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Pseudo-Riemannian manifolds with simple Jacobi operators

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Abstract. Osserman pseudo-Riemannian manifolds with diagonalizable Jacobi operators are studied. A classification of such manifolds is achieved under two conditions on the number of different eigenvalues of the Jacobi operators and their associated eigenspaces.

1. Introduction.

A Riemannian manifold (M, g) is said to be an *Osserman space* if the eigenvalues of the Jacobi operators are constant on the unit sphere bundle. Clearly, Euclidean spaces are Osserman and moreover, any rank-one symmetric space is so. The lack of other examples led Osserman to conjecture that any Osserman Riemannian manifold must be locally isometric to a two-point homogeneous space ([**21**]). It was proved by Chi that the above conjecture holds true in many cases. In particular, he showed that any n-dimensional Osserman Riemannian manifold is locally a space of constant curvature or a Kähler manifold of constant holomorphic sectional curvature, provided that $n \neq 4k$, $k > 1$ ([**9**]). (See also [**10**], [**11**], [**17**] for related work). A pseudo-Riemannian manifold (M, g) is said to be *Osserman* if the (possibly complex) eigenvalues of the Jacobi operators R_x are independent of the spacelike or timelike unit vectors x. It is shown in [**5**], [**14**] that a Lorentzian manifold (M, g) is Osserman at a point $p \in M$ if and only if the sectional curvature is constant at p and hence that Osserman Lorentzian manifolds are locally real space forms. The situation is however much more complicated when metrics of other signatures are considered. Indeed, the existence of nonsymmetric (even not locally homogeneous) Osserman pseudo-Riemannian manifolds of any signature (p, q), $p, q \geq 2$ is shown in [**7**], [**8**], [**15**]. A specific feature of those examples is that all of them have nondiagonalizable Jacobi operators. Therefore, in order to characterize

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Osserman pseudo-Riemannian spaces, it seemed natural to consider those with diagonalizable Jacobi operators at the first step. This has been done by Blažić, Bokan and Rakić, who showed that a four-dimensional Osserman manifold with metric of signature $(2,2)$ and diagonalizable Jacobi operators must be locally isometric to a real, complex or paracomplex space form [**6**].

The purpose of this paper is to study further Osserman pseudo-Riemannian manifolds of higher dimensions and arbitrary signature under the assumption of the diagonalizability of the Jacobi operators. It is clear that, if the Jacobi operator associated to each unit x is diagonalizable with a unique eigenvalue (which changes sign from spacelike to timelike vectors), then the manifold is a space of constant curvature. Therefore, we devote our attention to the first nontrivial case: the Jacobi operators having two distinct eigenvalues.

(I) *For each unit vector x, the Jacobi operator R_x is diagonalizable with exactly two distinct eigenvalues*: $\varepsilon_x\lambda$ *and* $\varepsilon_x\mu$, *where* $\varepsilon_x = g(x,x)$.

Simplest examples of pseudo-Riemannian manifolds satisfying (I) are indefinite Kähler manifolds of constant holomorphic sectional curvature and para-Kähler manifolds of constant paraholomorphic sectional curvature. (See [**1**], [**12**] and [**13**] for details and further references). Moreover, indefinite quaternionic and paraquaternionic space forms (cf. [**4**], [**16**], [**22**]) as well as the Cayley planes over the octonians and the anti-octonians (cf. [**23**]) satisfy (I) above.

It is well-known that not only the eigenvalues, but also the existence of distinguished eigenspaces for the Jacobi operators provides of insight geometric information. Hence, in what follows, we will make an additional hypothesis on the eigenspaces of the Jacobi operators which may be viewed as an infinitesimal version of Hopf fibration in terms of the eigenspace associated to one of the two eigenvalues of R_x. To describe this property, we introduce the following notation. For each unit vector x, let $E_\lambda(x)$ denote the subspace generated by x and the eigenspace associated to the eigenvalue $\varepsilon_x\lambda$ of the Jacobi operator R_x, i.e., $E_\lambda(x) = \langle x\rangle \oplus \ker(R_x - \varepsilon_x\lambda\,\mathrm{Id})$. Now, we put

(II) *If z is a unit vector in $E_\lambda(x)$, then $E_\lambda(x) = E_\lambda(z)$ and moreover, if* $y \in \ker(R_x - \varepsilon_x\mu\,\mathrm{Id})$, *then* $x \in \ker(R_y - \varepsilon_y\mu\,\mathrm{Id})$.

Let (M,g,J) be a nonflat indefinite complex or paracomplex space form. Then the Jacobi operator associated to each unit vector x has exactly two distinct eigenvalues $\varepsilon_x\lambda$ and $(\varepsilon_x/4)\lambda$. Moreover, $E_\lambda(x) = \langle\{x, Jx\}\rangle$ for each x, where J denotes the complex or paracomplex structure of M and (II) follows immediately from $g(x,Jy) + g(Jx,y) = 0$ (Furthermore, note that the induced inner product on $E_\lambda(x)$ is positive or negative definite for each unit $x \in T_pM$ if (M,g,J) is an indefinite complex space form, but it is of Lorentzian signature in the paracomplex case). Similarly, (II) is satisfied by indefinite quaternionic and paraquaternionic space forms, as well as the Cayley planes over the octonians and the anti-octonians. Note here that conditions (I) and (II) above are independent (see Remark 4.1 for examples of curvaturelike functions satisfying (I) but not (II)).

For the sake of simplicity, we state the following definition

Definition 1.1. A pseudo-Riemannian manifold (M, g) is said to be a special Osserman space if (I) and (II) above are satisfied.

The purpose of this paper is to prove the following result on the characterization of special Osserman manifolds:

Theorem 1.1. *Let (M, g) be a complete and simply connected special Osserman pseudo-Riemannian manifold. Then it is isometric to one of the following*:
(a) *an indefinite complex space form,*
(b) *an indefinite quaternionic space form,*
(c) *a paracomplex space form,*
(d) *a paraquaternionic space form, or*
(e) *a Cayley plane over the octonians with definite or indefinite metric, or a Cayley plane over the anti-octonians with indefinite metric of signature* $(8, 8)$.

The paper is organized as follows. In §2 we derive some identities for the curvature of special Osserman manifolds to be used through this paper. Section 3 is devoted to the study of the possible multiplicities of the distinguished eigenvalue λ. To do that we construct certain Clifford modules and, by means of the use of topological restrictions on their existence, we obtain all the possible multiplicities of λ as well as the dimensions and signatures of the corresponding tangent spaces where such curvature tensors are realized (cf. Theorem 3.2). In section 4 we provide a proof of the local version of (a)–(d) in Theorem 1.1. To do that, we firstly show that the curvature tensor of a special Osserman manifold with eigenvalue λ of multiplicity different from 7 and 15 can be expressed in terms of certain curvaturelike functions at each T_pM (cf. Theorem 4.2). Then, in §4.2, we show that any special Osserman manifold with eigenvalue λ of multiplicity different from 7 and 15 is locally symmetric and (a)–(d) in Theorem 1.1 are obtained. Finally, the study of the exceptional cases corresponding to the multiplicity of λ equal to 7 and 15 is developed in §5.

2. Preliminaries.

In this section we will restrict our attention to the study of the tangent space at an arbitrary point $p \in M$ of a special Osserman manifold (M^n, g). First of all, it is worth to recall from (II) that the subspace $E_\mu(x)$ does not satisfy a similar condition to that imposed on $E_\lambda(x)$. This shows that the eigenvalues $\varepsilon_x\lambda$ and $\varepsilon_x\mu$ play different roles in the geometry of special Osserman manifolds.

LEMMA 2.1. *Let* (M,g) *be a special Osserman manifold. Then,*

(i) *If* x, y *are unit vectors, with* $y \in E_\lambda(x)^\perp$, *then* $y \in \ker(R_x - \varepsilon_x \mu \, \mathrm{Id})$.

(ii) *If* x, y *are chosen as in* (i), *then* $E_\lambda(y) \perp E_\lambda(x)$.

(iii) T_pM *can be decomposed as a direct sum of orthogonal subspaces* $E_\lambda(\cdot)$.

PROOF. (i) follows immediately from the tangent space decomposition $T_pM = E_\lambda(x) \oplus \ker(R_x - \varepsilon_x \mu \, \mathrm{Id})$ since y is assumed to be orthogonal to $E_\lambda(x)$. To prove (ii), let $\bar{x}$ be a unit vector in $E_\lambda(x)$. Then (II) implies that $E_\lambda(\bar{x})$ coincides with $E_\lambda(x)$ and therefore, it follows from (i) that $y \in \ker(R_{\bar{x}} - \varepsilon_{\bar{x}} \mu \, \mathrm{Id})$. Then, again by (II), we have $\bar{x} \in \ker(R_y - \varepsilon_y \mu \, \mathrm{Id})$, and thus $\bar{x} \perp E_\lambda(y)$ for all unit vectors $\bar{x} \in E_\lambda(x)$. Finally, (iii) is obtained as a direct application of (ii). □

Now, it immediately follows from (I) and (II) that

LEMMA 2.2. *Let* $T_pM = E_\lambda(x) \oplus E_\lambda(y) \oplus E_\lambda(z) \oplus \cdots$ *be an orthogonal decomposition of* T_pM *as in Lemma* 2.1. *Then*

(i) $R(y, x_1)x_2 = -R(y, x_2)x_1 = -(1/2)R(x_1, x_2)y$,

(ii) $R(x, y)z = 0$,

(iii) $R(x_1, x_2)x_3 = 0$,

where x_1, x_2, x_3 *are orthogonal unit vectors in* $E_\lambda(x)$.

From now on we denote by τ the multiplicity of the distinguished eigenvalue λ. Therefore, $E_\lambda(\xi_0)$ is a $(\tau+1)$-dimensional subspace of T_pM, for each unit $\xi_0 \in T_pM$. Moreover, put $E_\lambda(\xi_0) = \langle\{\xi_0, \xi_1, \ldots, \xi_\tau\}\rangle$, where $\{\xi_1, \ldots, \xi_\tau\}$ is an orthonormal basis of $\ker(R_{\xi_0} - \varepsilon_{\xi_0} \lambda \, \mathrm{Id})$. Next we investigate the form of the Jacobi operator R_w associated to any unit vector w of the form $w = ax_0 + by_0$, where $y_0 \in E_\lambda(x_0)^\perp$. This will allow us to obtain (2.1) below, which will be extensively used in what follows.

LEMMA 2.3. *Let* x_0, y_0 *be unit vectors with* $y_0 \in E_\lambda(x_0)^\perp$. *The relation*

$$\sum_{j=1}^{\tau} R(x_i, y_0, x_0, y_j) R(x_k, y_0, x_0, y_j) \varepsilon_{y_j} = \delta_{ik} \left(\frac{\lambda - \mu}{3} \right)^2 \varepsilon_{x_0} \varepsilon_{x_i} \varepsilon_{y_0} \tag{2.1}$$

holds for all $i, k = 1, \ldots, \tau$, *where* $E_\lambda(x_0) = \langle\{x_0, x_1, \ldots, x_\tau\}\rangle$, $E_\lambda(y_0) = \langle\{y_0, y_1, \ldots, y_\tau\}\rangle$ *and* δ_{ik} *denotes the Kronecker's delta.*

PROOF. Let x_0, y_0 be unit vectors with $y_0 \in E_\lambda(x_0)^\perp$ and take nonzero a, b such that $w = ax_0 + by_0$ is a unit vector. Further, let $\{x_0, x_1, \ldots, x_\tau, y_0, y_1, \ldots, y_\tau, z_1, \ldots, z_{n-2(\tau+1)}\}$ be an orthonormal basis associated with the decomposition $T_pM = E_\lambda(x_0) \oplus E_\lambda(y_0) \oplus (E_\lambda(x_0) \oplus E_\lambda(y_0))^\perp$. Then one has

$$R_w(x_0) = -ab\mu\varepsilon_{x_0} y_0 + b^2 \mu \varepsilon_{y_0} x_0$$

$$R_w(x_i) = (a^2 \varepsilon_{x_0} \lambda + b^2 \varepsilon_{y_0} \mu) x_i + ab(R(x_i, y_0)x_0 + R(x_i, x_0)y_0).$$

Now, since $R(x_i, x_0)y_0 = 2R(x_i, y_0)x_0$, it follows from Lemma 2.2 that $R(x_i, y_0)x_0 \in \langle\{y_1, \dots, y_\tau\}\rangle$, and hence

$$R_w(x_i) = (a^2\varepsilon_{x_0}\lambda + b^2\varepsilon_{y_0}\mu)x_i + 3ab\sum_{j=1}^{\tau} R(x_i, y_0, x_0, y_j)\varepsilon_{y_j}y_j,$$

for all $i = 1, \dots, \tau$. In an analogous way

$$R_w(y_0) = a^2\mu\varepsilon_{x_0}y_0 - ab\mu\varepsilon_{y_0}x_0$$

$$R_w(y_i) = 3ab\sum_{j=1}^{\tau} R(x_j, y_0, x_0, y_i)\varepsilon_{x_j}x_j + (a^2\varepsilon_{x_0}\mu + b^2\varepsilon_{y_0}\lambda)y_i, \quad (i = 1, \dots, \tau).$$

Moreover, since $R(z_i, x_0)y_0 = R(z_i, y_0)x_0 = 0$, one has

$$R_w(z_i) = \mu\varepsilon_w z_i$$

for all $z_i \in (E_\lambda(x_0) \oplus E_\lambda(y_0))^\perp$.

