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# On a $p$-local stable splitting of Stiefel manifolds

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**Abstract.** H. Miller introduced a filtration of Stiefel manifolds which splits stably and proved that the starata of the filtration are vector bundles, so that Stiefel manifolds are stably equivalent to a wedge of the corresponding Thom spaces. In this paper, we shall give a finer $p$-local stable splitting of the Stiefel manifolds by using the Adams operations.

## 1. Introduction.

Let $U(n:m)$ be the Stiefel manifold of orthonormal $n$-frames in $\boldsymbol{C}^{n+m}$. A common notation for the Stiefel manifold is $V_{n+m,n}$, but we will use this notation $U(n:m)$. The reason we want to use this notation is that we regard this $U(n:m) \subset \mathrm{Hom}(\boldsymbol{C}^n, \boldsymbol{C}^{n+m})$ as the space of isometric linear maps from $\boldsymbol{C}^n$ to $\boldsymbol{C}^{n+m}$. Let $f: \boldsymbol{C}^n \to \boldsymbol{C}^{n+m}$ be an isometric linear map. Then there exists a unique matrix $A = (a_{ij})$, $1 \leq i \leq n+m$, $1 \leq j \leq n$ such that $f(z) = Az$ for $z \in \boldsymbol{C}^n$. By taking $g = (a_{ij})$, $h = (a_{kj})$, $1 \leq i, j \leq n$, $n+1 \leq k \leq n+m$ associating to the matrix $A$, we can represent each element of $U(n:m)$ as a pair $(g,h)$ where $g \in \mathrm{End}(\boldsymbol{C}^n)$, $h \in \mathrm{Hom}(\boldsymbol{C}^n, \boldsymbol{C}^m)$ satisfying $g^*g + h^*h = 1_n$. H. Miller [**2**] has introduced a filtration $\{(1,0)\} = R^0(\boldsymbol{C}^n : \boldsymbol{C}^m) \subset R^1(\boldsymbol{C}^n : \boldsymbol{C}^m) \subset \cdots \subset R^n(\boldsymbol{C}^n : \boldsymbol{C}^m) = U(n:m)$ and shown that there is a stable splitting

$$U(n:m) \simeq \bigvee_{k=1}^{n} R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+ \tag{1}$$

where $R_k(\boldsymbol{C}^n : \boldsymbol{C}^m) = R^k(\boldsymbol{C}^n : \boldsymbol{C}^m) - R^{k-1}(\boldsymbol{C}^n : \boldsymbol{C}^m)$ and $R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+ \cong R^k(\boldsymbol{C}^n : \boldsymbol{C}^m)/R^{k-1}(\boldsymbol{C}^n : \boldsymbol{C}^m)$ is the one-point compactification of $R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)$. Later M. Crabb [**1**] gave a simpler constrution of the stable splitting.

In particular, by taking $m = 0$ in (1), we have the following stable splitting of $U(n)$

$$U(n) \simeq \bigvee_{k=1}^{n} R_k(\boldsymbol{C}^n)^+ \tag{2}$$

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On the other hand it is known [**3**] that for a prime $p$ there is a $p$-local unstable decomposition

$$U(n) \underset{p}{\simeq} X_1(n) \times \cdots \times X_{p-1}(n)$$

into a product of $p-1$ spaces, using the unstable Adams operations with the $X_i(n)$'s satisfying $H^*(X_i(n); \boldsymbol{Z}_{(p)}) \cong \Lambda_{\boldsymbol{Z}_{(p)}}(x_i, x_{i+p-1}, \dots, x_{i+s(p-1)})$ where $s = [(n-i)/(p-1)]$. Then we obtain a $p$-local stable splitting

$$U(n) \underset{p}{\simeq} \bigvee X_{i_1}(n) \wedge \cdots \wedge X_{i_s}(n), \quad 1 \le i_1 < \cdots < i_s \le p-1. \tag{3}$$

In [**4**], Nishida and Yang have shown first, that the stable splitting map of unitary group $U(n)$ is homologically diagonal and then, by mixing (2) and (3), obtained a finer decomposition of $U(n)$.

In this paper, we will generalize the above argument to the case of Stiefel manifolds. Our main theorem is

THEOREM 3.5. *Let $n$ be a positive integer and let $p$ be an odd prime. For each pair $(t,k)$ of integers such as $1 \le t \le p-1$ and $1 \le k \le n$, $M_{t,k}$ denotes the submodule of $H_*(U(n:m); \boldsymbol{Z}/p)$ spanned by $x_{i_1} \cdots x_{i_k}$ such that $i_1 + \cdots + i_k \equiv t \bmod p-1$. Then there exists a finite spectrum $Y_{t,k}$ satisfying*

$$H_*(Y_{t,k}; \boldsymbol{Z}/p) \cong M_{t,k}$$

*as a module, and a stable $p$-equivalence*

$$U(n:m) \to \bigvee Y_{t,k}$$

*where the wedge sum is taken over $1 \le t \le p-1$ and $1 \le k \le n$.*

In §2, we will follow the same way to that of §2 in [**4**] and show that the stable splitting map of the Stiefel manifold is homologically diagonal.

