfaces of class VII_0 which contains no global spherical shells and with positive second betti number (cf. [**23**, 5.5]).

2) Typical examples which fall under the case 2) of Theorems 1 and 2 are primary Hopf surfaces (M, g) with natural conformally flat metric $g = (dz_1 \cdot d\bar{z}_1 + dz_2 \cdot d\bar{z}_2)/\|z\|^2$, where

$$M = (\boldsymbol{C}^2 - \{0\})/\langle h \rangle, \quad h(z_1, z_2) = (az_1, bz_2), \quad |a| = |b| \neq 1.$$

In this case M is homeomorphic to $S^1 \times S^3$. According to the choices of the complex numbers a and b the algebraic dimension of the associated twistor space can take any of the values zero, one and two (cf. [**13**], [**12**]). So the case 2) of both the theorems also occur actually. In the Hopf case we have $b_2 = 0$. However, LeBrun [**21**] constructed anti-self-dual metrics on certain blown-up Hopf surfaces and it turns out that the associated twistor spaces is always of algebraic dimension one. These give examples in which M is homeomorphic to $(S^1 \times S^3)\sharp(m\boldsymbol{CP}^2), m > 0$, in Theorem 1.3.

3) It is known that as a consequence of a general theorem of Taubes [**32**] any finitely presented group can be realized as the fundamental group of some compact self-dual manifold. In view of the above theorems the associated twistor space is in most cases necessarily of algebraic dimension zero.

Combined with a theorem of Kuiper [**19**] the following is a corollary of Theorems 1.1 and 1.3 except for the exceptional case in Theorem 1.3.

Theorem 1.4. *Let $(M, [g])$ be a compact oriented conformally flat* 4-*manifold. Suppose that the algebraic dimension $a(Z)$ of the twistor space Z of M is positive. Then $(M, [g])$ is conformally equivalent to one of the following*:

a) *the* 4-*sphere with standard conformally flat metric*,

b) *a flat manifold*,

c) *a finite quotient of a Hopf surface with standard conformally flat hermitian metric by a finite group of conformal transformations acting freely on the manifold.*

In fact, except for the exceptional case in Theorem 1.3 the fundamental group of M contains a normal abelian subgroup of finite index. So we can apply the theorem of Kuiper in [**19**] to conclude that M falls under one of the classes a), b) and c). We shall give a complex analytic proof of the theorem which is valid also in the above exceptional case in the final section 6.

The proofs of Theorems 1.1 and 1.3 proceed as follows. Let F be a general fiber of algebraic reduction of Z. Consider the natural image N of $\pi_1(F)$ in $\pi_1(Z)$ and the associated quotient $Q := \pi_1(Z)/N$, where π_1 denotes the fundamental group. The key point of the proof is to show that Q is a finite group. We shall show this by using Stein factorization and L^2 method given in Sections 2 and 3 respectively. Our result, however, is formulated in the frame work of a general compact complex manifold and its algebraic reduction (cf. Theorem 4.1 in Section 4) since it might be of independent interest. If this is combined with the estimate of the first betti number in Proposition 5.3, the rest of the proof is not difficult and is given in Section 5.

The referee has kindly pointed out that by using Theorem 2.2 of Campana [**5**] one can obtain simpler proofs of Theorems 1.1 and 1.3 which do not use the methods of Sections 2 and 3 of this paper, although the intermediate results obtained by the methods of those sections may be of some independent interest.

2. Stein factorization of certain holomorphic maps and fundamental groups.

Let $g : A \to B$ be a surjective holomorphic map of irreducible normal complex spaces. Then a *Stein factorization* of g is a pair of holomorphic maps $(h : A \to C, k : C \to B)$ with $g = kh$ where C is a normal complex space, h is a proper surjective holomorphic map with connected fibers and k is a (possibly ramified) covering. If g is proper, the existence and uniqueness of a Stein factorization as above is well-known.

We are interested in the existence of a Stein factorization in the following situation: Let X and Y be compact connected complex manifolds and $f : X \to Y$ a surjective holomorphic map with connected fibers. Let F be any smooth fiber of f. Let N be the natural image of $\pi_1(F)$ in $\pi_1(X)$, which is independent of the choice of F. In fact, the following is well-known:

Lemma 2.1. *N is a normal subgroup of $\pi_1(X)$.*

Proof. Let U be a Zariski open subset of Y over which f is smooth. Then, since f is topologically a fiber bundle over U, we get the associated homotopy exact sequence in the usual sense

$$\to \pi_1(F) \xrightarrow{\iota} \pi_1(f^{-1}(U)) \to \pi_1(U) \to 1.$$

In particular, the image $\mathrm{Im}\,\iota$ is a normal subgroup of $\pi_1(f^{-1}(U))$. Then, N also is a normal subgroup of $\pi_1(X)$ as it is the image of $\mathrm{Im}\,\iota$ in $\pi_1(X)$ by the natural surjection $\pi_1(f^{-1}(U)) \to \pi_1(X)$. $\square$

Let $Q := \pi_1(X)/N$ be the quotient group and $q : Q \to G$ an arbitrary quotient of Q. Consider an unramified covering

$$u : \tilde{X} \to X$$

corresponding to the natural quotient homomorphism $\pi_1(X) \to G$. Then we ask if a Stein factorization of the composite map $fu : \tilde{X} \to Y$ exists or not.

Proposition 2.2. *Let $f : X \to Y, G$ and $u : \tilde{X} \to X$ be as above. Suppose that there exists a divisor D in Y with only normal crossings such that f is smooth over $Y - D$. Then a Stein factorization $(\tilde{f} : \tilde{X} \to \tilde{Y}, v : \tilde{Y} \to Y)$ of fu exists so that the following diagram commutes*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{u} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \tilde{Y} & \xrightarrow{v} & Y. \end{array} \tag{1}$$

Moreover, v is a (possibly ramified) Galois covering with Galois group G whose branch locus on Y is contained in D, and any general fiber of $\tilde{f}$ is mapped isomorphically onto a general fiber of f by u.

The key to the proof of the proposition is a local lemma proved in [**11**, Lemma 2.1], which we shall state and prove in our simple situation here (and also make some minor corrections of [**11**]).

Let B be a polycylinder of sufficiently small multi-radii in $\boldsymbol{C}^n = \boldsymbol{C}^n(z_1, \dots, z_n)$ for some $n > 0$. Define a hypersurface A in B by the equation $z_1 \cdots z_l = 0$ for some $1 \le l \le n$. We set $U = B - A$. Let W be a complex manifold and $f : W \to B$ a proper surjective holomorphic map with connected fibers. We assume that f is smooth over U. Fixing any point $b \in U$ we set $F = f^{-1}(b)$.

In order to state the lemma exactly we introduce a terminology. Let $A_i := \{z_i = 0\}$, $1 \le i \le l$, be the irreducible components of A. We take a point $a^i = (a_1^i, \dots, a_n^i)$ of $A_i - \bigcup_{j \ne i} A_j$ and consider the restriction $f_i : W_i \to D_i$ of f over the 1-dimensional disc $D_i = \{z_j = a_j^i; j \ne i\}$, where $W_i = f^{-1}(D_i)$. If we take a^i sufficiently general, then W_i is nonsingular. Let $S_{i\mu}, 1 \le \mu \le d_i$, be the irreducible components of $f_i^{-1}(a^i)$, and $m_{i\mu}$ the multiplicity of $S_{i\mu}$ in $f_i^{-1}(a^i)$. Then we call the greatest common divisor m_i of $m_{i\mu}$ the *multiplicity* of f along A_i. It is standard to check that m_i is independent of the choice of a point a^i as above. Then our lemma is stated as follows.