Next consider the eigenspace of R_w associated to the eigenvalue $\varepsilon_w\lambda$. Since $\{b\varepsilon_{x_0}\varepsilon_{y_0}x_0 - ay_0, x_1, \dots, x_\tau, y_1, \dots, y_\tau, z_1, \dots, z_{n-2(\tau+1)}\}$ is an orthonormal basis of $\langle w\rangle^\perp$, for each $\xi \in \ker(R_w - \varepsilon_w\lambda\,\mathrm{Id})$, put

$$\xi = \alpha(b\varepsilon_{x_0}\varepsilon_{y_0}x_0 - ay_0) + \sum_{i=1}^{\tau}(\gamma_i x_i + \delta_i y_i) + \sum_{j=1}^{n-2(\tau+1)} \beta_j z_j. \tag{2.2}$$

Then $R_w(\xi) = \alpha\mu\varepsilon_w(b\varepsilon_{x_0}\varepsilon_{y_0}x_0 - ay_0) + \sum_{i=1}^{\tau}(\gamma_i' x_i + \delta_i' y_i) + \sum_{j=1}^{n-2(\tau+1)} \beta_j\mu\varepsilon_w z_j$ and, if ξ is an eigenvector of R_w associated to λ, one has

$$R_w(\xi) = \alpha\lambda\varepsilon_w(b\varepsilon_{x_0}\varepsilon_{y_0}x_0 - ay_0) + \sum_{i=1}^{\tau}(\gamma_i\lambda\varepsilon_w x_i + \delta_j\lambda\varepsilon_w y_i) + \sum_{j=1}^{n-2(\tau+1)} \beta_j\lambda\varepsilon_w z_j.$$

Hence

$$\alpha(\lambda - \mu)\varepsilon_w(b\varepsilon_{x_0}\varepsilon_{y_0}x_0 - ay_0) + \sum_{i=1}^{\tau}(\gamma_i'' x_i + \delta_i'' y_i) + \sum_{j=1}^{n-2(\tau+1)} \beta_j(\lambda - \mu)\varepsilon_w z_j = 0$$

from where $\alpha = 0$ and $\beta_j = 0$, $j = 1, \dots, n - 2(\tau + 1)$. Hence, $\xi \in \langle\{x_1, \dots, x_\tau, y_1, \dots, y_\tau\}\rangle$, and thus $\ker(R_w - \varepsilon_w\lambda\,\mathrm{Id}) \subset V = \langle\{x_1, \dots, x_\tau, y_1, \dots, y_\tau\}\rangle$. Next, let $\tilde{R}_w$ denote the restriction of R_w to V. Then, when expressing in the basis $\{x_1, \dots, x_\tau, y_1, \dots, y_\tau\}$, one has

$$\tilde{R}_w = \left(\begin{array}{c|c} (a^2\varepsilon_{x_0}\lambda + b^2\varepsilon_{y_0}\mu)\,\mathrm{Id}_\tau & 3abC \\ \hline 3abB & (a^2\varepsilon_{x_0}\mu + b^2\varepsilon_{y_0}\lambda)\,\mathrm{Id}_\tau \end{array}\right), \tag{2.3}$$

where B and C are the $(\tau \times \tau)$-matrices given by $(B_{ij}) = (\varepsilon_{y_i}R(x_j, y_0, x_0, y_i))$ and $(C_{ij}) = (\varepsilon_{x_i}R(x_i, y_0, x_0, y_j))$.

Note here that, since $\ker(R_w - \varepsilon_w \lambda \,\mathrm{Id}) \subset V$, $\tilde{R}_w$ has two eigenvalues, λ and μ, both with multiplicity τ. Moreover it follows from (2.3) that $\ker(\tilde{R}_w - \varepsilon_w \lambda \,\mathrm{Id})$ is determined by

$$\begin{cases} -b^2(\lambda-\mu)\varepsilon_{y_0}\vec{x} + 3abC\vec{y} = 0, \\ 3abB\vec{x} - a^2(\lambda-\mu)\varepsilon_{x_0}\vec{y} = 0, \end{cases} \tag{2.4}$$

where $\vec{x}$, $\vec{y}$ are vectors in $\langle\{x_1,\dots,x_\tau\}\rangle$ and $\langle\{y_1,\dots,y_\tau\}\rangle$, respectively. One can check directly that the solution of (2.4) is given by

$$\vec{y} = \frac{3}{\lambda-\mu}\frac{b}{a}\varepsilon_{x_0}B\vec{x} \quad \text{and} \quad CB\vec{x} = \left(\frac{\lambda-\mu}{3}\right)^2 \varepsilon_{x_0}\varepsilon_{y_0}\vec{x}.$$

Now, since $\ker(\tilde{R}_w - \varepsilon_w \lambda \,\mathrm{Id})$ has dimension τ and $\vec{x}$, $\vec{y}$ are $(\tau \times 1)$-matrices, it follows that $CB\vec{x} = ((\lambda-\mu)/3)^2 \varepsilon_{x_0}\varepsilon_{y_0}\vec{x}$ for all vectors $\vec{x}$ in $\langle\{x_1,\dots,x_\tau\}\rangle$, and then

$$CB = \left(\frac{\lambda-\mu}{3}\right)^2 \varepsilon_{x_0}\varepsilon_{y_0} \,\mathrm{Id}_\tau. \tag{2.5}$$

Finally, (2.1) is obtained from (2.5) just using the fact that the (i,k)-element of the product CB, is given by $\varepsilon_{x_i}\sum_{j=1}^{\tau} R(x_i, y_0, x_0, y_j)R(x_k, y_0, x_0, y_j)\varepsilon_{y_j}$. □

We close this section with the study of the eigenspaces corresponding to the eigenvalues $\varepsilon_w\lambda$ and $\varepsilon_w\mu$ of R_w.

Lemma 2.4. *Let x_0, y_0 be unit vectors with $y_0 \in E_\lambda(x_0)^\perp$, and a, b nonzero numbers such that $w = ax_0 + by_0$ is a unit vector. Further, let $\{x_1,\dots,x_\tau\}$ be an orthonormal basis of* $\ker(R_{x_0} - \varepsilon_{x_0}\lambda \,\mathrm{Id})$ *and put*

$$u_i = x_i - \frac{3}{2(\lambda-\mu)}\frac{b}{a}\varepsilon_{x_0}R(x_0, x_i)y_0, \quad i = 1,\dots,\tau,$$

$$v_i = \frac{b}{a}\varepsilon_{y_0}x_i + \frac{3}{2(\lambda-\mu)}R(x_0, x_i)y_0, \quad i = 1,\dots,\tau.$$

Then

(i) $\{u_1,\dots,u_\tau\}$ *is a basis of* $\ker(R_w - \varepsilon_w\lambda \,\mathrm{Id})$,

(ii) $\langle\{v_1,\dots,v_\tau\}\rangle$ *is a τ-dimensional subspace of* $\ker(R_w - \varepsilon_w\mu \,\mathrm{Id})$,

(iii) $R_{u_i}v_j = \mu g(u_i, u_i)v_j$, *for* $i, j = 1,\dots,\tau$.

Proof. Let x_0, y_0 be unit vectors with $y_0 \in E_\lambda(x_0)^\perp$ and take nonzero a, b such that $w = ax_0 + by_0$ is a unit vector. If $\{y_1,\dots,y_\tau\}$ is an orthonormal basis of $\ker(R_{y_0} - \varepsilon_{y_0}\lambda \,\mathrm{Id})$, we already obtained in the proof of previous lemma that

$$R_w(x_i) = (a^2\varepsilon_{x_0}\lambda + b^2\varepsilon_{y_0}\mu)x_i - \frac{3}{2}abR(x_0, x_i)y_0 \quad (i = 1,\dots,\tau), \tag{2.6}$$

$$R_w(y_i) = (a^2\varepsilon_{x_0}\mu + b^2\varepsilon_{y_0}\lambda)y_i + 3abR(y_i, x_0)y_0 \quad (i = 1,\dots,\tau). \tag{2.7}$$

Moreover, since

$$R(x_0, x_r)y_0 = \sum_{j=1}^{\tau} R(x_0, x_r, y_0, y_j)\varepsilon_{y_j} y_j, \quad r = 1, \dots, \tau, \tag{2.8}$$

it follows that $R_w(R(x_0, x_i)y_0) = \sum_{j=1}^{\tau} R(x_0, x_i, y_0, y_j)\varepsilon_{y_j} R_w(y_j)$, and from (2.7),

$$R_w(R(x_0, x_i)y_0) = +3ab \sum_{j=1}^{\tau} R(x_0, x_i, y_0, y_j)\varepsilon_{y_j} R(y_j, x_0)y_0.$$

Next, from (2.8) and $R(y_j, x_0)y_0 = \sum_{k=1}^{\tau} R(y_j, x_0, y_0, x_k)\varepsilon_{x_k} x_k$, we obtain

$$\begin{aligned} R_w(R(x_0, x_i)y_0) &= (a^2\varepsilon_{x_0}\mu + b^2\varepsilon_{y_0}\lambda)R(x_0, x_i)y_0 \\ &\quad + 3ab \sum_{k=1}^{\tau} \left\{ \sum_{j=1}^{\tau} R(x_0, x_i, y_0, y_j)R(y_j, x_0, y_0, x_k)\varepsilon_{y_j} \right\} \varepsilon_{x_k} x_k. \end{aligned}$$

Then, using that $R(x_0, x_i, y_0, y_j) = -2R(x_i, y_0, x_0, y_j)$ and $R(y_j, x_0, y_0, x_k) = R(x_k, y_0, x_0, y_j)$, it follows from Lemma 2.3, that

$$R_w(R(x_0, x_i)y_0) = (a^2\varepsilon_{x_0}\mu + b^2\varepsilon_{y_0}\lambda)R(x_0, x_i)y_0 - 6ab \sum_{k=1}^{\tau} \delta_{ik} \left(\frac{\lambda - \mu}{3}\right)^2 \varepsilon_{x_0}\varepsilon_{x_i}\varepsilon_{y_0}\varepsilon_{x_k} x_k,$$

where

$$R_w(R(x_0, x_i)y_0) = (a^2\varepsilon_{x_0}\mu + b^2\varepsilon_{y_0}\lambda)R(x_0, x_i)y_0 - \frac{2}{3}ab(\lambda - \mu)^2\varepsilon_{x_0}\varepsilon_{y_0} x_i. \tag{2.9}$$

Then, (2.6) and (2.9) show that $R_w(u_i) = \varepsilon_w \lambda u_i$ and $R_w(v_i) = \varepsilon_w \mu v_i$, $(i = 1, \dots, \tau)$. Hence $\{u_1, \dots, u_\tau\}$ and $\{v_1, \dots, v_\tau\}$ are eigenvectors of R_w corresponding to eigenvalues $\varepsilon_w\lambda$ and $\varepsilon_w\mu$, respectively. To finish the proof, we show that $\{u_i\}, \{v_i\}$, $i = 1, \dots, \tau$ are orthogonal nonnull vectors. To do this, note that (2.8) and Lemma 2.3 give

$$g(R(x_0, x_i)y_0, R(x_0, x_j)y_0) = 4\delta_{ij}\left(\frac{\lambda - \mu}{3}\right)^2 \varepsilon_{x_0}\varepsilon_{x_i}\varepsilon_{y_0}, \quad (i, j = 1, \dots, \tau),$$

and thus, since $R(x_0, x_i)y_0$ and $R(x_0, x_j)y_0$ are orthogonal to $\langle\{x_1, \dots, x_\tau\}\rangle$, one gets $g(u_i, u_j) = \delta_{ij}(\varepsilon_{x_0}\varepsilon_{x_i}\varepsilon_w)/a^2$, and $g(v_i, v_j) = \delta_{ij}(\varepsilon_{y_0}\varepsilon_{x_i}\varepsilon_w)/a^2$ for all $i, j = 1, \dots, \tau$, which proves (i) and (ii). To prove (iii) note that $\bar{u}_i = u_i/\|u_i\|$ satisfies $E_\lambda(w) = E_\lambda(\bar{u}_i)$ for all $i = 1, \dots, \tau$. Moreover, from (ii) v_j is orthogonal to $E_\lambda(w) = E_\lambda(\bar{u}_i)$, and hence $v_j \in \ker(R_{\bar{u}_i} - \varepsilon_{\bar{u}_i}\mu\,\mathrm{Id})$ which shows that $R_{\bar{u}_i} v_j = \varepsilon_{\bar{u}_i}\mu v_j$ for all $i, j = 1, \dots, \tau$. □

3. Multiplicities of the eigenvalue λ.

In this section we study the possible multiplicities of the eigenvalue λ. To do this, we endow each subspace $E_\lambda(\cdot)$ with a certain Clifford module structure and, by using some topological restrictions on the existence of such structures, we will obtain that the eigenvalue λ may only have multiplicity 1, 3, 7, or 15 (cf. Theorem 3.2). We begin with the following

DEFINITION 3.1. Let x_0 be a unit vector and $\{x_0, x_1, \dots, x_\tau\}$ an orthonormal basis of $E_\lambda(x_0)$. For each $i = 1, \dots, \tau$, define $\phi_i : E_\lambda(x_0)^\perp \to E_\lambda(x_0)^\perp$ by

$$\phi_i \xi = \frac{3}{2(\lambda - \mu)} R(x_0, x_i)\xi, \tag{3.1}$$

where ξ is any vector in $E_\lambda(x_0)^\perp$.

