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On Julia sets of postcritically finite branched coverings Part II—S^1-parametrization of Julia sets

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Abstract. We prove that for an expanding postcritically finite branched covering f, the Julia set is orientedly S^1-parametrizable if and only if f^n is combinatorially equivalent to the degenerate mating of two polynomials for some $n > 0$.

1. Introduction.

In the preceding paper [**3**], the author introduced the notion of Julia sets for (probably non-holomorphic) expanding postcritically finite branched coverings on the 2-dimensional sphere. It should be noted that all postcritically finite rational maps are expanding postcritically finite branched coverings. Moreover, there exists an expanding postcritically finite branched covering not equivalent to a rational map (see [**3**], Section 6). We have studied semiconjugacies from symbolic dynamics in [**3**]. The main purpose of the present paper is to investigate semiconjugacies from the d-fold maps on the circle.

For a polynomial map $f : \boldsymbol{C} \to \boldsymbol{C}$ there often exists a surjective map $\phi : \{|z| = 1\} \to J_f$ such that $\phi(z^d) = f(\phi(z))$, where d is the degree of f and J_f is the Julia set of f. This property will be called S^1-*parametrizability*. Recall that a post-critically finite polynomial f has this property. In fact, for a simple closed curve γ around the infinity, the inverse image $\gamma_i = f^{-i}(\gamma)$ uniformly converges to a closed curve in J_f as $i \to \infty$ (see [**1**]). We will consider the class of postcritically finite branched coverings with S^1-parametrizability. The main result is to give a connection between S^1-parametrizability and mating.

The paper is organized as follows. In Section 2 we recall results of [**3**] which we will use. In Section 3 we define S^1-parametrizability of Julia sets and show that it is equivalent to the existence of a closed curve which is homotopically invariant. In Section 4 we give an example of rational maps with Julia sets not S^1-parametrizable. In Section 5 we introduce a class of branched coverings, called nesting branched coverings, and give a sufficient condition for a nesting branched covering to be S^1-parametrizable. In Section 6 we show that for a postcritically finite branched covering f, the Julia set is orientedly S^1-parametrizable if and only if f is equivalent to the degenerate mating of two polynomials, where we say the Julia set is *orientedly* S^1-parametrizable if the homotopically invariant closed curve can be perturbed to a simple closed curve.

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2. Summary of basic facts.

In this section, we recall some results obtained in [**3**].

Definition 2.1. Suppose $f : S^2 \to S^2$ is a topological branched covering. We say the set C_f of critical points is the *critical set* of f, and

$$P_f = \overline{\{f^n(c) \mid c \in C_f, n > 0\}}$$

is the *postcritical set* of f. We say f is *postcritically finite* if P_f is a finite set.

Throughout this paper, we suppose that $f : S^2 \to S^2$ is a postcritically finite branched covering of degree $d \geq 2$.

Definition 2.2. Let f be a postcritically finite branched covering. A point in P_f is a *postcritical point*. We say a periodic cycle $\{x_1, x_2, \ldots, x_k\}$ is a *critical cycle* if it contains a critical point. A point of a critical cycle is called a critical periodic point. We divide P_f into P_f^a and P_f^r.

$$P_f^a = \{x \in P_f \mid \exists k > 0, f^k(x) \text{ is contained in a critical cycle}\}, \quad P_f^r = P_f - P_f^a.$$

Definition 2.3. A smooth postcritically finite branched covering f is said to be *expanding* if there exists a Riemannian metric $\|\cdot\|$ on $S^2 - P_f$ which satisfies:

1. Any compact piecewise smooth curve inside $S^2 - P_f^a$ has finite length.
2. The distance $d(\cdot,\cdot)$ on $S^2 - P_f^a$ determined by the curve length is complete.
3. For some constants $C > 0$ and $0 < \lambda < 1$,

$$\|v\| < C\lambda^k \|df^k(v)\|$$

for any $k > 0$ and any tangent vector $v \in T_p(S^2)$ if $f^k(p) \in S^2 - P_f$.