In §3, we will study how can the cohomology generators of the Stiefel manifold be represented by the Adams operation. To be precise, let $q$ be an integer such that $(q, m!) = 1$. Then we know that there exists a map $\psi^q : BU(m) \to BU(m)$ such that $(\psi^q)^*(c_r) = q^r c_r$, $1 \le r \le m$. By applying the loop functor, we obtain a map

$$\Omega\psi^q : U(m) \to U(m).$$

Now we consider the fibration $U(m) \to U(n+m) \to U(n:m)$, then from the following diagram

$$\begin{array}{ccccccccc}
U(m) & \longrightarrow & U(n+m) & \longrightarrow & U(n:m) & \longrightarrow & BU(m) & \longrightarrow & BU(n+m) \\
\downarrow{\scriptstyle \Omega\psi^q} & & \downarrow{\scriptstyle \Omega\psi^q} & & & & \downarrow{\scriptstyle \psi^q} & & \downarrow{\scriptstyle \psi^q} \\
U(m) & \longrightarrow & U(n+m) & \longrightarrow & U(n:m) & \longrightarrow & BU(m) & \longrightarrow & BU(n+m)
\end{array}$$

we can show that there exists a map $U(n:m) \to U(n:m)$ such that all square diagrams are homotopy commutative. We denote this map by the same notation $\psi^q$ and call it "Adams operation" too. Then we shall show that $\psi^q$ is cohomologically diagonal, and obtain a finer stable splitting of Stiefel manifolds.

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## 2. Property of stable splittings.

We define an increasing filtration of the Stiefel manifold $U(n:m)$

$$\{(1,0)\} = R^0(\boldsymbol{C}^n : \boldsymbol{C}^m) \subset R^1(\boldsymbol{C}^n : \boldsymbol{C}^m) \subset \cdots \subset R^n(\boldsymbol{C}^n : \boldsymbol{C}^m) = U(n:m)$$

by

$$R^k(\boldsymbol{C}^n : \boldsymbol{C}^m) = \{(g,h) \in U(n:m) \mid \dim_{\boldsymbol{C}}(\operatorname{Ker}(g-1))^{\perp} \leq k\}, \quad 0 \leq k \leq n$$

where 1 denotes the unit element of $\operatorname{End}(\boldsymbol{C}^n)$. The difference $R^k(\boldsymbol{C}^n : \boldsymbol{C}^m) - R^{k-1}(\boldsymbol{C}^n : \boldsymbol{C}^m) = \{(g,h) \in U(n:m) \mid \dim_{\boldsymbol{C}}(\operatorname{Ker}(g-1))^{\perp} = k\}$ is written by $R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)$. The one-point compactification $R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+$ is identified with the quotient space $R^k(\boldsymbol{C}^n : \boldsymbol{C}^m)/R^{k-1}(\boldsymbol{C}^n : \boldsymbol{C}^m)$. The pointed natural projection map is denoted by $\pi : R^k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+ \to R^k(\boldsymbol{C}^n : \boldsymbol{C}^m)/R^{k-1}(\boldsymbol{C}^n : \boldsymbol{C}^m) = R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+$.

H. Miller [**2**] and M. Crabb [**1**] have shown the following

Theorem 2.1. *There exists a stable splitting*

$$U(n:m) \simeq \bigvee_{k=1}^{n} R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+$$

We recall the construction of the splitting following [**1**]. Let $\operatorname{End}(\boldsymbol{C}^k)$ be the space of all $k \times k$ matrices. We regard $\operatorname{End}(\boldsymbol{C}^k)$ as a $U(k)$-space by the adjoint action. Then we have following $U(k)$-invariant subspaces of $\operatorname{End}(\boldsymbol{C}^k)$:

1) $\mathscr{H}(k)$: the space of Hermitian matrices,
2) $\mathfrak{u}(k)$: the space of skew-Hermitian matrices,
3) $U(k)$: the unitary group.

Also we regard $\operatorname{Hom}(\boldsymbol{C}^k, \boldsymbol{C}^{k+m})$ as a $U(k)$-space by pre-composition. Then we have $U(k)$-invariant subspace $U(k:m)_0 = \{(g,h) \in U(k:m) \mid g-1 \text{ is invertible}\}$. Occasionally, we shall write $\operatorname{Hom}(\boldsymbol{C}^k, \boldsymbol{C}^{k+m})$ by $M_{k,k+m}$ for the sake of convenience. Thus, $\operatorname{Hom}(\boldsymbol{C}^k, \boldsymbol{C}^{k+m}) = \operatorname{End}(\boldsymbol{C}^k) \oplus \operatorname{Hom}(\boldsymbol{C}^k, \boldsymbol{C}^m) = \mathfrak{u}(k) \oplus \mathscr{H}(k) \oplus M_{k,m}$. Let $G_{n,k}$ be the Grassman manifold of $k$-planes in $\boldsymbol{C}^n$ and let $U(k) \to U(k:n-k) \to G_{n,k}$ be the standard principal $U(k)$-bundle. We denote this bundle simply by $\zeta_{n,k}$. Let $F$ be a manifold with a $U(k)$-action. We denote the associate fibre bundle by $\zeta_{n,k}(F)$. The generalized Cayley transform