Lemma 2.3. *Let the notations and assumptions be as above. Then, possibly after restricting B around the origin, the cokernel of the natural homomorphism*

$\pi_1(F) \to \pi_1(W)$ is a finite abelian group R which is a quotient of the finite abelian group $\oplus_i \boldsymbol{Z}/m_i\boldsymbol{Z}$; we have thus an exact sequence of groups

$$\pi_1(F) \to \pi_1(W) \xrightarrow{q} R \to 1. \tag{2}$$

Moreover, when $n = l = 1$, we have $R = \boldsymbol{Z}/m\boldsymbol{Z}$, where $m = m_1$.

Proof. Let $V = f^{-1}(U)$. Then the natural map $t : \pi_1(V) \to \pi_1(W)$ is surjective. On the other hand, since the restriction $f_V : V \to U$ of f to V is topologically a fiber bundle, we get an exact sequence of groups

$$1 \to \pi_1(F) \to \pi_1(V) \xrightarrow{b} \pi_1(U) \to 1. \tag{3}$$

We claim that the image in $\pi_1(U)$ $(\cong \oplus_i \boldsymbol{Z})$ by b of the kernel K of t contains the subgroup $\oplus_i m_i\boldsymbol{Z}$. (Note that the class γ_i defined by a loop which turns once around the a^i counterclockwise in the punctured disc $D_i - a^i$ give canonical generators of $\pi_1(U)$.) Take a 1-dimensional disc $D_{i\mu}$ in W_i which intersect transversally with $S_{i\mu}$ at a general point $y_{i\mu}$ and then consider the class $\gamma_{i\mu}$ of $\pi_1(V)$ defined by a loop in $D_{i\mu}$ turning once around $y_{i\mu}$. Since the induced map $D_{i\mu} \to D_i$ is an $m_{i\mu}$-ple covering ramified at $y_{i\mu}$, the image by u of $\gamma_{i\mu}$ in $\pi_1(U)$ generate the subgroup $m_{i\mu}\boldsymbol{Z}$ of the i-th component of $\pi_1(U)$. This consideration for all i and μ gives our claim that $b(K)$ contains $\oplus_i m_i\boldsymbol{Z}$. Then by taking the quotient of the sequence (3) by K we get the exact sequence (2).

It remains to show that $R \cong \boldsymbol{Z}/m\boldsymbol{Z}$ assuming that $n = l = 1$. Define a covering map $h : B' \to B$ by $z = h(z') = z'^m$ for an appropriate polycylinder B' in $\boldsymbol{C}$. Then, if W' is the normalization of the fibered product $W \times_B B'$, the induced morphism $u' : W' \to W$ turns out to be unramified. Hence, the image $b(K)$ must coincide with $m\boldsymbol{Z}$. Then we have $R \cong \boldsymbol{Z}/m\boldsymbol{Z}$. □

Proof of Proposition 2.2. We define an equivalence relation on $\tilde{X}$ by the following rule: two points $\tilde{x}$ and $\tilde{x}'$ of $\tilde{X}$ are said to be *equivalent* if they belong to one and the same connected component of a fiber of $fu : \tilde{X} \to Y$. Let $\tilde{Y}$ be the quotient space of $\tilde{X}$ by this equivalent relation. Denote by $\tilde{f} : \tilde{X} \to \tilde{Y}$ the quotient map and we put the quotient topology on $\tilde{Y}$. As usual, make $\tilde{Y}$ a local ringed space $(\tilde{Y}, O_{\tilde{Y}})$, starting from the presheaf $U \to O_{\tilde{Y}}(U) = \{$continuous function ϕ on U such that $\tilde{f}^*\phi$ is holomorphic on $\tilde{f}^{-1}(U)\}$, where U are open subsets of $\tilde{Y}$ (cf. [**8**, p. 74 1.26]). Then $\tilde{f}$ is a morphism of local ringed spaces. Further, it is easy to see that we also have a natural morphism of local ringed spaces $v : \tilde{Y} \to Y$ such that $fu = v\tilde{f}$ and that there exists a naturally induced effective action on $\tilde{Y}$ of the Galois group G of u such that $\tilde{Y}/G \cong Y$ as a local ringed spaces.

We shall show that $\tilde{Y}$ has a natural structure of a normal complex space compatible with the given local ringed space structure. In fact, then $\tilde{f}: \tilde{X} \to \tilde{Y}$ and $v: \tilde{Y} \to Y$ above would automatically be holomorphic maps of complex spaces and $\tilde{Y}/G \cong Y$ in the complex category.

For any point y of Y take a sufficiently small coordinate neighborhood $B = \{(z_1, \dots, z_n); |z_i| < \varepsilon\}$ of y in Y. If $y \in D$, we assume that $z_1 \cdots z_l = 0$ is a local equation of D in B for some l with $1 \le l \le n$. Set $W = f^{-1}(B)$. Then applying Lemma 2.3 to the induced morphism $f|W : W \to B$, we have the following commutative diagram of exact sequences (of non-commutative groups)

$$\begin{array}{ccccccc} \pi_1(F) & \xrightarrow{c} & \pi_1(W) & \longrightarrow & R & \longrightarrow & 1 \\ \| & & \big\downarrow a & & \big\downarrow b & & \\ \pi_1(F) & \xrightarrow{d} & \pi_1(X) & \longrightarrow & Q & \longrightarrow & 1 \end{array} \tag{4}$$

where a is induced by the inclusion $W \hookrightarrow X$, b is induced by a, c (resp. d) is induced by the inclusion $F \hookrightarrow W$ (resp. $F \hookrightarrow X$), and R is a finite (abelian) group. This implies that the restriction of u to each connected component $\tilde{W}_i$, $i \in I$, of $u^{-1}(W)$ is a finite unramified (abelian) covering. Hence the composite map $fu \,|\, \tilde{W}_i : \tilde{W}_i \to B$ is proper and we obtain the associated Stein factorization $(\tilde{f}_i : \tilde{W}_i \to \tilde{B}_i, v_i : \tilde{B}_i \to B)$, and hence a commutative diagram

$$\begin{array}{ccc} \tilde{W}_i & \xrightarrow{u_i} & W \\ \tilde{f}_i \big\downarrow & & \big\downarrow f|W \\ \tilde{B}_i & \xrightarrow{v_i} & B \end{array} \tag{5}$$

where $v_i : \tilde{B}_i \to B$ is a finite abelian covering whose branch locus in B is contained inside $D \cap B$ and the fibers of $\tilde{f}_i$ are connected.