Then $|l| < C\lambda^k |f^k(l)|$ for any piecewise smooth curve l with $f^k(l) \subset S^2 - P_f^a$, where $|\cdot|$ means the length of a curve.

Theorem 2.4. *If f is expanding, then there uniquely exists a non-empty compact subset $J \subset S^2 - P_f^a$ such that $f^{-1}(J) = J = f(J)$.*

Definition 2.5. The subset in the previous theorem is called the *Julia set* of f, and denoted by J_f.

Proposition 2.6. *If f is expanding, then the following hold*:

1. *For $x \in S^2 - P_f^a$, we have $J_f = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=n}^{\infty} f^{-k}(x)}$.*
2. *For $x \in S^2 - J_f$, the sequence $\{f^n(x)\}_{n>0}$ is attracted to a critical cycle.*
3. *The Julia set is connected.*

3. S^1-parametrizability.

We denote by q_N the N-fold map on the circle $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$, that is, $q_N(\theta) = N\theta \bmod 1$ for $\theta \in [0,1]$. We identify $\boldsymbol{T}$ and $S^1 = \{|z| = 1\}$ by $\theta \to \exp(2\pi i\theta)$.

Definition 3.1. A dynamical system $f : X \to X$ is called *S^1-parametrizable* if there exists a continuous surjection $\phi : \boldsymbol{T} \to X$ such that

$$\begin{array}{ccc} \boldsymbol{T} & \xrightarrow{q_N} & \boldsymbol{T} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ X & \xrightarrow[f]{} & X \end{array}$$

commutes for some N.

Definition 3.2. Let f be an expanding postcritically finite branched covering. We say its Julia set J_f is *S^1-parametrizable* if the restriction $f|J_f$ is S^1-parametrizable.

Definition 3.3. Let f be a postcritically finite branched covering of degree d. A closed curve $\alpha : S^1 \to S^2 - P_f^a$ is said to be *f-invariant up to homotopy* if there exists a closed curve $\alpha_1 : S^1 \to f^{-1}(\alpha)$ such that there exists a homotopy $h : S^1 \times [0,1] \to S^2 - P_f^a$ from α to α_1 with P_f^r fixed (i.e. if $h(\theta,t) = p \in P_f^r$ for some t, then $h(\theta,t) = p$ for every $t \in [0,1]$) and that $f \circ \alpha_1 = \alpha \circ q_N$ for some $N \geq 2$. If $N = d$, then we say α is *fully f-invariant up to homotopy*.

Lemma 3.4. *Let α be f-invariant up to homotopy. If $\beta : S^1 \to S^2 - P_f^a$ is homotopic to α with P_f^r fixed, then β is f-invariant up to homotopy. More in detail, if H is a homotopy from α to β with P_f^r fixed, then we have a homotopy H' from α_1 to β_1 ($\alpha_1 : S^1 \to f^{-1}(\alpha)$ homotopic to α, and $\beta_1 : S^1 \to f^{-1}(\beta)$) with P_f^r fixed such that*

$$\begin{array}{ccc} S^1 \times [0,1] & \xrightarrow{q_N \times \mathrm{id}} & S^1 \times [0,1] \\ {\scriptstyle H'}\downarrow & & \downarrow{\scriptstyle H} \\ S^2 & \xrightarrow[f]{} & S^2 \end{array}$$

commutes.

Proof. The required homotopy H' is the lift of $H \circ (q_N \times \mathrm{id})$ by the branched covering f. The existence of the lift is guaranteed by the existence of α_1. More precisely, H' is constructed as follows. Let $\theta \in S^1$. If $\alpha q_N(\theta) \in P_f^r$, then set $H'(\theta,t) = \alpha_1(\theta)$ for each $t \in [0,1]$. Otherwise, set $H'(\theta,\cdot) : [0,1] \to S^2 - f^{-1}(P_f)$ to be the lift of $H(q_N(\theta),\cdot) : [0,1] \to S^2 - P_f$ by the covering $f : S^2 - f^{-1}(P_f) \to S^2 - P_f$ such that $H'(\theta,0) = \alpha_1(\theta)$. □