REMARK 3.1. Note that the maps ϕ_i are well-defined by Lemma 2.2-(iii). Moreover, if ξ_0 is a unit vector in $E_\lambda(x_0)^\perp$ and $E_\lambda(\xi_0) = \langle\{\xi_0, \xi_1, \dots, \xi_\tau\}\rangle$ then, since $R(x_0, x_i)\xi_0 \in \langle\{\xi_1, \dots, \xi_\tau\}\rangle$, it follows from Lemma 2.2 that $\phi_i \xi_0 = (3/(2(\lambda - \mu))) \sum_{j=1}^{\tau} R(x_0, x_i, \xi_0, \xi_j)\varepsilon_{\xi_j}\xi_j$. Once more, from Lemma 2.2-(i), $R(x_0, x_i, \xi_0, \xi_j) = -2R(x_i, \xi_0, x_0, \xi_j)$ and thus

$$\phi_i \xi_0 = -\frac{3}{\lambda - \mu} \sum_{j=1}^{\tau} R(x_i, \xi_0, x_0, \xi_j)\varepsilon_{\xi_j}\xi_j. \tag{3.2}$$

This shows that each subspace $E_\lambda(\cdot) \subset E_\lambda(x_0)^\perp$ remains invariant by the action of the ϕ_i's.

Let us recall at this point that a *complex structure* on a vector space W is an endomorphism J of W such that $J^2 = -\mathrm{Id}$. Moreover, an inner product $\langle\,,\rangle$ is called *Hermitian* if $\langle x, Jy\rangle + \langle Jx, y\rangle = 0$ for all $x, y \in W$. Also, an endomorphism J of W with $J^2 = \mathrm{Id}$ is called a *paracomplex structure* and an inner product $\langle\,,\rangle$ is said to be *para-Hermitian* if it satisfies $\langle x, Jy\rangle + \langle Jx, y\rangle = 0$ for all $x, y \in W$. Now, by using Lemma 2.3, we have the following

LEMMA 3.1. *The endomorphisms ϕ_i of $E_\lambda(x_0)^\perp$ defined by* (3.1) *satisfy the following*:

(i) $g(\phi_i\xi, \eta) + g(\xi, \phi_i\eta) = 0$, *for all vectors* $\xi, \eta \in E_\lambda(x_0)^\perp$,
(ii) $g(\phi_i\xi, \phi_j\xi) = \delta_{ij}\varepsilon_{x_0}\varepsilon_{x_i} g(\xi, \xi)$, *for all vectors* $\xi \in E_\lambda(x_0)^\perp$,
(iii) $\phi_i\phi_j + \phi_j\phi_i = -2\delta_{ij}\varepsilon_{x_0}\varepsilon_{x_i}\,\mathrm{Id}$,

where $i, j \in \{1, \dots, \tau\}$.

PROOF. If $\xi, \eta \in E_\lambda(x_0)^\perp$, then

$$g(\phi_i\xi, \eta) = \frac{3}{2(\lambda - \mu)} g(R(x_0, x_i)\xi, \eta) = \frac{-3}{2(\lambda - \mu)} g(R(x_0, x_i)\eta, \xi) = -g(\xi, \phi_i\eta),$$

which shows (i). To prove (ii), let ξ_0 be a unit vector in $E_\lambda(x_0)^\perp$ and put $E_\lambda(\xi_0) = \langle\{\xi_0, \xi_1, \dots, \xi_\tau\}\rangle$. By (3.2), we have

$$g(\phi_i\xi_0, \phi_j\xi_0) = \left(\frac{3}{\lambda - \mu}\right)^2 \sum_{k=1}^{\tau} R(x_i, \xi_0, x_0, \xi_k) R(x_j, \xi_0, x_0, \xi_k) \varepsilon_{\xi_k},$$

and thus, Lemma 2.3 shows that $g(\phi_i\xi_0, \phi_j\xi_0) = \delta_{ij}\varepsilon_{x_0}\varepsilon_{x_i} g(\xi_0, \xi_0)$, which proves (ii). Next, to prove (iii), let ξ be an arbitrary vector in $E_\lambda(x_0)^\perp$ and note that, from (i) and (ii),

$$g(\phi_i\phi_j\xi, \xi) = -g(\phi_i\xi, \phi_j\xi) = -\delta_{ij}\varepsilon_{x_0}\varepsilon_{x_i} g(\xi, \xi). \tag{3.3}$$

Now, if $\xi, \eta \in E_\lambda(x_0)^\perp$, so is $(\xi + \eta)$, and from (3.3), after linearization

$$\begin{aligned} &g(\phi_i\phi_j\xi, \xi) + g(\phi_i\phi_j\eta, \eta) + g(\phi_i\phi_j\xi, \eta) + g(\phi_i\phi_j\eta, \xi) \\ &\quad = -\delta_{ij}\varepsilon_{x_0}\varepsilon_{x_i} g(\xi, \xi) - \delta_{ij}\varepsilon_{x_0}\varepsilon_{x_i} g(\eta, \eta) - 2\delta_{ij}\varepsilon_{x_0}\varepsilon_{x_i} g(\xi, \eta). \end{aligned}$$

Hence $g(\phi_i\phi_j\xi, \eta) + g(\phi_i\phi_j\eta, \xi) = -2\delta_{ij}\varepsilon_{x_0}\varepsilon_{x_i} g(\xi, \eta)$, and the desired result follows from (i). □

Complex and paracomplex structures may only exist on even-dimensional spaces. Therefore, from Lemma 3.1-(iii) the multiplicity of λ is necessarily odd. Moreover note that indefinite Hermitian metrics are of signature $(2k, 2r)$, $(k, r \geq 0)$ but para-Hermitian metrics are necessarily of neutral signature (k, k). As an application, we obtain the following:

LEMMA 3.2. *Let $T_pM = E_\lambda(x) \oplus E_\lambda(y) \oplus \cdots$ be an orthogonal decomposition of the tangent space of a special Osserman manifold given by Lemma* 2.1. *Then, either of the following hold*:

(i) *The restriction of the metric to each $E_\lambda(\cdot)$ is definite of signature $(\tau + 1, 0)$ or $(0, \tau + 1)$.*
(ii) *The restriction of the metric to each $E_\lambda(\cdot)$ is of neutral signature $((\tau + 1)/2, (\tau + 1)/2)$.*

PROOF. Let $T_pM = E_\lambda(x) \oplus E_\lambda(y) \oplus \cdots$ be an orthogonal decomposition of T_pM as given by Lemma 2.1. If $\{x_0, \dots, x_\tau\}$ is an orthonormal basis of $E_\lambda(x_0)$ then the induced ϕ_i's satisfy $\phi_i^2 = \sigma_i \,\mathrm{Id}$, where $\sigma_i = -\varepsilon_{x_0}\varepsilon_{x_i}$ $(i = 1, \dots, \tau)$. Therefore they are τ complex structures on $E_\lambda(x_0)^\perp$ or exactly $((\tau - 1)/2)$-complex and $((\tau + 1)/2)$-paracomplex structures on $E_\lambda(x_0)^\perp$. Now, the first case above happens if the restriction of the metric to $E_\lambda(x_0)$ is definite (and thus, it is the case for all the subspaces $E_\lambda(\cdot)$) and the second case corresponds to neutral metric on $E_\lambda(x_0)$ (and thus in all the $E_\lambda(\cdot)$'s). □

Next, we will show that the multiplicity of λ strongly influences the dimension of a special Osserman manifold whenever such multiplicity is assumed to be greater than 3.

Lemma 3.3. *Let (M, g) be a special Osserman manifold. If the multiplicity of λ is strictly greater than* 3, *then* $\dim M = 2(\tau + 1)$.

Proof. Let $x_0, y_0 \in T_pM$ be unit vectors with $y_0 \in E_\lambda(x_0)^\perp$ and consider the endomorphisms $\phi_i : E_\lambda(x_0)^\perp \to E_\lambda(x_0)^\perp$, $(i = 1, \dots, \tau)$ defined by (3.1). It follows from previous results that $\{\xi, \phi_1\xi, \dots, \phi_\tau\xi\}$ is an orthonormal basis of $E_\lambda(y_0)$ for each unit $\xi \in E_\lambda(y_0)$, and thus we put

$$\phi_i\phi_j\xi = \alpha_{ij}^0(\xi)\xi + \sum_{s=1}^{\tau} \alpha_{ij}^s(\xi)\phi_s\xi,$$

where $i \neq j$, $i, j \in \{1, \dots, \tau\}$. Since for each $i \neq j$, $\phi_i\phi_j\xi \in \langle\{\xi, \phi_i\xi, \phi_j\xi\}\rangle^\perp$, the above expression reduces to

$$\phi_i\phi_j\xi = \sum_{s=1, s\neq i, j}^{\tau} \alpha_{ij}^s(\xi)\phi_s\xi. \tag{3.4}$$

Next, suppose that $\dim M > 2(\tau + 1)$, and take a unit vector η orthogonal to the subspaces $E_\lambda(x_0)$ and $E_\lambda(y_0)$. Choose nonzero a, b in such a way that $w = a\xi + b\eta$ and $t = b\xi - a\varepsilon_\xi\varepsilon_\eta\eta$ are unit vectors. Then, for $l, m, n \in \{1, \dots, \tau\}$ we have that $g(\phi_l\phi_m\phi_n w, t) = 0$, since $\phi_l\phi_m\phi_n w \in E_\lambda(w)$ and $t \in E_\lambda(w)^\perp$. Hence,

$$\begin{aligned} 0 &= ab(g(\phi_l\phi_m\phi_n\xi, \xi) - \varepsilon_\xi\varepsilon_\eta g(\phi_l\phi_m\phi_n\eta, \eta)) \\ &\quad - a^2\varepsilon_\xi\varepsilon_\eta g(\phi_l\phi_m\phi_n\xi, \eta) + b^2 g(\phi_l\phi_m\phi_n\eta, \xi), \end{aligned}$$

and since $g(\phi_l\phi_m\phi_n\xi, \eta) = g(\phi_l\phi_m\phi_n\eta, \xi) = 0$ (note that $E_\lambda(\xi) \perp E_\lambda(\eta)$, and both subspaces remain invariant under the action of the ϕ_i's) we obtain that $g(\phi_l\phi_m\phi_n\xi, \xi) = \varepsilon_\xi\varepsilon_\eta g(\phi_l\phi_m\phi_n\eta, \eta)$. This shows that the coefficients $\alpha_{ij}^s(\xi)$ in (3.4) are given by $\alpha_{ij}^s(\xi) = g(\phi_i\phi_j\xi, \phi_s\xi)g(\phi_s\xi, \phi_s\xi) = -\varepsilon_{x_0}\varepsilon_{x_s}\varepsilon_\eta g(\phi_s\phi_i\phi_j\eta, \eta)$ and therefore, they are independent of the unit vector ξ. Thus, we have (3.4) for all unit vectors $\xi \in E_\lambda(y_0)$, where the coefficients α_{ij}^s do not depend on ξ. Now, choose $k \in \{1, \dots, \tau\}$ in such a way that i, j, k are different. Then (3.4) leads to

$$\phi_i\phi_j(\phi_k\xi) = \sum_{s=1, s\neq i, j}^{\tau} \alpha_{ij}^s\phi_s(\phi_k\xi) = -\alpha_{ij}^k\varepsilon_{x_0}\varepsilon_{x_k}\xi + \sum_{s=1, s\neq i, j, k}^{\tau} \alpha_{ij}^s\phi_s\phi_k\xi.$$

On the other hand, (3.4) also gives

$$\phi_k(\phi_i\phi_j\xi) = -\alpha_{ij}^k\varepsilon_{x_0}\varepsilon_{x_k}\xi - \sum_{s=1, s\neq i, j, k}^{\tau} \alpha_{ij}^s\phi_s\phi_k\xi,$$

and, since $\phi_i\phi_j(\phi_k\xi) = \phi_i\phi_j\phi_k\xi = \phi_k\phi_i\phi_j\xi = \phi_k(\phi_i\phi_j\xi)$, we obtain $\phi_i\phi_j\phi_k\xi = -\alpha_{ij}^k\varepsilon_{x_0}\varepsilon_{x_k}\xi$, for all unit vectors $\xi \in E_\lambda(y_0)$. This shows that the composition $\phi_i\phi_j$ coincides with ϕ_k or $-\phi_k$ on $E_\lambda(y_0)$, whenever i, j, k are different, which is a contradiction. Therefore, $\dim M = 2(\tau+1)$. □

Next, we concern with the possible multiplicities of the eigenvalue λ. First of all, we will recall some known facts about Clifford modules. Let W^m be a m-dimensional real vector space endowed with an inner product $\langle\,,\rangle$. A real *Cliff*(v)*-module* structure C on W is a family c_i of endomorphisms of W with $c_ic_j + c_jc_i = -2\delta_{ij}$ for $i, j = 1, \dots, v$ (i.e., C determines an anticommuting family of complex structures on W). There exist topological restrictions to the existence of Clifford structures as follows.

