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Fourier integral representation of harmonic functions in terms of a current

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Abstract. We give a Fourier integral representation of harmonic functions in three variables in terms of the current of integration over $\{z_1^2 + z_2^2 + z_3^2 = 0\} \subset \boldsymbol{C}^3$.

1. Introduction.

Let B_n be the open unit ball of $\boldsymbol{R}_t^n$ and $u(t) \in \mathscr{C}^\infty(\bar{B}_n)$ be a harmonic function in B_n. Then there exists an exponentially decreasing measure $d\mu(z)$ supported on $\{z \in \boldsymbol{C}^n; z_1^2 + \cdots + z_n^2 = 0\}$ such that $u(t) = \int e^{-i\langle z,t\rangle}\, d\mu(z)$.

This fact is an example of the Ehrenpreis fundamental principle concerning an arbitrary system with constant coefficients, which was originally shown by using the Hahn-Banach theorem. The abstract nature of the proof led to the lack of any explicit representation formula.

On the other hand, integral formulas in Several Complex Variables have recently been employed to solve division problems of holomorphic functions. By Fourier transform and duality, one immediately obtains explicit versions of the fundamental principle: see [**1**], [**2**], [**6**], [**9**] and [**11**]. The authors obtained explicit formulas with currents instead of measures.

In the particular case of the Laplacian, the general formula of [**2**] has some redundancy in the sense that it involves not only the Dirichlet boundary value but also some other data; the former is enough to determine a harmonic function. It has motivated us to try to find a Fourier integral representation formula of harmonic functions free from any superfluous data.

In [**10**], we considered the two variables case and gave an explicit formula which represents a harmonic function in the unit disk B_2 in terms of its Dirichlet boundary value, exponential functions and a current supported on $\{z \in \boldsymbol{C}^2; z_1^2 + z_2^2 = 0\}$.

In the present paper, we deal with the three variables case and prove that

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$$u(t) = [V].\frac{1}{4\pi^2}\left(2 - \frac{1}{|y|}\right)v(y/|y|)e^{-i\langle z, t-y/|y|\rangle}(\bar{\partial}\partial|y|)^2,$$

where $u \in \mathscr{C}^0(\bar{B}_3)$ is harmonic in the unit ball B_3, $[V]$ is the current of integration over $V = \{z \in \boldsymbol{C}^3; z^2 = z_1^2 + z_2^2 + z_3^2 = 0\}$, $y = \operatorname{Im} z$, and v is the Dirichlet boundary value of $u(t)$. That is, we show that the right hand side coincides with the Poisson integral.

As is proved in [**4**] and [**8**], the residue current $\bar{\partial}[1/z^2]$ has the property that $\bar{\partial}[1/z^2] \wedge d(z^2) = 2\pi i[V]$. Therefore the above formula is a representation of $u(t)$ by $\bar{\partial}[1/z^2]$.

2. Geometry.

In $\boldsymbol{C}_z^3$, $z = (z_1, z_2, z_3)$, we consider the analytic set $V = \{z \in \boldsymbol{C}^3; z^2 = z_1^2 + z_2^2 + z_3^2 = 0\}$. It is singular at the origin. The current of integration over (the smooth locus of) V is denoted by $[V]$.

Set $x_j = \operatorname{Re} z_j, y_j = \operatorname{Im} z_j$ $(j = 1, 2, 3)$ and $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$. The equation $z^2 = |x|^2 - |y|^2 + 2i\langle x, y\rangle = 0$ is equivalent to saying that $|x| = |y|$ and that x and y are perpendicular to each other. For a fixed $y \neq 0$, the totality of the corresponding x's can be identified with $S^1 = \boldsymbol{R}/2\pi\boldsymbol{Z}$. We will give a parametrization of an open dense subset of V based on this observation.

Let $\hat{V}$ be the open dense subset of V defined by $\hat{V} = V \setminus \{y_1 = y_2 = 0\}$. Set $E = \boldsymbol{R}_y^3 \setminus \{y_1 = y_2 = 0\}$.

For $y = (y_1, y_2, y_3) \in E$, put

$$v = \frac{|y|}{\sqrt{y_1^2 + y_2^2}}(-y_2, y_1, 0), \quad w = \frac{1}{|y|}y \times v = \frac{(-y_1y_3, -y_2y_3, y_1^2 + y_2^2)}{\sqrt{y_1^2 + y_2^2}},$$

where $\times$ is the cross product. We see that $\langle y, v\rangle = \langle v, w\rangle = \langle w, y\rangle = 0$ and that $|y| = |v| = |w|$.