Theorem 3.5. *Let f be an expanding postcritically finite branched covering. If there exists a closed curve $\alpha : S^1 \to S^2 - P_f^a$ which is f-invariant up to homotopy, then there exists an invariant subset $K \subset J_f$ such that $f|K : K \to K$* is *S^1-parametrizable. In particular, if α is fully f-invariant up to homotopy, then J_f is S^1-parametrizable.*

Proof. Let h be a homotopy from α to α_1 as in Definition 3.3. By Lemma 3.4, there exists a homotopy $h_1 : S^1 \times [0,1] \to S^2$ such that $h(q_N(\theta),t) = fh_1(\theta,t)$ and

$h_1(\cdot,0)=\alpha_1$. Similarly, for each $k>0$ we inductively obtain a homotopy $h_k: S^1\times[0,1]\to S^2$ such that $h_k(q_N(\theta),t)=fh_{k+1}(\theta,t)$ and $h_k(\cdot,1)=h_{k+1}(\cdot,0)$. Write $\alpha_k=h_{k-1}(\cdot,1)$. Then $h_k(\theta,\cdot):[0,1]\to S^2-P_f^a$ is a curve joining $\alpha_k(\theta)$ and $\alpha_{k+1}(\theta)$. By the expandingness of f, we have $|h_k(\theta,\cdot)|\le C\lambda^k|h(q_N^k(\theta),\cdot)|$, and hence $\alpha_k:S^1\to S^2$ uniformly converges to a curve $\beta:S^1\to S^2$ as $k\to\infty$ such that $f\circ\beta=\beta\circ q_N$. Then $K=\beta(S^1)$ is an invariant set, which is included in the Julia set by Proposition 2.6-1. □

Theorem 3.6. *Let f be a postcritically finite branched covering. If there exists a closed curve $\alpha:S^1\to S^2-P_f^a$ which is f-invariant up to homotopy, then α can be continuously deformed so as to have at most finitely many self-intersections keeping f-invariant up to homotopy.*

In particular, if the Julia set of an expanding postcritically finite branched covering f is S^1-parametrizable, then there exists a closed curve $\alpha:S^1\to S^2-P_f^a$ which is fully f-invariant up to homotopy and has at most finitely many self-intersections.

Proof. Let $\alpha:[0,1]\to S^2-P_f^a$ be f-invariant up to homotopy. Fix $p\in P_f^r$. Assume $\alpha(0)=\alpha(1)\ne p$. We show that α can be deformed to a curve α' with $\#\alpha'^{-1}(p)<\infty$. For $a,b\in\alpha^{-1}(p)$ with $a<b$, we say that $[a,b]\subset[0,1]$ is *trivial* if the restriction $\alpha:[a,b]\to S^2-P_f^a$ is homotopic to a constant map $t\mapsto p$ relative to $\{a,b\}$ in $S^2-(P_f-\{p\})$. Note that for a trivial interval $J\subset[0,1]$ if we deform α to α' by $\alpha'(t)=\alpha(t)$ if $t\notin J$, $\alpha'(t)=p$ if $t\in J$ then α' is still f-invariant up to homotopy. Let U be a small neighborhood of p. Then for $a,b\in\alpha^{-1}(p)$ with $a<b$, the interval $[a,b]$ is trivial whenever $\alpha([a,b])\subset U$. Let A be the set of $x\in[0,1]$ such that there exist $a<b$ with $a,b\in\alpha^{-1}(p)$, $a<x<b$ and $\alpha([a,b])\subset U$. Since $[0,1]$ is compact, A is a finite union of open intervals and $\#([0,1]-A)\cap\alpha^{-1}(p)<\infty$. We define α' by $\alpha'(t)=\alpha(t)$ if $t\notin A$, $\alpha'(t)=p$ if $t\in A$. Since $\alpha'^{-1}(p)$ consists of at most finite connected components, we obtain the required curve by modifying α'.

