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On harmonic Hardy and Bergman spaces

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Abstract. In this paper we use an identity of Hardy-Stein type in investigations of the harmonic $\mathcal{H}^p(B)$ and Bergman $b^p(B)$ spaces.

1. Introduction and auxiliary results.

Throughout this paper n is an integer greater than 1, $B(a,r) = \{x \in \boldsymbol{R}^n \mid |x-a| < r\}$ denotes the open ball centered at a of radius r, where $|x|$ denotes the norm of $x \in \boldsymbol{R}^n$ and B is the open unit ball in $\boldsymbol{R}^n$. $S = \partial B = \{x \in \boldsymbol{R}^n \mid |x| = 1\}$ is the boundary of B.

Let dV denote the Lebesgue measure on $\boldsymbol{R}^n$, $d\sigma$ the surface measure on S, σ_n the surface area of a S, dV_N the normalized Lebesgue measure on B, $d\sigma_N$ the normalized surface measure on S.

Let $\mathcal{H}(B)$ denote the set of complex-valued harmonic functions on B, $\mathcal{H}^p(B)$ denote the set of harmonic functions on B such that:

$$\|u\|_{\mathcal{H}^p(B)} = \sup_{0<r<1}\left(\int_S |u(r\zeta)|^p \, d\sigma_N(\zeta)\right)^{1/p} < +\infty$$

and let $b^p(B)$ denote the Bergman space i.e., the set of harmonic functions u on B such that:

$$\|u\|_p = \left(\int_B |u(x)|^p \, dV(x)\right)^{1/p} < +\infty.$$

A function $f \in C^1(B)$ is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} = \sup_{x \in B}(1-|x|)|\nabla f(x)| < +\infty$$

where $|\nabla f(x)| = (\sum_1^n |\partial f(x)/\partial x_i|^2)^{1/2}$. The space of Bloch functions is denoted by $\mathcal{B}(B)$.

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Let $p > 0$. A Borel function f, locally integrable on B, is said to be a $BMO_p(B)$ function if

$$\|f\|_{BMO_p} = \sup_{B(a,r)\subset B}\left(\frac{1}{V(B(a,r))}\int_{B(a,r)}|f(x) - f_{B(a,r)}|^p\,dV(x)\right)^{1/p} < +\infty$$

where the supremum is taken over all balls $B(a,r)$ in B, and $f_{B(a,r)}$ is the mean value of f over $B(a,r)$.

In [**6**] for $p \geq 1$, Muramoto proved that $\mathscr{B}(B)\cap\mathscr{H}(B)$ is isomorphic to $BMO_p\cap\mathscr{H}(B)$ as Banach spaces. That paper inspired us to calculate exactly BMO norm which is the theme of [**8**]. We proved the following:

Theorem A. *Let* $u \in \mathscr{H}(B)$, $p > 1$. *Then*

a)

$$\|u\|^p_{BMO_p} = \sup_{a\in B,0<r<1-|a|}\frac{p(p-1)}{2n(n-2)}$$
$$\times\left(\int_B |u_{a,r}(x) - u_{a,r}(0)|^{p-2}|\nabla u_{a,r}(x)|^2(2|x|^{2-n} + (n-2)|x|^2 - n)\,dV_N(x)\right)$$

for $n \geq 3$

b)

$$\|u\|^p_{BMO_p} = \sup_{a\in B,0<r<1-|a|} p(p-1)$$
$$\times\left(\int_B |u_{a,r}(x) - u_{a,r}(0)|^{p-2}|\nabla u_{a,r}(x)|^2(\ln(1/|x|) - 1 + |x|)\,dV_N(x)\right)$$

for $n = 2$.

In the proof of this theorem we essentially also proved a generalization of Hardy-Stein identity, [**2**]. This identity is included in the following lemma.

Lemma 1. *Let* $1 < p < +\infty$, $u \in \mathscr{H}(B)$, *then*

$$\int_S |u(r\zeta)|^p\,d\sigma_N(\zeta)$$
$$= |u(0)|^p + \frac{p(p-1)}{n(n-2)}\int_{rB}|u(x)|^{p-2}|\nabla u(x)|^2(|x|^{2-n} - r^{2-n})\,dV_N(x), \quad n \geq 3. \quad (1)$$

It is interesting that Lemma 1 is a consequence of the Riesz decomposition theorem, see, for example [**4**], of the subharmonic function $|u|^p$ on rB into a difference between a harmonic function and a potential.