We define a diffeomorphism

$$\Phi : E \times S^1 \xrightarrow{\sim} \hat{V}, \quad (y, \theta) \mapsto z = x(y, \theta) + iy$$

by setting $x(y, \theta) = v\cos\theta + w\sin\theta$. It means that

$$x_1 = -|y|y_2\cos\theta/\sqrt{y_1^2 + y_2^2} - y_1y_3\sin\theta/\sqrt{y_1^2 + y_2^2},$$

$$x_2 = |y|y_1\cos\theta/\sqrt{y_1^2 + y_2^2} - y_2y_3\sin\theta/\sqrt{y_1^2 + y_2^2},$$

$$x_3 = \sqrt{y_1^2 + y_2^2}\sin\theta.$$

It is trivial that $\Phi^*(y_j) = y_j, \Phi^*(dy_j) = dy_j$ $(j = 1, 2, 3)$.

Since $\hat{V}$ is an open dense subset of V, the action of the current $[V]$ can be expressed as integration over $E \times S^1$.

PROPOSITION 1. *The* 4-*form* $(\Phi^{-1})^*(dy_1 \wedge dy_2 \wedge dy_3 \wedge d\theta)$ *is positive with respect to the natural orientation of* $\hat{V}$.

PROOF. In $\{z \in V; y_3 \neq 0\}$, (z_1, z_2) is a system of local holomorphic coordinates. So $dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$ is positive with respect to the natural orientation.

On the other hand, we can show that

$$\begin{aligned}
&\Phi^*(dx_1)\\
&\quad = -\frac{(-|y|y_2 \sin\theta + y_1 y_3 \cos\theta)\, d\theta}{\sqrt{y_1^2 + y_2^2}} + \frac{y_2 y_3(-|y|y_2 \sin\theta + y_1 y_3 \cos\theta)\, dy_1}{(y_1^2 + y_2^2)^{3/2}|y|}\\
&\qquad - \frac{\{(y_2^4 + 2y_1^2 y_2^2 + y_1^4 + y_1^2 y_3^2)\cos\theta - y_1 y_2 y_3 |y| \sin\theta\}\, dy_2}{(y_1^2 + y_2^2)^{3/2}|y|}\\
&\qquad - \frac{(y_2 y_3 \cos\theta + y_1 |y| \sin\theta)\, dy_3}{\sqrt{y_1^2 + y_2^2}|y|},
\end{aligned}$$

$$\begin{aligned}
&\Phi^*(dx_2)\\
&\quad = -\frac{(y_2 y_3 \cos\theta + y_1 |y| \sin\theta)\, d\theta}{\sqrt{y_1^2 + y_2^2}}\\
&\qquad + \frac{\{(2y_1^2 y_2^2 + y_1^4 + y_2^4 + y_2^2 y_3^2)\cos\theta + y_1 y_2 y_3 |y| \sin\theta\}\, dy_1}{(y_1^2 + y_2^2)^{3/2}|y|}\\
&\qquad - \frac{y_1 y_3 (y_2 y_3 \cos\theta + y_1 |y| \sin\theta)\, dy_2}{(y_1^2 + y_2^2)^{3/2}|y|} + \frac{(-|y|y_2 \sin\theta + y_1 y_3 \cos\theta)\, dy_3}{\sqrt{y_1^2 + y_2^2}|y|}.
\end{aligned}$$

It implies that

$$\Phi^*(dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2) = \frac{(y_1^2 + y_2^2)\sin^2\theta + y_3^2}{|y|} dy_1 \wedge dy_2 \wedge dy_3 \wedge d\theta.$$

Since $\{(y_1^2 + y_2^2)\sin^2\theta + y_3^2\}/|y|$ is positive, $dy_1 \wedge dy_2 \wedge dy_3 \wedge d\theta$ determines the same orientation as $dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$. □

REMARK. One may feel uneasy about the singularity along $y_1 = y_2 = 0$, but this kind of difficulty is inevitable. In fact, if there is a continuous mapping $\varphi : \boldsymbol{R}^3 \backslash \{0\} \to \boldsymbol{R}^3 \backslash \{0\}$ such that $\varphi(y)$ is perpendicular to y, then $\varphi|_{S^2}$ induces a nonvanishing vector field on S^2. It is known, however, that this is impossible because of Hopf's theorem.

3. Integral operators.

Let B_3 be the open unit ball in $\boldsymbol{R}_t^3$. Suppose that v is a continuous function on $\partial B_3 = S^2$.