Doing this deformation for all $p\in P_f^r$, we get $\tilde\alpha$ a curve f-invariant up to homotopy such that $\#\tilde\alpha^{-1}(P_f^r)<\infty$. The curve can be approximated by a piecewise analytic curve with P_f^r fixed. This completes the proof. □

Definition 3.7. Let f be a postcritically finite branched covering. We say a closed curve $\alpha:S^1\to S^2-P_f^a$ with at most finitely many self-intersections is *oriented* if α can be deformed to a simple closed curve by a small perturbation (i.e. there is a continuous map $h:S^1\times[0,1]\to S^2-P_f^a$ such that $h(\cdot,0)=\alpha$ and $h(\theta,t)\ne h(\theta',t)$ whenever $\theta\ne\theta'\in S^1$ and $t\in(0,1]$).

Suppose f is expanding. The Julia set J_f is *orientedly* S^1-parametrizable if there exists an oriented closed curve $\alpha:S^1\to S^2-P_f^a$ which is fully f-invariant up to homotopy. Note that the deformed simple closed curve is not necessarily f-invariant up to homotopy.

Example 3.8. Consider a rational map $f(z)=(z^2-2)/z^2$. The critical set $C_f=\{0,\infty\}$ and the postcritical set $P_f=\{\infty,1,-1\}$. The dynamics on $C_f\cup P_f$ is $0\to\infty\to 1\to -1\to -1$. Set

$$\gamma(t) = \begin{cases} -1 + \dfrac{t - 1/4}{t} & \left(0 \le t \le \dfrac{1}{4}\right) \\ -1 + i\dfrac{t - 1/4}{t - 1/2} & \left(\dfrac{1}{4} < t \le \dfrac{1}{2}\right) \\ 1 - \dfrac{t - 3/4}{t - 1/2} & \left(\dfrac{1}{2} < t \le \dfrac{3}{4}\right) \\ 1 - i\dfrac{t - 3/4}{t - 1} & \left(\dfrac{3}{4} < t \le 1\right) \end{cases}$$

Then γ is oriented and satisfies the condition of Theorem 3.5 for $N = 2$ (see Figures 1.1, 1.2 and 1.3). Thus the Julia set J_f, which is the whole sphere, is orientedly S^1-parametrizable.

Remark 3.9. A branched covering whose Julia set is S^1-parametrizable and not orientedly S^1-parametrizable is unknown.

4. A Julia set which is not S^1-parametrizable.

In this section we give one example of rational maps whose Julia sets are not S^1-parametrizable.

We use the notion of *branch group* which has been introduced in the preceding paper. See [**3**] Section 5. Let f be a postcritically finite branched covering of degree d. Choose a point $x \in S^2 - P_f$ and a *radial* r (see [**3**] Definition 3.2). We denote, by G_k, the k-th branch group, and denote, by $F_k : G_{k-1} \to G_k$, the induced homomorphism of f. Recall that $G_k = \pi_1(S^2 - P_f, x)^{W_k} \times \Lambda(W_k)$, where $W_k = \{1, 2, \dots, d\}^k$ is the set of words of length k and $\Lambda(W_k)$ is the set of permutations on W_k. Let $p_1 : G_k \to \pi_1(S^2 - P_f, x)^{W_k}$ and $p_2 : G_k \to \Lambda(W_k)$ be the projections.

Definition 4.1. For a permutation h, we say $(a_1, a_2, \dots, a_n)$ is an *orbit* of h if $a_i \ne a_j$ for $i \ne j$ and if $h(a_{i-1}) = a_i$ for $i = 2, \dots, n$ and $h(a_n) = a_1$. The number n is the *period* of the orbit.

A closed curve $\gamma : S^1 \to S^2 - P_f$ is *prime* if $l \circ q_n$ and γ are not homotopic in $S^2 - P_f$ for any closed curve l and any positive integer n.