Corollary 1. *Let* $1 < p < +\infty$, $u \in \mathscr{H}(B)$, $r \in (0,1)$ *then*

$$\frac{d}{dr}\int_S |u(r\zeta)|^p \, d\sigma_N(\zeta) = \frac{p(p-1)}{n} r^{1-n} \int_{rB} |u(x)|^{p-2} |\nabla u(x)|^2 \, dV_N(x). \tag{2}$$

Note that this identity is of Hardy-Stein type.

This lemma will be the main tool in this paper. We will use this lemma in investigations of spaces $\mathscr{H}^p(B)$ and $b^p(B)$. Similar identity exists in the case of holomorphic functions in [**9**], but the author does not use it. We shall keep our attention to the case $n \geq 3$. Analogous results hold in the case $n = 2$. Formulations and proofs of this results we leave to the reader.

We will need another lemma in our consideration.

Lemma 2. *Let* $I(r)$ *be a nonnegative nondecreasing function on the interval* $[0,1)$, $n \in \mathbf{N}$, $M > 0$, *and let*

$$r^n I(r) \leq n \int_0^r I(\rho)\rho^{n-1} \, d\rho + M, \tag{3}$$

then for each $\varepsilon \in (0,r)$ *the following inequality*

$$I(r) \leq \frac{M}{\varepsilon^n} + I(\varepsilon)$$

holds.

Proof. From (3) we have

$$\frac{nt^{n-1}I(t)}{(M + n\int_0^t I(\rho)\rho^{n-1}\,d\rho)} \leq n/t. \tag{4}$$

Integrating (4) from ε to r in the variable t, we obtain

$$\ln\frac{(M + n\int_0^r I(\rho)\rho^{n-1}\,d\rho)}{(M + n\int_0^\varepsilon I(\rho)\rho^{n-1}\,d\rho)} \leq n \ln\frac{r}{\varepsilon}$$

i.e.

$$M + n\int_0^r I(\rho)\rho^{n-1}\,d\rho \leq \left(M + n\int_0^\varepsilon I(\rho)\rho^{n-1}\,d\rho\right)\left(\frac{r}{\varepsilon}\right)^n.$$

Since $I(r)$ is nondecreasing we have

$$\left(M + n\int_0^\varepsilon I(\rho)\rho^{n-1}\,d\rho\right)\left(\frac{r}{\varepsilon}\right)^n \leq (M + I(\varepsilon)\varepsilon^n)\frac{r^n}{\varepsilon^n} = M\frac{r^n}{\varepsilon^n} + I(\varepsilon)r^n.$$

From which the result follows. □

In section 2 we prove several necessary and sufficient conditions for a harmonic function in the unit ball to be in $\mathcal{H}^p(B)$ (Theorems 1–4). In Theorem 5 we investigate the rate of growth of

$$E(r) = \int_{rB} |u(x)|^{p-2} |\nabla u(x)|^2 \, dV(x), \quad u \in \mathcal{H}^p(B),$$

as $r \to 1$. In Theorem 6 we give a local estimate of derivatives of such functions.

In section 3, we prove necessary and sufficient condition for a harmonic function in the unit ball to be in harmonic Bergman space $b^p(B)$ (Theorem 7). We also investigate the rate of growth of $E(r)$ as $r \to 1$, for $u \in b^p(B)$ (Theorem 8).

2. Hardy harmonic $\mathcal{H}^p(B)$ space.

In this section we consider Hardy harmonic $\mathcal{H}^p(B)$ space.

Theorem 1. *Let $1 < p < +\infty$. Function $u \in \mathcal{H}(B)$ belongs to $\mathcal{H}^p(B)$ if and only if*

$$\int_B |u(x)|^{p-2} |\nabla u(x)|^2 (1 - |x|^2) \, dV_N(x) < +\infty.$$

If $u \in \mathcal{H}^p(B)$, $1 < p < +\infty$, then

$$\|u\|^p_{\mathcal{H}^p} = \int_B |u(x)|^p \, dV_N(x) + \frac{p(p-1)}{2n} \int_B |u(x)|^{p-2} |\nabla u(x)|^2 (1 - |x|^2) \, dV_N(x).$$

Proof. Multiplying the formula (2) by r^n and integrating from 0 to r, we obtain

$$\int_0^r \rho^n \frac{d}{d\rho} \int_S |u(\rho\zeta)|^p \, d\sigma_N(\zeta) \, d\rho = \frac{p(p-1)}{n} \int_0^r \rho \int_{\rho B} |u(x)|^{p-2} |\nabla u(x)|^2 \, dV_N(x) \, d\rho.$$