This implies that the restriction of the quotient map $\tilde{f}$ to $\tilde{W}_i$ is naturally identified with $\tilde{f}_i$ and $\tilde{B}_i$ with an open subset of $\tilde{Y}$ as a local ringed space. Since every point of $\tilde{Y}$ is covered by an open subset of the form $\tilde{B}_i$ as above, this shows that $\tilde{Y}$ is naturally a normal complex space. (Note that since $\tilde{B}_i$ is normal, a holomorphic function on $\tilde{B}_i$ is identified with a continuous function which becomes holomorphic when pulled back to $\tilde{W}_i$.) Moreover, from the above description of v_i the branch locus of v is contained in D. Finally, the last assertion is clear from the obvious fact that the image of $\pi_1(F)$ in G is trivial. □

We call the diagram (1) in Proposition 2.2 the *canonical diagram associated to f and G*; in particular when $G = Q$, it is simply called the *canonical diagram associated to f*.

We shall show the finiteness of Q in a special case.

Proposition 2.4. *Suppose that* $\dim Y = 1$ *and that there exists a simply connected rational curve* C *in* X *which is mapped surjectively onto* Y. *Then* Q *is finite and isomorphic to a subgroup of the special orthogonal group* $SO(3)$. *The order of* Q *is a divisor of the intersection number* $C \cdot F$ *on* X.

Proof. Since C is simply connected, in the canonical diagram (1) associated to f, $u^{-1}(C)$ is a disjoint union of rational curves each of which is mapped isomorphically onto C by u. Moreover, each of them is mapped surjectively onto $\tilde{Y}$ by $\tilde{f}$. Hence, $\tilde{Y}$ is compact, and in fact is a nonsingular rational curve. In particular, v is a finite Galois covering with Galois group Q, which is thus isomorphic to a finite subgroup of $SO(3)$. Let d be the order of Q. Then the final assertion follows from the relation $d(C \cdot F) = d\tilde{C} \cdot d\tilde{F} = d^2(\tilde{C} \cdot \tilde{F})$, where $\tilde{C}$ (resp. $\tilde{F}$) is a connected component of $u^{-1}(C)$ (resp. $u^{-1}(F)$). □

Retaining the assumption that $\dim Y = 1$ we also note a relation of the ramification points of v and the multiple fibers of f. First of all, let $D = \{y_1, \dots, y_k\}$ be the subset of Y of branch points of v, and n_i the ramification index at y_i. On the other hand, let $N = \{p_1, \dots, p_s\}$ be the set of points of Y corresponding to multiple fibers of f, and m_j the multiplicity of $F_{p_j} := f^{-1}(p_j)$. (If m_j are the multiplicities of the irreducible components F_j, $1 \le j \le k$, of $F_y := f^{-1}(y)$, $m := \gcd(m_1, \dots, m_k)$ is by definition the *multiplicity* of F_y. When $m > 1$, F_y is called a *multiple fiber*.) Then we get:

Proposition 2.5. *Under the assumption of the previous proposition* D *is a subset of* N *and we have the bound* $0 \le s \le 3$ *for* s; *here if* $s = 0$ *or* $1, Q = \{e\}$, *if* $s = 2$, Q *is a cyclic group of order* $\gcd(m_1, m_2)$, *and if* $s = 3$, *then* $D = N$ *and we have* $n_j = m_i$ *if* $y_j = p_i$.

Proof. Since v is a Galois covering between nonsingular rational curves, the number of branch points k do not exceed 3. For any $y_j \in D$ consider the diagram (4) for a small neighborhood B of y_j. By the last assertion of Lemma 2.3 we have $R \cong \boldsymbol{Z}/m_j'\boldsymbol{Z}$ with m_j' the multiplicity of F_{y_j}. Hence, in the diagram (5) v_i is a cyclic ramified covering of degree n_j with $n_j | m_j'$, where in general $n|m$ means that n divides m. Therefore we have $m_j' > 1$ and $y_j \in N$; the inclusion $D \subseteq N$ is verified. After suitable renumbering of $\{p_i\}$ we may assume that $y_j = p_j$ for $1 \le j \le k$ so that $m_j = m_j'$.

Suppose now that $s \ge 3$. Then by the solution of Fenchel conjecture (cf. [**9**]) there exists a finite ramified Galois covering $v' : Y' \to Y$ which is ramified at p_i with ramification index m_i and otherwise unramified. Let X' be the normalization of the fibered product $X \times_Y Y'$. Then the induced map $X' \to X$ is unramified and the general fiber of $X' \to Y'$ is mapped isomorphically onto a general fiber of f. Thus by the definition of Q $u : \tilde{X} \to X$ factors through X';

$$\tilde{X} \to X' \to X.$$

Correspondingly, $v : \tilde{Y} \to Y$ factors through Y';

$$\tilde{Y} \to Y' \xrightarrow{v'} Y.$$

In particular, Y' also is a nonsingular rational curve. Combined with the relation $n_j | m_j$ obtained above we conclude that $m_j = n_j$ and that $\tilde{Y} \to Y'$ is isomorphic; hence $k = s$ and $D = N$. The rest of the assertions are immediate to see. $\square$

3. L^2-method and covering maps.

In order to obtain the finiteness of Q we need an L^2-method by Demailly [**6**] which will be applied in a situation similar to [**5**] and [**24**].

Lemma 3.1. *Let Y be a connected projective algebraic manifold. Let $v : \tilde{Y} \to Y$ be a (possibly ramified) Galois covering of Y with Galois group G, where $\tilde{Y}$ is an irreducible normal complex space. Then there exists a G-equivariant resolution $r : \hat{Y} \to \tilde{Y}$ of the singularities of $\tilde{Y}$ such that on $\hat{Y}$ we have a G-invariant complete Kähler metric $\hat{g}$ and a G-invariant hermitian holomorphic line bundle $(\hat{L}, \hat{h})$ whose chern form is positive.*

If G is finite, then for a suitable choice of r, $vr : \hat{Y} \to Y$ becomes a projective morphism with a G-invariant vr-ample line bundle F on $\hat{Y}$; in this case the assertion is well-known; in fact, the line bundle $\hat{L}_N := (rv)^* L^N \otimes F$ for a sufficiently large N admits a hermitian metric with positive chern form. The point here is simply that even if G is infinite, because of the compactness modulo G of $\hat{Y}$ together with the G-invariance of L and F, the same argument still works. So we shall give partly only a rough outline.

Proof. Take any holomorphic hermitian line bundle (L, h) on Y whose associated chern form $c_1(L, h) := -\sqrt{-1}/2\pi\partial\bar{\partial}\log h$ is positive. First, we shall construct on $\tilde{L} := v^* L$ a hermitian metric of the form $\tilde{h}e^{-\phi}, \tilde{h} = \hat{v}^* h$, on $\tilde{Y}$ with positive chern form where ϕ is some C^∞ function on $\tilde{Y}$. Take a finite open covering $\mathscr{V} = \{V_i\}$ of Y with each V_i sufficiently small so that

a) if $V_{i\mu}, \mu \in M_i$, are the connected components of $v^{-1}(V_i)$, the induced maps $V_{i\mu} \to V_i$ are mutually isomorphic finite Galois coverings whose Galois group $G_{i\mu}$ is a finite subgroup of G which are conjugate to each other, and

b) on each $V_{i\mu}$ there exists a $G_{i\mu}$-invariant C^∞ strictly plurisubharmonic function $\psi_{i\mu}$ which are mapped to each other by any isomorphism $V_{i\mu} \cong V_{i\mu'}$ over V_i. Take a C^∞ partition of unity $\{\rho_i\}$ subordinate to the covering $\mathscr{V}$, and set $\tilde{\rho}_i = v^*\rho_i$. Define a G-invariant C^∞ function ψ on $\tilde{Y}$ by $\psi = \sum_i \tilde{\rho}_i \psi_i$. Clearly,

$\sqrt{-1}\partial\bar{\partial}\psi$ is positive definite on the Zariski tangent spaces of each (0-dimensional) fiber of v.