$$\psi : \mathfrak{u}(k) \oplus M_{k,m} \to U(k:m)_0$$

defined by $\psi(x,y) = (g,h)$ where $c = x/2 + y^*y/4$, $g = (c+1)^{-1}(c-1)$, $h = y(1-g)/2$ is a $U(k)$-equivariant diffeomorphism. Consider the bundle $\zeta_{n,k}(U(k:m)) \to G_{n,k}$; the fibre over $E \in G_{n,k}$ is $\{(X,Y) \mid X \in \mathrm{End}(E), Y \in \mathrm{Hom}(E, \boldsymbol{C}^m), X^*X + Y^*Y = 1_k\}$ of a $k$-dimensional subspace $E \subseteq \boldsymbol{C}^n$, and we can regard an element of $\zeta_{n,k}(U(k:m))$ as a triple $(E,X,Y)$ where $E \in G_{n,k}$, $X \in \mathrm{End}(E)$, $Y \in \mathrm{Hom}(E, \boldsymbol{C}^m)$. Now we have a surjective map $p : \zeta_{n,k}(U(k:m)) \to R^k(\boldsymbol{C}^n : \boldsymbol{C}^m)$ given by

$$p(E,X,Y) = (g,h),$$

where $g = X \oplus 1 : E \oplus E^\perp \to E \oplus E^\perp$ and $h = Y \oplus 0 : E \oplus E^\perp \to \boldsymbol{C}^m$. It is clear that the restriction of $p$ gives a homeomorphism

$$\zeta_{n,k}(U(k:m)_0) \underset{\cong}{\to} R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)$$

The space $R_k(\boldsymbol{C}^n : \boldsymbol{C}^m) \subseteq R^k(\boldsymbol{C}^n : \boldsymbol{C}^m)$ fibres over $G_{n,k}$, by mapping $(g,h) \in R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)$ to $(\mathrm{Ker}(g-1))^\perp$. By using the generalized Cayley transform we can identify the bundle $R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)$ with $\zeta_{n,k}(\mathfrak{u}(k) \times M_{k,m})$. Thus we have

PROPOSITION 2.2. *There is a natural diffeomorphism between $R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)$ and the total space of the vector bundle $\zeta_{n,k}(\mathfrak{u}(k) \times M_{k,m})$ over $G_{n,k}$.*

Let $\mathrm{Mon}(\boldsymbol{C}^k, \boldsymbol{C}^{k+m}) \subset \mathrm{Hom}(\boldsymbol{C}^k, \boldsymbol{C}^{k+m})$ be the open subset of injective linear maps. We note the decomposition $\mathrm{Mon}(\boldsymbol{C}^k, \boldsymbol{C}^{k+m}) \cong U(k:m) \times P(k)$ where $P(k)$ is the space of positive definite Hermitian matrices. The exponential map $\exp : \mathscr{H}(k) \to P(k)$ is a diffeomorphism and hence we have a diffeomorphism

$$1 \times \exp : U(k:m) \times \mathscr{H}(k) \to \mathrm{Mon}(\boldsymbol{C}^k, \boldsymbol{C}^{k+m})$$

which is clearly $U(k)$-equivariant. We now construct a stable map $\varsigma : R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+ \to R^k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+$ as follows. We have a diffeomorphism

$$\zeta_{n,k}(U(k:m) \times \mathscr{H}(k)) \underset{\cong}{\to} \zeta_{n,k}(\mathrm{Mon}(\boldsymbol{C}^k, \boldsymbol{C}^{k+m}))$$

and an open embedding

$$\zeta_{n,k}(\mathrm{Mon}(\boldsymbol{C}^k, \boldsymbol{C}^{k+m})) \subset \zeta_{n,k}(\mathrm{Hom}(\boldsymbol{C}^k, \boldsymbol{C}^{k+m})) \cong \zeta_{n,k}(\mathfrak{u}(k) \oplus \mathscr{H}(k) \oplus M_{k,m}).$$

Note that $\zeta_{n,k}(U(k:m) \times \mathscr{H}(k))$ is identified with the fibre product $\zeta_{n,k}(U(k:m)) \underset{G_{n,k}}{\times} \zeta_{n,k}(\mathscr{H}(k))$ of the spaces over $G_{n,k}$, and similarly for $\zeta_{n,k}(\mathfrak{u}(k) \oplus \mathscr{H}(k) \oplus M_{k,m})$. Since $\zeta_{n,k}(\mathscr{H}(k))$ is a real vector bundle over a compact space $G_{n,k}$, there is an embedding $\zeta_{n,k}(\mathscr{H}(k)) \subset G_{n,k} \times \boldsymbol{R}^N$ into the product bundle for some integer $N$. Let $\gamma$ be the orthogonal complement of $\zeta_{n,k}(\mathscr{H}(k))$ so that $\zeta_{n,k}(\mathscr{H}(k)) \oplus \gamma = G_{n,k} \times \boldsymbol{R}^N$. Then applying $\underset{G_{n,k}}{\times} \gamma$ to the above diffeomorphism and the embedding, we have an open embedding $\zeta_{n,k}(U(k:m)) \times \boldsymbol{R}^N \subset \zeta_{n,k}(\mathfrak{u}(k) \times M_{k,m}) \times \boldsymbol{R}^N$ and applying the Pontrjagin-Thom construction we obtain a stable map $s : \zeta_{n,k}(\mathfrak{u}(k) \times M_{k,m})^+ \to \zeta_{n,k}(U(k:m))^+$. By a usual argument it is easy to see that the homotopy class of the stable map $s$ does not depend on a choice of an embedding $\zeta_{n,k}(\mathscr{H}(k)) \subset G_{n,k} \times \boldsymbol{R}^N$. Now we define the stable map $\varsigma$ as the following composition

$$\varsigma : R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+ \cong \zeta_{n,k}(\mathfrak{u}(k) \times M_{k,m})^+ \xrightarrow{s} \zeta_{n,k}(U(k;m))^+ \xrightarrow{p^+} R^k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+.$$