Applying integration by parts on the left hand side of previous equality and using Fubini's theorem on the right hand side we obtain

$$r^n \int_S |u(r\zeta)|^p \, d\sigma_N(\zeta) = \int_{rB} |u(x)|^p \, dV_N(x) + \frac{p(p-1)}{2n} \int_{rB} |u(x)|^{p-2} |\nabla u(x)|^2 (r^2 - |x|^2) \, dV_N(x). \tag{5}$$

If $u \in \mathcal{H}^p(B)$, then by polar coordinates easily follows $u \in b^p(B)$. From that by the monotone convergence theorem we have

$$\lim_{r \to 1-0} \int_{rB} |u(x)|^p \, dV_N(x) = \int_B |u(x)|^p \, dV_N(x).$$

Since functions r^n and $\int_S |u(r\zeta)|^p \, d\sigma_N(\zeta)$ are nondecreasing then function $r^n \int_S |u(r\zeta)|^p \, d\sigma_N(\zeta)$ is also such a function. From $u \in \mathscr{H}^p(B)$ also follows

$$\lim_{r\to 1-0} r^n \int_S |u(r\zeta)|^p \, d\sigma_N(\zeta) = \int_S |u^*(\zeta)|^p \, d\sigma_N(\zeta),$$

where $u^*(\zeta)$ is a usual radial limit for a function in $\mathscr{H}^p$.

From all of the above we have there exists

$$\lim_{r\to 1-0} \int_{rB} |u(x)|^{p-2} |\nabla u(x)|^2 (r^2 - |x|^2) \, dV_N(x).$$

By the monotone convergence theorem we have

$$\begin{aligned}\lim_{r\to 1-0} &\int_{rB} |u(x)|^{p-2} |\nabla u(x)|^2 (r^2 - |x|^2) \, dV_N(x) \\ &= \int_B \lim_{r\to 1-0} |u(x)|^{p-2} |\nabla u(x)|^2 (r^2 - |x|^2) \chi_{rB}(x) \, dV_N(x) \\ &= \int_B |u(x)|^{p-2} |\nabla u(x)|^2 (1 - |x|^2) \, dV_N(x)\end{aligned}$$

as required.

Conversely, if $M = \int_B |u(x)|^{p-2} |\nabla u(x)|^2 (1 - |x|^2) \, dV_N(x) < +\infty$, then by (5) we have

$$r^n \int_S |u(r\zeta)|^p \, d\sigma_N(\zeta) \le \int_{rB} |u(x)|^p \, dV_N(x) + M_1, \quad M_1 \in \boldsymbol{R}_+, \ r \in (0,1). \tag{6}$$

Let $I(r) = \int_S |u(r\zeta)|^p \, d\sigma_N(\zeta)$, then we may simply write (6) in the following form

$$r^n I(r) \le n \int_0^r \rho^{n-1} I(\rho) \, d\rho + M_1.$$

Taking $\varepsilon = 1/2$ in Lemma 2 we have $I(r) \le M_1 2^n + I(1/2)$, for $r \in [1/2, 1)$. Since $I(r)$ is nondecreasing function we have $I(r) \le I(1/2)$, for $r \in [0, 1/2]$. Thus the result follows. □

Theorem 2. *Let* $u \in \mathscr{H}^p(B)$, $1 < p < +\infty$, *then*

$$\|u\|^p_{\mathscr{H}^p} = |u(0)|^p + \frac{p(p-1)}{n(n-2)} \int_B |u(x)|^{p-2} |\nabla u(x)|^2 (|x|^{2-n} - 1) \, dV_N(x).$$

Proof. Since $u \in \mathscr{H}^p(B)$, letting $r \to 1$ in formula (1), and noting that the integrand is positive and increasing in r we obtain the above formula. □

Next theorem is a generalization of Lemma 1 and Lemma 2 in [**2**]. We use the following notation:

$$I_s(r,u) = \left(\int_S |u(r\zeta)|^s \, d\sigma_N(\zeta) \right)^{1/s}.$$

Theorem 3. *Let* $s > 0$. *If* $u \in \mathscr{H}^s(B)$, $1 < p \le 2$, *then*

$$\int_0^1 (1-\rho) I^2_{2s/(s-p+2)}(\rho, \nabla u)\, d\rho < +\infty. \tag{7}$$

If $2 \le p < s+2$, $s \le p$, *then* (7) *implies* $u \in \mathscr{H}^s(B)$.