Theorem 3.1 ([24]). *Let $m = 2^r \cdot m_0$, with m_0 odd.*
(i) *V^m admits a Cliff(v)-module structure if and only if $v \le v(r)$,*
(ii) *TS^{m-1} admits a q-dimensional distribution, for $2q \le m-1$, if and only if $q \le v(r)$,*
where v is given by $v(i+4) = v(i) + 8$ and $v(i) = 2^i - 1$ for $i = 0, 1, 2, 3$.

Now, we have the following

Theorem 3.2. *Let (M, g) be a special Osserman manifold. Then one of the following conditions holds:*
(i) *$\tau = 1$ and M is a $2n$-dimensional manifold with metric of signature (n, n) or $(2k, 2r)$, for some $k, r \ge 0$,*
(ii) *$\tau = 3$ and M is a $4n$-dimensional manifold with metric of signature $(2n, 2n)$ or $(4k, 4r)$, for some $k, r \ge 0$,*
(iii) *$\tau = 7$ and M is a 16-dimensional manifold with metric of signature $(8, 8)$, $(16, 0)$ or $(0, 16)$, or*
(iv) *$\tau = 15$ and M is a 32-dimensional manifold with metric of signature $(16, 16)$,*
where τ denotes the multiplicity of the eigenvalue λ.

Proof. Let $T_pM = E_\lambda(x_0) \oplus E_\lambda(y_0) \oplus \cdots$ be an orthogonal decomposition of T_pM as in Lemma 2.1 and define endomorphisms ϕ_i on $E_\lambda(y_0)$ by (3.1). By Lemmas 3.1 and 3.2, we have $\tilde{\tau}$ complex structures defining a Cliff($\tilde{\tau}$)-module structure on $E_\lambda(y_0)$. Note that $\dim E_\lambda(y_0) = \tilde{\tau}+1$ (if the restriction of the metric to $E_\lambda(y_0)$ is definite) or $\dim E_\lambda(y_0) = 2\tilde{\tau}+1$ (whenever the induced metric on $E_\lambda(y_0)$ is of neutral signature). Further note that a Cliff(τ)-module structure is available on a $(\tau+1)$-dimensional vector space if an only if the τ-dimensional sphere is parallelizable, which restricts to the cases of $\tau = 1$, 3, or 7.

Henceforth, we next concentrate on the case $\tilde{\tau} = (\tau-1)/2$. It follows from

Theorem 3.1-(i) that there exists such Cliff($\tilde{\tau}$)-module structure on $E_\lambda(y_0)$ if and only if

$$\tilde{\tau} \leq v(r), \tag{3.5}$$

where $2\tilde{\tau}+2=2^r\cdot m_0$, with m_0 odd.

First of all, suppose that $\tilde{\tau}$ is even ($\tilde{\tau}=2\alpha$). Then $2\tilde{\tau}+2=2^r\cdot m_0$ is given by $2\tilde{\tau}+2=4\alpha+2=2(2\alpha+1)$, and thus $r=1$ and $m_0=2\alpha+1$. Therefore, from (3.5), $2\alpha\leq v(1)=1$ and hence $\tilde{\tau}$ must be zero or odd. In what follows we will suppose that $\tilde{\tau}$ is odd and it can be written in the form $\tilde{\tau}=2^\alpha-1$ for some α. In this case, $2\tilde{\tau}+2=2^r\cdot m_0$ is given by $2\tilde{\tau}+2=2(2^\alpha-1)+2=2^{\alpha+1}$ and then $r=\alpha+1$ and $m_0=1$. Hence, (3.5) shows that there exists a Cliff($\tilde{\tau}$)-module structure on $E_\lambda(y_0)$ if and only if

$$2^\alpha-1\leq v(\alpha+1). \tag{3.6}$$

Next we consider this inequality. If one put $\alpha+1=4a+b$, $0\leq b\leq 3$, we have $v(\alpha+1)=v(b)+8a$ and since $v(b)=2^b-1$ and $a=(\alpha+1-b)/4$, it follows that $v(\alpha+1)=2^b-2b+2\alpha+1$. Then, (3.6) reduces to $2^\alpha-1\leq 2^b-2b+2\alpha+1$, that is, $2^\alpha-2\alpha\leq 2^b-2b+2$. Now, since $0\leq b\leq 3$, we have that $2^b-2b+2\leq 4$ and, therefore,

$$2^\alpha-2\alpha\leq 4. \tag{3.7}$$

Let $f(x)=2^x-2x$, with $x\in\boldsymbol{R}$. This function grows strictly in $(2,+\infty)$, and $f(4)=8$, which implies that $2^\alpha-2\alpha\geq 8$ whenever $\alpha\geq 4$. Thus, (3.7) gives a contradiction whenever $\alpha\geq 4$ and hence $\tilde{\tau}$ cannot be written in the form $2^\alpha-1$ for $\alpha\geq 4$. Let suppose that $\tilde{\tau}$ satisfies

$$2^\alpha-1<\tilde{\tau}<2^{\alpha+1}-1,\quad \alpha\geq 4, \tag{3.8}$$

and put $2\tilde{\tau}+2=2^r\cdot m_0$, with m_0 odd. Then (3.5) holds and (3.8) implies that $2^{\alpha+1}<2^r\cdot m_0<2^{\alpha+2}$, from where $r\leq\alpha+1$ and thus

$$v(r)\leq v(\alpha+1). \tag{3.9}$$

On the other hand, since $\alpha\geq 4$, (3.6) does not hold,

$$2^\alpha-1>v(\alpha+1). \tag{3.10}$$

Note now that (3.5), (3.8), (3.9) and (3.10) give

$$\tilde{\tau}\leq v(r)\leq v(\alpha+1)<2^\alpha-1<\tilde{\tau},$$

which means that $\tilde{\tau}$ cannot satisfy (3.8). Therefore, $\tilde{\tau}\in\{0,1,3,5,7,9,11,13\}$. Now, a direct calculation from (3.5) shows that $\tilde{\tau}\in\{0,1,3,7\}$ and hence $\tau\in\{1,3,7,15\}$. Now the result follows from Lemmas 3.2 and 3.3. □

4. Special Osserman manifolds with eigenvalue λ of multiplicity different from 7, 15.

The purpose of this section is to prove a local version of Theorem 1.1, in the characterization of special Osserman manifolds when the eigenvalue λ is of multiplicity different from 7 and 15 as in the below

THEOREM 4.1. *Let (M, g) be a special Osserman pseudo-Riemannian manifold. If the multiplicity of the distinguished eigenvalue λ is different from* 7 *and* 15 *then (M, g) is locally isometric to one of the following*
(a) *an indefinite complex space form,*
(b) *an indefinite quaternionic space form,*
(c) *a paracomplex space form, or*
(d) *a paraquaternionic space form.*

To prove the result above, we firstly obtain the expression of the curvature tensor of such manifolds in §4.1. Then, Theorem 4.1 follows from the second Bianchi identity in §4.2.

4.1. Pointwise expression of the curvature tensor.

A quadrilinear map $\tilde{F} : W \times W \times W \times W \to \boldsymbol{R}$ is said to be a *curvaturelike function* on a vector space W if it satisfies

$$\tilde{F}(x, y, z, w) = -\tilde{F}(y, x, z, w) = -\tilde{F}(x, y, w, z),$$
$$\tilde{F}(x, y, z, w) = \tilde{F}(z, w, x, y),$$
$$\tilde{F}(x, y, z, w) + \tilde{F}(y, z, x, w) + \tilde{F}(z, x, y, w) = 0,$$

for all vectors $x, y, z, w \in W$. Moreover, if $\langle\, ,\rangle$ denotes an inner product on W then the associated $(1,3)$-tensor defined by $\langle F(x, y)z, w\rangle = \tilde{F}(x, y, z, w)$ will be called the associated *curvaturelike tensor*. Next we recall the definition of two curvaturelike tensors which will play a basic role in what follows. Induced by the inner product $\langle\, ,\rangle$, define a curvaturelike tensor as follows

$$R^0(x, y)z = \langle y, z\rangle x - \langle x, z\rangle y.$$

Also, if J is a complex (resp., paracomplex) structure on W in such a way that $(V, \langle\, ,\rangle, J)$ is an indefinite Hermitian (resp., para-Hermitian) vector space,

$$R^J(x, y)z = \langle Jx, z\rangle Jy - \langle Jy, z\rangle Jx + 2\langle Jx, y\rangle Jz$$

defines a curvaturelike tensor on W.

Our purpose in this subsection is to show that the curvature tensor of any special Osserman manifold whose eigenvalue λ has multiplicity different from 7 and 15, can be writen at each point as a linear combination of R^0 and certain R^J's as follows

Theorem 4.2. *Let (M, g) be a special Osserman manifold with eigenvalue λ of multiplicity different from 7 and 15. Then, at each point $p \in M$ one of the following conditions holds*:

(i) *There exists a complex structure J such that (g, J) defines an indefinite Hermitian structure on T_pM and the curvature tensor satisfies*

$$R = \mu R^0 - \frac{\lambda - \mu}{3} R^J.$$

(ii) *There exists a paracomplex structure J such that (g, J) defines a para-Hermitian structure on T_pM and the curvature tensor is given by*

$$R = \mu R^0 + \frac{\lambda - \mu}{3} R^J.$$

(iii) *There exists a quaternionic structure $\boldsymbol{Q}$ such that $(g, \boldsymbol{Q})$ defines an indefinite Hermitian quaternionic structure on T_pM and the curvature tensor is given by*

$$R = \mu R^0 - \frac{\lambda - \mu}{3} \sum_{i=1}^{3} R^{J_i},$$

where $\{J_1, J_2, J_3\}$ is a canonical basis for $\boldsymbol{Q}$.

(iv) *There exists a paraquaternionic structure $\tilde{\boldsymbol{Q}}$ such that $(g, \tilde{\boldsymbol{Q}})$ defines a Hermitian paraquaternionic structure on T_pM and the curvature satisfies*

$$R = \mu R^0 + \frac{\lambda - \mu}{3} \sum_{i=1}^{3} \sigma_i R^{J_i},$$

where $\{J_1, J_2, J_3\}$ is a canonical basis for $\tilde{\boldsymbol{Q}}$ and $J_i^2 = \sigma_i \,\mathrm{Id}$, $i = 1, 2, 3$.

Remark 4.1. As pointed out in the Introduction, conditions (I) and (II) in Definition 1.1 are pointwise independent. In fact, it is not difficult to exhibit examples of curvaturelike functions satisfying (I) but not (II) as follows.

Let $(V, \langle\,,\rangle)$ be an inner product vector space and $\{J_1, J_2\}$ a pair of anticommuting complex structures such that $(V, \langle\,,\rangle, J_1)$ and $(V, \langle\,,\rangle, J_2)$ are Hermitian vector spaces. Next, consider the curvaturelike function $F = R^{J_1} + R^{J_2}$. It follows that the Jacobi operators F_x are diagonalizable with $F_x = \mathrm{diag}[3, 3, 0, \ldots, 0]$ for all unit vectors $x \in V$. Thus $F = R^{J_1} + R^{J_2}$ is an Osserman curvaturelike function satisfying condition (I). However, it does not satisfy condition (II). In fact, note that $E_3(x) = \langle\{x, J_1x, J_2x\}\rangle$ and $E_3(J_1x) = \langle\{x, J_1x, J_2J_1x\}\rangle$, which shows that $E_3(x) \neq E_3(J_1x)$ in contradiction to condition (II).

In order to prove Theorem 4.2, some technical lemmas are needed. As an immediate consequence of Lemma 2.4 and (3.1), we have

LEMMA 4.1. *Let* $x_0, \xi \in T_pM$ *be unit vectors with* $\xi \in E_\lambda(x_0)^\perp$, *and* a, b *nonzero constants such that* $ax_0 + b\xi$ *is a unit vector. Let* $\{x_1, \dots, x_\tau\}$ *be an orthonormal basis for* $\ker(R_{x_0} - \varepsilon_{x_0}\lambda \,\mathrm{Id})$ *and* $\phi_1, \dots, \phi_\tau$ *the associated endomorphisms defined by* (3.1). *Then*

$$u_i = x_i - \frac{b}{a}\varepsilon_{x_0}\phi_i\xi, \quad i = 1, \dots, \tau,$$

$$v_i = \frac{b}{a}\varepsilon_\xi x_i + \phi_i\xi, \quad i = 1, \dots, \tau,$$

satisfy

$$R_{u_i}v_j = g(u_i, u_i)\mu v_j, \quad i, j = 1, \dots, \tau.$$

LEMMA 4.2. *Let* $x_0 \in T_pM$ *be a unit vector. For any unit vector* ξ *orthogonal to* $E_\lambda(x_0)$, *one has*

$$R(x_i, x_j)\xi = -\frac{2}{3}(\lambda - \mu)\varepsilon_{x_0}\phi_i\phi_j\xi, \quad i \neq j, \ i, j \in \{1, \dots, \tau\},$$

where $\{x_1, \dots, x_\tau\}$ *is an orthonormal basis of* $\ker(R_{x_0} - \varepsilon_{x_0}\lambda \,\mathrm{Id})$ *and* $\phi_1, \dots, \phi_\tau$ *are the associated endomorphisms defined by* (3.1).

