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Selections and sandwich-like properties via semi-continuous Banach-valued functions

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Abstract. We introduce lower and upper semi-continuity of a map to the Banach space $c_0(\lambda)$ for an infinite cardinal λ. We prove that the following conditions (i), (ii) and (iii) on a T_1-space X are equivalent: (i) For every two maps $g, h : X \to c_0(\lambda)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \le h$, there exists a continuous map $f : X \to c_0(\lambda)$, with $g \le f \le h$. (ii) For every Banach space Y, with $w(Y) \le \lambda$, every lower semi-continuous set-valued mapping $\phi : X \to \mathscr{C}_c(Y)$ admits a continuous selection, where $\mathscr{C}_c(Y)$ is the set of all non-empty compact convex sets in Y. (iii) X is normal and every locally finite family $\mathscr{F}$ of subsets of X, with $|\mathscr{F}| \le \lambda$, has a locally finite open expansion provided it has a point-finite open expansion. We also characterize several paracompact-like properties by inserting continuous maps between semi-continuous Banach-valued functions.

1. Introduction.

Throughout this paper, by a space we mean a non-empty T_1-space. Our investigation was motivated by the following two theorems; the former was proved by Katětov [**14**], [**15**] and Tong [**29**], and the latter was proved by Kandô [**13**] and Nedev [**23**]:

THEOREM 1.1 (Katětov-Tong's insertion theorem). *A space X is normal if and only if for every two functions $g, h : X \to \boldsymbol{R}$ such that g is upper semi-continuous, h is lower semi-continuous and $g \le h$, there exists a continuous function $f : X \to \boldsymbol{R}$ such that $g \le f \le h$.*

For a Banach space Y, let $\mathscr{F}_c(Y)$ (resp., $\mathscr{C}_c(Y)$) denote the set of all non-empty closed (resp., non-empty compact) convex sets in Y. A map $f : X \to Y$ is called a *selection* of a mapping $\phi : X \to \mathscr{F}_c(Y)$ if $f(x) \in \phi(x)$ for every $x \in X$.

THEOREM 1.2 (Kandô-Nedev's selection theorem). *Let λ be an infinite cardinal. Then, the following conditions on a space X are equivalent:*

(1) *Every point-finite open cover $\mathscr{U}$ of X, with $|\mathscr{U}| \le \lambda$, is normal.*
(2) *For every Banach space Y, with $w(Y) \le \lambda$, every lower semi-continuous mapping $\phi : X \to \mathscr{C}_c(Y)$ admits a continuous selection.*
(3) *Every lower semi-continuous mapping $\phi : X \to \mathscr{C}_c(\ell_1(\lambda))$ admits a continuous selection.*

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Theorem 1.2 can be regarded as an essential part of Michael's selection theorem [**19**, Theorem 3.2′] (see, also, [**2**]) asserting that a space X is λ-collectionwise normal if and only if X satisfies the condition (2) with $\mathscr{C}_c(Y)$ replaced by $\mathscr{C}_c(Y) \cup \{Y\}$.

For a space Y, let $C_0(Y)$ denote the Banach space of all real-valued continuous functions s on Y such that for each $\varepsilon > 0$ the set $\{y \in Y : |s(y)| \geq \varepsilon\}$ is compact, where the linear operations are defined pointwise and $\|s\| = \sup_{y \in Y}|s(y)|$ for each $s \in C_0(Y)$. In particular, we use $c_0(\lambda)$ to denote the space $C_0(Y)$, where Y is the discrete space of cardinality λ, i.e. $c_0(\lambda)$ is the Banach space consisting of all points $s \in \boldsymbol{R}^{\lambda}$ such that the set $\{\alpha < \lambda : |s(\alpha)| \geq \varepsilon\}$ is finite for each $\varepsilon > 0$.

In this paper, we introduce lower and upper semi-continuity of a map to $C_0(Y)$. We prove that if the space $\boldsymbol{R}$ in Theorem 1.1 is replaced by $c_0(\lambda)$, then the resulting statement is equivalent to the conditions listed in Theorem 1.2, see Theorem 3.1. Thus, insertions and selections are connected via the space $c_0(\lambda)$. As a result, we obtain several sandwich-like analogues to selection theorems as well as selection theorems corresponding to sandwich-like properties, see Section 4.

For set-valued mappings φ and ψ defined on a space X, we say that φ is a *set-valued selection* of ψ, or ψ is an *expansion* of φ, if $\varphi(x) \subseteq \psi(x)$ for each $x \in X$. Let $\mathscr{C}(Y)$ denote the set of all non-empty compact sets in a space Y. In [**23**] Nedev has characterized several paracompact-like properties by the existence of set-valued selections of $\mathscr{C}(Y)$-valued mappings for completely metrizable spaces Y. In contrast to this, we characterize expandability and almost expandability in the sense of [**16**], [**27**] by insertion of $c_0(\lambda)$-valued maps, and by the existence of expansions of $\mathscr{C}(Y)$-valued mappings for completely metrizable spaces Y, see Section 5.

We often consider two kinds of maps in the same statement, i.e., a single-valued map to a space Y and a set-valued map to a hyperspace of Y. To distinguish them, we use the term *map* for the former one and the term *mapping* for the latter one. As usual, a cardinal is identified with the initial ordinal and an ordinal is the set of all smaller ordinals. The cardinality of a set A is denoted by $|A|$. Let ω denote the first infinite cardinal and $\boldsymbol{N}$ the set of non-negative integers. Other terms and notation will be used as in [**8**].

2. Semi-continuous $C_0(Y)$-valued functions and compact sets.

In this section, X and Y denote arbitrary spaces and λ stands for a cardinal. For a real-valued function $f : X \to \boldsymbol{R}$ and $r \in \boldsymbol{R}$, let $L(f, r) = \{x \in X : f(x) > r\}$ and $U(f, r) = \{x \in X : f(x) < r\}$. Recall that a function $f : X \to \boldsymbol{R}$ is *lower* (resp., *upper*) *semi-continuous* if $L(f, r)$ (resp., $U(f, r)$) is open in X for each $r \in \boldsymbol{R}$. Now, we extend these notions to $C_0(Y)$-valued maps as follows:

Definition 2.1. A map $f : X \to C_0(Y)$ is *lower* (resp., *upper*) *semi-continuous* if for every $x \in X$ and every $\varepsilon > 0$, there is a neighbourhood G of x in X such that if $x' \in G$, then $f(x')(y) > f(x)(y) - \varepsilon$ (resp., $f(x')(y) < f(x)(y) + \varepsilon$) for each $y \in Y$.

With every map $f : X \to C_0(Y)$ we associate another one $-f : X \to C_0(Y)$ defined by $(-f)(x)(y) = -f(x)(y)$ for each $x \in X$ and each $y \in Y$. The first lemma is a direct consequence of the definition.

LEMMA 2.2. *A map $f : X \to C_0(Y)$ is continuous if and only if it is both lower and upper semi-continuous. A map $f : X \to C_0(Y)$ is lower semi-continuous if and only if the map $-f$ is upper semi-continuous.*

The following three lemmas concern only the case of $c_0(\lambda)$. For each $\alpha < \lambda$, let $\pi_\alpha : \boldsymbol{R}^\lambda \to \boldsymbol{R}$ denote the α-th projection, i.e. $\pi_\alpha(s) = s(\alpha)$ for $s \in \boldsymbol{R}^\lambda$.

LEMMA 2.3. *For a map $f : X \to c_0(\lambda)$, the following are valid*:

(1) *f is lower semi-continuous if and only if $\pi_\alpha \circ f$ is lower semi-continuous for each $\alpha < \lambda$, and $\{U(\pi_\alpha \circ f, -\varepsilon) : \alpha < \lambda\}$ is locally finite in X for each $\varepsilon > 0$.*

(2) *f is upper semi-continuous if and only if $\pi_\alpha \circ f$ is upper semi-continuous for each $\alpha < \lambda$, and $\{L(\pi_\alpha \circ f, \varepsilon) : \alpha < \lambda\}$ is locally finite in X for each $\varepsilon > 0$.*

PROOF. Note that $L(\pi_\alpha \circ f, \varepsilon) = U(\pi_\alpha \circ (-f), -\varepsilon)$ for every $\alpha < \lambda$. Hence, by Lemma 2.2, (2) is a consequence of (1). Thus, it only suffices to prove (1). Suppose that f is lower semi-continuous. Clearly, $\pi_\alpha \circ f$ is lower semi-continuous for each $\alpha < \lambda$. Let $\varepsilon > 0$ be fixed, and let $x \in X$. Since f is lower semi-continuous, there is a neighbourhood G of x such that if $y \in G$, then $f(y)(\alpha) > f(x)(\alpha) - \varepsilon/2$ for each $\alpha < \lambda$. We show that G intersects only finitely many $U(\pi_\alpha \circ f, -\varepsilon)$'s. By the definition of $c_0(\lambda)$ the set $A = \{\alpha < \lambda : f(x)(\alpha) < -\varepsilon/2\}$ is finite. If $y \in G$ and $\alpha \in \lambda \backslash A$, then $f(y)(\alpha) > f(x)(\alpha) - \varepsilon/2 \geq -\varepsilon/2 - \varepsilon/2 = -\varepsilon$, i.e. $y \notin U(\pi_\alpha \circ f, -\varepsilon)$. Hence, $G \cap U(\pi_\alpha \circ f, -\varepsilon) = \varnothing$ for each $\alpha \in \lambda \backslash A$.

Conversely, suppose that each $\pi_\alpha \circ f$, $\alpha < \lambda$, is lower semi-continuous and the family $\{U(\pi_\alpha \circ f, -\varepsilon) : \alpha < \lambda\}$ is locally finite in X for each $\varepsilon > 0$. Let $x \in X$ and $\varepsilon > 0$ be fixed. Then, there exist a neighbourhood H of x and a finite set $B \subseteq \lambda$ such that $H \cap U(\pi_\alpha \circ f, -\varepsilon/2) = \varnothing$ for each $\alpha \in \lambda \backslash B$. Since $f(x) \in c_0(\lambda)$, the set $C = \{\alpha < \lambda : f(x)(\alpha) \geq \varepsilon/2\}$ is finite. Put $D = B \cup C$. If $y \in H$ and $\alpha \in \lambda \backslash D$, then $f(y)(\alpha) \geq -\varepsilon/2 > f(x)(\alpha) - \varepsilon$. For each $\alpha \in D$, since $\pi_\alpha \circ f$ is lower semi-continuous, there exists a neighbourhood H_α of x such that $f(y)(\alpha) > f(x)(\alpha) - \varepsilon$ for every $y \in H_\alpha$. Therefore, if $y \in H \cap \bigcap_{\alpha \in D} H_\alpha$, then $f(y)(\alpha) > f(x)(\alpha) - \varepsilon$ for each $\alpha < \lambda$. Hence, f is lower semi-continuous. $\square$

LEMMA 2.4. *Let $f : X \to \boldsymbol{R}^\lambda$ be a map. Then, $f[X] \subseteq c_0(\lambda)$ if and only if both $\{L(\pi_\alpha \circ f, \varepsilon) : \alpha < \lambda\}$ and $\{U(\pi_\alpha \circ f, -\varepsilon) : \alpha < \lambda\}$ are point-finite in X for each $\varepsilon > 0$.*

PROOF. This follows from the definition of $c_0(\lambda)$. $\square$

For $s \in C_0(Y)$ and $\varepsilon > 0$, let $B(s, \varepsilon) = \{t \in C_0(Y) : \|s - t\| < \varepsilon\}$. For $s, t \in C_0(Y)$, we write $s \leq t$ if $s(y) \leq t(y)$ for each $y \in Y$. Further, if $s \leq t$, then we define $[s, t] = \{u \in \boldsymbol{R}^Y : s \leq u \leq t\}$. Obviously, $[s, t]$ is a closed convex subset of $C_0(Y)$. In the case of $c_0(\lambda)$, we have a stronger result:

LEMMA 2.5. *For every $s, t \in c_0(\lambda)$, with $s \leq t$, the subspace topology σ on $[s, t]$ coincides with the subspace topology induced from the product topology τ on $\boldsymbol{R}^\lambda$. Hence, in particular, $[s, t]$ is a compact convex subset of $c_0(\lambda)$.*

PROOF. Obviously, the topology σ is finer than the subspace topology induced from τ. To prove the converse, let $u \in [s, t]$ and consider an ε-neighbourhood $B(u, \varepsilon)$

of u. It suffices to find $V \in \tau$ such that $u \in V \cap [s,t] \subseteq B(u,\varepsilon)$. Since $s - t \in c_0(\lambda)$, there is a finite set $A \subseteq \lambda$ such that $|s(\alpha) - t(\alpha)| < \varepsilon$ for each $\alpha \in \lambda \backslash A$. Let

$$V = \prod_{\alpha \in A} \{x \in \boldsymbol{R} : |u(\alpha) - x| < \varepsilon\} \times \boldsymbol{R}^{\lambda \backslash A}.$$

Then, $V \in \tau$ and it is easy to check that $u \in V \cap [s,t] \subseteq B(u,\varepsilon)$. Thus, we have the first statement. Clearly, $[s,t]$ is convex. Since $[s,t]$ is a compact subset of $(\boldsymbol{R}^{\lambda}, \tau)$, it finally follows that $[s,t]$ is a compact subset of $c_0(\lambda)$ too. $\square$

We now recall the definitions of upper and lower semi-continuity of set-valued mappings. Let $\phi : X \to \mathscr{S}$ be a set-valued mapping, where $\mathscr{S}$ is a family of non-empty subsets of a space Y. For a subset $U \subseteq Y$, let $\phi^{-1}[U] = \{x \in X : \phi(x) \cap U \neq \varnothing\}$ and $\phi^{\#}[U] = \{x \in X : \phi(x) \subseteq U\}$. The mapping $\phi : X \to \mathscr{S}$ is called *lower* (resp., *upper*) *semi-continuous* if $\phi^{-1}[U]$ (resp., $\phi^{\#}[U]$) is open in X for every open set U in Y. Also, ϕ is called *continuous* if it is both lower and upper semi-continuous.