Then by the G-invariance of ψ we see immediately that

$$\tilde{\omega} := c_1(\tilde{L}, \tilde{h}) + \varepsilon \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\psi$$

is positive if ε is a sufficiently small positive number. Then $\tilde{h}e^{-\phi}$ with $\phi = \varepsilon\psi$ is the desired hermitian metric on $\tilde{L}$.

From now on we denote this new metric again by $\tilde{h}$ so that the associated chern form $c_1(\tilde{L}, \tilde{h})$ is the Kähler form $\tilde{\omega}$.

From the definition of $\tilde{\omega}$ it follows readily that, if we take ε smaller if necessary, the length of a path on $\tilde{Y}$ with respect to the Kähler metric $\tilde{g}$ associated to $\tilde{\omega}$ is greater or equal to that of its image in Y with respect to the Kähler metric associated to $\omega := c_1(L, h)$. This shows that $(\tilde{Y}, \tilde{g})$ is complete.

Now we take a resolution $r : \hat{Y} \to \tilde{Y}$ of the singularities of $\tilde{Y}$ according to Hironaka [**2**, Theorem III], which is necessarily G-invariant and is locally a finite succession of monoidal transformations with nonsingular centers. By the G-invariance, however, it is in fact a finite succession of monoidal transformations. Then a suitable tensor product F of the holomorphic line bundles corresponding to the ideal sheaves of exceptional divisors of r is ample on each fiber of r, or more precisely r-ample over $\tilde{Y}$. Furthermore, the G-action on $\hat{Y}$ naturally lifts to F. Then we can find a G-invariant hermitian metric k on F whose chern form $c_1(F, k)$ is positive on (the Zariski tangent spaces of) each fiber of r by using a C^∞ partition of unity subordinate to a suitable open covering of $\tilde{Y}$.

Let $\hat{L} := r^*\tilde{L}$. Then by the G-invariance of k we see that for a sufficiently large n the hermitian metric $\hat{h}_n := \tilde{h}^n \otimes k$ on $\hat{L}_n := \hat{L}^n \otimes F$ has the positive chern form $\hat{\omega}_n$. Thus, $(\hat{L}_n, \hat{h}_n)$ for such an n is a desired hermitian line bundle, and $\hat{\omega}_n$ is then a desired Kähler form. Indeed, by our construction we conclude that the length of a C^∞ path in $\hat{Y}$ with respect to the associated Kähler metric $\hat{g}_n$ is greater or equal to that of its image in $\tilde{Y}$ with respect to the Kähler metric associated to $\tilde{\omega}$. Hence $\hat{g}_n$ is also complete. □

Using the L^2 theory of Hörmander-Demailly (cf. [**6**]) in the same way as in Campana [**5**, A.B.1] or Napier-Ramachandran [**24**] we shall prove the following:

Lemma 3.2. *Let Y and $v : \tilde{Y} \to Y$ be as in the previous lemma. We consider the resolution $r : \hat{Y} \to \tilde{Y}$, the complete Kähler metric $\hat{g}$, and the positive line bundle $(\hat{L}, \hat{h})$ on $\hat{Y}$ obtained in that lemma. Then, if v is an infinite covering, for all sufficiently large integer N, $H^0(\hat{Y}, K_{\hat{Y}} \otimes \hat{L}^N)$ is infinite dimensional, where $K_{\hat{Y}}$ is the canonical bundle of $\hat{Y}$.*

Proof. Let U be any open subset of Y over which $\hat{v} := vr : \hat{Y} \to Y$ is an unramified Galois covering. Let G be the corresponding Galois group. Because of the G-invariance, the restrictions $(\hat{L}, \hat{h}) \mid \hat{U}$ and $\hat{g}|\hat{U}$ descend to a hermitian line bundle $(\bar{L}, \bar{h})$ and a Kähler metric $\bar{g}$ respectively on U, where $\hat{U} = \hat{v}^{-1}(U)$. Then take any point y of U, and for a positive integer N fix a unit vector α in the fiber $(K_Y \otimes \bar{L}^N)_y$ with respect to the metric induced by $\bar{g}$ and $\bar{h}^N$. Consider the set $\hat{v}^{-1}(y) = \{\hat{y}_g\}_{g \in G}$ parametrized by G, and let α_g be the unit vector of $(K_{\hat{Y}} \otimes \hat{L}^N)_{\hat{y}_g}$ induced by α. Denote by $l^2(G)$ the Hilbert space of square-summable sequences of complex numbers parametrized by G. Then for the lemma we have only to prove the following

Claim. *If we take N sufficiently large, for any element $(a_g)_{g \in G} \in l^2(G)$ there exists an element s of $H^0(\hat{Y}, K_{\hat{Y}} \otimes \hat{L}^N)$ which restricts to $a_g \alpha_g$ at $\hat{y}_g$.*

Proof. Take a coordinate neighborhood W of y in Y contained in U such that $\hat{W} := \hat{v}^{-1}(W)$ is a disjoint union of neighborhoods $\hat{W}_g$ of $\hat{y}_g$, each mapped isomorphically onto W. Then, choose a non-negative C^∞ function ρ on Y whose support is contained in W and which is identically equal to 1 in a neighborhood of y. For the coordinates $z_1, \dots, z_n$ on W consider the function $\varphi := \rho \log(\sum_{i=1}^n |z_i|^2)$ as a function defined on the whole Y with singularity at y and with support in W, where $n = \dim X$. Set $\hat{\varphi} := \hat{v}^* \varphi$. Then take N so large that

$$N\omega - \frac{\sqrt{-1}}{2\pi} n \partial\bar{\partial}\varphi \geq \omega$$

as a current on Y. In particular, the singular hermitian metric $\hat{h}'_N := \hat{h}^N e^{-n\hat{\varphi}}$ has a positive chern form.