Proposition 4.2. *Let $\gamma : S^1 \to S^2 - P_f$ be a closed curve with basepoint $\gamma(0) = x$. Let $[\gamma]$ be the element of $\pi_1(S^2 - P_f, x)$ with a representative γ. The permutation $p_2(F_j \circ F_{j-1} \circ \cdots \circ F_1([\gamma]))$ has an orbit of period n if and only if there exists a prime closed curve $l : S^1 \to S^2 - P_f$ such that $f^j \circ l = \gamma \circ q_n$.*

Proof. In view of [**3**] Theorem 3.4, we have a mapping $W_j \ni w \mapsto x_w \in f^{-j}(x)$. For $w \in W_j$, we denote by $\omega_w : [0, 1] \to S^2 - P_f$ the curve such that $f^j \circ \omega_w = \gamma$ and $\omega_w(0) = x_w$. We use the mappings $\tilde{L}_w$ and e of [**3**] Section 5. Write $\tau = p_1(F_j \circ \cdots \circ F_1([\gamma]))$ and $h = p_2(F_j \circ \cdots \circ F_1([\gamma]))$. It follows from the definition of F_k that

$$\tau(a_1 a_2 \cdots a_j) = \tilde{L}_{a_1 a_2 \cdots a_j}([\gamma]), \quad h(a_1 a_2 \cdots a_j) = b_1 b_2 \cdots b_j,$$

where $b_k = e(\tilde{L}_{a_{k+1} a_{k+2} \cdots a_j}([\gamma]), a_k)$. Thus $h(w) = w'$ if and only if $\omega_w(1) = x_{w'}$.

Let $(w^1, w^2, \dots, w^n = w^0)$ be an orbit of $p_2(F_j \circ \cdots \circ F_1(\gamma))$. Then $\omega_{w^i}(1) = \omega_{w^{i+1}}(0)$. Hence $\gamma' = \omega_{w^1}\omega_{w^2}\cdots\omega_{w^n}$ satisfies $f^j \circ \gamma' = \gamma \circ q_n$.

Conversely, if a closed curve l satisfies $f^j \circ l = \gamma \circ q_n$, then $l = \omega_{w^1}\omega_{w^2}\cdots\omega_{w^n}$ for some $w^1, w^2, \dots, w^n \in W_j$. □

Consider the rational map $f(z) = (z^2+1)/(z^2-1)$. The critical set $C_f = \{0, \infty\}$ and the postcritical set $P_f = P_f^a = \{-1, \infty, 1\}$. The dynamics on $C_f \cup P_f$ is $0 \mapsto -1 \mapsto \infty \mapsto 1 \mapsto \infty$. We show that $f^j|J_f$ is not S^1-parametrizable for every j.

Assume that $f^j|J_f$ is S^1-parametrizable. Then there exists $\gamma : S^1 \to S^2 - P_f$ such that $f^j \circ \gamma = \gamma \circ q_{2^j}$.

Take a radial r and generators A, B of $\pi_1(S^2 - P_f, x)$ as Figures 2.1 and 2.2. We denote by $(a_1\ a_2)$ the permutation interchanging a_1 and a_2. Then

$$p_2F_1(A) = (1\ 2), \quad p_2F_1(B) = (1\ 2) \tag{1}$$

and

$$p_1F_1(A)(1) = B^{-1}, \quad p_1F_1(A)(2) = A^{-1}, \quad p_1F_1(B)(1) = 1, \quad p_1F_1(B)(2) = 1. \tag{2}$$

Therefore,

$$p_2F_2F_1(A) = (11\ 22)(21\ 12), \quad p_2F_2F_1(B) = (11\ 12)(21\ 22).$$

Since the image of $p_2F_2F_1$ is generated by $\{(11\ 22)(21\ 12), (11\ 12)(21\ 22)\}$, it is

$$\{\mathrm{id}, (11\ 22)(21\ 12), (11\ 12)(21\ 22), (11\ 21)(12\ 22)\}.$$

By Proposition 4.2, for every closed curve $\gamma \subset S^2 - P_f$, there is no closed curve γ' such that $f^j \circ \gamma' = \gamma \circ q_{2^j}$ if $j \geq 2$.