PROOF. Take $i, j \in \{1, \dots, \tau\}$, $i \neq j$. Since ξ is a unit vector orthogonal to $E_\lambda(x_0)$, so is $\phi_j\xi$ and therefore, if a and b are nonzero such that $ax_0 + b\phi_j\xi$ is a unit vector, Lemma 4.1 gives

$$R_{u_j}v_i = g(u_j, u_j)\mu v_i, \tag{4.1}$$

where $u_j = x_j + \frac{b}{a}\varepsilon_{x_j}\xi$ and $v_i = \frac{b}{a}\varepsilon_{x_0}\varepsilon_{x_j}\varepsilon_\xi x_i + \phi_i\phi_j\xi$. After linearization,

$$\begin{aligned} R_{u_j}v_i &= \frac{b}{a}\varepsilon_{x_0}\varepsilon_{x_j}\varepsilon_\xi R(x_i, x_j)x_j + \frac{b^3}{a^3}\varepsilon_{x_0}\varepsilon_{x_j}\varepsilon_\xi R(x_i, \xi)\xi \\ &\quad + \frac{b^2}{a^2}\varepsilon_{x_0}\varepsilon_\xi(R(x_i, x_j)\xi + R(x_i, \xi)x_j) + R(\phi_i\phi_j\xi, x_j)x_j \\ &\quad + \frac{b^2}{a^2}R(\phi_i\phi_j\xi, \xi)\xi + \frac{b}{a}\varepsilon_{x_j}(R(\phi_i\phi_j\xi, x_j)\xi + R(\phi_i\phi_j\xi, \xi)x_j), \end{aligned}$$

and, from Lemmas 2.1 and 2.2,

$$R_{u_j}v_i = \left(\frac{b}{a}\varepsilon_{x_0}\varepsilon_\xi\lambda + \frac{b^3}{a^3}\varepsilon_{x_0}\varepsilon_{x_j}\mu\right)x_i + \frac{3}{2}\frac{b}{a}\varepsilon_{x_j}R(\phi_i\phi_j\xi,\xi)x_j$$
$$+\left(\varepsilon_{x_j}\mu + \frac{b^2}{a^2}\varepsilon_\xi\lambda\right)\phi_i\phi_j\xi + \frac{3}{2}\frac{b^2}{a^2}\varepsilon_{x_0}\varepsilon_\xi R(x_i,x_j)\xi.$$

Therefore, since $g(u_j,u_j) = \varepsilon_{x_j} + \frac{b^2}{a^2}\varepsilon_\xi$, (4.1) is equivalent to

$$\left(\frac{b}{a}\varepsilon_{x_0}\varepsilon_\xi\lambda + \frac{b^3}{a^3}\varepsilon_{x_0}\varepsilon_{x_j}\mu\right)x_i + \frac{3}{2}\frac{b}{a}\varepsilon_{x_j}R(\phi_i\phi_j\xi,\xi)x_j$$
$$+\left(\varepsilon_{x_j}\mu + \frac{b^2}{a^2}\varepsilon_\xi\lambda\right)\phi_i\phi_j\xi + \frac{3}{2}\frac{b^2}{a^2}\varepsilon_{x_0}\varepsilon_\xi R(x_i,x_j)\xi$$
$$= \left(\frac{b}{a}\varepsilon_{x_0}\varepsilon_\xi\mu + \frac{b^3}{a^3}\varepsilon_{x_0}\varepsilon_{x_j}\mu\right)x_i + \left(\varepsilon_{x_j}\mu + \frac{b^2}{a^2}\varepsilon_\xi\mu\right)\phi_i\phi_j\xi,$$

and hence

$$\frac{b^2}{a^2}\varepsilon_\xi(\lambda-\mu)\phi_i\phi_j\xi + \frac{3}{2}\frac{b^2}{a^2}\varepsilon_{x_0}\varepsilon_\xi R(x_i,x_j)\xi$$
$$= -\frac{b}{a}\varepsilon_{x_0}\varepsilon_\xi(\lambda-\mu)x_i - \frac{3}{2}\frac{b}{a}\varepsilon_{x_j}R(\phi_i\phi_j\xi,\xi)x_j.$$

Now, since $\phi_i\phi_j\xi$ and $R(x_i,x_j)\xi$ are orthogonal to $E_\lambda(x_0)$, and moreover x_i and $R(\phi_i\phi_j\xi,\xi)x_j$ belong to $E_\lambda(x_0)$, previous expression gives

$$\frac{b^2}{a^2}\varepsilon_\xi(\lambda-\mu)\phi_i\phi_j\xi + \frac{3}{2}\frac{b^2}{a^2}\varepsilon_{x_0}\varepsilon_\xi R(x_i,x_j)\xi = 0,$$

from where it follows that $(3/2)\varepsilon_{x_0}R(x_i,x_j)\xi = -(\lambda-\mu)\phi_i\phi_j\xi$ and the desired result is obtained. □

Next we prove the main result of this subsection.

Proof of Theorem 4.2. Since the multiplicity of λ is assumed to be different from 7 and 15, Theorem 3.2 shows that only multiplicity 1 or 3 may occur. We will analyze each case separately.

(a): λ of multiplicity $\tau = 1$.

Let x_0 be a unit vector in T_pM and take x_1 a unit vector in $\ker(R_{x_0} - \varepsilon_{x_0}\lambda\,\mathrm{Id})$. Let $\phi : E_\lambda(x_0)^\perp \to E_\lambda(x_0)^\perp$ be defined by (3.1), so that $\phi^2 = \sigma\,\mathrm{Id}$, where $\sigma = -\varepsilon_{x_0}\varepsilon_{x_1}$. Next, if y_0 is a unit vector orthogonal to $E_\lambda(x_0)$, take $y_1 = -\varepsilon_{y_0}\phi y_0$ and define $\psi : E_\lambda(x_0) \to E_\lambda(x_0)$ by $\psi = (3/(2(\lambda-\mu)))R(y_0,y_1)$, in such a way that $\psi^2 = -\varepsilon_{y_0}\varepsilon_{y_1}\,\mathrm{Id} = \sigma\,\mathrm{Id}$. Hence $J = \psi \oplus \phi$ defines a complex (resp., para-complex) structure on T_pM if $\sigma = -1$ (resp., $\sigma = +1$).

Next we will show that the curvature tensor R in T_pM can be expressed in terms of the curvaturelike tensors R^0 and R^J. (R^J being defined by the Hermitian or para-Hermitian structure (g,J), where g is the metric tensor in T_pM and J is the complex or paracomplex structure above). To do this, we will show that $F = R - \mu R^0 - ((\lambda-\mu)/3)\sigma R^J$ vanishes identically.

First of all we study the action of F on vectors in $E_\lambda(x_0)$. It is easy to check that $F(E_\lambda(x_0), E_\lambda(x_0))E_\lambda(x_0) \subset E_\lambda(x_0)$ and then, if $\tilde{F}, \tilde{R}, \tilde{R}^0, \tilde{R}^J$ denote the restrictions of F, R, R^0, R^J to $E_\lambda(x_0)$, one has

$$\tilde{R}_x = \varepsilon_x \lambda \,\mathrm{Id}_1, \quad \tilde{R}^0_x = \varepsilon_x \,\mathrm{Id}_1, \quad \tilde{R}^J_x = 3\sigma\varepsilon_x \,\mathrm{Id}_1,$$

for all unit vectors $x \in E_\lambda(x_0)$. Thus $\tilde{F}_x = 0$, which shows that F vanishes when restricted to $E_\lambda(x_0)$. Next, note that $F(E_\lambda(x_0)^\perp, E_\lambda(x_0)^\perp)E_\lambda(x_0)^\perp \subset E_\lambda(x_0)^\perp$, and denote by $\hat{F}, \hat{R}, \hat{R}^0, \hat{R}^J$ the restrictions of F, R, R^0, R^J to $E_\lambda(x_0)^\perp$. If ξ is a unit vector in $E_\lambda(x_0)^\perp$ then the associated Jacobi operators, when expressed in the basis $\{J\xi, \eta_1, \dots, \eta_{n-4}\}$ of $\langle\xi\rangle^\perp \cap E_\lambda(x_0)^\perp$ are given by

$$\hat{R}_\xi = \mathrm{diag}[\varepsilon_\xi\lambda, \varepsilon_\xi\mu, \overset{n-4}{\dots}, \varepsilon_\xi\mu], \quad \hat{R}^0_\xi = \mathrm{diag}[\varepsilon_\xi, \overset{n-3}{\dots}, \varepsilon_\xi], \quad \hat{R}^J_\xi = \mathrm{diag}[3\sigma\varepsilon_\xi, 0, \overset{n-4}{\dots}, 0],$$

and thus $\hat{F}_\xi = 0$. This shows that F is identically zero when applied to vectors in $E_\lambda(x_0)^\perp$. So far, we have proved that considering the decomposition $T_pM = E_\lambda(x_0) \oplus E_\lambda(x_0)^\perp$, the curvaturelike tensor F vanishes when restricted to anyone of the subspaces. Thus, by Lemma 2.2, to show that F is identically zero it suffices to prove that $F(x_0, x_1) : E_\lambda(x_0)^\perp \to E_\lambda(x_0)^\perp$ vanishes. For, if ξ is a unit vector in $E_\lambda(x_0)^\perp$,

$$R(x_0, x_1)\xi = \frac{2}{3}(\lambda - \mu)J\xi, \quad R^0(x_0, x_1)\xi = 0. \tag{4.2}$$

Also, since $g(Jx_0, x_1) = (3/(2(\lambda-\mu)))g(R(y_0, y_1)x_0, x_1) = (3/(2(\lambda-\mu))) \cdot g(R(x_0, x_1)y_0, y_1) = g(Jy_0, y_1)$ and $y_1 = -\varepsilon_{y_0}Jy_0$, it follows that $g(Jx_0, x_1) = \sigma$, and thus

$$R^J(x_0, x_1)\xi = 2g(Jx_0, x_1)J\xi = 2\sigma J\xi. \tag{4.3}$$

Now, (4.2) and (4.3) give $F(x_0, x_1)\xi = (2/3)(\lambda-\mu)J\xi - ((\lambda-\mu)/3)\sigma(2\sigma J\xi) = 0$, which proves Theorem 4.2 when the multiplicity of λ is equal to 1.

(b): λ of multiplicity $\tau = 3$.

Let x_0 be a unit vector in T_pM and fix an orthonormal basis $\{x_1, x_2, x_3\}$ of $\ker(R_{x_0} - \varepsilon_{x_0}\lambda\,\mathrm{Id})$. Define $\phi_i : E_\lambda(x_0)^\perp \to E_\lambda(x_0)^\perp$ by (3.1), such that $\phi_i^2 = \sigma_i\,\mathrm{Id}$, where $\sigma_i = -\varepsilon_{x_0}\varepsilon_{x_i}$ $(i = 1,2,3)$. Next, let y_0 be a unit vector orthogonal to $E_\lambda(x_0)$ and take $y_i = -\varepsilon_{y_0}\phi_i y_0$. Then $\{y_0, y_1, y_2, y_3\}$ is an orthonormal basis of $E_\lambda(y_0)$, and define $\psi_i : E_\lambda(x_0) \to E_\lambda(x_0)$, by $\psi_i = (3/(2(\lambda-\mu)))R(y_0, y_i)$, so that $\psi_i^2 = -\varepsilon_{y_0}\varepsilon_{y_i}\,\mathrm{Id} = \sigma_i\,\mathrm{Id}$. (Note that, either $\sigma_1 = \sigma_2 = \sigma_3 = -1$ or $\sigma_1 = -\sigma_2 = -\sigma_3 =$

-1 as a consequence of Lemma 3.2. See also [**23**] for algebras with a certain number of generators). Now, for each $i = 1, 2, 3$, the endomorphisms $J_i : T_pM \to T_pM$ defined by $J_i = \psi_i \oplus \phi_i$ are complex or paracomplex structures on T_pM depending on whether σ_i is -1 or $+1$, respectively.

First of all we will show that there is no loss of generality in assuming $J_1J_2 = J_3$. Consider the decomposition $T_pM = E_\lambda(x_0) \oplus E_\lambda(x_0)^\perp$. If ξ is a unit vector in anyone of the subspaces, then $J_1J_2\xi$ belongs to $E_\lambda(\xi)$ and, since $J_1J_2\xi \in \langle\{\xi, J_1\xi, J_2\xi\}\rangle^\perp$, it follows that $J_1J_2\xi \in \langle J_3\xi\rangle$. This proves that $J_1J_2 = \pm J_3$ and, changing x_0 by $-x_0$ if necessary, we have that $J_1J_2 = J_3$.