Proof. By Theorem 1 condition $u \in \mathscr{H}^p(B)$ is equivalent with

$$\int_B |u(x)|^{p-2} |\nabla u(x)|^2 (1-|x|^2)\, dV(x)$$
$$= \int_0^1 r^{n-1}(1-r^2) \int_S |u(r\zeta)|^{p-2} |\nabla u(r\zeta)|^2 \, d\sigma(\zeta)\, dr < +\infty. \tag{8}$$

Let $J_p(r) = \int_S |u(r\zeta)|^{p-2} |\nabla u(r\zeta)|^2 \, d\sigma_N(\zeta)$.
If $p \le 2$, we can use Jensen's inequality to the integral

$$\frac{J_p(r)}{I_s^s(r,u)} = \int_S \frac{|\nabla u(r\zeta)|^2}{|u(r\zeta)|^{s-p+2}} \frac{|u(r\zeta)|^s}{I_s^s(r,u)} d\sigma_N(\zeta).$$

Since in that case $0 < s/(s-p+2) \le 1$, we have

$$\left(\frac{J_p(r)}{I_s^s(r,u)} \right)^{s/(s-p+2)} \ge \int_S \frac{|\nabla u(r\zeta)|^{2s/(s-p+2)}}{|u(r\zeta)|^s} \frac{|u(r\zeta)|^s}{I_s^s(r,u)} d\sigma_N(\zeta)$$
$$= \int_S |\nabla u(r\zeta)|^{2s/(s-p+2)} \, d\sigma_N(\zeta) I_s^{-s}(r,u), \tag{9}$$

i.e.

$$J_p(r) \ge I_s^{p-2}(r,u) I^2_{2s/(s-p+2)}(r, \nabla u). \tag{10}$$

From that we have

$$J_p(r) \ge \|u\|^{p-2}_{\mathscr{H}^s} I^2_{2s/(s-p+2)}(r, \nabla u). \tag{11}$$

By (8) and (11) we get

$$\int_0^1 r^{n-1}(1-r^2) I^2_{2s/(s-p+2)}(r, \nabla u)\, dr < +\infty.$$

It is easy to see that the last integral is equiconvergent with integral in (7).

If $2 \le p < s+2$, $s \le p$, then in (10) the converse inequality holds. Therefore from (1) we obtain, using polar coordinates,

$$I_p^p(r,u) \le |u(0)|^p + \frac{p(p-1)}{(n-2)}\int_0^r J_p(\rho)(\rho^{2-n} - r^{2-n})\rho^{n-1}\,d\rho$$
$$\le |u(0)|^p + \frac{p(p-1)}{(n-2)}\int_0^r I_s^{p-2}(\rho,u) I_{2s/(s-p+2)}^2(\rho,\nabla u)(\rho - r^{2-n}\rho^{n-1})\,d\rho.$$

Since $u \in \mathscr{H}(B)$, function $I_p(r,u)$ is nondecreasing in r; see, for example [**4**]. By Jensen's inequality we get $I_s(r,u) \le I_p(r,u)$ for $s \le p$. Thus we have

$$I_p^2(r,u) \le |u(0)|^2 + \frac{p(p-1)}{(n-2)}\int_0^r I_{2s/(s-p+2)}^2(\rho,\nabla u)(\rho - r^{2-n}\rho^{n-1})\,d\rho$$
$$\le |u(0)|^2 + \frac{p(p-1)}{(n-2)}\int_0^r I_{2s/(s-p+2)}^2(\rho,\nabla u)\rho(1-\rho^{n-2})\,d\rho$$
$$\le |u(0)|^2 + p(p-1)\int_0^r I_{2s/(s-p+2)}^2(\rho,\nabla u)(1-\rho)\,d\rho.$$

From that follows second part of this theorem. □

Corollary 2. *Let* $p \ge 2$, $u \in \mathscr{H}(B)$. *If* $\int_B (1-|x|)|\nabla u(x)|^p\,dV(x) < +\infty$, *then* $u \in \mathscr{H}^p(B)$.

Proof. Since $u \in \mathscr{H}(B)$, function $|\nabla u|^p$ is subharmonic, see [**7**], and therefore $I_p(r,\nabla u)$ is nondecreasing in r. If $\sup_{0<r<1} I_p^p(r,\nabla u)$ is finite, then $\sup_{0<r<1} I_p^2(r,\nabla u)$ is also finite. Thus we have

$$\int_0^1 (1-\rho)I_p^2(\rho,\nabla u)\,d\rho < +\infty.$$

By Theorem 3, we obtain $u \in \mathscr{H}^p(B)$.