Suppose there exists a closed curve γ such that $[\gamma] = A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots A^{n_k}B^{m_k}$ and $f \circ \gamma' = \gamma \circ q_2$ for some γ' homotopic to γ in $S^2 - P_f$. By Proposition 4.2, $p_2F_1([\gamma]) = (1\ 2)$. Therefore $\sum_{i=1}^k (n_i + m_i)$ is odd by (1). Let

$$p_1F_1([\gamma])(1) = A^{n_1'}B^{m_1'}A^{n_2'}B^{m_2'}\cdots A^{n_k'}B^{m_k'}$$

and

$$p_1F_1([\gamma])(2) = A^{n_1''}B^{m_1''}A^{n_2''}B^{m_2''}\cdots A^{n_k''}B^{m_k''}.$$

By (2), $\sum_{i=1}^k (n_i' + m_i' + n_i'' + m_i'')$ is even. Since $\alpha^{-1}p_1F_1([\gamma])(1)p_1F_1([\gamma])(2)\alpha = [\gamma]$ for some $\alpha \in \pi(S^2 - P_f, x)$, this is a contradiction.

5. Nesting branched coverings.

In this section we state a sufficient condition for the Julia set of a nesting expanding postcritically finite branched covering to be orientedly S^1-parametrizable.

DEFINITION 5.1. A postcritically finite branched covering f is called *nesting* if there exists a topological graph $H \subset S^2$ satisfying:

1. $f : H \to H$ is a homeomorphism,
2. f^{-1} has d branches defined on $S^2 - f^{-n}(H)$ for some n (i.e. there exist maps $g_1, g_2, \dots, g_d$ on $S^2 - f^{-n}(H)$ such that $f \circ g_k = \mathrm{id}$ and $\bigcup_{i=1}^{d} g_i(S^2 - f^{-n}(H)) = S^2 - f^{-n-1}(H)$).

Recall that a topological graph means a 1-dim finite simplicial complex. We say H is a *cut graph* of f.

Remark 5.2. 1. A postcritically finite polynomial is nesting. Indeed, we can easily make a cut graph by joining some external rays.
2. If f is expanding and nesting, then f is topologically conjugate to a nesting rational map in some neighborhoods of their Julia sets ([**3**] Corollary 6.7).

Proposition 5.3. *Let f be a nesting expanding postcritically finite branched covering with cut graph H. Then* (1) *$H \cap J_f$ is a finite set and* (2) *$f^{-k}(H)$ is connected for each $k \geq 0$.*

Proof. Let $\mathscr{B}$ be the set of connected components of $H - J_f$ which intersect P_f^a. By the expandingness of f and the injectivity of $f|H$, we see that $H - \bigcup_{L \in \mathscr{B}} L$ consists of at most finitely many points. Thus (1) is proved.

Suppose that $f^{-K}(H)$ is not connected for some $K \geq 0$. Then $f^{-k}(H)$ is not connected for every $k \geq K$. We may assume that f^{-1} has d branches on $S^2 - f^{-K}(H)$. Since $f^{-k}(H) \subset f^{-k-1}(H)$, f^{-1} has d branches on $S^2 - f^{-k}(H)$ for every $k \geq K$. We say a collection Γ of disjoint simple closed curves in $S^2 - f^{-k}(H)$ *separates* $f^{-k}(H)$ if each connected component of $S^2 - \bigcup_{\gamma \in \Gamma} \gamma$ includes at most one connected component of $f^{-k}(H)$. If Γ_K separates $f^{-K}(H)$, then $\Gamma_{K+j} = \{\text{a component of } f^{-j}(\gamma) \mid \gamma \in \Gamma_K\}$ separates $f^{-K-j}(H)$. Then $\max_{\gamma \in \Gamma_{K+j}} |\gamma| \to 0$ as $j \to \infty$ by the expandingness of f. This contradicts the fact

$$0 < \min\left\{\max_{\gamma \in \Gamma} |\gamma| : \Gamma \text{ separates } f^{-K}(H)\right\} \leq \min\left\{\max_{\gamma \in \Gamma} |\gamma| : \Gamma \text{ separates } f^{-K+j}(H)\right\},$$

and hence completes the proof. □

Corollary 5.4. *If an expanding postcritically finite branched covering f is nesting, then the following are satisfied*:
1. *for two points $y_1, y_2 \in S^2 - J_f$, there exists an arc γ joining y_1 and y_2 such that $\#(J_f \cap \gamma) < \infty$,*
2. *for $y \in P_f$, there exists a component U of $S^2 - J_f$ such that $y \in \overline{U}$.*