The next theorem is due to M. Crabb [**1**]

THEOREM 2.3. *$\pi \circ \varsigma$ is homotopic to the identity.*

We denote the inclusion maps $R^k(\boldsymbol{C}^n : \boldsymbol{C}^m) \to R^{k+1}(\boldsymbol{C}^n : \boldsymbol{C}^m)$ by $j'$. Then we have a stable map $j' \circ \varsigma : R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+ \to U(n:m)^+$ and by taking a wedge sum we have

$$\bigvee_{k=0}^{n} R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+ \to \bigvee U(n:m)^+ \to U(n:m)^+$$

which is clearly a homotopy equivalence. Since $R_0(\boldsymbol{C}^n : \boldsymbol{C}^m)^+ \simeq S^0$, we finally obtain a stable homotopy equivalence

$$\bigvee_{k=1}^{n} R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+ \to U(n:m).$$

Now we recall basic facts about the homology of $R^k(\boldsymbol{C}^n : \boldsymbol{C}^m)$, see, e.g., [**5**]. Cosider a map $\rho : \Sigma(\boldsymbol{CP}_+^{n+m-1}) \to U(n+m)$ defined by

$$\rho(\lambda, z) = (\delta_{i,j} + (\lambda - 1)z_i\bar{z}_j), \quad 1 \le i, j \le n+m$$

for $\lambda \in S^1$, $z = [z_1; \dots; z_{n+m}]$ and $z = (z_1, \dots, z_{n+m}) \in S^{2n+2m-1} \subset \boldsymbol{C}^{n+m}$ and let $q : U(n+m) \to U(n+m)/U(m) = U(n:m)$ be the standard projection. Now let us consider the composition $\Sigma(\boldsymbol{CP}_+^{n+m-1}) \xrightarrow{\rho} U(n+m) \xrightarrow{q} U(n+m)/U(m) = U(n:m)$. By Lemma 3.2 (*cf.* §3) we have

$$H_*(U(n:m); \boldsymbol{Z}) \cong \boldsymbol{Z}\{x_{i_1} \cdots x_{i_s} \mid m+1 \le i_1 < \cdots < i_s \le m+n\}$$

where $x_i = (q \circ \rho)_*(\sigma(s_{i-1}))$ and $s_{i-1}$ is a generator of $H_{2i-2}(\boldsymbol{CP}^{n+m-1}; \boldsymbol{Z})$. Let $j : R^k(\boldsymbol{C}^{n+m}) \to U(n+m)$ and $j' : R^k(\boldsymbol{C}^n : \boldsymbol{C}^m) \to U(n:m)$ be the inclusion maps. Consider the following commutative diagram

$$\begin{array}{ccc} R^k(\boldsymbol{C}^{n+m}) & \xrightarrow[j]{} & U(n+m) \\ \Big\downarrow q & & \Big\downarrow q \\ R^k(\boldsymbol{C}^n : \boldsymbol{C}^m) & \xrightarrow[j']{} & U(n:m) \end{array}$$

Then from Proposition 2.4 [**4**], we have

PROPOSITION 2.4. *The homomorphism*

$$j'_* : H_*(R^k(\boldsymbol{C}^n : \boldsymbol{C}^m); \boldsymbol{Z}) \to H_*(U(n:m); \boldsymbol{Z})$$

*is injective and* $\operatorname{Im} j'_*$ *is spanned by* $x_{i_1} \cdots x_{i_s}$, $m+1 \le i_1 < \cdots < i_s \le m+n$, $s \le k$.

We identify $H_*(R^k(\boldsymbol{C}^n : \boldsymbol{C}^m); \boldsymbol{Z})$ with its image in $H_*(U(n:m); \boldsymbol{Z})$. Now consider the following commutative diagram of the cofibrations

$$\begin{array}{ccccc} R^{k-1}(\boldsymbol{C}^{n+m}) & \longrightarrow & R^k(\boldsymbol{C}^{n+m}) & \longrightarrow & R_k(\boldsymbol{C}^{n+m})^+ \\ \Big\downarrow q & & \Big\downarrow q & & \Big\downarrow q \\ R^{k-1}(\boldsymbol{C}^n : \boldsymbol{C}^m) & \longrightarrow & R^k(\boldsymbol{C}^n : \boldsymbol{C}^m) & \xrightarrow{\pi} & R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+ \end{array}$$

Then from Proposition 2.5 [**4**], we easily obtain the following

PROPOSITION 2.5. *The homomorphism*

$$\pi_* : H_*(R^k(\boldsymbol{C}^n : \boldsymbol{C}^m); \boldsymbol{Z}) \to \tilde{H}_*(R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+; \boldsymbol{Z})$$

*is surjective and* $\operatorname{Ker} \pi_*$ *is spanned by* $x_{i_1} \cdots x_{i_s}$, $m+1 \le i_1 < \cdots < i_s \le m+n$, $s \le k-1$.