For maps $g, h : X \to C_0(Y)$, we shall write $g \leq h$ if $g(x) \leq h(x)$ for every $x \in X$. With every two such maps we associate a set-valued mapping $[g,h] : X \to \mathscr{F}_c(C_0(Y))$ defined by $[g,h](x) = [g(x), h(x)]$ for $x \in X$. Also, we associate two mappings $[g, +\infty)$ and $(-\infty, h]$ from X to $\mathscr{F}_c(C_0(Y))$ by $[g,+\infty)(x) = \{s \in C_0(Y) : s \geq g(x)\}$ and $(-\infty, h](x) = \{s \in C_0(Y) : s \leq h(x)\}$ for $x \in X$, respectively. Finally, for $S \subseteq C_0(Y)$ and $\varepsilon > 0$, let $B(S,\varepsilon)$ denote the ε-neighbourhood of S in $C_0(Y)$, i.e. $B(S,\varepsilon) = \bigcup_{s \in S} B(s,\varepsilon)$.

Lemma 2.6. *Let $g, h : X \to C_0(Y)$ be maps such that $g \leq h$.*

(1) *If g is upper semi-continuous, then $[g,+\infty)$ is lower semi-continuous.*
(2) *If h is lower semi-continuous, then $(-\infty, h]$ is lower semi-continuous.*
(3) *If g is upper semi-continuous and h is lower semi-continuous, then the mapping $[g,h]$ is lower semi-continuous.*
(4) *If g is lower semi-continuous, h is upper semi-continuous and Y is discrete, then the mapping $[g,h]$ is upper semi-continuous.*

Proof. In order to prove (1), let U be an open set in $C_0(Y)$ and $x \in [g,+\infty)^{-1}[U]$. Since $[g,+\infty)(x) \cap U \neq \varnothing$, there exists $s \in U$ with $g(x) \leq s$. Choose $\varepsilon > 0$ such that $B(s,\varepsilon) \subseteq U$. Since g is upper semi-continuous, there exists a neighbourhood G of x such that $x' \in G$ implies

$$g(x')(y) < g(x)(y) + \varepsilon \leq s(y) + \varepsilon, \quad \text{for each } y \in Y. \tag{2.1}$$

Now, we show that $G \subseteq [g,+\infty)^{-1}[U]$. Take a point $x' \in G$ and define $t(y) = \max\{g(x')(y), s(y)\}$ for $y \in Y$. Then, $t \in C_0(Y)$ and, by (2.1), $t \geq g(x')$ and $\|s - t\| < \varepsilon$. Hence, $t \in [g,+\infty)(x') \cap U$, which implies that $x' \in [g,+\infty)^{-1}[U]$. Consequently, $G \subseteq [g,+\infty)^{-1}[U]$, and it follows that $[g,+\infty)^{-1}[U]$ is open in X.

To prove (2), note that $s \in (-\infty, h](x)$ if and only if $-s \in [-h, +\infty)(x)$. Hence, this follows from (1) because, by Lemma 2.2, the map $-h$ is upper semi-continuous.

To prove (3), let us observe that $[g,h]^{-1}[U] = [g,+\infty)^{-1}[U] \cap (-\infty,h]^{-1}[U]$ for every $U \subseteq C_0(Y)$. Hence, the statement follows from (1) and (2).

We finally prove (4). Let V be an open set in $C_0(Y)$ and $z \in [g,h]^{\#}[V]$. Then, $[g,h](z) \subseteq V$. Since Y is discrete, it follows from Lemma 2.5 that $[g,h](z)$ is compact. Thus, we can find $\delta > 0$ such that $B([g,h](z),\delta) \subseteq V$. Since g is lower semi-continuous and h is upper semi-continuous, there exists a neighbourhood H of z in X such that if $z' \in H$, then $g(z')(y) > g(z)(y) - \delta$ and $h(z')(y) < h(z)(y) + \delta$ for each $y \in Y$. Then, it is easy to check that $H \subseteq [g,h]^{\#}[V]$. Hence $[g,h]$ is upper semi-continuous. □

For a non-empty bounded set $K \subseteq C_0(Y)$, we define points $\sup K$ and $\inf K$ of $\boldsymbol{R}^Y$ by $(\sup K)(y) = \sup\{s(y) : s \in K\}$ and $(\inf K)(y) = \inf\{s(y) : s \in K\}$, respectively, for each $y \in Y$.

Lemma 2.7. *If K is a non-empty compact set in $C_0(Y)$, then $\sup K \in C_0(Y)$ and $\inf K \in C_0(Y)$. Hence, $K \subseteq [\inf K, \sup K]$.*

Proof. We write $s = \sup K$ for short. Since K is compact, K is equicontinuous, which implies that s is continuous. To show that $s \in C_0(Y)$, let $\varepsilon > 0$ be fixed. If the set $A_- = \{y \in Y : s(y) \le -\varepsilon\}$ is not compact, then every point $u \in K$ is not in $C_0(Y)$ since A_- is closed in $\{y \in Y : |u(y)| \ge \varepsilon\}$. This contradiction proves that A_- is compact. Next, suppose that the set $A_+ = \{y \in Y : s(y) \ge \varepsilon\}$ is not compact. For each $y \in A_+$, choose $u_y \in K$ with $u_y(y) \ge 2\varepsilon/3$ and let $B_y = \{y' \in Y : u_y(y') \ge \varepsilon/3\}$. Then, A_+ can not be covered by finitely many B_y's since each B_y is compact. Hence, we can find a sequence $\{y(n) : n < \omega\} \subseteq A_+$ such that $y(n) \notin \bigcup_{i<n} B_{y(i)}$ for each $n > 0$. Then, $\{u_{y(n)} : n < \omega\}$ is discrete closed in K, because $\|u_{y(m)} - u_{y(n)}\| \ge \varepsilon/3$ whenever $m \ne n$. This however contradicts the compactness of K. Thus, A_+ is also compact, and hence, $s \in C_0(Y)$. This also implies that $\inf K \in C_0(Y)$ because $-(\inf K) = \sup(-K)$. □

For a mapping $\phi : X \to \mathscr{C}(C_0(Y))$, we define single-valued maps $\sup\phi : X \to C_0(Y)$ and $\inf\phi : X \to C_0(Y)$ by $(\sup\phi)(x) = \sup\phi(x)$ and $(\inf\phi)(x) = \inf\phi(x)$, respectively, for each $x \in X$.

Lemma 2.8. *Let $\phi : X \to \mathscr{C}(C_0(Y))$ be a mapping.*

(1) *If ϕ is lower semi-continuous, then $\sup\phi$ is lower semi-continuous and $\inf\phi$ is upper semi-continuous.*
(2) *If ϕ is upper semi-continuous, then $\sup\phi$ is upper semi-continuous and $\inf\phi$ is lower semi-continuous.*

Proof. We prove both (1) and (2) only for $\sup\phi$, since the proofs for $\inf\phi$ are similar. First, assume that ϕ is lower semi-continuous and put $s = \sup\phi$. To show that s is lower semi-continuous, take a point $x \in X$ and $\varepsilon > 0$. Fix a point $t \in \phi(x)$. Then the set $A = \{y \in Y : |s(x)(y) - t(y)| \ge \varepsilon/2\}$ is compact. Since $\phi(x)$ is compact, $\phi(x)$ is equicontinuous. Hence, for each $y \in Y$, there exists an open neighbourhood U_y of y in Y such that

$$|u(y) - u(y')| < \varepsilon/4 \quad \text{for all } u \in \phi(x) \text{ and for all } y' \in U_y. \tag{2.2}$$

This implies that

$$|s(x)(y) - s(x)(y')| \le \varepsilon/4 \quad \text{for all } y' \in U_y. \tag{2.3}$$

Since A is compact, there exists a finite set $B \subseteq A$ such that $\bigcup\{U_y : y \in B\} \supseteq A$. For each $y \in B$, take $u_y \in \phi(x)$ such that

$$s(x)(y) \geq u_y(y) > s(x)(y) - \varepsilon/4. \tag{2.4}$$

Since ϕ is lower semi-continuous, the set

$$G = \phi^{-1}[B(t, \varepsilon/2)] \cap \bigcap\{\phi^{-1}[B(u_y, \varepsilon/4)] : y \in B\}$$

is a neighbourhood of x in X. Let $x' \in G$. It suffices to show that $s(x')(z) > s(x)(z) - \varepsilon$ for each $z \in Y$. If $z \in A$, then $z \in U_y$ for some $y \in B$. Since $\phi(x') \cap B(u_y, \varepsilon/4) \neq \emptyset$, $s(x')(z) \geq u_y(z) - \varepsilon/4$. On the other hand, by (2.3), (2.4) and (2.2), we have that

$$\begin{aligned} |s(x)(z) - u_y(z)| &\leq |s(x)(z) - s(x)(y)| + |s(x)(y) - u_y(y)| + |u_y(y) - u_y(z)| \\ &< \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = 3\varepsilon/4. \end{aligned}$$

Hence, $s(x')(z) > s(x)(z) - \varepsilon$. Otherwise, $z \notin A$ implies $|s(x)(z) - t(z)| < \varepsilon/2$. Since $\phi(x') \cap B(t, \varepsilon/2) \neq \emptyset$, it follows that $s(x')(z) \geq t(z) - \varepsilon/2 > s(x)(z) - \varepsilon$. Thus, we have proved that s is lower semi-continuous.

Next, assume that ϕ is upper semi-continuous. Let $x \in X$, $\varepsilon > 0$ and put $U = B(\phi(x), \varepsilon)$. Since ϕ is upper semi-continuous, $\phi^{\#}[U]$ is a neighbourhood of x in X. If $x' \in \phi^{\#}[U]$, $(\sup \phi)(x')(y) < (\sup \phi)(x)(y) + \varepsilon$ for each $y \in Y$, because $\phi(x') \subseteq U$. Hence, $\sup \phi$ is upper semi-continuous. □

3. Extension of Theorem 1.2.

For two families $\mathscr{F}$ and $\mathscr{G}$ of subsets of a space X, we call $\mathscr{G}$ an *expansion* of $\mathscr{F}$ if there exists a bijection $G : \mathscr{F} \to \mathscr{G}$ such that $F \subseteq G(F)$ for each $F \in \mathscr{F}$. An *open expansion* is an expansion consisting of open sets. For real-valued functions f_α, $\alpha < \lambda$, on a space X, let $\triangle_{\alpha<\lambda} f_\alpha$ denote the map $f : X \to \boldsymbol{R}^\lambda$ such that $\pi_\alpha \circ f = f_\alpha$ for each $\alpha < \lambda$.

In this section, we find a natural relationship between insertions and selections by proving the following theorem which extends Theorem 1.2. The equivalence of (1) and (2) is due to Kandô [**13**] and Nedev [**23**] as was stated in the introduction.