Now choose any holomorphic section s of $(K_Y \otimes \bar{L}^N)_y$ on W with $s(y) = \alpha$, which induces by pulling back a holomorphic section $\hat{s}_g$ of $K_{\hat{Y}} \otimes \hat{L}^N$ on each $\hat{W}_g$, giving α_g at $\hat{y}_g$. Define a holomorphic section $\hat{s}$ on $\hat{W}$ by $\hat{s}|\hat{W}_g = a_g \hat{s}_g$ for $g \in G$. Then $\sigma := \hat{\rho}\hat{s}$ defines a smooth section of $K_{\hat{Y}} \otimes \hat{L}^N$ on $\hat{W}$ giving $a_g \hat{s}_g$ at each y_g. We may extend this by zero to a C^∞ section on the whole $\hat{Y}$ with support contained in $\hat{W}$. Set $\theta := \bar{\partial}\sigma$. Then θ is an $\hat{L}^N$-valued $(n,1)$-form on $\hat{Y}$ with finite L^2 norm with respect to the singular metric $\hat{h}'_N$ and the Kähler metric $\hat{g}$. Then by Demailly [**6**, Theorem 5.1], there exists an $\hat{L}^N$-valued n-form η on $\hat{Y}$ with finite L^2 norm such that $\bar{\partial}\eta = \theta$. Thus, we get an $\hat{L}^N$-valued holomorphic n-form $\beta := \sigma - \eta$ which gives on each fiber $(K_Y \otimes \hat{L}^N)_{y_g}$ over y_g the element $a_g \alpha_g$. (Note that $\eta(y_g) = 0$ by the L^2-condition.) Thus the claim, and hence the lemma also, is proved. □

4. Algebraic reduction and fundamental group.

Let Z be a compact connected complex manifold. We first recall the notion of algebraic reduction. An *algebraic reduction* of Z is a diagram

$$\begin{array}{ccc} \hat{Z} & \xrightarrow{\mu} & Z \\ {\scriptstyle f}\downarrow & & \\ Y & & \end{array} \tag{6}$$

of compact complex manifolds with the following properties:

1) μ is a bimeromorphic morphism,
2) Y is a projective algebraic manifold,
3) f is a surjective morphism with connected fibers, and
4) we have natural isomorphisms of meromorphic function fields

$$\boldsymbol{C}(Z) \xrightarrow{\mu^*} \boldsymbol{C}(\hat{Z}) \xleftarrow{f^*} \boldsymbol{C}(Y).$$

The *algebraic dimension* $a(Z)$ of Z is by definition the transcendence degree over $\boldsymbol{C}$ of the algebraic function field $\boldsymbol{C}(Z)$, and thus coincides with the dimension of Y. The diagram (6) is up to bimeromorphic equivalence determined uniquely by Z. Let F be a general fiber of f, which is a compact connected complex manifold. Then the fundamental group of F is independent of the choice of algebraic reductions, as the fundamental group is in general a bimeromorphic invariant of a complex manifold. Thus the natural image N of $\pi_1(F)$ in $\pi_1(Z)$ is a normal subgroup (cf. Lemma 2.1). The group N, and the corresponding quotient group $Q := \pi_1(Z)/N$, also is an invariant of the complex manifold Z itself. We call Q the *algebraic reduction* of $\pi_1(Z)$. By definition we have the obvious exact sequence of groups

$$\pi_1(F) \xrightarrow{\beta} \pi_1(Z) \to Q \to 1 \tag{7}$$

where the image of β is N.

Theorem 4.1. *Let Z be a compact connected complex manifold. Let A be an analytic subspace of Z. Suppose that A admits a fundamental system $\{V_n\}_{n=1,2,\dots}$ of neighborhoods V_n of A in Z such that* $\dim H^0(V_n, \mathscr{F}) < \infty$ *for all torsion-free coherent analytic sheaves $\mathscr{F}$ on X. Then the cokernel of the composite map $\pi_1(A) \to \pi_1(Z) \to Q$ is finite. In particular if $\pi_1(A)$ is finite, the algebraic reduction Q of $\pi_1(Z)$ also is finite.*

By [**29**, Cor. 1,ii)] together with Théorème 1 and Proposition 10 of [**1**] a submanifold A with ample normal bundle always admits fundamental system of neighborhoods as above. Thus we obtain the following:

Corollary 4.2. *Suppose that A is a compact complex submanifold with an ample normal bundle in Z. Then the conclusion of the theorem holds true.*

Proof. Take an algebraic reduction (6) of Z in such a way that f is smooth over a Zariski open subset of Y whose complement is a divisor with only normal crossings. Let G be the cokernel of the composite map $\pi_1(A) \to \pi_1(Z) \to Q$. Take an unramified covering $u : \tilde{Z} \to \hat{Z}$ corresponding to the natural quotient map $\pi_1(\hat{Z}) \cong \pi_1(Z) \to G$. By Proposition 2.2 we can find a Stein factorization $(\tilde{f} : \tilde{Z} \to \tilde{Y}, v : \tilde{Y} \to Y)$ of fu, where v is a Galois covering with Galois group G. It suffices to show that v is a finite covering.

Suppose on the contrary that v is an infinite covering. By Lemmas 3.1 and 3.2 there exist a G-equivariant resolution $r : \hat{Y} \to \tilde{Y}$ of $\tilde{Y}$, a holomorphic line bundle $\hat{L}$, and a positive integer N such that $H^0(\hat{Y}, K_{\hat{Y}} \otimes \hat{L}^N)$ is infinite dimensional. Then, $\mathscr{F} := r_* \mathcal{O}_{\hat{Y}}(K_{\hat{Y}} \otimes \hat{L}^N)$ is a torsion-free coherent analytic sheaf on $\tilde{Y}$ with $H^0(\tilde{Y}, \mathscr{F}) \cong H^0(\hat{Y}, K_{\hat{Y}} \otimes L^N)$. Thus, if we define $\tilde{\mathscr{F}}$ to be $\tilde{f}^* \mathscr{F}$ modulo torsion, $H^0(\tilde{Z}, \tilde{\mathscr{F}})$ also becomes infinite dimensional.

On the other hand, if W is a tubular neighborhood of A in Z, by our definition of u, $u^{-1}(W)$ is a disjoint union of open subsets which are mapped isomorphically onto W. Let $\tilde{W}$ be such an open subset and $\tilde{A} = \tilde{W} \cap u^{-1}(A)$. Then by our assumption there exists a neighborhood $\tilde{V}$ of $\tilde{A}$ in $\tilde{W}$ such that $H^0(\tilde{V}, \tilde{\mathscr{F}})$ is finite dimensional. Since the space $H^0(\tilde{Z}, \tilde{\mathscr{F}})$ injects into $H^0(\tilde{V}, \tilde{\mathscr{F}})$, this is a contradiction. Thus v is a finite covering as desired. □

When Q is finite, the sequence (7) shows that the main part of $\pi_1(Z)$ is the image N of $\pi_1(F)$. We summarize some known facts on the structure of a general fiber F in the following lemma (cf. Ueno [**33**, 12.5]) in the case where the dimension of Z equals three.

Lemma 4.3. *Suppose that* $\dim Z = 3$. *As above, let F be a general fiber of f in* (6).

1) *If $a(Z) = 2$, any smooth fiber of f is a nonsingular elliptic curve.*

2) *If $a(Z) = 1$, a smooth fiber of f is bimeromorphic* (*or isomorphic*) *to one of the following surfaces*:

a) *a Kähler surface with vanishing real first chern class,*

b) *a ruled surface of genus one,*

c) *a rational surface,*

d) *an elliptic bundle over an elliptic curve with trivial canonical bundle* (*Kodaira surface*),

e) *a surface with $b_1 = 1$* (*surface of class VII*).

Remark. The surface is Kähler in the cases a)–c) and not Kähler in the cases d) and e). The exceptional case of Theorem 1.3 falls under the special case of e) above.