We write $\pi_*(x_{i_1} \cdots x_{i_k}) \in \tilde{H}_*(R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+; \boldsymbol{Z})$ by the same symbol $x_{i_1} \cdots x_{i_k}$. Let $\Delta^k_{\boldsymbol{Z}}(x_{m+1}, \dots, x_{m+n}) = \boldsymbol{Z}\{x_{i_1} \cdots x_{i_k}\}$ be the module generated by $x_{i_1} \cdots x_{i_k}$, $m+1 \le i_1 < \cdots < i_k \le m+n$. Then we have an isomorphism

$$\tilde{H}_*(R_k(\boldsymbol{C}^n : \boldsymbol{C}^m)^+; \boldsymbol{Z}) \cong \Delta^k_{\boldsymbol{Z}}(x_{m+1}, \dots, x_{m+n})$$

as an abelian group.

Now let $A \in R^k(\boldsymbol{C}^{n+m})$ and $B \in R^l(\boldsymbol{C}^n : \boldsymbol{C}^m)$, then it is clear that the composition $AB \in R^{k+l}(\boldsymbol{C}^n : \boldsymbol{C}^m)$. Thus we obtain a pairing

$$\tilde{\mu} : R^k(\boldsymbol{C}^{n+m}) \times R^l(\boldsymbol{C}^n : \boldsymbol{C}^m) \to R^{k+l}(\boldsymbol{C}^n : \boldsymbol{C}^m).$$

Note that $\tilde{\mu}(R^{k-1}(\boldsymbol{C}^{n+m}) \times R^l(\boldsymbol{C}^n : \boldsymbol{C}^m)) \subset R^{k+l-1}(\boldsymbol{C}^n : \boldsymbol{C}^m)$ and $\tilde{\mu}(R^k(\boldsymbol{C}^{n+m}) \times R^{l-1}(\boldsymbol{C}^n : \boldsymbol{C}^m)) \subset R^{k+l-1}(\boldsymbol{C}^n : \boldsymbol{C}^m)$. Therefore, by identifying $R_k(\boldsymbol{C}^{n+m})^+$ with $R^k(\boldsymbol{C}^{n+m})/R^{k-1}(\boldsymbol{C}^{n+m})$ and $R_l(\boldsymbol{C}^n : \boldsymbol{C}^m)^+$ with $R^l(\boldsymbol{C}^n : \boldsymbol{C}^m)/R^{l-1}(\boldsymbol{C}^n : \boldsymbol{C}^m)$, we have an induced pairing

$$\tilde{\mu} : R_k(\boldsymbol{C}^{n+m})^+ \wedge R_l(\boldsymbol{C}^n : \boldsymbol{C}^m)^+ \to R_{k+l}(\boldsymbol{C}^n : \boldsymbol{C}^m)^+.$$

Now we have

PROPOSITION 2.6. *The diagram*

$$\begin{array}{ccc}
R_k(\boldsymbol{C}^{n+m})^+ \wedge R_l(\boldsymbol{C}^n : \boldsymbol{C}^m)^+ & \xrightarrow[\tilde{\mu}]{} & R_{k+l}(\boldsymbol{C}^n : \boldsymbol{C}^m)^+ \\
\Big\downarrow{\scriptstyle \sigma \wedge \varsigma} & & \Big\downarrow{\scriptstyle \varsigma} \\
R^k(\boldsymbol{C}^{n+m})^+ \wedge R^l(\boldsymbol{C}^n : \boldsymbol{C}^m)^+ & \xrightarrow[\tilde{\mu}]{} & R^{k+l}(\boldsymbol{C}^n : \boldsymbol{C}^m)^+
\end{array}$$

*is homotopy commutative where $\sigma$ is the stable splitting map of $U(n+m)$.*

PROOF. First note that the natural diagram

$$\begin{array}{ccc}
GL(k, \boldsymbol{C}) \times \mathrm{Mon}(\boldsymbol{C}^l, \boldsymbol{C}^{l+m}) & \longrightarrow & \mathrm{End}(\boldsymbol{C}^k) \times (\mathrm{End}(\boldsymbol{C}^l) \oplus M_{l,m}) \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{Mon}(\boldsymbol{C}^{k+l}, \boldsymbol{C}^{k+l+m}) & \longrightarrow & \mathrm{End}(\boldsymbol{C}^{k+l}) \oplus M_{k+l,m}
\end{array}$$

is a pull-back diagram, where all maps are inclusion maps. The horizontal maps are open embeddings and the vertical maps are proper. The upper horizontal and lower horizontal maps are $U(k) \times U(l)$ and $U(k+l)$ equivariant and vertical maps are equivariant with respect to the $U(k) \times U(l)$-action on $\mathrm{End}(\boldsymbol{C}^{k+l})$ induced by the inclusion $U(k) \times U(l) \subset U(k+l)$. Therefore we have an induced commutative diagram

$$\begin{array}{ccc}
\zeta_{n,k}(GL(k, \boldsymbol{C})) \times \zeta_{n,l}(\mathrm{Mon}(\boldsymbol{C}^l, \boldsymbol{C}^{l+m})) & \longrightarrow & \zeta_{n,k}(\mathrm{End}(\boldsymbol{C}^k)) \times \zeta_{n,l}(\mathrm{End}(\boldsymbol{C}^l) \oplus M_{l,m}) \\
\Big\downarrow & & \Big\downarrow \\
\zeta_{2n,k+l}(\mathrm{Mon}(\boldsymbol{C}^{k+l}, \boldsymbol{C}^{k+l+m})) & \longrightarrow & \zeta_{2n,k+l}(\mathrm{End}(\boldsymbol{C}^{k+l}) \oplus M_{k+l,m})
\end{array}$$