THEOREM 3.1. *For an infinite cardinal λ, the following conditions on a space X are equivalent*:

(1) *Every point-finite open cover $\mathscr{U}$ of X, with $|\mathscr{U}| \leq \lambda$, is normal.*
(2) *For every Banach space Y, with $w(Y) \leq \lambda$, every lower semi-continuous mapping $\varphi : X \to \mathscr{C}_c(Y)$ admits a continuous selection.*
(3) *Every lower semi-continuous mapping $\varphi : X \to \mathscr{C}_c(c_0(\lambda))$ admits a continuous selection.*
(4) *For every two maps $g, h : X \to c_0(\lambda)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exists a continuous map $f : X \to c_0(\lambda)$ such that $g \leq f \leq h$.*
(5) *For every two maps $g, h : X \to c_0(\lambda)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exist a lower semi-continuous map $f_\ell : X \to c_0(\lambda)$ and an upper semi-continuous map $f_u : X \to c_0(\lambda)$ such that $g \leq f_\ell \leq f_u \leq h$.*

(6) *X is normal, and every locally finite family $\mathscr{F}$ of closed sets in X, with $|\mathscr{F}| \leq \lambda$, has a locally finite open expansion provided it has a point-finite open expansion.*
(7) *Every discrete family $\mathscr{F}$ of closed sets in X, with $|\mathscr{F}| \leq \lambda$, has a disjoint open expansion provided it has a point-finite open expansion.*

Proof. The implication (1) $\Rightarrow$ (2) is due to Nedev, while (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4): Let $g, h : X \to c_0(\lambda)$ be as in (4). Then, by Lemma 2.6, the mapping $[g, h] : X \to \mathscr{C}_c(c_0(\lambda))$ is lower semi-continuous. Hence, by (3), $[g, h]$ admits a continuous selection $f : X \to c_0(\lambda)$. The map f satisfies that $g \leq f \leq h$.

(4) $\Rightarrow$ (5): Obvious.

(5) $\Rightarrow$ (6): Let, for some $\mu \leq \lambda$, $\mathscr{F} = \{F_\alpha : \alpha < \mu\}$ be a locally finite family of closed sets in X and $\mathscr{U} = \{U_\alpha : \alpha < \mu\}$ be a point-finite open expansion of $\mathscr{F}$, i.e. $F_\alpha \subseteq U_\alpha$ for each $\alpha < \mu$. Also, let g_α (resp., h_α) be the characteristic functions of F_α (resp., U_α) for each $\alpha < \mu$. In case $\mu < \lambda$, we define $g_\beta(x) = h_\beta(x) = 0$ for each $x \in X$ and each β, with $\mu \leq \beta < \lambda$. Finally, define $g = \triangle_{\alpha<\lambda} g_\alpha$ and $h = \triangle_{\alpha<\lambda} h_\alpha$. Then, by Lemma 2.4, we can consider both g and h to be maps to $c_0(\lambda)$. Since each g_α (resp., h_α) is upper (resp., lower) semi-continuous, it follows from Lemma 2.3 that g is upper semi-continuous and h is lower semi-continuous. Since $g \leq h$, by (5), there exist a lower semi-continuous map $f_\ell : X \to c_0(\lambda)$ and an upper semi-continuous map $f_u : X \to c_0(\lambda)$ such that $g \leq f_\ell \leq f_u \leq h$. Put $V_\alpha = L(\pi_\alpha \circ f_\ell, 1/2)$ and $W_\alpha = L(\pi_\alpha \circ f_u, 1/2)$ for each $\alpha < \mu$. Then, each V_α is open, and $\{W_\alpha : \alpha < \lambda\}$ is locally finite in X by Lemma 2.3. Since $F_\alpha \subseteq V_\alpha \subseteq W_\alpha \subseteq U_\alpha$ for each $\alpha < \mu$, $\{V_\alpha : \alpha < \mu\}$ is a locally finite open expansion of $\mathscr{F}$. Moreover, since $\mathrm{cl}_X V_\alpha \subseteq U_\alpha$, the above proof for $\mu = 1$ provides that X is normal.

(6) $\Rightarrow$ (7): Let $\mathscr{F}$ be a discrete family of closed sets in X, with $|\mathscr{F}| \leq \lambda$, having a point-finite open expansion. Then, by (6), there exists a locally finite family $\mathscr{G} = \{G(F) : F \in \mathscr{F}\}$ of open sets in X such that $F \subseteq G(F)$ for each $F \in \mathscr{F}$. Since $\mathscr{F}$ is discrete, we may assume that $G(F) \cap F' = \varnothing$ whenever $F \neq F'$. Let $\mathscr{U} = \mathscr{G} \cup \{X \backslash \bigcup \mathscr{F}\}$. Then, $\mathscr{U}$ is a locally finite open cover of X. Since every locally finite open cover of a normal space is normal (see [**8**, p. 305]), $\mathscr{U}$ has an open star-refinement $\mathscr{V}$. Then, $\{St(F, \mathscr{V}) : F \in \mathscr{F}\}$ is a disjoint open expansion of $\mathscr{F}$.

(7) $\Rightarrow$ (1): Notice that X is normal by (7). Hence, this can be proved quite similarly to the proof of the Michael-Nagami theorem asserting that every metacompact collectionwise normal space is paracompact (cf. [**8**, Theorem 5.3.3]). $\square$

Remark 3.2. The following conditions (8) and (9) are also equivalent to the conditions listed in Theorem 3.1. For two mappings $\varphi, \psi : X \to \mathscr{C}(Y)$, we write $\varphi \subseteq \psi$ if $\varphi(x) \subseteq \psi(x)$ for each $x \in X$.

(8) For every metrizable space Y, with $w(Y) \leq \lambda$, and every lower semi-continuous mapping $\phi : X \to \mathscr{C}(Y)$, there exist a lower semi-continuous mapping $\varphi : X \to \mathscr{C}(Y)$ and an upper semi-continuous mapping $\psi : X \to \mathscr{C}(Y)$ such that $\varphi \subseteq \psi \subseteq \phi$.
(9) There exist a space Y and a disjoint family $\mathscr{G}$ of non-empty open sets in Y, with $|\mathscr{G}| = \lambda$, such that for every lower semi-continuous mapping $\phi : X \to \mathscr{C}(Y)$, there exists an upper semi-continuous mapping $\psi : X \to \mathscr{C}(Y)$ such that $\psi \subseteq \phi$.

The equivalence of (1) and (8) was proved by Nedev in [**23**, Theorem 3], while (8) $\Rightarrow$ (9) is obvious. To show that (9) $\Rightarrow$ (7), let $\mathscr{F}$ be a discrete family of closed sets in X, with $|\mathscr{F}| \le \lambda$, and $\mathscr{U} = \{U(F) : F \in \mathscr{F}\}$ be a point-finite open expansion of $\mathscr{F}$. We may assume that $\mathscr{U}$ covers X and $U(F) \cap F' = \varnothing$ whenever $F \neq F'$. On the other hand, there exists a disjoint family $\mathscr{G} = \{G(F) : F \in \mathscr{F}\}$ of non-empty open sets in Y. Fix a point $y_F \in G(F)$ for each $F \in \mathscr{F}$ and define $\phi : X \to \mathscr{C}(Y)$ by $\phi(x) = \{y_F : x \in U(F), F \in \mathscr{F}\}$ for $x \in X$. Then, ϕ is lower semi-continuous because $\mathscr{U}$ is an open cover of X (see Lemma 5.4 below). Hence, by (9), there exists an upper semi-continuous mapping $\psi : X \to \mathscr{C}(Y)$ such that $\psi \subseteq \phi$. Let $V(F) = \psi^{\#}[G(F)]$ for each $F \in \mathscr{F}$. Then $\{V(F) : F \in \mathscr{F}\}$ is a disjoint open expansion of $\mathscr{F}$.

Let λ-$\mathscr{PN}$ be the class of all spaces satisfying one of (and hence, all of) the conditions listed in Theorem 3.1. Define the class $\mathscr{PN}$ by $X \in \mathscr{PN}$ if and only if $X \in \lambda$-$\mathscr{PN}$ for every cardinal λ. Then, $\mathscr{PN}$ is included in the class $\mathscr{N}$ of all normal spaces and contains the class $\mathscr{CN}$ of all collectionwise normal spaces, i.e. $\mathscr{CN} \subseteq \mathscr{PN} \subseteq \mathscr{N}$. Michael [**18**] has shown that both inclusions are proper by giving the examples which we now sketch below:

The example showing that $\mathscr{PN} \neq \mathscr{CN}$ is the standard Bing's example (cf. [**8**, Example 5.1.23]). The product space $X = D^{2^{\mathfrak{c}}}$ of the discrete space $D = \{0, 1\}$ contains a discrete subspace M, with $|M| = \mathfrak{c}$. Bing's space Z is obtained from the space X by making all points of $X \backslash M$ isolated. It is known that $Z \in \mathscr{N} \backslash \mathscr{CN}$. Notice that every point-finite family of non-empty open sets in X is at most countable; this follows from the fact that the Šanin number of X is countable (cf. [**8**, 2.7.11, p. 116]). Hence, it follows that $Z \in \mathscr{PN}$. Next, consider the subspace $Y = M \cup D$ of Z, where $D = \{x \in X : \{\alpha < 2^{\mathfrak{c}} : x(\alpha) \neq 0\}$ is finite$\}$. Michael [**18**] has shown that the space Y is normal metacompact but not paracompact. Hence, $Y \in \mathscr{N} \backslash \mathscr{PN}$ because every metacompact space in $\mathscr{PN}$ must be paracompact.

Since the space $Y = M \cup D$ is closed in Bing's space Z, the example above also shows that the class $\mathscr{PN}$ is not closed under taking closed subspaces unlike $\mathscr{N}$ and $\mathscr{CN}$. From this fact, it is natural to ask whether a space X is in $\mathscr{CN}$ if every closed subspace of X is in $\mathscr{PN}$. Now, we show that the answer is negative if there exists a Q-set. To this end, let us recall that a subset A of the real line $\boldsymbol{R}$ is called a *Q-set* if A is uncountable and every subset of A is a G_δ-set in A with respect to the subspace topology on A inherited from the usual topology on $\boldsymbol{R}$. It is known that every uncountable subset $A \subseteq \boldsymbol{R}$, with $|A| < \mathfrak{c}$, is a Q-set under assuming Martin's axiom and the negation of the continuum hypothesis (see [**20**] for details).

Example 3.3. If there exists a Q-set in $\boldsymbol{R}$, then there exists a perfectly normal space X such that every subspace is in $\mathscr{PN}$ but $X \notin \mathscr{CN}$.

Proof. Let L be the Niemytzki plane (cf. [**8**, Example 1.2.4]). All we need here is the fact that L is the closed upper half-plane, the x-axis $L_1 = \boldsymbol{R} \times \{0\}$ is closed discrete and nowhere dense in L and the open subspace $L \backslash L_1$ has the usual Euclidean topology. Now, assuming the existence of a Q-set $A \subseteq \boldsymbol{R}$, we consider the subspace $X = (A \times \{0\}) \cup (L \backslash L_1)$ of L. It is known that X is perfectly normal but $X \notin \mathscr{CN}$ (cf. [**28**, Example F]). We show that every subspace S of X is in $\mathscr{PN}$. For $S \subseteq X$, let

$S_0 = S \cap \mathrm{cl}_L(S \backslash L_1)$ and $S_1 = S \backslash S_0$. Then, the subspace S is the topological sum of S_0 and S_1. Since X is hereditarily normal, S_0 is a separable normal space, and hence, $S_0 \in \mathscr{PN}$. On the other hand, $S_1 \in \mathscr{PN}$ because S_1 is discrete. Thus, $S \in \mathscr{PN}$. $\square$

Problem 3.4. Does there exist an example in ZFC of a space $X \notin \mathscr{CN}$ such that every closed subspace of X is in $\mathscr{PN}$?

4. Sandwich-like characterizations of paracompact-like properties.

A space X is called *λ-collectionwise normal* if every discrete family $\mathscr{F}$ of closed sets in X, with $|\mathscr{F}| \le \lambda$, has a discrete open expansion. In what follows, for a Banach space Y, we put $\mathscr{C}_c'(Y) = \mathscr{C}_c(Y) \cup \{Y\}$.

Our first result is an insertion-like theorem which characterizes λ-collectionwise normality.

Theorem 4.1. *Let λ be an infinite cardinal. For a space X the following conditions are equivalent*:

(1) *X is λ-collectionwise normal.*

(2) *For every Banach space Y, with $w(Y) \le \lambda$, every lower semi-continuous mapping $\varphi : X \to \mathscr{C}_c'(Y)$ has a continuous selection.*

(3) *Every lower semi-continuous mapping $\phi : X \to \mathscr{C}_c'(c_0(\lambda))$ has a continuous selection.*

(4) *For every closed subspace A of X and for every two maps $g, h : A \to c_0(\lambda)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \le h$, there exists a continuous map $f : X \to c_0(\lambda)$ such that $g \le f|_A \le h$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from [**19**, Theorem 3.2$'$] (see, also, [**2**]), while (2) $\Rightarrow$ (3) is obvious. To show (3) $\Rightarrow$ (4), let $A \subseteq X$ and $g, h : A \to c_0(\lambda)$ be as in (4). By Lemma 2.6, $[g, h] : A \to \mathscr{C}_c(c_0(\lambda))$ is lower semi-continuous. Then, by a result of [**19**], the mapping $\phi : X \to \mathscr{C}_c'(c_0(\lambda))$, defined by $\phi(x) = [g, h](x)$ if $x \in A$ and $\phi(x) = c_0(\lambda)$ otherwise, is lower semi-continuous too. Hence, by (3), ϕ has a continuous selection f. This f is as in (4).