The condition $J_1J_2 = J_3$ may be generalized to have $J_\alpha J_\beta = J_{\alpha\beta}$ for $\alpha, \beta = 1, 2, 3$, where the product $\alpha \cdot \beta$ is equal to the value $\pm\gamma$ such that $\pm e_\gamma = e_\alpha e_\beta \{e_0, e_1, e_2, e_3\}$ is a standard basis for the multiplication given by the following table:

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$\sigma_1 e_0$	e_3	$\sigma_1 e_2$
e_2	e_2	$-e_3$	$\sigma_2 e_0$	$-\sigma_2 e_1$
e_3	e_3	$-\sigma_1 e_2$	$\sigma_2 e_1$	$\sigma_3 e_0$

Note that this product corresponds to the quaternion one if $\sigma_1 = \sigma_2 = \sigma_3 = -1$, and to the paraquaternion one if $\sigma_1 = -\sigma_2 = -\sigma_3 = -1$ (cf. [**19**]). Moreover, put $J_0 = \mathrm{Id}$ and denote by $J_{-\alpha}$ the tensor $-J_\alpha$.

Next we will prove that the curvature tensor R on T_pM can be expressed according to Theorem 4.2 by showing that $F = R - \mu R^0 - ((\lambda - \mu)/3) \sum_{i=1}^{3} \sigma_i R^{J_i}$ vanishes identically. Considering again the orthogonal decomposition $T_pM = E_\lambda(x_0) \oplus E_\lambda(x_0)^\perp$, and proceeding in the same way as in the case of multiplicity 1, we have that the curvaturelike tensor F vanishes when restricted to anyone of the subspaces $E_\lambda(x_0)$ and $E_\lambda(x_0)^\perp$. Therefore, it will suffice to show that for each $\alpha, \beta \in \{0, 1, 2, 3\}$, $\alpha < \beta$, $F(x_\alpha, x_\beta) : E_\lambda(x_0)^\perp \to E_\lambda(x_0)^\perp$ vanishes identically.

If ξ is a unit vector in $E_\lambda(x_0)^\perp$, then

$$R(x_0, x_\alpha)\xi = \frac{2}{3}(\lambda - \mu)J_\alpha\xi, \quad R^0(x_0, x_\alpha)\xi = 0, \tag{4.4}$$

and $g(J_ix_0, x_\alpha) = (3/(2(\lambda - \mu)))g(R(y_0, y_i)x_0, x_\alpha) = (3/(2(\lambda - \mu))) \cdot g(R(x_0, x_\alpha)y_0, y_i) = g(J_\alpha y_0, y_i)$. Moreover, since $y_i = -\varepsilon_{y_0}J_iy_0$, it follows that $g(J_ix_0, x_\alpha) = \delta_{i\alpha}\sigma_\alpha$ and thus,

$$R^{J_i}(x_0, x_\alpha)\xi = 2g(J_ix_0, x_\alpha)J_i\xi = 2\delta_{i\alpha}\sigma_\alpha J_\alpha\xi. \tag{4.5}$$

Now, (4.4) and (4.5) give

$$F(x_0, x_\alpha)\xi = \frac{2}{3}(\lambda - \mu)J_\alpha\xi - \frac{\lambda - \mu}{3}\sum_{i=1}^{3}\sigma_i(2\delta_{i\alpha}\sigma_\alpha J_\alpha\xi)$$
$$= \frac{2}{3}(\lambda - \mu)J_\alpha\xi - \frac{\lambda - \mu}{3}2J_\alpha\xi,$$

which proves that $F(x_0, x_\alpha)$ vanishes ($\alpha = 1, 2, 3$).

Finally if ξ is a unit vector in $E_\lambda(x_0)^\perp$ and $\alpha, \beta \in \{1, 2, 3\}$, $\alpha < \beta$, then using that $J_\alpha J_\beta = J_{\alpha\beta}$ and Lemma 4.2 we get

$$R(x_\alpha, x_\beta)\xi = -\frac{2}{3}(\lambda - \mu)\varepsilon_{x_0}J_{\alpha\beta}\xi, \tag{4.6}$$

and $g(J_i x_\alpha, x_\beta) = (3/(2(\lambda - \mu)))g(R(y_0, y_i)x_\alpha, x_\beta) = -\varepsilon_{x_0}g(J_{\alpha\beta}y_0, y_i)$. Therefore, since $y_i = -\varepsilon_{y_0}J_i y_0$, it follows that $g(J_i x_\alpha, x_\beta) = -\delta_{i,\alpha\beta}\sigma_{\alpha\beta}\varepsilon_{x_0}$, and thus

$$R^{J_i}(x_\alpha, x_\beta)\xi = 2g(J_i x_\alpha, x_\beta)J_i\xi = -2\delta_{i,\alpha\beta}\sigma_{\alpha\beta}\varepsilon_{x_0}J_{\alpha\beta}\xi. \tag{4.7}$$

Moreover, since $R^0(x_\alpha, x_\beta)\xi = 0$, from (4.6) and (4.7) one has

$$F(x_\alpha, x_\beta)\xi = -\frac{2}{3}(\lambda - \mu)\varepsilon_{x_0}J_{\alpha\beta}\xi - \frac{\lambda - \mu}{3}\sum_{i=1}^{3}\sigma_i(-2\delta_{i,\alpha\beta}\sigma_{\alpha\beta}\varepsilon_{x_0}J_{\alpha\beta}\xi)$$
$$= -\frac{2}{3}(\lambda - \mu)\varepsilon_{x_0}J_{\alpha\beta}\xi + \frac{\lambda - \mu}{3}2\varepsilon_{x_0}J_{\alpha\beta}\xi = 0,$$

which completes the proof of Theorem 4.2. □

4.2. Local classification.

In what remains of this subsection, (M, g) is assumed to be a special Osserman manifold with λ of multiplicity distinct of 7, 15. Therefore, from Theorem 4.2, the curvature tensor of M is locally given by

$$R = \mu R^0 + \frac{\lambda - \mu}{3}\sum_{i=1}^{\tau}\sigma_i R^{J_i}, \tag{4.8}$$

where $\tau = 1$ and (g, J) defines an indefinite almost Hermitian structure in a neigbordhood U_p of each $p \in M$ (if $\sigma = -1$) or (g, J) is an almost para-Hermitian structure on U_p (for $\sigma = 1$). In the case $\tau = 3$, $(g, \langle\{J_1, J_2, J_3\}\rangle)$ defines an indefinite quaternionic structure on U_p when $\sigma_1 = \sigma_2 = \sigma_3 = -1$ and a paraquaternionic structure if $\sigma_1 = -1, \sigma_2 = \sigma_3 = 1$.

Next, we state some technical lemmas involving the covariant derivatives of the tensor fields J_i.

Lemma 4.3. *Let* (g, J) *be an indefinite almost Hermitian or almost para-Hemitian structure on a manifold* M. *Then*

(i) $(\nabla_X J)JY = -J(\nabla_X J)Y$,

(ii) $g((\nabla_X J)Y, Z) = -g(Y, (\nabla_X J)Z)$,

(iii) $g((\nabla_X J)Y, Y) = g((\nabla_X J)Y, JY) = 0$,

for all vector fields X, Y, Z *on* M.

Lemma 4.4. *Let* (M, g) *be a pseudo-Riemannian manifold whose curvature tensor is given by* (4.8). *Then,*

$$(\nabla_X R)(Y,Z)W = \frac{\lambda-\mu}{3}\sum_{i=1}^{\tau}\sigma_i\{g(Y,J_iW)(\nabla_X J_i)Z + g(Y,(\nabla_X J_i)W)J_iZ - g(Z,J_iW)(\nabla_X J_i)Y - g(Z,(\nabla_X J_i)W)J_iY + 2g(Y,J_iZ)(\nabla_X J_i)W + 2g(Y,(\nabla_X J_i)Z)J_iW\}$$

for all vector fields X, Y, Z, W *on* M.

Lemma 4.5. *Let* (M, g) *be a pseudo-Riemannian manifold whose curvature tensor is given by* (4.8). *Then,*

$$(\nabla_X J_s)X \in \langle\{J_iX; i \in \{1,\dots,\tau\}, i \neq s\}\rangle, \quad s = 1,\dots,\tau,$$

for all vector fields X *on* M.

Proof. Let Y be a unit vector field orthogonal to $E_\lambda(X)$. Then from Lemma 4.4 and $(\nabla_Y R)(X, J_sY)X + (\nabla_X R)(J_sY, Y)X + (\nabla_{J_sY} R)(Y, X)X = 0$, it follows that

$$0 = \sum_{i=1}^{\tau}\sigma_i\{g(X,(\nabla_Y J_i)X)g(J_iJ_sY, Y) - g(Y,(\nabla_X J_i)X)g(J_iJ_sY, Y) + 2g(J_sY, J_iY)g((\nabla_X J_i)X, Y)\}.$$

Now, since $g(J_iY, J_sY) = -\delta_{is}\sigma_s\varepsilon_Y$ and $g(J_iJ_sY, Y) = -g(J_iY, J_sY)$, one has $g((\nabla_Y J_s)X, X) = 3g((\nabla_X J_s)X, Y)$ and thus Lemma 4.3-(iii) shows that $g((\nabla_X J_s)X, Y) = 0$, from where it follows that $(\nabla_X J_s)X \in E_\lambda(X)$ and the result follows from Lemma 4.3. □

Now we are ready to prove the following

Theorem 4.3. *Let* (M, g) *be a special Osserman pseudo-Riemannian manifold with eigenvalue* λ *of multiplicity different from* 7, 15. *Then, it is a locally symmetric space.*

Proof. In what follows we will show that any pseudo-Riemannian manifold with curvature tensor given by (4.8) is locally symmetric. Let $X_0 \in T_pM$ be a locally defined unit vector field and consider the local decomposition $TM = E_\lambda(X_0) \oplus E_\lambda(Y_0) \oplus \cdots$ given by Lemma 2.1. We will show that

$$(\nabla_{X_0}R)(T, X_0, X_0, W) = 0, \tag{4.9}$$

for all vector fields T, W in the orthonormal local frame induced by the decomposition above.

First of all note that, from Lemma 4.4, since X_0 is orthogonal to J_iX_0 and $(\nabla_{X_0}J_i)X_0$,

$$\begin{aligned}(\nabla_{X_0}R)(T, X_0, X_0, W) = (\lambda - \mu)\sum_{i=1}^{\tau}\sigma_i\{&g(T, J_iX_0)g(W, (\nabla_{X_0}J_i)X_0)\\ &+ g(T, (\nabla_{X_0}J_i)X_0)g(W, J_iX_0)\}.\end{aligned}$$

Now, by Lemma 4.5, it follows that previous expression vanishes whenever at least one of the vector fields T, W is orthogonal to $E_\lambda(X_0)$ and thus (4.9) holds for such a choice of T and W. To finish the proof we analize the case of $T, W \in E_\lambda(X_0)$. Since

$$\begin{aligned}&\nabla_{X_0}(R(T, X_0, X_0, W)) - R(\nabla_{X_0}T, X_0, X_0, W) - R(T, X_0, X_0, \nabla_{X_0}W)\\ &\quad = \nabla_{X_0}(\lambda\varepsilon_{X_0}g(T, W)) - \lambda\varepsilon_{X_0}g(W, \nabla_{X_0}T) - \lambda\varepsilon_{X_0}g(T, \nabla_{X_0}W)\\ &\quad = \lambda\varepsilon_{X_0}(\nabla_{X_0}g)(T, W) = 0,\end{aligned}$$

it follows that

$$(\nabla_{X_0}R)(T, X_0, X_0, W) = -R(T, \nabla_{X_0}X_0, X_0, W) - R(T, X_0, \nabla_{X_0}X_0, W). \tag{4.10}$$

Next, put $T = X_i$ and $W = X_j$, with $i, j \in \{1, \ldots, \tau\}$. If $i \neq j$, Lemma 2.2-(iii) shows that (4.10) vanishes. If $i = j$, from (4.10) we get

$$(\nabla_{X_0}R)(X_i, X_0, X_0, X_i) = -2R(X_0, X_i, X_i, \nabla_{X_0}X_0) = -2\lambda\varepsilon_{X_i}g(X_0, \nabla_{X_0}X_0),$$

which also vanishes since $g(X_0, \nabla_{X_0}X_0) = 0$. This shows that (4.9) holds and therefore, (M, g) is locally symmetric. □

Proof of Theorem 4.1. Since the multiplicity of λ is different from 7 and 15, (M, g) is locally symmetric and its curvature tensor is given by (4.8). Further note that, as an immediate consequence of the definition of the R^{J_i}'s, one has

$$(\nabla_X R^{J_i})(Y, Z)Z = 3\{g((\nabla_X J_i)Z, Y)J_iZ + g(Y, J_iZ)(\nabla_X J_i)Z\}, \tag{4.11}$$

for all vector fields X, Y and Z.

(a): $\tau = 1$ and (g, J) defines an indefinite almost Hermitian structure.

Since (M, g) is locally symmetric, (4.8) implies that $\nabla R^J = 0$ and, in particular, $(\nabla_X R^J)(JY, Y)Y = 0$ for all vector fields X, Y. Now, (4.11) and Lemma 4.3 give $(\nabla_X R^J)(JY, Y)Y = 3\varepsilon_Y(\nabla_X J)Y$, and thus $(\nabla_X J)Y = 0$. This shows that (g, J) is an indefinite Kähler structure. On the other hand, if $x, y \in T_pM$ with $y \in \langle\{x, Jx\}\rangle^\perp$, since $R^0(x, Jx, x, Jy) = R^J(x, Jx, x, Jy) = 0$, (4.8) implies that $R(x, Jx, x, Jy) = 0$. Now, the constancy of the holomorphic sectional curvature follows from [**1**, Theorem 5.1].