We can directly check that the diagram is pull-back. We can identify $\zeta_{n,k}(GL(k, \boldsymbol{C}))$ with $\zeta_{n,k}(U(k) \times \mathscr{H}(k))$ and $\zeta_{n,k}(\mathrm{Mon}(\boldsymbol{C}^l, \boldsymbol{C}^{l+m}))$ with $\zeta_{n,k}(U(l : m) \times \mathscr{H}(l))$ similarly for $\zeta_{2n,k+l}(\mathrm{Mon}(\boldsymbol{C}^{k+l}, \boldsymbol{C}^{k+l+m}))$. Let $\gamma_{k+l}$ be a vector bundle over $G_{2n,k+l}$ such that $\zeta_{2n,k+l}(\mathscr{H}(k+l)) \oplus \gamma_{k+l} \cong G_{2n,k+l} \times \boldsymbol{R}^N$. Let $f : G_{n,k} \times G_{n,l} \to G_{2n,k+l}$ be the natural map. Note that $\mathscr{H}(k+l) \cong \mathscr{H}(k) \oplus \mathscr{H}(l) \oplus \boldsymbol{C}^{kl}$ as $U(k) \times U(l)$-module where $\boldsymbol{C}^{kl}$ is a trivial $U(k) \times U(l)$-module. Hence $f^*(\zeta_{2n,k+l}(\mathscr{H}(k+l))) \cong \zeta_{n,k}(\mathscr{H}(k)) \times \zeta_{n,l}(\mathscr{H}(l)) \oplus (2kl)\varepsilon$, where $(2kl)\varepsilon$ is the trivial real bundle of dimension $2kl$. Therefore $\zeta_{n,k}(\mathscr{H}(k)) \times \zeta_{n,l}(\mathscr{H}(l)) \oplus f^*\gamma_{k+l} \oplus (2kl)\varepsilon$ is isomorphic to a trivial bundle. Hence we have a pull-back diagram

$$\begin{array}{ccc} \zeta_{n,k}(U(k)) \times \zeta_{n,l}(U(l:m)) \times \boldsymbol{R}^N & \subset & \zeta_{n,k}(\mathfrak{u}(k)) \times \zeta_{n,l}(\mathfrak{u}(l) \times M_{l,m}) \times \boldsymbol{R}^N \\ \downarrow & & \downarrow \\ \zeta_{2n,k+l}(U(k+l:m)) \times \boldsymbol{R}^N & \subset & \zeta_{2n,k+l}(\mathfrak{u}(k+l) \times M_{k+l,m}) \times \boldsymbol{R}^N \end{array}$$

of open embeddings and proper maps. Thus we obtain a homotopy commutative diagram of the Pontrjagin-Thom construction. The rest of the proof is similar to Lemma 2.6 [**4**]. □

THEOREM 2.7. *The homomorphism*

$$\varsigma_* : \tilde{H}_*(R_l(\boldsymbol{C}^n : \boldsymbol{C}^m)^+; \boldsymbol{Z}) \to H_*(R^l(\boldsymbol{C}^n : \boldsymbol{C}^m); \boldsymbol{Z})$$

*is given by*

$$\varsigma_*(x_{j_1} \cdots x_{j_l}) = x_{j_1} \cdots x_{j_l}.$$

PROOF. We prove the theorem by induction on $l$. It is clear for $l = 1$. Suppose that it is true up to $l$. By Proposition 2.6 we have the following commutative diagram

$$\begin{array}{ccc} \tilde{H}_*(R_k(\boldsymbol{C}^{n+m})^+; \boldsymbol{Z}) \otimes \tilde{H}_*(R_l(\boldsymbol{C}^n : \boldsymbol{C}^m)^+; \boldsymbol{Z}) & \xrightarrow[\tilde{\mu}_*]{} & \tilde{H}_*(R_{k+l}(\boldsymbol{C}^n : \boldsymbol{C}^m)^+; \boldsymbol{Z}) \\ \downarrow{\scriptstyle \sigma_* \otimes \varsigma_*} & & \downarrow{\scriptstyle \varsigma_*} \\ H_*(R^k(\boldsymbol{C}^{n+m}); \boldsymbol{Z}) \otimes H_*(R^l(\boldsymbol{C}^n : \boldsymbol{C}^m); \boldsymbol{Z}) & \xrightarrow[\tilde{\mu}_*]{} & H_*(R^{k+l}(\boldsymbol{C}^n : \boldsymbol{C}^m); \boldsymbol{Z}) \end{array}$$