In order to study the structure of N we review the structure of $\pi_1(F)$ in respective cases;

	F is bimeromorphic to:	structure of π_1
1)	abelian surface or hyperelliptic surface	$\boldsymbol{Z}^4 < \pi_1$
	K3 surface or Enriques surface	$\{e\}$ or $\boldsymbol{Z}/2\boldsymbol{Z}$
2)	rational surface	$\{e\}$
3)	ruled surface of genus one	$\boldsymbol{Z}^2$
4)	Kodaira surface	$\boldsymbol{Z} \times H$, $1 \to \boldsymbol{Z} \to H \to \boldsymbol{Z}^2 \to 1$
5)	surface in VII ($\Leftrightarrow b_1 = 1$)	?
	in case $\exists$ a global spherical shell	$\boldsymbol{Z} < \pi_1$

Here, $<$ shall denote a normal subgroup of finite index. In 4) the extension is central and there exists no proper normal subgroup of H of finite index which are mapped to a subgroup of finite index in $\boldsymbol{Z}^2$. On the other hand, F is said to contain a global spherical shell if it contains an open subset W which is isomorphic to the annulus B_ε for some small $\varepsilon > 0$ such that $F - W$ is connected, where $B_\varepsilon = \{(z, w) \in \boldsymbol{C}^2; 1 - \varepsilon < |z|^2 + |w|^2 < 1 + \varepsilon\}$. In this case F is known to be homeomorphic to a blown-up Hopf surface (cf. [**22**]).

5. Application to twistor spaces.

Let M be a compact (connected) oriented smooth 4-manifold and $[g]$ a self-dual conformal structure on M represented by a smooth Riemannian metric g on M (cf. [**3**]). We call $(M, [g])$ *of positive* (resp. *zero*) *type* if the conformal class $[g]$ contains a Riemannian metric of constant positive (resp. zero) scalar curvature s. ($[g]$ contains always such a metric cf. [**30**].)

First we note a topological result needed later.

Lemma 5.1. *Let M be an oriented compact self-dual 4-manifold of positive type.*

1) *If M is simply connected, M is homeomorphic to $m\boldsymbol{CP}^2$, where $m = b_2(M)$.*

2) *If the fundamental group of M is infinite cyclic, M is homeomorphic to $(S^1 \times S^3) \sharp m\boldsymbol{CP}^2$, where $m = b_2(M)$.*

Proof. Since M is of positive type, by Le Brun [**20**] the intersection form on $H^2(M, \boldsymbol{Z})$ is positive definite; then by Donaldson [**7**] the form is diagonalizable. On the other hand, in case 1) (resp. 2) by a theorem of Freedman [**10**] (resp. of Kawauchi [**17**]) the topological type is determined uniquely by the intersection form. Thus M is homeomorphic to $m\boldsymbol{CP}^2$ (resp. $(S^1 \times S^3) \sharp m\boldsymbol{CP}^2$). □

Let Z be the twistor space associated to the self-dual manifold M; it is a compact complex manifold of dimension 3 with a natural smooth fiber bundle structure $t : Z \to M$ with typical fiber the complex projective line $\boldsymbol{P}^1$ and with the following properties:

For each $x \in M$, $L_x := t^{-1}(x)$ is a complex submanifold of Z, called a (real) *twistor line*, with its normal bundle $N_{L_x/Z}$ isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$;

$$N_{L_x/Z} \cong \mathcal{O}(1) \oplus \mathcal{O}(1) \tag{8}$$

where $\mathcal{O}(1)$ is the line bundle of degree one on $L_x \cong \boldsymbol{P}$.

We study the fundamental group $\pi_1(M)$ of M with respect to some base point. Since Z is a $\boldsymbol{P}^1$-bundle over M, we have a natural isomorphism $\pi_1(Z) \cong \pi_1(M)$. Thus we have only to study $\pi_1(Z)$. In view of this and (8), as an immediate consequence of Corollary 4.2 we obtain the following:

Lemma 5.2. *For a twistor space Z as above the algebraic reduction Q of $\pi_1(Z)$ is finite.*

Remark. In case $a(Z) = 1$ we can alternatively use Propositions 2.4 and 2.5 without appealing to the L^2-method by noting that the proper transform of a general twistor line in $\hat{Z}$ is mapped surjectively onto Y in (6) (cf. Lemma 5.5 below).

Next we study the first betti number of Z. For this purpose first we note the following: If G is a subgroup of $\pi_1(M)$ with the corresponding unramified covering $M' \to M$ and if $\hat{G}$ is the corresponding subgroup of $\pi_1(Z)$ with the corresponding unramified covering $Z' \to Z$, or vice versa, then we have a commutative diagram

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "t'"'] & Z \arrow[d, "t"] \\ M' \arrow[r] & M \end{tikzcd} \tag{9}$$

where Z' is identified with the twistor space of M' with twistor fibration t', where the self-dual structure on M' is naturally induced from M.

We also recall that a *profinite completion* of a group G is the group which is obtained as the projective limit $\hat{G} := \lim_{\leftarrow} G/H$, where H runs through all the normal subgroups of finite indices. $\hat{G}$ reduces to the identity if and only if G contains no subgroup of finite index.

Proposition 5.3. *Suppose that the algebraic dimension $a(Z)$ of Z is positive and that M is of positive type. Then for the first betti number of Z we have $b_1(Z) \leq 1$. Moreover, if $b_1(Z) = 0$, then $\hat{\pi}_1(Z) = \{e\}$, where $\hat{\pi}_1$ denotes the profinite completion of π_1 as above.*

Proof. First we show that

$$b_1(Z) \leq b_1(F) \tag{10}$$

where F is a general fiber of f in an algebraic reduction of Z as in (6). By the natural surjection $\pi_1(Z) \to H_1(Z, \boldsymbol{Z})$, N is mapped surjectively onto the image I of the natural homomorphism $H_1(F, \boldsymbol{Z}) \to H_1(Z, \boldsymbol{Z})$. Thus if $b_1(Z) > \operatorname{rank} I$, then Q would not be finite since Q is mapped surjectively onto the abelian group $H_1(Z, \boldsymbol{Z})/I$ of positive rank, which would contradict Lemma 5.2. Thus (10) is proved.

Now let $Z' \to Z$ be any finite unramified covering. Denote its degree by d. Then we have

$$\chi(O_{Z'}) = d\chi(O_Z)$$

where in general $\chi(O_X)$ denotes the arithmetic genus of X. On the other hand, we also have

$$\begin{aligned}
\chi(O_Z) &= \frac{c_1 c_2}{24} \quad \text{(by Riemann-Roch)} \\
&= (\chi - \tau)/2 \quad \text{(cf. Hitchin [}\mathbf{15}, (1.5)\text{])} \\
&= 1 - b_1 + b_- \\
&= 1 - b_1 \quad \text{(cf. Le Brun [}\mathbf{20}\text{])}
\end{aligned}$$

where c_i are chern classes of Z, $b_1 = b_1(Z)$, χ and τ are the topological Euler characteristic and the signature of M respectively, and finally, b_- is the dimension of any maximal subspace of $H^2(M, \boldsymbol{R})$ on which the intersection form is negative definite. Let $(M', [g'])$ be the self-dual manifold corresponding to Z' (cf. (9)). Then it is again of positive type, and the same conclusion also holds true for Z':

$$\chi(O_{Z'}) = 1 - b_1'$$

where $b_1' = b_1(Z')$. Then combining the above equalities we get

$$b_1' - 1 = d(b_1 - 1). \tag{11}$$

Thus, if $b_1 = 0$, we must have $d = 1$. So there exists no non-trivial finite unramified covering of Z and $\hat{\pi}_1(Z) = \{e\}$.