(b): $\tau = 1$ and (g, J) defines an almost para-Hermitian structure.

Proceeding in the same way as before, it follows that (g, J) is a para-Kähler structure. Moreover, it also follows from (4.8) that $R(x, Jx, x, Jy) = 0$ for all $x, y \in T_pM$ with $y \in \langle\{x, Jx\}\rangle^\perp$. Hence, the paraholomorphic sectional curvature is constant (cf. [**13**, Theorem]).

(c): $\tau = 3$ and $(g, \langle\{J_1, J_2, J_3\}\rangle)$ defines an indefinite quaternionic structure.

Since M is locally symmetric and the curvaturelike tensor R^0 is parallel, if follows from (4.8) that $\sum_{i=1}^3 (\nabla_X R^{J_i})(Z, Y)Y = 0$ for all vector fields X, Y, Z. Now, if $Z \in E_\lambda(Y)^\perp$, (4.11) and Lemma 4.3 give $(\nabla_X R^{J_i})(Z, Y)Y = 3g((\nabla_X J_i)Y, Z)J_iY$ and thus $g((\nabla_X J_i)Y, Z) = 0$ whenever $Z \in E_\lambda(Y)^\perp$. This shows that $\boldsymbol{Q} = \langle\{J_1, J_2, J_3\}\rangle$ is parallel and thus, the indefinite quaternionic structure is Kähler. Now, if $x, y \in T_pM$ with $y \in \langle\{x, J_1x, J_2x, J_3x\}\rangle^\perp$, it follows from (4.8) that $R(x, J_ix, x, J_iy) = 0$ for all $i = 1, 2, 3$ and therefore, the quaternionic sectional curvature is constant (cf. [**22**, Lemma 5.4]).

(d): $\tau = 3$ and $(g, \langle\{J_1, J_2, J_3\}\rangle)$ defines a paraquaternionic structure.

In the same way as in (c) we prove that the paraquaternionic structure $\tilde{\boldsymbol{Q}} = \langle\{J_1, J_2, J_3\}\rangle$ is Kähler. Moreover, if x, y are vectors with $y \in \langle\{x, J_1x, J_2x, J_3x\}\rangle^\perp$, then, as in (c), we get $R(x, J_ix, x, J_iy) = 0$, $i = 1, 2, 3$, and thus the paraquaternionic sectional curvature is constant (cf. [**16**, Theorem 4.1]). □

5. Exceptional cases: multiplicities $\tau = 7$ and 15.

In this section we will prove the remaining part of Theorem 1.1. That is, those cases when the multiplicity of λ is equal to 7 ($\dim M = 16$) and 15 ($\dim M = 32$). Let x, y be unit vectors with $y \in E_\lambda(x)^\perp$ and take $\{x_1, \dots, x_\tau\}$ an orthonormal basis of $\ker(R_x - \varepsilon_x\lambda\,\mathrm{Id})$. Further, let $\{y_0 = y, y_1 = \phi_1 y_0, \dots, y_\tau = \phi_\tau y_0\}$ be an orthonormal basis of $E_\lambda(y)$, where the ϕ_i's are given by (3.1). Next, define a product on $E_\lambda(y)$ by

$$y_0 \cdot y_i = y_i \cdot y_0 = y_i, \quad y_i \cdot y_j = \phi_i \phi_j y_0, \quad i, j = 1, \dots, \tau,$$

and let $\{e_0, e_1, \dots, e_\tau\}$ denote a standard basis for the product above (i.e., a basis of $E_\lambda(y)$ such that $e_\alpha \cdot e_\beta$ is a basic element, say $e_{\alpha\beta}$). Note that, in the case $\tau = 7$, such a product is given by the following table

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	$-e_5$	e_4	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	$-e_6$	e_7	e_4	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	$-e_7$	$-e_6$	e_5	e_4
e_4	e_4	e_5	e_6	e_7	$-\varepsilon e_0$	$-\varepsilon e_1$	$-\varepsilon e_2$	$-\varepsilon e_3$
e_5	e_5	$-e_4$	$-e_7$	e_6	εe_1	$-\varepsilon e_0$	$-\varepsilon e_3$	εe_2
e_6	e_6	e_7	$-e_4$	$-e_5$	εe_2	εe_3	$-\varepsilon e_0$	$-\varepsilon e_1$
e_7	e_7	$-e_6$	e_5	$-e_4$	εe_3	$-\varepsilon e_2$	εe_1	$-\varepsilon e_0$

It corresponds to the product of the octonians (resp., the anti-octonians) if $\varepsilon = \varepsilon_{y_0}\varepsilon_x = 1$ (resp., $\varepsilon = \varepsilon_{y_0}\varepsilon_x = -1$) ([**19**]).

Lemma 5.1. *Let (M, g) be a special Osserman manifold with λ of multiplicity $\tau = 7, 15$, and let x, y be unit vectors with $y \in E_\lambda(x)^\perp$. Then, there exists an orthonormal basis $\{x_1, \dots, x_\tau\}$ of* $\ker(R_x - \varepsilon_x \lambda \,\mathrm{Id})$ *such that*

$$\phi_\alpha \phi_\beta y_0 = \phi_{\alpha\beta} y_0, \quad \alpha, \beta \in \{1, \dots, \tau\}$$

where $\phi_0 = \mathrm{Id}$ and $\phi_{-\alpha} = -\phi_\alpha$.

Proof. Let $\{x_1, \dots, x_\tau\}$ be an orthonormal basis for $\ker(R_{x_0} - \varepsilon_{x_0}\lambda\,\mathrm{Id})$ and consider the associated endomorphisms ϕ_i defined by (3.1). Moreover let $\{y_0, y_1, \dots, y_\tau\}$ denote the induced basis for $E_\lambda(y_0)$ given by $y_i = \phi_i y_0$, $i = 1, \dots, \tau$. If $\{e_0 = y_0, e_1, \dots, e_\tau\}$ is a standard basis for the product in $E_\lambda(y_0)$, then expressing the e_i's in the basis $\{y_0, \dots, y_\tau\}$, one has

$$e_0 = y_0, \quad e_i = \sum_{j=0}^{\tau} a_{ij} y_j, \quad i = 1, \dots, \tau.$$

Next, define

$$\bar{x}_0 = x_0, \quad \bar{x}_i = \sum_{j=0}^{\tau} a_{ij} x_j, \quad i = 1, \dots, \tau$$

and show that $\{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_\tau\}$ is the desired basis. Note that, if $\bar{\phi}_1, \dots, \bar{\phi}_\tau$ denote the associated endomorphisms defined by such basis, then

$$\begin{aligned}\bar\phi_\alpha\bar\phi_\beta y_0 &= \left(\frac{3}{2(\lambda-\mu)}\right)^2 R(\bar x_0,\bar x_\alpha)R(\bar x_0,\bar x_\beta)y_0\\ &= \frac{3}{2(\lambda-\mu)}\sum_{r,s=1}^{\tau} a_{\alpha r}a_{\beta s}R(x_0,x_r)\phi_s y_0\\ &= \sum_{r,s=1}^{\tau} a_{\alpha r}a_{\beta s}(y_r\cdot y_s) = e_\alpha\cdot e_\beta,\end{aligned}$$

and, on the other hand,

$$\begin{aligned}\bar\phi_{\alpha\beta}y_0 &= \frac{2}{3(\lambda-\mu)}R(\bar x_0,\bar x_{\alpha\beta})y_0\\ &= \frac{3}{2(\lambda-\mu)}\sum_{t=0}^{\tau} a_{(\alpha\beta)t}\frac{2(\lambda-\mu)}{3}\phi_t y_0 = \sum_{t=0}^{\tau} a_{(\alpha\beta)t}y_t = e_{\alpha\beta}.\end{aligned}$$

Now, the result follows from previous expressions, since $e_\alpha\cdot e_\beta = e_{\alpha\beta}$. □

Theorem 5.1. *Let (M,g) be a special Osserman manifold with λ of multiplicity $\tau = 7, 15$. Then it is locally symmetric.*

Proof. Let X be a locally defined unit vector field, and show that $(\nabla_X R)(T,X,X,W) = 0$ for all W, T. Let Y be a unit vector field in $E_\lambda(X)^\perp$ and take a basis $\{X_1,\dots,X_\tau\}$ of $\ker(R_{X_0} - \varepsilon_{X_0}\lambda\,\mathrm{Id})$ given by Lemma 5.1. In what follows we will show that $(\nabla_X R)(T,X,X,W) = 0$ for all T and W chosen in $\{X_0 = X, X_1,\dots,X_\tau, Y_0 = Y, Y_1,\dots,Y_\tau\}$, where $Y_i = \phi_i Y_0$, $(i = 1,\dots,\tau)$.

First of all, if $T \neq W \in E_\lambda(X)$, then

$$\begin{aligned}(\nabla_X R)(T,X,X,W) &= \nabla_X \lambda\varepsilon_X g(T,W) - \lambda\varepsilon_X g(W,\nabla_X T) - \lambda\varepsilon_X g(T,\nabla_X W)\\ &\quad - R(T,\nabla_X X,X,W) - R(T,X,\nabla_X X,W)\\ &= \lambda\varepsilon_X(\nabla_X g)(T,W) = 0\end{aligned}$$

by Lemma 2.2-(iii). Also, if $T = W \in E_\lambda(X)$, one has

$$\begin{aligned}(\nabla_X R)(W,X,X,W) &= \lambda\varepsilon_X(\nabla_X g)(W,W) - 2R(X,W,W,\nabla_X X)\\ &= -2\lambda\varepsilon_W g(X,\nabla_X X) = 0.\end{aligned}$$

In an analogous way, if $T \neq W \in E_\lambda(Y)$ then

$$\begin{aligned}(\nabla_X R)(T,X,X,W) &= \nabla_X \mu\varepsilon_X g(T,W) - \mu\varepsilon_X g(W,\nabla_X T) - \mu\varepsilon_X g(T,\nabla_X W)\\ &\quad - R(T,\nabla_X X,X,W) - R(T,X,\nabla_X X,W)\\ &= -R(X,W,T,\nabla_X X) + R(T,X,W,\nabla_X X) = 0\end{aligned}$$

by Lemma 2.2-(ii) and, if $T = W \in E_\lambda(Y)$, then

$$(\nabla_X R)(W, X, X, W) = \mu\varepsilon_X(\nabla_X g)(W, W) - 2R(X, W, W, \nabla_X X)$$
$$= -2\mu\varepsilon_W g(X, \nabla_X X) = 0.$$

To complete the proof, let consider the case $T = X_i$ and $W = Y_j$. Then, from Lemma 2.2-(ii) and Lemma 5.1,

$$(\nabla_X R)(X_i, X, X, Y_j) = (\lambda - \mu)\{\varepsilon_X g(Y_j, \nabla_X X_i) - g(Y_{ij}, \nabla_X X)\}.$$

On the other hand, once again from Lemma 2.2-(ii)

$$(\nabla_X R)(Y_{ij}, Y_j, Y_j, X) = \mu\varepsilon_{Y_j} g(Y_{ij}, \nabla_X X) - \lambda\varepsilon_{Y_j} g(Y_{ij}, \nabla_X X)$$
$$-\frac{3}{2}R(Y_j, Y_{ij}, X, \nabla_X Y_j).$$

Now, by Lemma 2.2-(i), $R(Y_j, Y_{ij})X = \sum_{k=0}^{\tau} R(Y_j, Y_{ij}, X, X_k)\varepsilon_{X_k} X_k$ and thus, since $R(Y_j, Y_{ij}, X, X_k) = R(Y_j, Y_{ij}, X, X_k) = \sum_{k=0}^{\tau}(2/3)(\lambda - \mu)g(Y_{kj}, Y_{ij})$, it follows that $R(Y_j, Y_{ij})X = (2/3)(\lambda - \mu)\varepsilon_X \varepsilon_{Y_j} X_i$. Then

$$(\nabla_X R)(Y_{ij}, Y_j, Y_j, X) = (\lambda - \mu)\varepsilon_{Y_j}\{\varepsilon_X g(Y_j, \nabla_X X_i) - g(Y_{ij}, \nabla_X X)\}$$

and therefore $(\nabla_X R)(X_i, X, X, Y_j) = \varepsilon_{Y_j}(\nabla_X R)(Y_{ij}, Y_j, Y_j, X)$. Next, by the second Bianchi identity,

$$(\nabla_X R)(Y_{ij}, Y_j, Y_j, X) = -(\nabla_{Y_{ij}} R)(Y_j, X, X, Y_j) - (\nabla_{Y_j} R)(Y_{ij}, X, X, Y_j)$$

and this expression vanishes (proceeding in an analogous way as for the previous case $T, W \in E_\lambda(Y)$), from where it follows that $(\nabla_X R)(X_i, X, X, Y_j) = 0$ and thus, (M, g) is locally symmetric. □

Now, we are ready to complete the announced

Proof of Theorem 1.1. First of all note that from Theorem 4.1 only those cases corresponding to multiplicity of λ equal to 7, 15 need to be considered. As a first observation, note that the action of the holonomy group of a special Osserman manifold with λ of multiplicity 7, 15 is irreducible. Indeed, take an orthonormal basis of the tangent space T_pM induced by the decomposition $T_pM = E_\lambda(x) \oplus E_\lambda(y)$ and Lemma 5.1. Then, after a straightforward calculation, using (3.1) and Lemmas 2.2 and 4.2, one obtains the following expressions of the curvature endomorphisms $R(x, y)$.