We introduce two integral operators $Q_0, Q_1 : \mathscr{C}^0(S^2) \to \mathscr{C}^\infty(B_3)$ by

$$Q_0[v](t) = [V].v(y/|y|)e^{-i\langle z, t-y/|y|\rangle}(-2i\bar{\partial}\partial|y|)^2,$$

$$Q_1[v](t) = [V].\frac{v(y/|y|)}{|y|}e^{-i\langle z, t-y/|y|\rangle}(-2i\bar{\partial}\partial|y|)^2$$

for $t \in B_3$.

LEMMA 1. *If f is a function of y, we have*

$$-2i\bar{\partial}\partial f = \sum_{j=1}^{3}\frac{\partial^2 f}{\partial y_j^2}dx_j \wedge dy_j + \sum{}' \frac{\partial^2 f}{\partial y_j \partial y_k}(dx_j \wedge dy_k + dx_k \wedge dy_j),$$

where $\sum'$ denotes a sum with respect to $(j,k) = (1,2),(1,3),(2,3)$. In particular, for $f(y) = |y|$, we have

$$-2i\bar{\partial}\partial|y| = \sum_{j=1}^{3}\left(\frac{1}{|y|} - \frac{y_j^2}{|y|^3}\right)dx_j \wedge dy_j - \sum{}' \frac{y_j y_k}{|y|^3}(dx_j \wedge dy_k + dx_k \wedge dy_j).$$

PROOF. By direct calculation. □

PROPOSITION 2. *On $E \times S^1$, we have*

$$\Phi^*(-i\langle z, t - y/|y|\rangle) = \langle y, t\rangle - |y| - i\langle x(y,\theta), t\rangle,$$

$$\Phi^*((-2i\bar{\partial}\partial|y|)^2) = -\frac{2}{|y|}dy_1 \wedge dy_2 \wedge dy_3 \wedge d\theta.$$

PROOF. The first equality follows from $\langle x, y\rangle = 0$.

By Lemma 1, we get

$$\begin{aligned}
&|y|^4(-2i\bar{\partial}\partial|y|)^2 \\
&\quad = 2(y_1^2\,dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3 + y_2^2\,dx_1 \wedge dy_1 \wedge dx_3 \wedge dy_3 \\
&\qquad + y_3^2\,dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 - y_1 y_3\,dx_2 \wedge dy_2 \wedge dx_1 \wedge dy_3 \\
&\qquad - y_1 y_2\,dx_1 \wedge dy_2 \wedge dx_3 \wedge dy_3 - y_1 y_3\,dx_3 \wedge dy_1 \wedge dx_2 \wedge dy_2
\end{aligned}$$

$$+ y_2 y_3\, dx_3 \wedge dy_1 \wedge dx_1 \wedge dy_2 - y_2 y_3\, dx_2 \wedge dy_3 \wedge dx_1 \wedge dy_1$$
$$+ y_1 y_2\, dx_2 \wedge dy_3 \wedge dx_3 \wedge dy_1).$$

We have already calculated $\Phi^*(dx_1)$ and $\Phi^*(dx_2)$ in the proof of Proposition 1. Moreover we have

$$\Phi^*(dx_3) = \sqrt{y_1^2 + y_2^2}\cos\theta\, d\theta + \frac{y_1 \sin\theta}{\sqrt{y_1^2 + y_2^2}} dy_1 + \frac{y_2 \sin\theta}{\sqrt{y_1^2 + y_2^2}} dy_2.$$

The evaluation of $\Phi^*((-2i\bar{\partial}\partial|y|)^2)$ involves very long and tedious calculations. We omit the details. (The difforms package of Maple helps a lot.) □

The integrals defining Q_0 and Q_1 will be calculated in terms of $(y, \theta) \in E \times S^1$. Notice that the set $\{y_1 = y_2 = 0\}$ is of measure 0.

Propositions 1 and 2 imply that

$$Q_0[v](t) = \int_{\boldsymbol{R}^3 \times S^1} e^{\langle y, t\rangle - |y| - i\langle x(y,\theta), t\rangle} \frac{-2v(y/|y|)}{|y|} dy_1 dy_2 dy_3 d\theta,$$
$$Q_1[v](t) = \int_{\boldsymbol{R}^3 \times S^1} e^{\langle y, t\rangle - |y| - i\langle x(y,\theta), t\rangle} \frac{-2v(y/|y|)}{|y|^2} dy_1 dy_2 dy_3 d\theta.$$