(4) $\Rightarrow$ (1): Let $\mathscr{F} = \{F_\alpha : \alpha < \lambda\}$ be a discrete family of closed sets in X, and let $A = \bigcup\{F_\alpha : \alpha < \lambda\}$. Note that there exists a disjoint family $\{G_\alpha : \alpha < \lambda\}$ of non-empty open sets in $c_0(\lambda)$. Pick a point $y_\alpha \in G_\alpha$ for each $\alpha < \lambda$ and define maps $g, h : A \to c_0(\lambda)$ by $g(x) = h(x) = y_\alpha$ if $x \in F_\alpha$. Since both g and h are continuous, it follows from (4) that there exists a continuous map $f : X \to c_0(\lambda)$ such that $g \le f \,|\, A \le h$. Then, $\{f^{-1}[G_\alpha] : \alpha < \lambda\}$ is a disjoint open expansion of $\mathscr{F}$ in X. Hence, X is λ-collectionwise normal. $\square$

Our next result is a characterization of countably paracompact and λ-collectionwise normal spaces. To prepare for this, we need two lemmas, the first of which may have an independent interest.

Let (Y, d) be a metric space, and let $2^Y = \{S \subset Y : S \ne \varnothing\}$. In what follows, as in the case of Banach spaces, for $S \in 2^Y$ and $\varepsilon > 0$ we use $B(S, \varepsilon)$ to denote the ε-neighbourhood of S in (Y, d) (i.e., $B(S, \varepsilon) = \{y \in Y : d(y, S) < \varepsilon\}$). Let us recall that a mapping $\varphi : X \to 2^Y$ is *d-upper semi-continuous* if $\varphi^{\#}[B(\varphi(x), \varepsilon)]$ is a neighbourhood of x, for every $x \in X$ and $\varepsilon > 0$. A mapping $\varphi : X \to 2^Y$ is called *d-proximal continuous* (see [**11**]) if it is both lower semi-continuous and d-upper semi-continuous. It should be mentioned that, for a metric space (Y, d), every continuous $\varphi : X \to 2^Y$ is d-proximal continuous but the converse is not true, see [**11**, Proposition 2.5]. On the other hand, the d-proximal continuity depends on the metric d of the range Y. To avoid this, for a metrizable Y, let us agree that a mapping $\varphi : X \to 2^Y$ is *proximal continuous* if it is d-proximal continuous with respect to some metric d on Y compatible with the topology of Y.

Lemma 4.2. *Let X be a λ-collectionwise normal space, Y be a Banach space with $w(Y) \leq \lambda$, $\varphi : X \to \mathscr{F}_c(Y)$ be a proximal continuous mapping, and let $\psi : X \to \mathscr{F}_c(Y)$ be a lower semi-continuous set-valued selection for φ such that $\psi(x)$ is compact whenever $\psi(x) \neq \varphi(x)$. Then, ψ has a continuous selection.*

Proof. Let us recall that a mapping $\phi : X \to \mathscr{F}_c(Y)$ has the *Selection Factorization Property* [**23**] if for every closed subset F of X and every locally finite collection $\mathscr{U}$ of open subsets of Y such that $\phi^{-1}[\mathscr{U}] = \{\phi^{-1}[U] : U \in \mathscr{U}\}$ covers F, there exists a locally finite open (in F) covering of F which refines $\phi^{-1}[\mathscr{U}]$. According to [**23**, Proposition 4.3], it now suffices to show that ψ has the Selection Factorization Property. Towards this end, take a closed set $F \subseteq X$ and a locally finite family $\mathscr{U}$ of non-empty open sets in Y such that $F \subseteq \bigcup \psi^{-1}[\mathscr{U}]$. Since φ is proximal continuous, by [**11**, Theorem 3.1], there exists a locally finite open cover $\{V_U : U \in \mathscr{U}\}$ of F such that $V_U \subseteq \varphi^{-1}[U]$ for every $U \in \mathscr{U}$. Set $\mathscr{W}_1 = \{W_U : U \in \mathscr{U}\}$, where $W_U = V_U \cap \psi^{-1}[U]$, $U \in \mathscr{U}$, and let $W_1 = \bigcup \mathscr{W}_1$. Then, $\psi(x)$ is compact for every $x \in F \backslash W_1$. Indeed, take a point $x \in F$ and $U \in \mathscr{U}$ such that $x \in V_U$ and $\psi(x) = \varphi(x)$. Since $\psi(x) \cap U = \varphi(x) \cap U$, we get that $x \in W_U \subseteq W_1$. Hence, $x \in F \backslash W_1$ implies $\psi(x) \neq \varphi(x)$ and, by hypothesis, $\psi(x)$ is compact. As a result, we now have that $\psi^{-1}[\mathscr{U}]$ is point-finite at every point of $F \backslash W_1$. On the other hand, $|\mathscr{U}| \leq \lambda$ because $\mathscr{U}$ is locally finite in Y and $w(Y) \leq \lambda$. Therefore, there exists a locally finite open (in F) cover $\mathscr{W}_2$ of $F \backslash W_1$ which refines $\psi^{-1}[\mathscr{U}]$ because X is λ-collectionwise normal. Then, $\mathscr{W} = \mathscr{W}_1 \cup \mathscr{W}_2$ is as required. $\square$

In our next lemma, a set-valued mapping $\varphi : X \to 2^Y$ has an *open graph* if the set $\{(x, y) \in X \times Y : y \in \varphi(x)\}$ is open in $X \times Y$.

Lemma 4.3. *Let $\varphi : X \to 2^Y$ be a mapping with an open graph, and let $\phi : X \to 2^Y$ be a lower semi-continuous mapping such that $\varphi(x) \cap \phi(x) \neq \varnothing$ for every $x \in X$. Define another mapping $\psi : X \to 2^Y$ by $\psi(x) = \varphi(x) \cap \phi(x)$, $x \in X$. Then, ψ is lower semi-continuous.*

Proof. Let U be a non-empty open set in Y, and let $x_0 \in \psi^{-1}[U]$. Take a point $y_0 \in \psi(x_0) \cap U$. Since φ has an open graph, there exists a neighbourhood $U_0 \subseteq U$ of y_0 and a neighbourhood V_0 of x_0 such that $U_0 \subseteq \varphi(x)$ for every $x \in V_0$. Then, $V = V_0 \cap \phi^{-1}[U_0]$ is a neighbourhood of x_0 such that $\psi(x) \cap U \neq \varnothing$ whenever $x \in V$. $\square$

To state our characterization of countably paracompact and λ-collectionwise normal spaces, we need also some terminology about Banach spaces.

Let Y be a space and let $e : Y \to \boldsymbol{R}^\lambda$ be a map. Then we define $e_\alpha = \pi_\alpha \circ e$, where $\pi_\alpha : \boldsymbol{R}^\lambda \to \boldsymbol{R}$ is the projection to the α-th factor of $\boldsymbol{R}^\lambda$, for each $\alpha < \lambda$. Thus, we have $e = \triangle\{e_\alpha : \alpha < \lambda\}$.

Suppose that Y is a Banach space. Let us recall that a sequence $\{\boldsymbol{e}_n \in Y : n < \omega\}$ is a *Schauder basis* for Y if any point $y \in Y$ has a unique representation $y = \sum_{n<\omega} y_n \boldsymbol{e}_n$ for some scalars (i.e., coordinates) $y_n \in \boldsymbol{R}$, $n < \omega$. Here, $y = \sum_{n<\omega} y_n \boldsymbol{e}_n$ means that $\lim_{n\to\infty} \|y - \sum_{k\le n} y_k \boldsymbol{e}_k\| = 0$, where $\|.\|$ is the norm of Y.

Note that any Schauder basis $\{\boldsymbol{e}_n \in Y : n < \omega\}$ for a Banach space Y defines a natural linear continuous injection $e : Y \to \boldsymbol{R}^\omega$, see [**3**, Exercise III.14.10] and [**26**, Theorem 3.1]. Namely, one may define $e : Y \to \boldsymbol{R}^\omega$ by $e_n(y) = y_n$, $n < \omega$, where $y = \sum_{n<\omega} y_n \boldsymbol{e}_n \in Y$. It should be mentioned that, with respect to this map $e = \triangle\{e_n : n < \omega\}$, we have $e_n(\boldsymbol{e}_n) = 1$ and $e_m(\boldsymbol{e}_n) = 0$ for $m \ne n$. Motivated by this, we shall say that a map $e : Y \to \boldsymbol{R}^\lambda$ is a *generalized Schauder basis* for a Banach space Y if it is a continuous linear injection such that, whenever $y \in Y$ and $\alpha < \lambda$, there is a point $y_\alpha \in Y$, with $e_\beta(y_\alpha) = e_\beta(y)$ if $\beta = \alpha$ and $e_\beta(y_\alpha) = 0$ otherwise. Clearly, the natural linear injection $e : Y \to \boldsymbol{R}^\omega$ determined by a Schauder basis for Y is a generalized Schauder basis but the converse does not hold. For instance, consider the Banach space ℓ^∞ of bounded sequences. Then the natural injection $e : \ell^\infty \to \boldsymbol{R}^\omega$ is a generalized Schauder basis but the space ℓ^∞ does not have a Schauder one since it is not separable.

The generalized Schauder basises will be used in the following special situation.

Definition 4.4. We shall say that a generalized Schauder basis $e : Y \to \boldsymbol{R}^\lambda$ for a Banach space Y is a $c_0(\lambda)$*-basis* for Y if $e[Y] \subset c_0(\lambda)$ and it is continuous as a map from Y to $c_0(\lambda)$. Also, we shall say that Y is a *generalized* $c_0(\lambda)$*-space* if it is a Banach space, with $w(Y) \le \lambda$, which has a $c_0(\lambda)$-basis.

Note that $c_0(\lambda)$ is a generalized $c_0(\lambda)$-space. Also, every Euclidean space is a generalized $c_0(\lambda)$-space for every infinite cardinal λ. Finally, the Banach spaces $\ell_p(\lambda)$, for $p \ge 1$, are another important example of generalized $c_0(\lambda)$-spaces.

In what follows, for a convex set K of a Banach space Y, we consider a *weak convex interior* $\mathrm{wci}(K)$ of K defined by

$$\mathrm{wci}(K) = \{x \in K : x = \delta x_1 + (1-\delta)x_2 \text{ for some } x_1, x_2 \in K\backslash\{x\} \text{ and } 0 < \delta < 1\}.$$

Also, for $s, t \in \boldsymbol{R}^\lambda$, we shall write $s < t$ if $s \le t$ and $s(\alpha) < t(\alpha)$ for some $\alpha < \lambda$. Finally, for maps $g, h : X \to \boldsymbol{R}^\lambda$, we write $g < h$ if $g(x) < h(x)$ for every $x \in X$.

Theorem 4.5. *Let λ be an infinite cardinal. For a space X the following conditions are equivalent*:

(1) *X is countably paracompact and λ-collectionwise normal.*

(2) *Whenever Y is a generalized $c_0(\lambda)$-space and $\phi : X \to \mathscr{C}'_c(Y)$ is a lower semi-continuous mapping such that $|\phi(x)| > 1$ for every $x \in X$, there exists a continuous map $f : X \to Y$ such that $f(x) \in \mathrm{wci}(\phi(x))$ for all $x \in X$.*

(3) *For every lower semi-continuous mapping $\phi : X \to \mathscr{C}'_c(c_0(\lambda))$, with $|\phi(x)| > 1$ for every $x \in X$, there exists a continuous map $f : X \to c_0(\lambda)$ such that $f(x) \in \mathrm{wci}(\phi(x))$ for all $x \in X$.*

(4) *For every closed subspace A of X and for every two maps $g, h : A \to c_0(\lambda)$ such that g is upper semi-continuous, h is lower semi-continuous and $g < h$, there exists a continuous map $f : X \to c_0(\lambda)$ such that $g < f|_A < h$.*

PROOF. (1) $\Rightarrow$ (2): Let Y and $\phi : X \to \mathscr{C}_c'(Y)$ be as in (2). Let $e : Y \to c_0(\lambda)$ be a $c_0(\lambda)$-basis for Y. For each $n < \omega$, let

$$U_n = \{x \in X : \text{diameter}(e_\alpha[\phi(x)]) > 2^{-n} \text{ for some } \alpha < \lambda\}.$$

Note that each U_n is open in X because e is continuous and ϕ is lower semi-continuous. Hence, the family $\{U_n : n < \omega\}$ is an open cover of X because e is injective and $|\phi(x)| > 1$ for every $x \in X$. Then, the countable paracompactness of X implies the existence of a locally finite open cover $\{G_n : n < \omega\}$ of X with $\mathrm{cl}_X G_n \subseteq U_n$, $n < \omega$.