It is clear that

$$\tilde{\mu}_*(x_{i_1} \cdots x_{i_k} \otimes x_{j_1} \cdots x_{j_l}) = \begin{cases} x_{i_1} \cdots x_{i_k} x_{j_1} \cdots x_{j_l}; & m+1 \le i_a, i_a \ne j_b \ \forall a, b \\ 0; & \text{otherwise} \end{cases}$$

We take $k = 1$. Then $x_{j_1} \cdots x_{j_l} x_{j_{l+1}} = \tilde{\mu}_*(x_{j_1} \otimes x_{j_l} \cdots x_{j_{l+1}})$. Then $\varsigma_*(x_{j_1} \cdots x_{j_l} x_{j_{l+1}}) = x_{j_1} \cdots x_{j_l} x_{j_{l+1}}$ by the assumption of induction. □

## 3. Adams operation and $p$-local stable splittings.

First of all, we recall the cohomology of $U(n:m)$. Let $y_i \in H^{2i-1}(U(n+m); \boldsymbol{Z})$ be a class transgressive to $c_i \in H^{2i}(BU(n+m); \boldsymbol{Z})$. Then it is well known that $y_i$ is primitive and we have

$$H^*(U(n+m); \boldsymbol{Z}) = \Lambda_{\boldsymbol{Z}}(y_1, \ldots, y_{n+m}).$$

Now consider the standard projection $p : U(n+m) \to U(n:m)$. Then the homomorphism $p^* : H^*(U(n:m); \boldsymbol{Z}) \to H^*(U(n+m); \boldsymbol{Z})$ is injective and $\operatorname{Im} p^*$ is spanned by $y_{i_1} \cdots y_{i_s}$, $m+1 \le i_1 < \cdots < i_s \le m+n$. If we identify

$H^*(U(n:m);\boldsymbol{Z})$ with its image in $H^*(U(n+m);\boldsymbol{Z})$, then we may write $H^*(U(n:m);\boldsymbol{Z}) = \Lambda_{\boldsymbol{Z}}(y_{m+1},\dots,y_{n+m})$.

Let $q$ be an integer such that $(q,m!)=1$. Then we know that there exists a map $\psi^q : BU(m) \to BU(m)$ such that $(\psi^q)^*(c_r) = q^r c_r$, $1 \le r \le m$. By applying the loop functor, we obtain a map

$$\Omega\psi^q : U(m) \to U(m).$$

Now we consider the fibration $U(m) \to U(n+m) \to U(n:m)$, then from the following diagram

$$\begin{array}{ccccccccc} U(m) & \longrightarrow & U(n+m) & \longrightarrow & U(n:m) & \longrightarrow & BU(m) & \longrightarrow & BU(n+m) \\ \downarrow{\scriptstyle \Omega\psi^q} & & \downarrow{\scriptstyle \Omega\psi^q} & & & & \downarrow{\scriptstyle \psi^q} & & \downarrow{\scriptstyle \psi^q} \\ U(m) & \longrightarrow & U(n+m) & \longrightarrow & U(n:m) & \longrightarrow & BU(m) & \longrightarrow & BU(n+m) \end{array}$$

we can show that there exists a map $U(n:m) \to U(n:m)$ such that all square diagrams are homotopy commutative. We denote this map by the same notation $\psi^q$.

About the generators $y_{m+1},\dots,y_{m+n}$, we have

Proposition 3.1. *The homomorphism*

$$(\psi^q)^* : H^*(U(n:m);\boldsymbol{Z}) \to H^*(U(n:m);\boldsymbol{Z})$$

*is given by*

$$(\psi^q)^*(y_r) = q^r y_r.$$

Proof. Since $(\Omega\psi^q) = q^r y_r$, the proposition follows from the above diagram. □

Recall that the homology of $U(n:m)$. Consider the standard projection $p : U(n+m) \to U(n:m)$. Let $s_{i-1}$ be a generator of $H_{2i-2}(\boldsymbol{CP}^{n+m-1};\boldsymbol{Z})$ and let $\sigma$ denote the homology suspension. Then we have

$$H_*(U(n+m);\boldsymbol{Z}) = \Lambda_{\boldsymbol{Z}}(x_1,\dots,x_{n+m})$$

where $x_i = \rho_*(\sigma(s_{i-1})) \in H_{2i-1}(U(n+m);\boldsymbol{Z})$. (See [**4**].)

Lemma 3.2. $\{p_*(x_{i_1}\cdots x_{i_s}) \mid m+1 \le i_1 < \cdots < i_s \le m+n\}$ *is a basis of* $H_*(U(n:m);\boldsymbol{Z})$.