On the other hand, in general we may apply (10) to Z' instead of to Z and obtain $b_1' \leq b_1(F)$, where we note that F is common to both Z and Z'. On the other hand, since $a(Z) = a(Z') > 0$, by Lemma 4.3 and Table 4 we have the bound $b_1(F) \leqq 4$. Therefore from (11) we have

$$4 \geq 1 + d(b_1 - 1). \tag{12}$$

Now if $b_1 \geq 1$, there exists an abelian unramified covering of Z of arbitrarily high degree d. Thus (12) forces b_1 to be either equal to 0 or 1. $\square$

Remark. The result is not true if one only assumes that $(M, [g])$ is of positive type (cf. [**18**]).

Recall that N is the image of the natural homomorphism $\pi_1(F) \to \pi_1(Z)$ (cf. (7)). From the above proposition we deduce immediately the following:

Lemma 5.4. *Suppose that $a(Z) > 0$ as above. Then for any subgroup N' of finite index of N its abelianization $N'/[N', N']$ is at most of rank one.*

Proof. N' is identified with the fundamental group of a suitable finite unramified covering manifold Z' of Z and we can apply Proposition 5.3 to Z'. $\square$

We are now ready to prove Theorem 1.1, the structure theorem when the algebraic dimension of Z equals two.

Proof of Theorem 1.1. By Lemma 5.2 $\pi_1(Z)$ is naturally an extension of the finite group Q by the quotient group N of $\pi_1(F) \cong \boldsymbol{Z}^2$, since F is a nonsingular elliptic curve in this case (cf. [**33**, 12.4]). By Lemma 5.4 the rank of the abelian group N is either zero (Case 1) or one (Case 2). In Case 1 the fundamental group of Z is finite, and hence by Proposition 5.3 Z is simply connected. In Case 2 clearly $\pi_1(Z)$ contains an infinite cyclic group as a normal subgroup of finite index. Let $M' \to M$ be the Galois covering associated to this subgroup of $\pi_1(M) \cong \pi_1(Z)$ so that $\pi_1(M') \cong \boldsymbol{Z}$.

On the other hand, in view of Lemma 1.2 our assumption that $a(Z) = 2$ implies that the self-dual manifold $(M, [g])$ is of positive type (cf. [**25**, 3.5]). Hence, by applying Lemma 5.1 to M we get the topological conclusion of Theorem 1.1. $\square$

Remark. In the situation of Theorem 1.1 the general fiber F of algebraic reduction f is a smooth elliptic curve and the image of the natural map $\pi_1(F) \to \pi_1(Z)$ is an infinite cyclic group with finite cokernel. The map becomes surjective after passing to a suitable finite unramified covering of M.

In passing we also note the structure of Y in the algebraic reduction (6) of Z.

Lemma 5.5. *If $a(Z) = 1$, Y is a* (*nonsingular*) *rational curve and if $a(Z) = 2$, Y is a* (*nonsingular*) *rational surface.*

Proof. Take a general twistor line L on Z. Then there exists a neighborhood W of L in Z such that $\mu \mid \mu^{-1}(W) : \mu^{-1}(W) \to W$ is isomorphic since the fundamental locus of μ^{-1} is of codimension at least two (cf. (6)). Fix a point p on L. Then as follows readily from the standard deformation theory, the union of all the complex twistor lines passing through the point p and contained in W contains an open subset, say V, of W. Identifying W with $\mu^{-1}(W)$ via the above isomorphism, we obtain an analytic family of nonsingular rational curves in $\hat{Z}$

which passes through one point and whose union contains an open subset of $\hat{Z}$. Taking the images of the members of this family by f we see that on Y we get an analytic family of rational curves with similar properties. This implies that Y is a rational variety. □

Finally we treat the case where the algebraic dimension of Z equals one and prove Theorem 1.3. So in what follows we assume that $a(Z) = 1$ and consider the algebraic reduction (6). We call Z *exceptional* if a general fiber F of $f : \hat{Z} \to Y$ is a surface of class VII which does not contain any global spherical shell and whose minimal model has the positive betti number $b_2 > 0$.

Recall the exact sequence (7):

$$\pi_1(F) \xrightarrow{\beta} \pi_1(Z) \to Q \to 1.$$

On the structure of the image of β we note the following:

Lemma 5.6. *Let B be any quotient group of $\pi_1(F)$ such that for any subgroup B' of B of finite index its abelianization $B'/[B', B']$ is at most of rank one. Unless Z is exceptional, B is either finite or contains an infinite cyclic group as a normal subgroup of finite index.*

Proof. This in fact can be checked case by case to be true according to Table 4. (The verification is immediate except possibly for the case 4.) □

Lemma 5.7. *Suppose that Z is not exceptional.*

1) *If $b_1(Z) = 0$, then Z is simply connected.*

2) *If $b_1(Z) = 1$, then $\pi_1(Z)$ contains an infinite cyclic group as a normal subgroup of finite index.*

Proof. 1) By Proposition 5.3 $\hat{\pi}_1 = \{e\}$. So in the exact sequence

$$1 \to N \to \pi_1(Z) \to Q \to 1 \tag{13}$$

Q, being finite by Lemma 5.2, reduces to the identity and hence the profinite completion $\hat{N}$ of N also reduces to the identity.

From Table 4 it follows easily that any nontrivial quotient of $\pi_1(F)$ always admits a subgroup of finite index > 1 if Z is not exceptional. Thus, N must reduce to the identity, and hence $\pi_1(Z) = \{e\}$.

2) In view of the sequence (13) and Lemma 5.2, after passing to a finite unramified covering we may assume that

$$N \cong \pi_1(Z).$$

Then the lemma follows immediately from Lemma 5.6. □

Proof of Theorem 1.3. By Proposition 5.3 $b_1 = b_1(Z) \leqq 1$. Suppose that Z is not exceptional. Then if $b_1 = 0$, Z is simply connected by Lemma 5.7. If $b_1 = 1$, $\pi_1(Z)$ contains an infinite cyclic group as a normal subgroup of finite index by the same lemma. Thus we may apply Lemma 5.1 to obtain the topological result of the theorem in the non-exceptional case.

Finally, we consider the case $b_2(M) = 0$. Since $(M, [g])$ is of positive type, this implies that $(M, [g])$ is conformally flat. Then we use Theorem 1.4 to conclude that in our case of $a(Z) = 1$ and $b_2(M) = 0$ M is conformal to a finite quotient of a Hopf surface, and hence, falls under the class 2) of the theorem. (The proof of Theorem 1.4 is of course independent.) □

Remark. In case $a(Z) = 1$, certain restrictions on the structure of Q is read off from Propositions 2.5 and 2.4. For instance if the real structure σ on Y induced from the canonical one on Z has a fixed point, the chern class $c_1(F)$ of the bundle $[F]$ is of the form $(k/2)c_1(Z)$ for some positive integer k. Then the intersection number is: $L \cdot F = (k/2)L \cdot F = k$. Thus from Proposition 2.5 we get that the order of Q is a divisor of k.