Now, for every $\alpha < \lambda$ and $n < \omega$, define an open subset U_α^n of X by

$$U_\alpha^n = \{x \in X : \text{diameter}(e_\alpha[\phi(x)]) > 2^{-n}\}.$$

Since Y is not a singleton, the same is true for $e[Y]$. Then, without loss of generality, we may suppose that $e_0[Y]$ is not a singleton. We claim that $\{U_\alpha^n : 0 < \alpha < \lambda\}$ is point-finite in $X_0^n = X \setminus U_0^n$. To show this, first let us observe that $e[\phi(x)]$ is compact for each $x \in X_0^n$. Namely, if $\phi(x) = Y$ for some $x \in X$, then $e_0[\phi(x)] = \boldsymbol{R}$ and, in particular, $\text{diameter}(e_0[\phi(x)]) > 2^{-n}$, so $x \in U_0^n$. Therefore $e[\phi(x)]$ is compact for each $x \in X_0^n$ because ϕ is $\mathscr{C}'(Y)$-valued and e is continuous. Thus, we can define a mapping $\phi_0^n : X_0^n \to \mathscr{C}(c_0(\lambda))$ which carries x to $e[\phi(x)]$. Hence, by Lemma 2.7, two maps $\inf \phi_0^n, \sup \phi_0^n : X_0^n \to c_0(\lambda)$ can also be defined. Then, the required property of $\{U_\alpha^n : 0 < \alpha < \lambda\}$ follows from Lemma 2.4 because

$$\{U_\alpha^n \cap X_0^n : 0 < \alpha < \lambda\} = \{L(\pi_\alpha \circ (\sup \phi_0^n - \inf \phi_0^n), 2^{-n}) : 0 < \alpha < \lambda\}.$$

Next, note that $\{U_\alpha^n : 0 < \alpha < \lambda\}$ covers $X_0^n \cap \mathrm{cl}_X G_n$. Since X is λ-collectionwise normal, there exists a locally finite open (in X) cover $\{V_\alpha^n : 0 < \alpha < \lambda\}$ of $X_0^n \cap \mathrm{cl}_X G_n$ such that $\mathrm{cl}_X V_\alpha^n \subseteq U_\alpha^n$, whenever $0 < \alpha < \lambda$. Finally, note that

$$\mathrm{cl}_X G_n \setminus \bigcup \{V_\alpha^n : 0 < \alpha < \lambda\} \subseteq U_0^n.$$

Hence, there exists an open set $V_0^n \subseteq X$ such that

$$\mathrm{cl}_X G_n \setminus \bigcup \{V_\alpha^n : 0 < \alpha < \lambda\} \subseteq V_0^n \subseteq \mathrm{cl}_X V_0^n \subseteq U_0^n.$$

Then, let $W_\alpha^n = V_\alpha^n \cap G_n$ for every $\alpha < \lambda$. Thus, we get a locally finite open cover $\{W_\alpha^n : \alpha < \lambda \text{ and } n < \omega\}$ of X because so is $\{G_n : n < \omega\}$. Therefore, the same is true for $\{W_\alpha : \alpha < \lambda\}$, where $W_\alpha = \bigcup \{W_\alpha^n : n < \omega\}$, $\alpha < \lambda$.

Now, for every $\alpha < \lambda$, set $X_\alpha = \mathrm{cl}_X W_\alpha$. Next, take a fixed $\alpha < \lambda$ such that $X_\alpha \neq \varnothing$. Note that the mapping $\varphi_\alpha : X_\alpha \to \mathscr{C}_c'(\boldsymbol{R})$ defined by $\varphi_\alpha(x) = e_\alpha[\phi(x)]$ for $x \in X_\alpha$ is lower semi-continuous. Let $\overline{\boldsymbol{R}}$ be the extended real line $[-\infty, +\infty]$. Then, we may consider $g_\alpha = \inf \varphi_\alpha : X_\alpha \to \overline{\boldsymbol{R}}$ and $h_\alpha = \sup \varphi_\alpha : X_\alpha \to \overline{\boldsymbol{R}}$. As a result, g_α is upper semi-continuous, h_α is lower semi-continuous (see, e.g., Lemma 2.8), and $g_\alpha < h_\alpha$ because $X_\alpha \subseteq \bigcup \{U_\alpha^n : n < \omega\}$. Since X_α is countably paracompact and normal, by a result of [**4**], [**5**], [**14**], there exists a continuous $r_{(g,\alpha)} : X_\alpha \to \overline{\boldsymbol{R}}$ such that $g_\alpha < r_{(g,\alpha)} < h_\alpha$. Note that, in fact, $r_{(g,\alpha)}[X_\alpha] \subseteq \boldsymbol{R}$. Now, according to the same result, we may find also continuous functions $r_{(1,\alpha)}, r_{(2,\alpha)}, r_{(h,\alpha)} : X_\alpha \to \boldsymbol{R}$ such that

$$g_\alpha < r_{(g,\alpha)} < r_{(1,\alpha)} < r_{(2,\alpha)} < r_{(h,\alpha)} < h_\alpha.$$

Define a set-valued mapping $\varphi_{(1,\alpha)} : X_\alpha \to 2^Y$ by

$$\varphi_{(1,\alpha)}(x) = e_\alpha^{-1}[(r_{(g,\alpha)}(x), r_{(1,\alpha)}(x))], \quad x \in X_\alpha.$$

Note that $\varphi_{(1,\alpha)}$ has an open graph, and $\varphi_{(1,\alpha)}(x) \cap \phi(x) \neq \emptyset$ for every $x \in X_\alpha$. Indeed, take a point $x \in X_\alpha$ and $y \in \varphi_{(1,\alpha)}(x)$. Since $e_\alpha(y) \in (r_{(g,\alpha)}(x), r_{(1,\alpha)}(x))$, there exists an $\varepsilon > 0$ with $[e_\alpha(y) - \varepsilon, e_\alpha(y) + \varepsilon] \subset (r_{(g,\alpha)}(x), r_{(1,\alpha)}(x))$. Then, $U = e_\alpha^{-1}[(e_\alpha(y) - \varepsilon, e_\alpha(y) + \varepsilon)]$ is a neighbourhood of y in Y because e_α is continuous. Since the functions $r_{(g,\alpha)}$ and $r_{(1,\alpha)}$ are continuous, there also exists a neighbourhood V of x in X_α such that $z \in V$ implies

$$r_{(g,\alpha)}(z) < e_\alpha(y) - \varepsilon < e_\alpha(y) + \varepsilon < r_{(1,\alpha)}(z).$$

Hence, $z \in V$ implies $U \subset \varphi_{(1,\alpha)}(z)$. That is, $\varphi_{(1,\alpha)}$ has an open graph.

Now, according to Lemma 4.3, the mapping $\psi_{(1,\alpha)}(x) = \varphi_{(1,\alpha)}(x) \cap \phi(x)$, $x \in X_\alpha$, is lower semi-continuous. In what follows, with every set-valued mapping $\theta : X_\alpha \to 2^Y$ we associate another one $\bar{\theta} : X_\alpha \to 2^Y$ by $\bar{\theta}(x) = \mathrm{cl}_Y\, \theta(x)$, $x \in X_\alpha$. Thus, by a result of Michael [**19**, Proposition 2.3], $\bar{\psi}_{(1,\alpha)} : X_\alpha \to \mathscr{F}_c(Y)$ is lower semi-continuous. Also, $\bar{\psi}_{(1,\alpha)}$ is a set-valued selection of $\bar{\varphi}_{(1,\alpha)}$, and $\bar{\psi}_{(1,\alpha)}(x) \neq \bar{\varphi}_{(1,\alpha)}(x)$ for some $x \in X_\alpha$ implies the compactness of $\bar{\psi}_{(1,\alpha)}(x)$.

In order to apply Lemma 4.2, let us show that $\bar{\varphi}_{(1,\alpha)}$ is $\|.\|$-proximal continuous, where $\|.\|$ is the norm of Y. In fact, it only suffices to show that $\bar{\varphi}_{(1,\alpha)}$ is $\|.\|$-upper semi-continuous because $\varphi_{(1,\alpha)}$ is lower semi-continuous as a mapping with an open graph. Towards this end, note that $e_\alpha[Y] = \boldsymbol{R}$ because $e_\alpha[Y] \neq \{0\}$. Then, by the open mapping theorem, e_α is an open map, which implies that

$$\begin{aligned}\bar{\varphi}_{(1,\alpha)}(x) &= \mathrm{cl}_Y\, e_\alpha^{-1}[(r_{(g,\alpha)}(x), r_{(1,\alpha)}(x))] \\ &= e_\alpha^{-1}[\mathrm{cl}_{\boldsymbol{R}}(r_{(g,\alpha)}(x), r_{(1,\alpha)}(x))] = e_\alpha^{-1}[[r_{(g,\alpha)}(x), r_{(1,\alpha)}(x)]]\end{aligned}$$

for each $x \in X_\alpha$. Now, take a point $x_0 \in X_\alpha$ and $\varepsilon > 0$. Also, let $Y_\alpha = \{y_\alpha : y \in Y\}$, where y_α's are as in the definition of a generalized Schauder basis with respect to our map $e : Y \to \boldsymbol{R}^\lambda$. Then, Y_α is isomorphic to the real line $\boldsymbol{R}$, so there exists a constant $c > 0$ such that

$$\frac{1}{c} \cdot |e_\alpha(y_\alpha)| \leq \|y_\alpha\| \leq c \cdot |e_\alpha(y_\alpha)|, \quad \text{for every } y \in Y. \tag{4.1}$$

Consider the neighbourhood

$$U = \left\{ x \in X_\alpha : |r_{(g,\alpha)}(x) - r_{(g,\alpha)}(x_0)| < \frac{\varepsilon}{c} \text{ and } |r_{(1,\alpha)}(x) - r_{(1,\alpha)}(x_0)| < \frac{\varepsilon}{c} \right\}$$

of x_0 in X_α. Finally, take a point $x_1 \in U$ and $y \in \bar{\varphi}_{(1,\alpha)}(x_1)$, and let us check that $y \in B(\bar{\varphi}_{(1,\alpha)}(x_0), \varepsilon)$. Turning to this last purpose, note that there are points $y^i_{(g,\alpha)}, y^i_{(1,\alpha)} \in Y_\alpha$ such that $e_\alpha(y^i_{(g,\alpha)}) = r_{(g,\alpha)}(x_i)$ and $e_\alpha(y^i_{(1,\alpha)}) = r_{(1,\alpha)}(x_i)$, $i = 0, 1$. Since $e_\alpha(y) \in [r_{(g,\alpha)}(x_1), r_{(1,\alpha)}(x_1)]$, there now exists $\delta \in [0, 1]$ such that

$$y_\alpha = \delta \cdot y^1_{(g,\alpha)} + (1 - \delta) \cdot y^1_{(1,\alpha)}.$$

Then, for $i = 0, 1$, let us consider the points

$$y_g^i = (y - y_\alpha) + y_{(g,\alpha)}^i \quad \text{and} \quad y_1^i = (y - y_\alpha) + y_{(1,\alpha)}^i.$$

In this way, we get a nice representation of y, namely $y = \delta \cdot y_g^1 + (1 - \delta) \cdot y_1^1$, while $z = \delta \cdot y_g^0 + (1 - \delta) \cdot y_1^0 \in \bar{\varphi}_{(1,\alpha)}(x_0)$ because $e_\alpha(y - y_\alpha) = 0$. Let us calculate the distance between these two points of Y:

$$\begin{aligned}
\|y - z\| &= \|\delta \cdot y_g^1 + (1 - \delta) \cdot y_1^1 - \delta \cdot y_g^0 - (1 - \delta) \cdot y_1^0\| \\
&= \|\delta \cdot (y_g^1 - y_g^0) + (1 - \delta) \cdot (y_1^1 - y_1^0)\| \\
&= \|\delta \cdot (y_{(g,\alpha)}^1 - y_{(g,\alpha)}^0) + (1 - \delta) \cdot (y_{(1,\alpha)}^1 - y_{(1,\alpha)}^0)\| \\
&\le \delta \cdot \|y_{(g,\alpha)}^1 - y_{(g,\alpha)}^0\| + (1 - \delta) \cdot \|y_{(1,\alpha)}^1 - y_{(1,\alpha)}^0\|.
\end{aligned}$$

Then, according to (4.1), we finally get that

$$\begin{aligned}
\|y - z\| &\le \delta \cdot \|y_{(g,\alpha)}^1 - y_{(g,\alpha)}^0\| + (1 - \delta) \cdot \|y_{(1,\alpha)}^1 - y_{(1,\alpha)}^0\| \\
&\le \delta \cdot c \cdot |e_\alpha(y_{(g,\alpha)}^1) - e_\alpha(y_{(g,\alpha)}^0)| + (1 - \delta) \cdot c \cdot |e_\alpha(y_{(1,\alpha)}^1) - e_\alpha(y_{(1,\alpha)}^0)| \\
&= \delta \cdot c \cdot |r_{(g,\alpha)}(x_1) - r_{(g,\alpha)}(x_0)| + (1 - \delta) \cdot c \cdot |r_{(1,\alpha)}(x_1) - r_{(1,\alpha)}(x_0)| \\
&< \delta \cdot c \cdot \frac{\varepsilon}{c} + (1 - \delta) \cdot c \cdot \frac{\varepsilon}{c} = \delta \cdot \varepsilon + (1 - \delta) \cdot \varepsilon = \varepsilon.
\end{aligned}$$

That is, $y \in B(\bar{\varphi}_{(1,\alpha)}(x_0), \varepsilon)$, so $\bar{\varphi}_{(1,\alpha)}$ is $\|.\|$-proximal continuous.