If $\sup_{0<r<1} I_p^p(r,\nabla u) = +\infty$ we have there exists r_0 such that for $r \in [r_0,1)$, $I_p^p(r,\nabla u) > 1$. Since $p \ge 2$ we have $I_p^2(r,\nabla u) \le I_p^p(r,\nabla u)$ for $r \in [r_0,1)$. Therefore

$$\int_{r_0}^1 (1-\rho)I_p^2(\rho,\nabla u)\,d\rho \le \int_{r_0}^1 (1-\rho)I_p^p(\rho,\nabla u)\,d\rho < +\infty.$$

Hence

$$\int_0^1 (1-\rho)I_p^2(\rho,\nabla u)\,d\rho < +\infty.$$

Applying Theorem 3 we obtain our result. □

By Theorem 3, as in Corollary 2 we can prove this result. Details we leave to the reader.

Corollary 3. *Let* $1 < p \le 2$, $u \in \mathscr{H}^p(B)$. *Then*

$$\int_B (1-|x|)|\nabla u(x)|^p \, dV(x) < +\infty.$$

We shall give an elementary proof of the following theorem (i.e. without using maximal theorem).

Theorem 4. *Let* $u \in \mathscr{H}(B)$ *and* $\sup_{0<r<1} \int_S |\nabla u(r\zeta)|^p \, d\sigma_N(\zeta) < +\infty$, $p \ge 2$, *then* $u \in \mathscr{H}^p(B)$.

Proof. Let $I(r) = \int_S |u(r\zeta)|^p \, d\sigma_N(\zeta)$ and

$$J_p(r) = J(r) = \int_S |u(r\zeta)|^{p-2} |\nabla u(r\zeta)|^2 \, d\sigma_N(\zeta)$$

in polar coordinates (2) becomes

$$I'(r) = p(p-1) r^{1-n} \int_0^r \rho^{n-1} J(\rho) \, d\rho. \tag{12}$$

Let us estimate $J(r)$ by $I(r)$. For that purpose we write $J(r)$ as follows

$$J(r) = \int_S \frac{|u(r\zeta)|^{p-2}}{|\nabla u(r\zeta)|^{p-2}} \frac{|\nabla u(r\zeta)|^p}{\|\nabla u(r\,\cdot)\|_p^p} d\sigma_N(\zeta) \cdot \|\nabla u(r\,\cdot)\|_p^p,$$

where

$$\|\nabla u(r\,\cdot)\|_p^p = \int_S |\nabla u(r\zeta)|^p \, d\sigma_N(\zeta).$$

For $p > 2$ we have $p/(p-2) > 1$. Applying Jensen's inequality we obtain

$$\begin{aligned} J(r)^{p/(p-2)} &\le \int_S \frac{|u(r\zeta)|^p}{|\nabla u(r\zeta)|^p} \frac{|\nabla u(r\zeta)|^p}{\|\nabla u(r\,\cdot)\|_p^p} d\sigma_N(\zeta) \cdot \|\nabla u(r\,\cdot)\|_p^{p^2/(p-2)} \\ &= I(r) \left(\int_S |\nabla u(r\zeta)|^p \, d\sigma_N(\zeta) \right)^{2p/(p-2)}. \end{aligned}$$

Thus

$$J(r) \le I(r)^{(p-2)/p} \left(\sup_{0<r<1} \int_S |\nabla u(r\zeta)|^p \, d\sigma_N(\zeta) \right)^2 = C_{\nabla u} I(r)^{(p-2)/p}. \tag{13}$$

Combining (12) and (13) we obtain

$$I'(r) \leq C_{\nabla u} p(p-1) r^{1-n} \int_0^r \rho^{n-1} I(\rho)^{(p-2)/p} \, d\rho.$$

Integrating from 0 to r we get

$$\begin{aligned} I(r) &\leq I(0) + C_{\nabla u} p(p-1) \int_0^r \rho^{1-n} \int_0^\rho s^{n-1} I(s)^{(p-2)/p} \, ds d\rho \\ &= I(0) + C_{\nabla u} p(p-1) \cdot \int_0^r s^{n-1} I(s)^{(p-2)/p} \int_s^r \rho^{1-n} \, d\rho ds \end{aligned}$$

i.e.

$$\begin{aligned} I(r) &\leq I(0) + \frac{C_{\nabla u} p(p-1)}{n-2} \int_0^r s^{n-1} I(s)^{(p-2)/p} (s^{2-n} - r^{2-n}) \, ds \\ &\leq I(0) + \frac{C_{\nabla u} p(p-1)}{n-2} \int_0^r s I(s)^{(p-2)/p} \, ds. \end{aligned} \tag{14}$$