6. Classification in the conformally flat case.

The puropose of this section is to prove Theorem 1.4 of Section 1. We follow the idea of Pontecorvo [**27**] and in fact reduce the proof to his result. Recall a general fact that if M admits a conformally flat metric, then the second betti number b_2 of M vanishes.

First we show the existence of an elementary divisor on Z, i.e., a divisor D on Z with $L \cdot D = 1$ for any twistor line L, after possibly passing to a finite unramified covering of M.

Since the metric is conformally flat, we have the developing map $d : \tilde{M} \to S^4$ from the universal covering $\tilde{M}$ of M to S^4. Correspondingly, we obtain the holomorphic developing mapping $h : \tilde{Z} \to \boldsymbol{P}^3$ from the universal covering $\tilde{Z}$ of Z, which is identified with the twistor space of the induced conformally flat manifold $(\tilde{M}, [\tilde{g}])$. Here, on Z and $\tilde{Z}$ we have the natural flat $PGL(4, \boldsymbol{C})$-structure inherited from the conformal structure of the bases M and $\tilde{M}$ respectively, and h is considered to be the developing map associated to this structure. We have also the associated monodromy representation $\rho : \pi_1(Z) \to PGL(4, \boldsymbol{C})$ such that

$$h(\gamma\tilde{z}) = \rho(\gamma)h(\tilde{z}), \quad \gamma \in \pi_1(Z), \, \tilde{z} \in \tilde{Z}$$

with respect to the natural action of $\pi_1(M)$ on $\tilde{Z}$ as the covering transformation group of $\tilde{Z} \to Z$.

Then by virtue of Lemma 5.9 of Ma. Kato [**16**], under our assumption that there exists a nonconstant meromorphic function on Z, we can find a subgroup, say G, of finite index in $\pi_1(M)$ which leaves invariant a hyperplane H of $\boldsymbol{P}^3$ with respect to the representation ρ. (Note that the assumption of Kato's lemma is that of [**16**, Theorem 5.2]).

We now set $\tilde{S} = h^{-1}(H)$, which is a $\pi_1(M)$-invariant smooth hypersurface in $\tilde{Z}$. In fact, if $\tilde{S}$ is empty, then $\tilde{Z}$ is mapped locally biholomorphically onto a domain contained in $\boldsymbol{P}^3 - H \cong \boldsymbol{C}^3$, which is impossible since $\tilde{Z}$ contains twistor lines which are compact curves. The image S of $\tilde{S}$ in $Z' := \tilde{Z}/G$ is then a closed smooth complex surface in Z'.

We show that S is an elementary divisor in Z' which is considered as the twistor space of the finite unramified covering $M' = \tilde{M}/G$ of M. Let L be a twistor line in Z'. There exists a neighborhood U of L which is evenly covered by the natural projection $u : \tilde{Z} \to Z'$. Let $V \subseteq \tilde{Z}$ be a connected component of $u^{-1}(U)$ and $\tilde{L} \subseteq V$ the inverse image of L in V. Since we have $u^{-1}(S) = \tilde{S}$ and $V \cap \tilde{S} = (u|V)^{-1}(S)$, it suffices to show that $\tilde{S} \cdot \tilde{L} = 1$ in the sense that $\tilde{S}$ and $\tilde{L}$ intersect transversally at exactly one point. (Note that $\tilde{S}$ is in general open.) We know that $h|V : V \to \boldsymbol{P}^3$ is an embedding ([**16**, Lemma 3.1]); hence $\tilde{S} \cdot \tilde{L} = H \cdot l = 1$ as desired, where l is the line of $\boldsymbol{P}^3$, which is the image of $\tilde{L}$ by h (and which is also a twistor line of $\boldsymbol{P}^3$ considered as a twistor space of S^4). Thus we have obtained an elementary divisor S on Z'. Suppose first that S contains a twistor line $L \cong \boldsymbol{P}^1$. In the normal bundle exact sequence

$$0 \to N_{L/S} \to N_{L/Z'} \to N_{S/Z'} \,|\, L \to 0, \tag{14}$$

associated to the inclusions $L \subseteq S \subseteq Z$ we have $N_{L/Z'} \cong O(1)^2$. Moreover from the equality of chern classes $c_1([S]) = (1/4)c_1(Z')$ as $c_1(H) = (1/4)c_1(\boldsymbol{P}^3)$ together with the adjunction formula we get that $N_{S/Z'} \,|\, L \cong O(1)$. Then it follows from (14) that $N_{L/S} \cong O(1)$, which, as in [**28**], implies that S is a rational surface. In particular S is simply connected, and then M' also is simply connected so that $M' = M$ and the developping map $Z = Z' \to \boldsymbol{P}^3$ is isomorphic. In particular, Z', and Z also, is Moishezon and thus M must be homeomorphic to S^4 by [**4**] or [**28**]. Then it is immediate to see that $M = M'$ (with its induced conformally flat structure) is isomorphic to S^4 with standard conformal structure; thus we are in the first case.

On the other hand, if S contains no twistor line, then the twistor fibration $t : Z \to M$ gives an orientation-reversing diffeomorphism $S \cong M'$ and gives a complex structure on M', which is easily seen to define, together with the given conformal structure $[g']$ on M', a (conformally flat) hermitian structure. Thus we may assume from the beginning that $(M', [g'])$ is a conformally flat hermitian surface with the orientation reversed. In this case the classification is already known by Pontecorvo [**27**]. Under our assumption his result yields that M' is one of the following: i) either a complex torus or a hyperelliptic surface with flat structure ii) Hopf surface with standard conformal flat metric. These cases corresponds to the cases b) and c) of our theorem respectively. □

Remark. 1) Suppose that either $a(Z) \geq 2$ or $a(Z) = 1$ and the general fiber of the algebraic reduction of Z is not a compact surface of class VII which does not contain a global spherical shell. Then by Theorems 1.1, 1.3 and Lemma 1.2 above $\pi_1(Z)$ contains an abelian subgroup, say G, as a normal subgroup of finite index. Then the existence of a G-invariant hyperplane follows at once without using the above result of Ma. Kato [**16**], whose proof is done by case-by-case checking.

In fact, if the above condition of existence of an abelian normal subgroup of finite index is satisfied, the conclusion of the theorem follows by the same argument; the argument can also be used to give a complex analytic proof of the classical result of Kuiper [**19**] which classifies conformally flat manifolds with abelian fundamental groups in any dimension.

2) A non-hermitian example in the class c) is given as follows. Consider the fixed-point-free anti-holomorphic involutive isometry σ of the Hopf surface M with $h(z_1, z_2) = (az_1, \bar{a}z_2)$ induced by the corresponding isometry of $\boldsymbol{C}^2 - \{(0,0)\}$ defined by $(z, w) \to (\bar{w}, -\bar{z})$. Then the quotient of M by σ gives a desired surface.

3) When the conformal class is of positive or of zero type, by a theorem of Schoen-Yau [**31**] $h : \tilde{Z} \to \boldsymbol{P}^3$ is always an embedding; so in this case we need not use Lemma 3.1 of [**16**].

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