Going back to our construction, by Lemma 4.2, $\bar{\psi}_{(1,\alpha)}$ has a continuous selection $f_{(1,\alpha)} : X_\alpha \to Y$ because $\bar{\varphi}_{(1,\alpha)}$ is proximal continuous, and $\bar{\psi}_{(1,\alpha)}(x) \neq \bar{\varphi}_{(1,\alpha)}(x)$ for some $x \in X_\alpha$ implies the compactness of $\bar{\psi}_{(1,\alpha)}(x)$.

Next, we repeat the same trick with the second pair of maps. Namely, we may define another set-valued mapping $\varphi_{(2,\alpha)} : X_\alpha \to 2^Y$ by

$$\varphi_{(2,\alpha)}(x) = e_\alpha^{-1}[(r_{(2,\alpha)}(x), r_{(h,\alpha)}(x))], \quad x \in X_\alpha.$$

Just like before, $\varphi_{(2,\alpha)}$ has an open graph, and $\varphi_{(2,\alpha)}(x) \cap \phi(x) \neq \varnothing$ for every $x \in X_\alpha$. Hence, by Lemma 4.3, the mapping $\psi_{(2,\alpha)}(x) = \varphi_{(2,\alpha)}(x) \cap \phi(x)$, $x \in X_\alpha$, is lower semi-continuous, and therefore so is $\bar{\psi}_{(2,\alpha)}$. Also, $\bar{\psi}_{(2,\alpha)}(x)$ is compact for every point $x \in X_\alpha$ with $\bar{\psi}_{(2,\alpha)}(x) \neq \bar{\varphi}_{(2,\alpha)}(x)$, while $\bar{\varphi}_{(2,\alpha)}$ is proximal continuous. Then, by Lemma 4.2, $\bar{\psi}_{(2,\alpha)}$ has a continuous selection $f_{(2,\alpha)} : X_\alpha \to Y$. Thus, we get two pointwise disjoint selections $f_{(i,\alpha)}$, $i = 1, 2$, for $\phi|_{X_\alpha}$ because $\bar{\varphi}_{(1,\alpha)}(x) \cap \bar{\varphi}_{(2,\alpha)}(x) = \varnothing$ for every $x \in X_\alpha$. Hence, $f_\alpha = (f_{(1,\alpha)} + f_{(2,\alpha)})/2$ is a continuous map such that $f_\alpha(x) \in \mathrm{wci}(\phi(x))$ for every $x \in X_\alpha$.

We now complete the proof of this implication as follows. Take a partition of unity $\{p_\alpha : \alpha < \lambda\}$ index-subordinated to the cover $\{W_\alpha : \alpha < \lambda\}$ of X. Then, the map $f : X \to Y$, defined by

$$f(x) = \sum\{p_\alpha(x) \cdot f_\alpha(x) : \alpha < \lambda\}, \quad x \in X,$$

is the required selection for ϕ. To see this, we need only to check that $f(x) \in \mathrm{wci}(\phi(x))$ for every $x \in X$. Take a point $x \in X$, and let $\mathscr{A}(x) = \{\alpha < \lambda : p_\alpha(x) > 0\}$. Note that $\mathscr{A}(x)$ is a finite set. In case $\mathscr{A}(x) = \{\alpha\}$, we have $f(x) = f_\alpha(x) \in \mathrm{wci}(\phi(x))$. Otherwise, pick a fixed $\beta \in \mathscr{A}(x)$, and then set

$$\delta = p_\beta(x) \quad \text{and} \quad y_\beta(x) = \sum\{p_\alpha(x) \cdot f_\alpha(x) : \alpha \in \mathscr{A}(x)\backslash\{\beta\}\}/(1-\delta).$$

Note that $y_\beta(x) \in \phi(x)$ because $\sum\{p_\alpha(x)/(1-\delta) : \alpha \in \mathscr{A}(x)\backslash\{\beta\}\} = 1$. Hence, $f(x) = \delta \cdot f_\beta(x) + (1-\delta) \cdot y_\beta(x) \in \mathrm{wci}(\phi(x))$. Indeed, $f_\beta(x) = y_\beta(x)$ implies $f(x) = f_\beta(x) \in \mathrm{wci}(\phi(x))$. Otherwise, by definition, $f(x) \in \mathrm{wci}(\phi(x))$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4): Let $g, h : A \to c_0(\lambda)$ be as in (4). Then, by Lemmas 2.5 and 2.6, $[g,h] : A \to \mathscr{C}_c(c_0(\lambda))$ is lower semi-continuous. Hence, the mapping $\phi : X \to \mathscr{C}_c'(c_0(\lambda))$, defined by $\phi(x) = [g,h](x)$ if $x \in A$ and $\phi(x) = c_0(\lambda)$ otherwise, is lower semi-continuous too. Also, $|\phi(x)| > 1$ for every $x \in X$ because $g < h$. Therefore, by (3), there exists a continuous $f : X \to c_0(\lambda)$ with $f(x) \in \mathrm{wci}(\phi(x))$ for every $x \in X$. Clearly, this f is as required in (4).

(4) $\Rightarrow$ (1): First of all, let us observe that X is a countably paracompact normal space. Indeed, take maps $g_0, h_0 : X \to \boldsymbol{R}$ such that g_0 is upper semi-continuous, h_0 is lower semi-continuous and $g_0(x) < h_0(x)$ for every $x \in X$. Next, for every $0 < \alpha < \lambda$, define $g_\alpha(x) = h_\alpha(x) = 0$, $x \in X$. Thus, we get an upper semi-continuous map $g = \triangle\{g_\alpha : \alpha < \lambda\} : X \to c_0(\lambda)$ and a lower semi-continuous one $h = \triangle\{h_\alpha : \alpha < \lambda\} : X \to c_0(\lambda)$ such that $g < h$. Hence, by (4), there exists a continuous map $f : X \to c_0(\lambda)$ with $g < f < h$. In particular, $f_0 = \pi_0 \circ f : X \to \boldsymbol{R}$ is continuous and $g_0(x) < f_0(x) < h_0(x)$ for every $x \in X$. Then, by a result of [**4**], [**5**], [**14**], X is countably paracompact and normal. Thus, it only remains to show that X is λ-collectionwise normal. Towards this end, take a closed set $A \subseteq X$ and two maps $g, h : A \to c_0(\lambda)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \le h$. By Theorem 4.1 it suffices to show the existence of a continuous map $f : X \to c_0(\lambda)$ such that $g \le f|_A \le h$. Let $\xi : \lambda \to \lambda\backslash\{0\}$ be the bijection defined by $\xi(\alpha) = \alpha + 1$ for $\alpha < \omega$ and $\xi(\alpha) = \alpha$ for $\alpha \ge \omega$. For each $x \in X$ and each $\alpha < \lambda$, define $g'(x)(\alpha) = 0$ for $\alpha = 0$ and $g'(x)(\alpha) = g(x)(\xi^{-1}(\alpha))$ for $\alpha \ne 0$, and also define $h'(x)(\alpha) = 1$ for $\alpha = 0$ and $h'(x)(\alpha) = h(x)(\xi^{-1}(\alpha))$ for $\alpha \ne 0$. Then, we have two maps $g', h' : A \to c_0(\lambda)$ such that g' is upper semi-continuous, h' is lower semi-continuous and $g' < h'$. Hence, by (4), there exists a continuous map $f' : X \to c_0(\lambda)$ such that $g < f'|_A < h$. Finally, define a map $f : X \to c_0(\lambda)$ by $f(x)(\alpha) = f'(x)(\xi(\alpha))$ for $x \in X$ and $\alpha < \lambda$. Then f is continuous and $g \le f|_A \le h$. $\square$

From one hand, Theorem 4.5 might be read as a possible extension of the Dowker-Katětov characterization of countably paracompact normal spaces [**5**], [**14**], see also [**4**]. From another hand, Theorem 4.5 should be compared with Michael's characterization [**19**, Theorem 3.1$'''$] of perfectly normal spaces by selections avoiding *supporting* points of convex sets. More precisely, in the Michael's terminology [**19**], if Y is a Banach space and $K \in \mathscr{F}_c(Y)$, then a *supporting* set of K is a closed convex subset S of K, $S \ne K$, such that if an interior point of a segment in K is in S, then the whole segment is in S. The set of all elements of K which are not in any supporting set of K is denoted by $\mathrm{I}(K)$ (suggesting "Inside of K"). Finally, as in [**19**], one may consider

$$\mathscr{D}(Y) = \{B \in 2^Y : B \text{ is convex and } \mathrm{I}(\mathrm{cl}_Y B) \subseteq B\}.$$

It is well known (see [**19**]) that $\mathscr{F}_c(Y) \subset \mathscr{D}(Y)$; that every convex $B \in 2^Y$ with a non-empty interior belongs to $\mathscr{D}(Y)$; and that every finite-dimensional convex $B \in 2^Y$ belongs to $\mathscr{D}(Y)$.

As for our weak convex interior, it is clear that $\mathrm{I}(K) \subseteq \mathrm{wci}(K)$ for every $K \in \mathscr{F}_c(Y)$ but the converse is not true. In fact, the Michael's [**19**, Theorem 3.1‴] states that a space X is perfectly normal if and only if for every separable Banach space Y, every lower semi-continuous $\phi : X \to \mathscr{D}(Y)$ has a continuous selection.

Our next result presents another possible characterization of perfectly normal spaces in terms of selections.