Proof. For simplicity, we write $x_I = x_{i_1}\cdots x_{i_s}$ and $y_J = y_{j_1}\cdots y_{i_l}$ for multi-indices $I=(i_1,\dots,i_s)$ and $J=(j_1,\dots,j_l)$. It is shown in [**4**] that $x_I$ in $H_*(U(n+m);\boldsymbol{Z})$ is a dual to $y_J$ in $H^*(U(n+m);\boldsymbol{Z})$. Then we see that

$p_*(x_{i_1}\cdots x_{i_s})$ in $H_*(U(n:m);\boldsymbol{Z})$ for $m+1\le i_1<\cdots<i_s\le m+n$ is dual to $y_J$ in $H^*(U(n:m);\boldsymbol{Z})$. $\square$

We write $p_*(x_{i_1}\cdots x_{i_s})$ by the same symbol $x_{i_1}\cdots x_{i_s}$. Now let $p$ be an odd prime and $n+m$ be a positive integer. Then we can choose a prime $l$ such that $(l,(n+m)!)=1$ and $l$ generates the multiplicative group $\boldsymbol{Z}_p^\times$. In the ring $\{U(n:m),U(n:m)\}$ of homotopy classes of stable self maps, we can define, for each $t$ $(1\le t\le p-1)$, a stable map $\phi_t:U(n:m)\to U(n:m)$ by

$$\phi_t=\prod(\psi^l-l^i\,\mathrm{id}),\quad m+1\le i\le m+n\quad\text{and}\quad i\not\equiv t \bmod p-1$$

where the product is taken by means of composition.

PROPOSITION 3.3. *$(\phi_t)_*:H_*(U(n:m);\boldsymbol{Z})\to H_*(U(n:m);\boldsymbol{Z})$ is given by*

$$(\phi_t)_*(x_{i_1}\cdots x_{i_s})=\begin{cases}ax_{i_1}\cdots x_{i_s}; & i_1+\cdots+i_s\equiv t \bmod p-1\\ 0; & \textit{otherwise}\end{cases}$$

*where $a$ is a certain integer $\not\equiv 0 \bmod p$.*

PROOF. Consider $(\psi^l)^*:H^*(U(n:m);\boldsymbol{Z})\to H^*(U(n:m);\boldsymbol{Z})$. Since in general, in the ring $\{Z,Z\}$ of stable self maps of a spectrum $Z$, $(f+g)^*(z)=f^*(z)+g^*(z)$, we see $(\psi^l-l^i\,\mathrm{id})^*(y_{i_1}\cdots y_{i_s})=(l^{i_1+\cdots+i_s}-l^i)(y_{i_1}\cdots y_{i_s})$. Then clearly $(\phi_t)^*(y_{i_1}\cdots y_{i_s})=\prod(l^{i_1+\cdots+i_s}-l^i)(y_{i_1}\cdots y_{i_s})$. Since $l^k-1\equiv 0 \bmod p$ if and only if $k\equiv 0 \bmod p-1$, we see that $(\phi_t)^*$ satisfies the required property for the cohomology basis. Then the proposition follows from the duality. $\square$

Let $q_1,q_2,\ldots$ be all primes except $p$ and put $d_k=q_1\cdots q_k$. Consider a sequence

$$U(n:m)\xrightarrow{d_1}U(n:m)\xrightarrow{\phi_t}U(n:m)\xrightarrow{d_2}U(n:m)\xrightarrow{\phi_t}\cdots$$

where $d_k$ means the $d_k$-times of the identity. We denote by $Y_t$ the telescope of the sequence. Note that the map $d_k:U(n:m)\to U(n:m)$ is homologically diagonal. Let $\mu_t:U(n:m)\to Y_t$ be the natural inclusion. Then we have $(\mu_t)_*(x_{i_1}\cdots x_{i_s})=0$ for $i_1+\cdots+i_s\not\equiv t \bmod p-1$ and writing $(\mu_t)_*(x_{i_1}\cdots x_{i_s})$ also by $x_{i_1}\cdots x_{i_s}$ for $i_1+\cdots+i_s\equiv t \bmod p-1$. Then we have

THEOREM 3.4. *The map*

$$\bigvee\mu_t:U(n:m)\to\bigvee_{t=1}^{p-1}Y_t$$

*is a $p$-local equivalence.*

Thus by mixing the above stable splitting and Miller's stable splitting, we have

THEOREM 3.5. *Let $n$ be a positive integer and let $p$ be an odd prime. For each pair $(t,k)$ of integers such as $1 \le t \le p-1$ and $1 \le k \le n$, $M_{t,k}$ denotes the submodule of $H_*(U(n:m);\boldsymbol{Z}/p)$ spanned by $x_{i_1}\cdots x_{i_k}$ such that $i_1+\cdots+i_k \equiv t \bmod p-1$. Then there exists a finite spectrum $Y_{t,k}$ satisfying*

$$H_*(Y_{t,k};\boldsymbol{Z}/p) \cong M_{t,k}$$

*as a module, and a stable p-equivalence*

$$U(n:m) \to \bigvee Y_{t,k}$$

*where the wedge sum is taken over $1 \le t \le p-1$ and $1 \le k \le n$.*

PROOF. For $1 \le k \le n$ and $1 \le t \le p-1$, let $e_k$ and $f_t$ be the idempotents of the ring $\{U(n:m), U(n:m)\}$ coming from Miller's splitting and the one given by the Adams operation, respectively. Then we easily see that $(e_k)_*(f_t)_* = (f_t)_*(e_k)_*$ for all $t,k$. Let $Y_{t,k}$ be the telescope of the map $f_t \circ e_k$. Then Clearly $H_*(Y_{t,k};\boldsymbol{Z}) \cong \boldsymbol{Z}\{y_{i_1}\cdots y_{i_k} \mid i_1+\cdots i_k \equiv t \bmod p-1\}$. The latter half follows easily from what we have shown. □

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