Let us introduce the polar coordinates in $\boldsymbol{R}^3$. Set $q = |y|, s_j = y_j/q$ $(j = 1, 2, 3)$, $s = (s_1, s_2, s_3) \in S^2 \backslash \{(0, 0, \pm 1)\}$. Let $d\sigma(s)$ be the surface-area measure on S^2. Then the integration with respect to q is of Laplace type and we get

$$\begin{aligned} Q_0[v](t) &= \int_{S^2} d\sigma(s) \int_{S^1} d\theta \int_0^\infty -2q e^{-q\{1 - \langle s, t\rangle + i\langle x/q, t\rangle\}} v(s)\, dr \\ &= -2 \int_{S^2} d\sigma(s) \int_{S^1} \frac{v(s)}{\{1 - \langle s, t\rangle + i\langle x/q, t\rangle\}^2} d\theta, \\ Q_1[v](t) &= \int_{S^2} d\sigma(s) \int_{S^1} d\theta \int_0^\infty -2 e^{-q\{1 - \langle s, t\rangle + i\langle x/q, t\rangle\}} v(s)\, dr \\ &= -2 \int_{S^2} d\sigma(s) \int_{S^1} \frac{v(s)}{1 - \langle s, t\rangle + i\langle x/q, t\rangle} d\theta. \end{aligned}$$

Here we write $x = x(y, \theta)$ for brevity and x/q is independent of q.

Now we give some elementary formulas which will be used later.

Lemma 2. *Assume that a is positive and that b is real, then we have*

$$\int_0^{2\pi} \frac{d\theta}{a + ib\sin\theta} = \frac{2\pi}{\sqrt{a^2 + b^2}}, \quad \int_0^{2\pi} \frac{d\theta}{(a + ib\sin\theta)^2} = \frac{2\pi a}{(a^2 + b^2)^{3/2}}.$$

PROOF. The former can be proved by residue calculus. We obtain the latter by differentiation with respect to the parameter a. $\square$

LEMMA 3. *Assume that a is positive and that b and c are real, then we have*

$$\int_0^{2\pi} \frac{d\theta}{a + i(b\sin\theta + c\cos\theta)} = \frac{2\pi}{\sqrt{a^2+b^2+c^2}},$$

$$\int_0^{2\pi} \frac{d\theta}{\{a + i(b\sin\theta + c\cos\theta)\}^2} = \frac{2\pi a}{(a^2+b^2+c^2)^{3/2}}.$$

PROOF. We have $b\sin\theta + c\cos\theta = \sqrt{b^2+c^2}\sin(\theta+\alpha)$, $\alpha = \arg(b+ic)$. For any function $f = f(\theta)$ of period 2π, it holds that $\int_0^{2\pi} f(\theta+\alpha)\,d\theta = \int_0^{2\pi} f(\theta)\,d\theta$ and it implies that

$$\int_0^{2\pi} \frac{d\theta}{\{a + i\sqrt{b^2+c^2}\sin(\theta+\alpha)\}^k} = \int_0^{2\pi} \frac{d\theta}{(a + i\sqrt{b^2+c^2}\sin\theta)^k}$$

for $k = 1, 2$. Then apply the previous lemma. $\square$

Now set

$$a = 1 - \langle s, t\rangle = 1 - s_1t_1 - s_2t_2 - s_3t_3 > 0,$$

$$b = -\frac{s_1s_3t_1}{\sqrt{s_1^2+s_2^2}} - \frac{s_2s_3t_2}{\sqrt{s_1^2+s_2^2}} + \sqrt{s_1^2+s_2^2}\,t_3,$$

$$c = -\frac{s_2t_1}{\sqrt{s_1^2+s_2^2}} + \frac{s_1t_2}{\sqrt{s_1^2+s_2^2}},$$

then $\langle x/q, t\rangle = b\sin\theta + c\cos\theta$ and we have

$$Q_0[v](t) = -2\int_{S^2} \frac{2\pi a v(s)}{(a^2+b^2+c^2)^{3/2}}\,d\sigma(s),$$

$$Q_1[v](t) = -2\int_{S^2} \frac{2\pi v(s)}{\sqrt{a^2+b^2+c^2}}\,d\sigma(s).$$

LEMMA 4. *For $s \in S^2\backslash\{(0,0,\pm 1)\}, t \in B_3$ and (a,b,c) above, we have*

$$\sqrt{a^2+b^2+c^2} = |s-t|.$$

PROOF. We can show that

$$(a^2+b^2+c^2-|s-t|^2)(s_1^2+s_2^2)$$
$$= (s_1^2+s_2^2+s_3^2-1)(s_1^2t_1^2+s_1^2t_3^2-s_1^2+2s_1s_2t_1t_2+s_2^2t_2^2-s_2^2+s_2^2t_3^2).$$