Let $\alpha = (p-2)/p$ then (14) can be written

$$I(r)^\alpha \leq \left(I(0) + C \int_0^r s I(s)^\alpha \, ds \right)^\alpha, \quad C = \frac{C_{\nabla u} p(p-1)}{n-2}. \tag{15}$$

From that we have

$$\frac{Cr I(r)^\alpha}{(I(0) + C \int_0^r s I(s)^\alpha \, ds)^\alpha} \leq Cr, \quad 0 < r < 1.$$

Integrating from 0 to r we get

$$\left(I(0) + C \int_0^r s I(s)^\alpha \, ds \right)^{1-\alpha} - I(0)^{1-\alpha} \leq (1-\alpha) \frac{Cr^2}{2}. \tag{16}$$

By (15) and (16) we have

$$I(r) \leq \left((1-\alpha) \frac{Cr^2}{2} + I(0)^{1-\alpha} \right)^{1/(1-\alpha)} \leq \left(\frac{(1-\alpha)C}{2} + I(0)^{1-\alpha} \right)^{1/(1-\alpha)} < +\infty$$

thus $u \in \mathscr{H}^p(B)$.

Remark 1. Theorem 4 is a consequence of Minkowski's inequality and maximal theorem, also in the case $p \in (1,2)$.

Theorem 5. *Let* $1 < p < +\infty$, $u \in \mathscr{H}^p(B)$, *then*

$$\lim_{r \to 1-0} (1-r) \int_{rB} |u(x)|^{p-2} |\nabla u(x)|^2 \, dV(x) = 0.$$

Proof. Integrating the following formula

$$r^{n-1}\frac{d}{dr}\int_S |u(r\zeta)|^p\,d\sigma_N(\zeta) = \frac{p(p-1)}{n}\int_{rB}|u(x)|^{p-2}|\nabla u(x)|^2\,dV_N(x)$$

from 0 to r we get

$$\begin{aligned} r^{n-1}\int_S |u(r\zeta)|^p\,d\sigma_N(\zeta) - (n-1)\int_0^r \rho^{n-2}\int_S |u(\rho\zeta)|^p\,d\sigma_N(\zeta)\,d\rho \\ = \frac{p(p-1)}{n}\int_0^r\int_{\rho B}|u(x)|^{p-2}|\nabla u(x)|^2\,dV_N(x)\,d\rho. \end{aligned}$$

Since $u \in \mathscr{H}^p(B)$ we have $u \in b^p(B)$. On the other hand

$$\begin{aligned} &\int_0^r \rho^{n-2}\int_S |u(\rho\zeta)|^p\,d\sigma_N(\zeta)\,d\rho \\ &= \int_0^{1/2}\rho^{n-2}\int_S |u(\rho\zeta)|^p\,d\sigma_N(\zeta)\,d\rho + \int_{1/2}^r \rho^{n-1}\int_S |u(\rho\zeta)|^p\,d\sigma_N(\zeta)\frac{d\rho}{\rho} \\ &\le c_1 + c_2\int_{rB}|u(x)|^p\,dV(x) < +\infty. \end{aligned}$$

Therefore, there exists

$$\lim_{r\to 1-0}\int_0^r \rho^{n-2}\int_S |u(\rho\zeta)|^p\,d\sigma_N(\zeta)\,d\rho,$$

from which it follows that

$$\lim_{r\to 1-0}\int_0^r\int_{\rho B}|u(x)|^{p-2}|\nabla u(x)|^2\,dV_N(x)\,d\rho$$

exists.

Thus by Cauchy's criterion we have

$$\lim_{r\to 1-0}\int_r^1\int_{\rho B}|u(x)|^{p-2}|\nabla u(x)|^2\,dV_N(x)\,d\rho = 0.$$

Since function $J(\rho) = \int_{\rho B}|u(x)|^{p-2}|\nabla u(x)|^2\,dV(x)$ is nondecreasing we obtain

$$(1-r)J(r) \le \int_r^1 J(\rho)\,d\rho \to 0, \quad \text{as } r \to 1-0,$$

which is what we needed to prove. □

In the following theorem we generalize Lemma 2 in [**5**]. In that paper Luecking proved theorem in the case of $n = 2$. Also Luecking reproved Hardy-Stein identity in that case and applied it in proving Littlewood-Paley inequality.