THEOREM 4.6. *Let λ be an infinite cardinal. For a space X the following conditions are equivalent*:

(1) *X is perfectly normal and λ-collectionwise normal.*

(2) *Whenever Y is a generalized $c_0(\lambda)$-space, every lower semi-continuous mapping $\phi : X \to \mathscr{C}_c'(Y)$ has a continuous selection f such that $f(x) \in \mathrm{wci}(\phi(x))$ for every $x \in X$ with $|\phi(x)| > 1$.*

(3) *Every lower semi-continuous mapping $\phi : X \to \mathscr{C}_c'(c_0(\lambda))$ has a continuous selection f such that $f(x) \in \mathrm{wci}(\phi(x))$ for every $x \in X$ with $|\phi(x)| > 1$.*

(4) *For every closed subspace A of X and for every two maps $g, h : A \to c_0(\lambda)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \le h$, there exists a continuous map $f : X \to c_0(\lambda)$ such that $g \le f|_A \le h$ and $g(x) < f(x) < h(x)$ whenever $x \in A$ with $g(x) < h(x)$.*

PROOF. (1) $\Rightarrow$ (2): Let Y and $\phi : X \to \mathscr{C}_c'(Y)$ be as in (2). Since X is λ-collectionwise normal, ϕ has a continuous selection $\ell : X \to Y$, see, for instance, Theorem 4.1. Let $Z = \{x \in X : |\phi(x)| > 1\}$. Since ϕ is lower semi-continuous, Z is an open subset of X. Hence, it is an F_σ-set because X is perfectly normal. Therefore, by a result of [**25**], Z is λ-collectionwise normal too. On the other hand, X is a countably paracompact space as a perfectly normal one. Hence, Z is also countably paracompact, see [**30**]. Thus, by Theorem 4.5, $\phi|_Z$ has a continuous selection $u : Z \to Y$ such that $u(z) \in \mathrm{wci}(\phi(z))$ for every $z \in Z$. Finally, take a continuous function $r : X \to [0,1]$ such that $X \setminus Z = r^{-1}(0)$. We define the required selection f for ϕ in the following way. First, define another continuous function $k : Z \to (0, +\infty)$ by $k(z) = \max\{r(z), \|u(z) - \ell(z)\|\}$, $z \in Z$, where $\|.\|$ is the norm of Y. Then, the function $\delta : Z \to [0,1]$, defined by $\delta(z) = r(z)/k(z)$, $z \in Z$, is also continuous. Now we may define our $f : X \to Y$ by $f(x) = \delta(x) \cdot u(x) + (1 - \delta(x)) \cdot \ell(x)$ if $x \in Z$ and $f(x) = \ell(x)$ otherwise. First, let us check that f is continuous. Clearly, $f|_Z$ and $f|_{X\setminus Z}$ are continuous. Hence, it suffices to check this for the points of $\mathrm{cl}_X Z \cap (X \setminus Z)$ if they exist. So, take a point $x_0 \in \mathrm{cl}_X Z \cap (X \setminus Z)$ and an $\varepsilon > 0$. Since $r(x_0) = 0$, there exists a neighbourhood V of x_0 such that $r(x) < \varepsilon/2$ for every $x \in V$. Since ℓ is continuous, there also exists a neighbourhood W of x_0 such that $\|\ell(x) - \ell(x_0)\| < \varepsilon/2$ for every

$x \in W$. Then, $U = V \cap W$ works as a desired neighborhood of x_0. Indeed, take a point $x \in U \cap Z$. Then,

$$\begin{aligned}\|f(x) - f(x_0)\| &\le \|f(x) - \ell(x)\| + \|\ell(x) - f(x_0)\| \\ &= \|\delta(x) \cdot u(x) + (1 - \delta(x)) \cdot \ell(x) - \ell(x)\| + \|\ell(x) - \ell(x_0)\| \\ &< \delta(x) \cdot \|u(x) - \ell(x)\| + \varepsilon/2 \\ &= r(x) \cdot \frac{\|u(x) - \ell(x)\|}{k(x)} + \varepsilon/2 \\ &\le r(x) + \varepsilon/2 < \varepsilon.\end{aligned}$$

That is, f is continuous. Clearly, f is a selection for ϕ because ϕ is convex-valued. Finally, let us check that $f(x) \in \mathrm{wci}(\phi(x))$ for every $x \in Z$. So, take a point $x \in Z$. If $u(x) = \ell(x)$, we have $f(x) = u(x) \in \mathrm{wci}(\phi(x))$. If $u(x) \ne \ell(x)$, then, by definition, $f(x) \in \mathrm{wci}(\phi(x))$.

The implication (2) $\Rightarrow$ (3) is obvious, while (3) $\Rightarrow$ (4) repeats precisely the corresponding implication in the proof of Theorem 4.5.

(4) $\Rightarrow$ (1): By Theorem 4.5, X is λ-collectionwise normal. To show that X is perfectly normal, we repeat the arguments suggested in [**23**, Question 2]. Take a closed set $A \subseteq X$, and let $g_0 : X \to \boldsymbol{R}$ be the constant 0, while h_0 be the characteristic function of $X \backslash A$. Next, for every $0 < \alpha < \lambda$, let $g_\alpha, h_\alpha : X \to \boldsymbol{R}$ be the constant functions whose value are equal to 0. Then, $g = \triangle\{g_\alpha : \alpha < \lambda\} : X \to c_0(\lambda)$ is upper semi-continuous, $h = \triangle\{h_\alpha : \alpha < \lambda\} : X \to c_0(\lambda)$ is lower semi-continuous, and $g \le h$. Hence, by (4), there exists a continuous $f : X \to c_0(\lambda)$ such that $g \le f \le h$ and $g(x) < f(x) < h(x)$ whenever $g(x) < h(x)$. Then, in particular, $f_0 = \pi_0 \circ f : X \to \boldsymbol{R}$ is continuous, $g_0(x) \le f_0(x) \le h_0(x)$ for every $x \in X$, and $g_0(x) < f_0(x) < h_0(x)$ whenever $g_0(x) < h_0(x)$. According to the definition of g_0 and h_0, the last implies that $A = f_0^{-1}(0)$. $\square$

Returning back to Theorem 4.5, a word should be said about condition (2) of this theorem. In fact, the reader may wonder if this condition holds for all Banach spaces. The authors do not know if this is true, which suggests the following natural question.

Problem 4.7. Let X be a countably paracompact and λ-collectionwise normal space for some infinite cardinal λ, Y be a Banach space with $w(Y) \le \lambda$, and $\phi : X \to \mathscr{C}'_c(Y)$ be lower semi-continuous such that $|\phi(x)| > 1$ for every $x \in X$. Does there exist a continuous map $f : X \to Y$ such that $f(x) \in \mathrm{wci}(\phi(x))$ for every $x \in X$?

5. Characterizations of expandable spaces.

Let λ be an infinite cardinal. A space X is called *λ-expandable* (resp., *almost λ-expandable*) if every locally finite family $\mathscr{F}$ of closed sets in X, with $|\mathscr{F}| \le \lambda$, has a locally finite (resp., point-finite) open expansion (cf. [**16**], [**27**]). We state the results, then proceed to the proofs.

Theorem 5.1. *For an infinite cardinal λ, the following conditions on a space X are equivalent*:

(1) *X is λ-expandable.*
(2) *For every completely metrizable space Y, with $w(Y) \leq \lambda$, and every upper semi-continuous mapping $\phi : X \to \mathscr{C}(Y)$, there exist two mappings $\varphi, \psi : X \to \mathscr{C}(Y)$ such that φ is lower semi-continuous, ψ is upper semi-continuous and $\phi \subseteq \varphi \subseteq \psi$.*
(3) *There exists a space Y and a locally finite family $\mathscr{G}$ of non-empty open sets in Y, with $|\mathscr{G}| = \lambda$, such that for every upper semi-continuous mapping $\phi : X \to \mathscr{C}(Y)$, there exist two mappings $\varphi, \psi : X \to \mathscr{C}(Y)$ such that φ is lower semi-continuous, ψ is upper semi-continuous and $\phi \subseteq \varphi \subseteq \psi$.*
(4) *For every upper semi-continuous map $f : X \to c_0(\lambda)$, there exist two maps $g, h : X \to c_0(\lambda)$ such that g is lower semi-continuous, h is upper semi-continuous and $f \leq g \leq h$.*

Theorem 5.2. *For an infinite cardinal λ, the following conditions on a space X are equivalent*:
(1) *X is almost λ-expandable.*
(2) *For every completely metrizable space Y, with $w(Y) \leq \lambda$, and every upper semi-continuous mapping $\phi : X \to \mathscr{C}(Y)$, there exists a lower semi-continuous mapping $\varphi : X \to \mathscr{C}(Y)$ such that $\phi \subseteq \varphi$.*
(3) *There exists a space Y and a locally finite family $\mathscr{G}$ of non-empty open sets in Y, with $|\mathscr{G}| = \lambda$, such that for every upper semi-continuous mapping $\phi : X \to \mathscr{C}(Y)$, there exists a lower semi-continuous mapping $\varphi : X \to \mathscr{C}(Y)$ such that $\phi \subseteq \varphi$.*
(4) *For every upper semi-continuous map $f : X \to c_0(\lambda)$, there exists a lower semi-continuous map $g : X \to c_0(\lambda)$ such that $f \leq g$.*

Miyazaki [**21**] has proven the equivalence (1) and (2) in Theorem 5.1 assuming that X is normal, and has shown that every metacompact space satisfies (2) in Theorem 5.2.

To prove Theorems 5.1 and 5.2, we need some definitions and lemmas. First, let us recall that, for a metric space (Y, d), the *Hausdorff distance* d_H on $\mathscr{C}(Y)$ associated with d is defined by

$$H(d)(S, T) = \sup\{d(S, y) + d(y, T) : y \in S \cup T\}, \quad S, T \in \mathscr{C}(Y).$$

Lemma 5.3 (Fort [**9**]). *Let (Y, d) be a metric space, and let $\{\varphi_n\}$ be a sequence of mappings from a space X to $\mathscr{C}(Y)$ which is uniformly convergent to a mapping $\varphi : X \to \mathscr{C}(Y)$ with respect to the Hausdorff distance d_H on $\mathscr{C}(Y)$ associated with d. Then, φ is lower* (*resp., upper*) *semi-continuous if φ_n is lower* (*resp., upper*) *semi-continuous for each n.*

Lemma 5.4. *Let $\mathscr{U} = \{U_\alpha : \alpha \in \mathscr{A}\}$ be a point-finite cover of a space X, and let $\{y_\alpha : \alpha \in \mathscr{A}\}$ be a subset of a space Y. Define a set-valued mapping $\phi : X \to \mathscr{C}(Y)$ by $\phi(x) = \{y_\alpha : x \in U_\alpha, \alpha \in \mathscr{A}\}$ for each $x \in X$. Then, ϕ is lower semi-continuous provided that $\mathscr{U}$ is an open cover, and ϕ is upper semi-continuous provided that $\mathscr{U}$ is a locally finite closed cover.*

Proof. This follows from the fact that $\phi^{-1}[V] = \bigcup\{U_\alpha : y_\alpha \in V, \alpha \in \mathscr{A}\}$ and $\phi^{\#}[V] = X \setminus \bigcup\{U_\alpha : y_\alpha \notin V, \alpha \in \mathscr{A}\}$ for every open set V in Y. □

Let X and Y be spaces, and $\phi : X \to \mathscr{C}(Y)$. We say that ϕ has the *locally finite* (resp., *point-finite*) *lifting property* if for every family $\mathscr{E}$ of subsets of Y admitting a locally finite open expansion, there exists a locally finite (resp., point-finite) open expansion of $\{\phi^{-1}[E] : E \in \mathscr{E}\}$ in X (see [**12**], [**21**]).

Lemma 5.5. *Let X and Y be spaces, and $\phi : X \to \mathscr{C}(Y)$. Then*:

(1) *If there exist mappings $\varphi, \psi : X \to \mathscr{C}(Y)$ such that φ is lower semi-continuous, ψ is upper semi-continuous and $\phi \subseteq \varphi \subseteq \psi$, then ϕ has the locally finite lifting property. The converse is also true if Y is completely metrizable.*

(2) *If there exists a lower semi-continuous mapping $\varphi : X \to \mathscr{C}(Y)$ such that $\phi \subseteq \varphi$, then ϕ has the point-finite lifting property. The converse is also true if Y is completely metrizable.*

Proof. We only prove (1) since (2) can be proved similarly. First, assume that there exist mappings $\varphi, \psi : X \to \mathscr{C}(Y)$ such that φ is lower semi-continuous, ψ is upper semi-continuous and $\phi \subseteq \varphi \subseteq \psi$. Let $\mathscr{E}$ be a locally finite family of subsets of Y with a locally finite open expansion $\{G(E) : E \in \mathscr{E}\}$, i.e. $E \subseteq G(E)$ for each $E \in \mathscr{E}$. Set $U(E) = \varphi^{-1}[G(E)]$ and $V(E) = \psi^{-1}[G(E)]$ for each $E \in \mathscr{E}$. Then, we have $\phi^{-1}[E] \subseteq U(E) \subseteq V(E)$ for each $E \in \mathscr{E}$. Since φ is lower-semi-continuous, $U(E)$ is open in X for each $E \in \mathscr{E}$ and, since ψ is upper semi-continuous, $\{V(E) : E \in \mathscr{E}\}$ is locally finite in X. Thus, $\{U(E) : E \in \mathscr{E}\}$ is a locally finite open expansion of $\{\phi^{-1}[E] : E \in \mathscr{E}\}$ in X. Hence, ϕ has the locally finite lifting property.