The right hand side vanishes because $s \in S^2 \backslash \{(0,0,\pm 1)\}$. Notice that $s_1^2 + s_2^2 \neq 0$. □

By this lemma we find that $Q_0[v](t)$ is similar to the Poisson integral and $Q_1[v](t)$ is a simple layer potential. More precisely, we have

Proposition 3.

$$Q_0[v](t) = -4\pi \int_{S^2} \frac{1 - \langle s, t\rangle}{|s-t|^3} v(s)\, d\sigma(s),$$

$$Q_1[v](t) = -4\pi \int_{S^2} \frac{1}{|s-t|} v(s)\, d\sigma(s).$$

4. Main result.

Let B_3 and v be as in §3.
We define another integral operator $Q : \mathscr{C}^0(S^2) \to \mathscr{C}^\infty(B_3)$ by

$$Q[v](t) = [V].\frac{1}{4\pi^2}\left(2 - \frac{1}{|y|}\right) v(y/|y|) e^{-i\langle z, t - y/|y|\rangle} (\bar{\partial}\partial |y|)^2, \quad t \in B_3.$$

Proposition 4. *The operator Q coincides with the Poisson integral*:

$$Q[v](t) = \frac{1}{4\pi} \int_{S^2} \frac{1 - |t|^2}{|s-t|^3} v(s)\, d\sigma(s), \quad t \in B_3.$$

Proof. Since $|s| = 1$, we find that $2(1 - \langle s, t\rangle) - |s-t|^2 = 1 - |t|^2$ and that

$$\frac{2}{-4\pi} Q_0[v](t) - \frac{1}{-4\pi} Q_1[v](t) = \int_{S^2} \frac{1 - |t|^2}{|s-t|^3} v(s)\, d\sigma(s).$$

The proposition follows from the equality

$$Q[v](t) = \frac{2}{-16\pi^2} Q_0[v](t) - \frac{1}{-16\pi^2} Q_1[v](t).$$ □

This proposition implies the following theorem, which is our main result:

Theorem 1. *If $u(t) \in \mathscr{C}^0(\bar{B}_3)$ is harmonic in B_3 and $v \in \mathscr{C}^0(S^2)$ is its Dirichlet boundary value, then $u(t) = Q[v](t)$ holds in B_3. In particular, $u(t)$ is a superposition of the exponentials* $\exp(-i\langle z, t\rangle)$ *with* $z^2 = z_1^2 + z_2^2 + z_3^2 = 0$ *and* $y/|y| \in \operatorname{supp} v$.

This Fourier integral representation is still valid in the complex domain. Indeed, we have the following result:

Corollary 1. *In the situation of the above theorem, u is holomorphic and satisfies $u(t+i\tau) = Q[v](t+i\tau)$ in the Levi ball $\{t+i\tau \in \boldsymbol{C}^3; |t|+|\tau| < 1\}$.*

Proof. First notice that

$$\mathrm{Re}(-i\langle z, t+i\tau - y/|y|\rangle) = \langle y, t\rangle + \langle x, \tau\rangle - |y| \leq -\varepsilon|y|$$

if $0 < \varepsilon < 1, |t|+|\tau| < 1-\varepsilon, |x| = |y|$.

The rapid decrease of the exponential factor guarantees the convergence of the integral in $\{t+i\tau \in \boldsymbol{C}^3; |t|+|\tau| < 1-\varepsilon\}$. Since ε is arbitrary, it is convergent in the Levi ball. □

Remark. It is easy to show that a harmonic function $u(t)$ in B_3 can be analytically continued to the Levi ball without assuming that it is continuous up to the boundary. Indeed, consider $u(t/(1-\delta))$, $0 < \delta < 1$, which is harmonic in B_3 and is continuous up to the boundary. By Corollary 1, we find that $u(t)$ can be analytically continued to $\{t+i\tau; |t|+|\tau| < 1-\delta\}$. Since δ is arbitrary, u is holomorphic in the Levi ball.

However, this is not the best result: u is holomorphic in the Lie ball $\{t+i\tau; L(t+i\tau) < 1\}$, where

$$L(t+i\tau) = \sqrt{|t|^2 + |\tau|^2 + 2\sqrt{|t|^2|\tau|^2 - \langle t, \tau\rangle^2}} \leq |t|+|\tau|.$$

See p. 69 of [**7**] for the proof of this fact.

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