THEOREM 6. *Let* $p \geq 2$ *and* $u \in \mathscr{H}^p(B)$, *then*

$$|\nabla u(0)|^p \leq \frac{n^{p/2}p(p-1)}{(n-2)n}\int_B |u(x)|^{p-2}|\nabla u(x)|^2(|x|^{2-n}-1)\,dV_N(x), \quad n \geq 3.$$

PROOF. It is well-known that if $u \in \mathscr{H}(B)$ then $u(x) = \sum_{m=0}^{+\infty} p_m(x)$, where $p_m(x)$ is harmonic homogeneous polynomial of degree m on B. By Hölder inequality we have $\|u\|_{\mathscr{H}^2} \leq \|u\|_{\mathscr{H}^p}$. For $u \in \mathscr{H}^2(B)$, the following formula

$$\|u\|^2_{\mathscr{H}^2} = \sum_{m=0}^{+\infty}\int_S |p_m(\zeta)|^2\,d\sigma_N(\zeta) \tag{17}$$

holds; see, for example [**1**, p. 122].

On the other hand, for homogeneous polynomial of degree m the following formula $\langle \nabla p_m(x), x\rangle = mp_m(x)$, $x \in \boldsymbol{R}^n$, holds. From (17) we have

$$\begin{aligned}\|u\|_{\mathscr{H}^2} - |u(0)|^2 &= \sum_{m=1}^{+\infty}\int_S |p_m(\zeta)|^2\,d\sigma_N(\zeta) \geq \int_S |p_1(\zeta)|^2\,d\sigma_N(\zeta)\\ &= \int_S |\langle \nabla p_1(\zeta), \zeta\rangle|^2\,d\sigma_N(\zeta)\end{aligned}$$

since $u(0) = p_0(0)$. Let $p_1(x) = a_1x_1 + \cdots + a_nx_n$, $a_1, \ldots, a_n \in \boldsymbol{C}$, then

$$\begin{aligned}\int_S |\langle \nabla p_1(\zeta), \zeta\rangle|^2\,d\sigma_N(\zeta) &= \int_S |a_1\zeta_1 + \cdots + a_n\zeta_n|^2\,d\sigma_N(\zeta)\\ &= \int_S \left(\sum_{i=1}^n |a_i|^2\zeta_i^2 + 2\sum_{i\neq j} a_i\bar{a}_j\zeta_i\zeta_j\right) d\sigma_N(\zeta).\end{aligned}$$

By symmetry of the sphere and subintegral functions we have $\int_S \zeta_i\zeta_j\,d\sigma_N(\zeta) = 0$, $i \neq j$, and $\int_S \zeta_i^2\,d\sigma_N(\zeta) = 1/n$, $i = 1, \ldots, n$.
Thus

$$\int_S |\langle \nabla p_1(\zeta), \zeta\rangle|^2\,d\sigma_N(\zeta) = \frac{1}{n}\sum_{i=1}^n |a_i|^2.$$

Since $\sum_{i=1}^n |a_i|^2 = |\nabla p_1(x)|^2 = |\nabla p_1(0)|^2 = |\nabla u(0)|^2$ we have

$$\frac{1}{n}|\nabla u(0)|^2 \leq \|u\|^2_{\mathscr{H}^2} - |u(0)|^2 \leq \|u\|^2_{\mathscr{H}^p} - |u(0)|^2.$$

By the inequality $(a-b)^q + b^q \le a^q$, $a \ge b > 0$, $q \ge 1$, we obtain

$$|\nabla u(0)|^p \le n^{p/2}(\|u\|^2_{\mathscr{H}^p} - |u(0)|^2)^{p/2} \le n^{p/2}(\|u\|^p_{\mathscr{H}^p} - |u(0)|^p). \tag{18}$$

By the following formula

$$\|u\|^p_{\mathscr{H}^p} = |u(0)|^p + \frac{p(p-1)}{n(n-2)} \int_B |u(x)|^{p-2}|\nabla u(x)|^2(|x|^{2-n} - 1)\,dV_N(x), \tag{19}$$

(18) and (19) the result follows. □

3. Bergman $b^p(B)$ space.

In this section we consider harmonic Bergman space $b^p(B)$.