Next, assume that ϕ has the locally finite lifting property and Y is a completely metrizable space. Fix a complete metric d on Y. Then, there exist locally finite open covers $\mathscr{G}_n = \{G_\alpha : \alpha \in \mathscr{A}_n\}$, $n \in \boldsymbol{N}$, of Y and a chain of maps

$$\mathscr{A}_1 \overset{p_1}{\leftarrow} \mathscr{A}_2 \overset{p_2}{\leftarrow} \mathscr{A}_3 \leftarrow \cdots \leftarrow \mathscr{A}_n \overset{p_n}{\leftarrow} \mathscr{A}_{n+1} \leftarrow \cdots$$

such that $G_\alpha = \bigcup\{G_\beta : \beta \in p_n^{-1}(\alpha)\}$ and $\operatorname{diameter}(G_\alpha) < 1/2^n$ for each $\alpha \in \mathscr{A}_n$ and $n \in \boldsymbol{N}$. Since ϕ has the locally finite lifting property, $\{\phi^{-1}[G_\alpha] : \alpha \in \mathscr{A}_n\}$ has a locally finite open expansion for each $n \in \boldsymbol{N}$. Moreover, $\phi^{-1}[G_\alpha] = \bigcup\{\phi^{-1}[G_\beta] : \beta \in p_n^{-1}(\alpha)\}$ for each $\alpha \in \mathscr{A}_n$ and $n \in \boldsymbol{N}$. Thus, we can construct inductively locally finite open covers $\mathscr{U}_n = \{U_\alpha : \alpha \in \mathscr{A}_n\}$, $n \in \boldsymbol{N}$, of X with the same index set as $\mathscr{G}_n$ such that

$$\phi^{-1}[G_\alpha] \subseteq U_\alpha = \bigcup\{U_\beta : \beta \in p_n^{-1}(\alpha)\}$$

for each $\alpha \in \mathscr{A}_n$ and each $n \in \boldsymbol{N}$. Whenever $n \in \boldsymbol{N}$, fix a point $y_\alpha \in G_\alpha$ for each $\alpha \in \mathscr{A}_n$, and define mappings $\varphi_n, \psi_n : X \to \mathscr{C}(Y)$ by $\varphi_n(x) = \{y_\alpha : x \in U_\alpha, \alpha \in \mathscr{A}_n\}$ and $\psi_n(x) = \{y_\alpha : x \in \operatorname{cl}_X U_\alpha, \alpha \in \mathscr{A}_n\}$, respectively, for $x \in X$. Then, it follows from Lemma 5.4 that φ_n is lower semi-continuous and ψ_n is upper semi-continuous. Let d_H be the Hausdorff metric on $\mathscr{C}(Y)$ associated to the metric d of Y. According to the definitions of φ_n and ψ_n,

$$d_H(\varphi_n(x), \varphi_{n+1}(x)) < 1/2^n \quad \text{and} \quad d_H(\psi_n(x), \psi_{n+1}(x)) < 1/2^n$$

for each $x \in X$ and each $n \in \boldsymbol{N}$, i.e. $\{\varphi_n\}$ and $\{\psi_n\}$ are Cauchy sequences in the uniform space of all maps from X to $(\mathscr{C}(Y), d_H)$. Since $(\mathscr{C}(Y), d_H)$ is complete by [**8**, 4.5.23 (d), p. 298], $\{\varphi_n\}$ uniformly converges to a map $\varphi : X \to \mathscr{C}(Y)$, and $\{\psi_n\}$ uniformly converges to a map $\psi : X \to \mathscr{C}(Y)$. By Lemma 5.3, φ is lower semi-continuous and ψ is upper semi-continuous. Finally, it is easy to check that $\phi \subseteq \varphi \subseteq \psi$. □

Now, we are ready to prove Theorems 5.1 and 5.2.

PROOF OF THEOREM 5.1. (1) $\Rightarrow$ (2): Let $\phi : X \to \mathscr{C}(Y)$ be an upper semi-continuous mapping. Then, $\{\phi^{-1}[E] : E \in \mathscr{E}\}$ is locally finite in X for every locally finite family $\mathscr{E}$ of subsets in Y. Thus, (1) implies that ϕ has the locally finite lifting property. Hence, we have (2) by Lemma 5.5.

(2) $\Rightarrow$ (4): Let $f : X \to c_0(\lambda)$ be an upper semi-continuous map. Then, the map $h : X \to c_0(\lambda)$, defined by $h(x) = \max\{f(x), 0\}$, $x \in X$, is also upper semi-continuous by Lemma 2.3. Hence, it follows from Lemma 2.6 that the mapping $[0, h] : X \to \mathscr{C}(c_0(\lambda))$, which carries x to $[0, h(x)]$, is upper semi-continuous. By (2), there exist two mappings $\varphi, \psi : X \to \mathscr{C}(c_0(\lambda))$ such that φ is lower semi-continuous, ψ is upper semi-continuous and $[0, h] \subseteq \varphi \subseteq \psi$. Finally, by Lemma 2.8, $\sup \varphi : X \to c_0(\lambda)$ is lower semi-continuous, $\sup \psi : X \to c_0(\lambda)$ is upper semi-continuous and $f \le h \le \sup \varphi \le \sup \psi$.

(4) $\Rightarrow$ (3): We show that $c_0(\lambda)$ satisfies the conditions on Y stated in (3). Clearly, $\{\pi_\alpha^{-1}[(1, +\infty)] : \alpha < \lambda\}$ is a locally finite family of non-empty open sets in $c_0(\lambda)$ with cardinality λ. Let $\phi : X \to \mathscr{C}(c_0(\lambda))$ be an upper semi-continuous mapping. Then, it follows from Lemma 2.8 that $\sup \phi$ is upper semi-continuous and $\inf \phi$ is lower semi-continuous. Thus, by (4) and the second statement of Lemma 2.2, we can find lower semi-continuous maps $g_\ell, h_\ell : X \to c_0(\lambda)$ and upper semi-continuous maps $g_u, h_u : X \to c_0(\lambda)$ such that

$$h_\ell \le g_u \le \inf \phi \le \sup \phi \le g_\ell \le h_u. \tag{5.1}$$

Define $\varphi = [g_u, g_\ell]$ and $\psi = [h_\ell, h_u]$. Then, φ is lower semi-continuous and ψ is upper semi-continuous by Lemma 2.6, and $\phi \subseteq \varphi \subseteq \psi$ by (5.1).

(3) $\Rightarrow$ (1): To show that X is λ-expandable, let $\mathscr{F}$ be a locally finite family of closed sets in X with $|\mathscr{F}| \le \lambda$. We may assume that $\mathscr{F}$ covers X. Also, let Y be as in (3). Then, Y has a locally finite family $\mathscr{G} = \{G(F) : F \in \mathscr{F}\}$ of non-empty open sets in Y. Choose a point $y_F \in G(F)$ for each $F \in \mathscr{F}$, and define a mapping $\phi : X \to \mathscr{C}(Y)$ by $\phi(x) = \{y_F : x \in F \in \mathscr{F}\}$ for $x \in X$. Then, ϕ is upper semi-continuous by Lemma 5.4. According to the properties of Y and Lemma 5.5, ϕ has the locally finite lifting property. Hence, the family $\{\phi^{-1}[\{y_F\}] : F \in \mathscr{F}\}$ has a locally finite open expansion $\mathscr{U}$ in X, because $\mathscr{G}$ is an open expansion of $\{\{y_F\} : F \in \mathscr{F}\}$. Since $F \subseteq \phi^{-1}[\{y_F\}]$ for each $F \in \mathscr{F}$, $\mathscr{U}$ is also an expansion of $\mathscr{F}$. $\square$

The proof of Theorem 5.2 is left to the reader since it is almost same as that of Theorem 5.1.

It is known ([**16**]) that a space X is ω-expandable if and only if it is countably paracompact. Hence, by the definitions, a space X is λ-collectionwise normal and countably paracompact if and only if X satisfies one of the following two conditions: (i) X is λ-expandable and $X \in \lambda\text{-}\mathscr{PN}$; (ii) X is almost λ-expandable and $X \in \lambda\text{-}\mathscr{PN}$. Thus, we get several characterizations of a λ-collectionwise normal and countably paracompact space by combining one of the conditions in Theorems 5.1 and 5.2 with one of the conditions (1)–(9) in Theorem 3.1 and Remark 3.2. In particular, we have the following consequence which is a mapping analogue of the Dowker's characterization [**6**] of collectionwise normal and countably paracompact spaces.

COROLLARY 5.6. *For an infinite cardinal λ, the following conditions on a normal space X are equivalent*:

(1) *X is λ-collectionwise normal and countably paracompact.*
(2) *For every upper semi-continuous map $g : X \to c_0(\lambda)$, there exists a continuous map $f : X \to c_0(\lambda)$ such that $g \leq f$.*

For other characterizations of collectionwise normal countably paracompact spaces, see [**21**].

We complete this paper with the following characterization of paracompact spaces which is just like Corollary 5.6, only it deals with maps to $C_0(\lambda)$, where λ is the space of all ordinals less than λ with the usual order topology.

Theorem 5.7. *For an infinite cardinal λ, the following conditions on a normal space X are equivalent*:
(1) *X is λ-paracompact.*
(2) *For every space Y, with $w(Y) \leq \lambda$, and for every upper semi-continuous map $g : X \to C_0(Y)$, there exists a continuous map $f : X \to C_0(Y)$ such that $g \leq f$.*
(3) *For every upper semi-continuous map $g : X \to C_0(\lambda)$, there exists a continuous map $f : X \to C_0(\lambda)$ such that $g \leq f$.*

Proof. The implication $(1) \Rightarrow (2)$ is a consequence of [**19**, Theorem 3.2″]. Namely, let Y and $g : X \to C_0(Y)$ be as in (2). Then, the mapping $[g, +\infty) : X \to \mathscr{F}_c(C_0(Y))$ is lower semi-continuous by Lemma 2.6. Hence, by the mentioned Michael's result, ϕ has a continuous selection f. This f is as required in (2).

Since $(2) \Rightarrow (3)$ is obvious, we complete the proof showing that $(3) \Rightarrow (1)$. To this end, take a monotone increasing open cover $\mathscr{U} = \{U_\alpha : \alpha < \lambda\}$ of X. By [**17**, Theorem 5], it suffices to show that $\mathscr{U}$ has a locally finite open refinement. For each $x \in X$, let $\alpha(x) = \min\{\alpha < \lambda : x \in U_\alpha\}$. Finally, define a map $g : X \to C_0(\lambda)$ by

$$g(x)(\alpha) = \begin{cases} 1 & \text{for } \alpha \leq \alpha(x) \\ 0 & \text{for } \alpha > \alpha(x). \end{cases}$$

Let us show that g is upper semi-continuous. Let $x \in X$ and $\varepsilon > 0$ be fixed. For every $x' \in U_{\alpha(x)}$, if $\alpha \leq \alpha(x)$, then $g(x')(\alpha) \leq 1 < g(x)(\alpha) + \varepsilon$, and if $\alpha > \alpha(x)$, then $g(x')(\alpha) = 0 < g(x)(\alpha) + \varepsilon$ because $\alpha(x') \leq \alpha(x)$. Since $U_{\alpha(x)}$ is a neighbourhood of x, this means that g is upper semi-continuous. As a result, by (3), there exists a continuous map $f : X \to C_0(\lambda)$ with $g \leq f$. Take a locally finite open cover $\mathscr{V}$ of X such that diameter $f[V] \leq 1/2$ for each $V \in \mathscr{V}$. To show that $\mathscr{V}$ refines $\mathscr{U}$, let $V \in \mathscr{V}$ and fix a point $x \in V$. Since $f(x) \in C_0(\lambda)$, $f(x)(\alpha) < 1/2$ for some $\alpha < \lambda$. If there exists a point $y \in V \backslash U_\alpha$, then $\alpha(y) > \alpha$ and hence $f(y)(\alpha) \geq g(y)(\alpha) = 1$ by the definition of g. Thus, $\|f(x) - f(y)\| > 1/2$, which contradicts the assumption that diameter $f[V] \leq 1/2$. Hence, V must be included in U_α. That is, $\mathscr{V}$ is a locally finite open refinement of $\mathscr{U}$. □

In the proof of Theorem 5.7, the normality of X is only used to apply Michael's result in the implication $(1) \Rightarrow (2)$. Thus, we have the following corollary.

Corollary 5.8. *The following conditions on a Hausdorff space X are equivalent*:

(1) *X is paracompact.*
(2) *For every space Y and every upper semi-continuous map $g : X \to C_0(Y)$, there exists a continuous map $f : X \to C_0(Y)$ such that $g \leq f$.*
(3) *For every infinite cardinal λ and every upper semi-continuous map $g : X \to C_0(\lambda)$, there exists a continuous map $f : X \to C_0(\lambda)$ such that $g \leq f$.*

Concerning the statements of Corolary 5.8, the following question naturally arises.

Problem 5.9. Is a space X paracompact provided for every space Y and every two maps $g, h : X \to C_0(Y)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exists a continuous map $f : X \to C_0(Y)$ with $g \leq f \leq h$?

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