For each $i = 1, \dots, \tau$, the endomorphisms $R(x_0, x_i)$ satisfy

$$R(x_0,x_i)x_\alpha = \begin{cases} -\varepsilon_{x_0}\lambda x_i & \alpha = 0 \\ \varepsilon_{x_i}\lambda x_0 & \alpha = i \\ 0 & \alpha \neq 0, i. \end{cases} \quad \text{and} \quad R(x_0,x_i)y_\alpha = \frac{2(\lambda-\mu)}{3} y_{i\alpha}$$

Also, $R(x_0, y_0)$ is given by

$$R(x_0,y_0)x_\alpha = \begin{cases} -\varepsilon_{x_0}\mu y_0 & \alpha = 0 \\ ((\lambda-\mu)/3)y_\alpha & \alpha \neq 0, \end{cases}$$

$$R(x_0,y_0)y_\alpha = \begin{cases} \varepsilon_{y_0}\mu x_0 & \alpha = 0 \\ -((\lambda-\mu)/3)\varepsilon_{x_0}\varepsilon_{y_0}x_\alpha & \alpha \neq 0. \end{cases}$$

Furthermore, for each $i, j = 1, \ldots, \tau,\ i \neq j,\ R(x_i, x_j)$ satisfies

$$R(x_i,x_j)x_\alpha = \begin{cases} -\varepsilon_{x_i}\lambda x_j & \alpha = i \\ \varepsilon_{x_j}\lambda x_i & \alpha = j \\ 0 & \alpha \neq i,j. \end{cases} \quad \text{and} \quad R(x_i,x_j)y_\alpha = -\frac{2(\lambda-\mu)}{3}\varepsilon_{x_0} y_{i(j\alpha)}$$

Also, for each $i = 1, \ldots, \tau,\ R(x_i, y_0)$ is given by

$$R(x_i,y_0)x_\alpha = \begin{cases} -((\lambda-\mu)/3)y_i & \alpha = 0 \\ -\varepsilon_{x_i}\mu y_0 & \alpha = i \\ -((\lambda-\mu)/3)\varepsilon_{x_0}y_{i\alpha} & \alpha \neq 0, i \end{cases}$$

and

$$R(x_i,y_0)y_\alpha = \begin{cases} \varepsilon_{y_0}\mu x_i & \alpha = 0 \\ ((\lambda-\mu)/3)\varepsilon_{x_0}\varepsilon_{y_i}x_0 & \alpha = i \\ -((\lambda-\mu)/3)\varepsilon_{y_0}x_{i\alpha} & \alpha \neq 0, i. \end{cases}$$

Now, it follows from previous expressions that the action of the holonomy group on each tangent space is irreducible and thus, if (M, g) is assumed to be complete and simply connected, then it follows that the special Osserman pseudo-Riemannian manifolds with eigenvalue λ of multiplicity $\tau = 7, 15$ must correspond to one of the symmetric spaces in Berger's list [**2**, p. 157]. In order to complete the proof of Theorem 1.1, we will consider separately the different possibilities corresponding to $\tau = 7$ and $\tau = 15$.

(a): The 16-dimensional case; $\tau = 7$.

First of all we note that, after a long but straightforward calculation, it can be shown that the curvature endomorphisms $R(x_i, x_j)$, $(i, j = 0, \ldots, 7, i < j)$ and $R(x_i, y_0)$, $(i = 1, \ldots, 7)$ are linearly independent. Thus, the dimension of the isotropy group of a special Osserman manifold with eigenvalue λ of multiplicity 7 must be ≥ 36. Moreover, any special Osserman manifold with eigenvalue λ of multiplicity 7 must be of signature $(16, 0)$, $(0, 16)$, or $(8, 8)$ (cf. Theorem 3.2).

Thus, the only candidates in Berger's list are $SL(9,\boldsymbol{R})/(SL(8,\boldsymbol{R})+\boldsymbol{R})$, $SO(9,\boldsymbol{C})/SO(8,\boldsymbol{C})$, $Sp(5,\boldsymbol{R})/(Sp(1,\boldsymbol{R})+Sp(4,\boldsymbol{R}))$, $F_4/SO(9)$, $F_4^2/SO(9)$, $F_4^2/SO^1(9)$, $F_4^1/SO^4(9)$, and

$$SU^i(n)/(SU^k(k+h)+SU^{i-k}(n-k-h)+T), \quad (k+h)(n-k-h)=8,$$

$$SO^i(n)/(SO^k(k+h)+SO^{i-k}(n-k-h)), \quad (k+h)(n-k-h)=16,$$

$$k+h>2, n-k-h>2,$$

$$Sp^i(n)/(Sp^k(k+h)+Sp^{i-k}(n-k-h)+T), \quad (k+h)(n-k-h)=4.$$

Now, note that $SL(9,\boldsymbol{R})/(SL(8,\boldsymbol{R})+\boldsymbol{R})$, and $Sp(5,\boldsymbol{R})/(Sp(1,\boldsymbol{R})+Sp(4,\boldsymbol{R}))$, correspond to the paracomplex and paraquaternionic space forms, respectively. Moreover, $SO(9,\boldsymbol{C})/SO(8,\boldsymbol{C})$, corresponds to the complex sphere $\boldsymbol{C}S^8$, which can be viewed as a hypersurface in the indefinite sphere. It easily follows from [**20**] that such complex spheres are not Osserman spaces. Now, it also follows that $SO^i(n)/(SO^k(k+h)+SO^{i-k}(n-k-h))$ occurs only if $n=8$, but it must be excluded, since in this case the dimension of the holonomy group is <36. The symmetric spaces $Sp^i(n)/(Sp^k(k+h)+Sp^{i-k}(n-k-h)+T)$ can only occur if $n=5$ and then they correspond to the indefinite quaternionic projective or hyperbolic spaces. Also, by an argument on the dimension of the holonomy group, $SU^i(n)/(SU^k(k+h)+SU^{i-k}(n-k-h)+T)$ may only occur if $n=9$, but in this case, they correspond to the indefinite complex projective or hyperbolic spaces.

The remaining spaces, $F_4/SO(9)$, $F_4^2/SO(9)$, $F_4^2/SO^1(9)$ and $F_4^1/SO^4(9)$, correspond to the Cayley planes over the octonians and the anti-octonians (see also [**23**]).

(b): The 32-dimensional case; $\tau=15$.

Proceeding as in the previous case, the curvature endomorphisms $R(x_i,x_j)$, $(i,j=0,\dots,15, i<j)$ and $R(x_i,y_0)$, $(i=1,\dots,15)$ are linearly independent and thus, the dimension of the isotropy group of any special Osserman manifold with eigenvalue λ of multiplicity 15 must be ≥ 136. Moreover, since any such special Osserman manifold must be of neutral signature $(16,16)$ (cf. Theorem 3.2), an examination of Berger's list shows that, the only candidates are $SL(17,\boldsymbol{R})/(SL(16,\boldsymbol{R})+\boldsymbol{R})$, $SO(17,C)/SO(16,C)$, $Sp(9,\boldsymbol{R})/(Sp(1,\boldsymbol{R})+Sp(8,\boldsymbol{R}))$ and

$$SU^i(n)/(SU^k(k+h)+SU^{i-k}(n-k-h)+T), \quad (k+h)(n-k-h)=16,$$

$$SO^i(n)/(SO^k(k+h)+SO^{i-k}(n-k-h)), \quad (k+h)(n-k-h)=32,$$

$$k+h>2, n-k-h>2,$$

$$Sp^i(n)/(Sp^k(k+h)+Sp^{i-k}(n-k-h)+T), \quad (k+h)(n-k-h)=8.$$

Now, $SL(17,\boldsymbol{R})/(SL(16,\boldsymbol{R})+\boldsymbol{R})$, and $Sp(9,\boldsymbol{R})/(Sp(1,\boldsymbol{R})+Sp(8,\boldsymbol{R}))$, correspond to the paracomplex and paraquaternionic space forms, with eigenvalue λ of multiplicity one and three, respectively. Proceeding as in the previous case, the other symmetric spaces listed above are not special Osserman, or they correspond to the indefinite complex or quaternionic space forms. Therefore, it follows the nonexistence of special Osserman manifolds with eigenvalue λ of multiplicity 15.

This finishes the proof of Theorem 1.1. □

Remark 5.1. Note that (I) and (II) reduces to Axioms 1, 2 in [**10**] if the metric g is assumed to be positive definite. Moreover, Theorem 1.1 reduces to [**10**, Theorem 1] for Riemannian metrics.

Remark 5.2. Finally, note that Theorem 1.1 lists the simplest pseudo-Riemannian manifolds (besides the spaces of constant curvature) from the point of view of their curvature.

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References

[1] M. Barros and A. Romero, Indefinite Kähler manifolds, Math. Ann., **261** (1982), 55–62.

[2] M. Berger, Les espaces symétriques non compacts, Ann. Eco. Norm. Sup., **12** (1957), 85–177.

[3] A. Besse, Einstein manifolds, Ergeb. Math. Grenzgeb., **10**, Springer-Verlag, Berlin-Heidelberg-New York, 1987.

[4] N. Blažić, Paraquaternionic projective space and pseudo-Riemannian geometry, Publ. Inst. Math. (Beograd), **60** (74) (1996), 101–107.

[5] N. Blažić, N. Bokan and P. Gilkey, A note on Osserman Lorentzian manifolds, Bull. London Math. Soc., **29** (1997), 227–230.

[6] N. Blažić, N. Bokan and Z. Rakić, Characterization of 4-dimensional Osserman pseudo-Riemannian manifolds, J. Austral. Math. Soc., **71** (2001), 367–395.

[7] N. Blažić, N. Bokan and Z. Rakić, A note on Osserman conjecture and isotropic covariant derivative of curvature, Proc. Amer. Math. Soc., **128** (2000), 245–253.

[8] A. Bonome, R. Castro, E. García-Río, L. Hervella and R. Vázquez-Lorenzo, Nonsymmetric Osserman indefinite Kähler manifolds, Proc. Amer. Math. Soc., **126** (1998), 2763–2769.

[9] Q. S. Chi, A curvature characterization of certain locally rank-one symmetric spaces, J. Differential Geom., **28** (1988), 187–202.

[10] Q. S. Chi, Curvature characterization and classification of locally rank-one symmetric spaces, Pacific J. Math., **150** (1991), 31–42.

[11] Q. S. Chi, Quaternionic Kähler manifolds and a curvature characterization of two-point homogenenous spaces, Illinois J. Math., **35** (1991), 408–418.

[12] V. Cruceanu, P. Fortuny and P. M. Gadea, A survey on paracomplex geometry, Rocky Mountain J. Math., **26** (1996), 83–115.

[13] P. M. Gadea and A. Montesinos, Spaces of constant para-holomorphic sectional curvature, Pacific J. Math., **136** (1989), 85–101.

[14] E. García-Río, D. N. Kupeli and M. E. Vázquez-Abal, On a problem of Osserman in Lorentzian geometry, Differential Geom. Appl., **7** (1997), 85–100.
[15] E. García-Río, M. E. Vázquez-Abal and R. Vázquez-Lorenzo, Nonsymmetric Osserman pseudo-Riemannian manifolds, Proc. Amer. Math. Soc., **126** (1998), 2771–2778.
[16] E. García-Río, Y. Matsushita and R. Vázquez-Lorenzo, Paraquaternionic Kähler manifolds, Rocky Mountain J. Math., **31** (2001), 237–260.
[17] P. B. Gilkey, A. Swann and L. Vanhecke, Isoparametric geodesic spheres and a conjecture of Osserman concernig the Jacobi operator, Quart. J. Math. Oxford, **46** (1995), 299–320.
[18] F. R. Harvey, Spinors and calibrations, Perspectives in Mathematics, **9**, Academic Press, Boston, MA, 1990.
[19] P. Libermann, Sur le problème d'équivalence de certaines structures infinitésimales, Ann. Mat. Pura Appl., **36**, (1954), 27–120.
[20] M. A. Magid, Indefinite Einstein hypersurfaces with imaginary principal curvatures, Houston J. Math., **10** (1984), 57–61.
[21] R. Osserman, Curvature in the eighties, Amer. Math. Monthly, **97** (1990), 731–756.
[22] J. D. Pérez and F. G. Santos, Indefinite quaternion space forms, Ann. Mat. Pura Appl., **132** (1982), 383–398.
[23] B. A. Rosenfeld, Geometry of Lie Groups, Math. Appl., **393**, Kluwer Acad. Publ., Dordrecht, 1997.
[24] N. Steenrod, The topology of fibre bundles, Princeton, 1965.
[25] J. Wolf, Spaces of constant curvature, Publish or Perish, Boston, MA, 1974.
[26] H. Wu, On the de Rham decomposition theorem, Illinois J. Math., **8** (1964), 291–311.

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