THEOREM 7. *Let* $u \in \mathscr{H}(B)$, $p > 1$, *then* $u \in b^p(B)$ *if and only if*

$$I_u = \int_B |u(x)|^{p-2}|\nabla u(x)|^2(2|x|^{2-n} + (n-2)|x|^2 - n)\,dV_N(x) < +\infty.$$

PROOF. Multiplying the formula (1) by nr^{n-1}, and integrating from 0 to 1, applying Fubini's theorem, for $u \in \mathscr{H}(B)$, we obtain the following formula

$$\|u\|^p_{b^p} = |u(0)|^p + \frac{p(p-1)}{2n(n-2)} \int_B |u(x)|^{p-2}|\nabla u(x)|^2(2|x|^{2-n} + (n-2)|x|^2 - n)\,dV_N(x),$$

thus the result follows. □

COROLLARY 4. $u \in b^p(B)$, $p > 1$ *if and only if*

$$\int_B |u(x)|^{p-2}|\nabla u(x)|^2(1-|x|)^2\,dV(x) < +\infty.$$

PROOF. Let

$$I_u = \int_B |u(x)|^{p-2}|\nabla u(x)|^2((n-2)|x|^n - n|x|^{n-2} + 2)/|x|^{n-2}\,dV_N(x).$$

We leave to the reader that I_u is equiconvergent with $\int_B |u(x)|^{p-2}|\nabla u(x)|^2 \cdot (1-|x|)^2/|x|^{n-2}\,dV(x)$.

Using polar coordinates it is easy to see that the last integral is equiconvergent with

$$\int_B |u(x)|^{p-2}|\nabla u(x)|^2(1-|x|)^2\,dV(x).$$

If $n \ge 3$, the Green function for the ball B with pole in the origin is given by $G(|x|, 1) = |x|^{2-n} - 1$. □

COROLLARY 5. *Let* $1 < p < +\infty$, *then* $u \in b^p(B)$ *if and only if*

$$L_u = \int_B |u(x)|^{p-2} |\nabla u(x)|^2 G(|x|, 1)(1 - |x|)\, dV(x) < +\infty.$$

PROOF. By Theorem 7 it is enough to show that integrals L_u and I_u are equiconvergent. We know that

$$I_u \sim \int_B |u(x)|^{p-2} |\nabla u(x)|^2 (1 - |x|)^2 / |x|^{n-2}\, dV(x)$$
$$= \int_B |u(x)|^{p-2} |\nabla u(x)|^2 G(|x|, 1)(1 - |x|)^2 / (1 - |x|^{n-2})\, dV(x).$$

It is clear that expression

$$\frac{1 - |x|}{1 - |x|^{n-2}} = \frac{1}{1 + |x| + \cdots + |x|^{n-3}}$$

for $x \in B$ takes values from the interval $(1/(n-2), 1]$. Thus we have the required result. □

REMARK 2. Note that from previous estimates we have

$$|u(0)|^p + \frac{p(p-1)}{2n(n-2)^2} c_1 I_u \le \|u\|_{b_p}^p \le |u(0)|^p + \frac{p(p-1)}{2n(n-2)} c_2 I_u,$$

for positive $c_1 = c_1(n)$ and $c_2 = c_2(n)$.

THEOREM 8. *Let* $u \in b^p(B)$, $p > 1$, *then*

$$\lim_{r \to 1-0} (1 - r)^2 \int_{rB} |u(x)|^{p-2} |\nabla u(x)|^2\, dV(x) = 0.$$

PROOF. By Fubini's theorem we get

$$\int_0^1 \int_{\rho B} |u(x)|^{p-2} |\nabla u(x)|^2 (1 - |x|)\, dV(x)\, d\rho = \int_B |u(x)|^{p-2} |\nabla u(x)|^2 (1 - |x|)^2\, dV(x).$$

Thus, by Corollary 4 we have

$$\lim_{r \to 1-0} \int_r^1 \int_{\rho B} |u(x)|^{p-2} |\nabla u(x)|^2 (1 - |x|)\, dV(x)\, d\rho = 0.$$

It is obvious that

$$(1 - r) \int_{rB} |u(x)|^{p-2} |\nabla u(x)|^2 (1 - |x|)\, dV(x)$$
$$\le \int_r^1 \int_{\rho B} |u(x)|^{p-2} |\nabla u(x)|^2 (1 - |x|)\, dV(x)\, d\rho.$$

Therefore

$$\lim_{r\to 1-0}(1-r)\int_{rB}|u(x)|^{p-2}|\nabla u(x)|^2(1-|x|)\,dV(x)=0.$$

For $|x|<r$ we have $1-|x|>1-r$, so we have

$$(1-r)^2\int_{rB}|u(x)|^{p-2}|\nabla u(x)|^2\,dV(x)\le(1-r)\int_{rB}|u(x)|^{p-2}|\nabla u(x)|^2(1-|x|)\,dV(x),$$

which finish the proof of the theorem. □

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