Proof. Let H be the cut graph. Since f^{-1} has d branches on $S^2 - f^{-n}(H)$ for some n, all critical values are contained in $f^{-n}(H)$, and hence $P_f \subset f^{-n}(H)$. Suppose $y_1, y_2 \in S^2 - J_f$. Let U_1, U_2 be the components of $S^2 - J_f$ such that $y_i \in U_i$, $i = 1, 2$. Since $f^k(U_i)$ contains a critical periodic point for some k, $U_i \cap f^{-k}(H) \neq \emptyset$. By the connectedness of $f^{-k}(H)$, we have an arc γ joining y_1 and y_2 with $\#(J_f \cap \gamma) < \infty$. Suppose $y \in J_f \cap P_f$. Let V be a small neighborhood of y. Since $\#(H \cap J_f) < \infty$, $V \cap H \cap J_f = \{p\}$. Take a component U of $S^2 - J_f$ which includes a component of $V \cap H - \{p\}$. □

Remark 5.5. The converse of Corollary 5.4 is also true for rational maps, that is, a postcritically finite rational map with the two conditions is nesting. The proof is left to the reader.

Example 5.6. We again consider the rational map $f(z) = (z^2+1)/(z^2-1)$. Since the interval $[1, \infty]$ makes a cut graph, f is nesting.

Theorem 5.7. *Let f be a nesting expanding branched covering of degree d. Let H be the cut graph. Suppose there exist a subgraph $H_0 \subset f^{-n}(H)$ and $A \subset P_f$ such that $P_f \subset H_0$, $H_0 - A \subset f^{-1}(H_0 - A)$, $f(A) \subset A$, and both of $H_0 - A$ and $f^{-1}(H_0 - A)$ are connected and simply connected. Then J_f is orientedly S^1-parametrizable.*

Proof. Let U be a small neighborhood of A. Let γ be a simple closed curve in $S^2 - (H_0 - U)$ such that $B \cap P_f = A$, where B is one disc bounded by γ. Since $H_0 - A$ is connected and simply connected, γ is uniquely determined up to homotopy in $S^1 - P_f$. Since $f^{-1}(H_0 - A)$ is connected and simply connected, $\gamma' = f^{-1}(\gamma)$ consists of only one connected component, and hence $f : \gamma' \to \gamma$ is of degree d. It is easily seen that γ and γ' are homotopic in $S^2 - P_f$. Thus J_f is orientedly S^1-parametrizable by Theorem 3.5. □

Example 5.8. Consider a rational map $f(z) = (z^3 - 16/27)/z$. The critical set and the postcritical set are

$$C_f = \{-2/3, -(2/3)\omega, -(2/3)\omega^2, \infty\}, \quad P_f = \{4/3, (4/3)\omega, (4/3)\omega^2, \infty\},$$

where ω is a cubic root of 1. The dynamics on $C_f \cup P_f$ is $-2/3 \to 4/3 \to 4/3$, $-(2/3)\omega^s \to (4/3)\omega^t \to (4/3)\omega^s$, $\infty \to \infty$, where $(s, t) = (1, 2)$ or $(2, 1)$. The Julia set J_f is homeomorphic to the Sierpinski gasket (see [**5**], [**2**]).

Denote by l the interval $[4/3, \infty]$. Set $H = l \cup \omega l \cup \omega^2 l$. Then $f(H) = H$, and f^{-1} has three branches on $\boldsymbol{C} - H$. So f is a nesting branched covering with cut graph H. Since H and $A = \{(4/3)\omega, (4/3)\omega^2\}$ satisfy the condition of Theorem 5.7, J_f is orientedly S^1-parametrizable.

6. Mating.

In this section we show that J_f is orientedly S^1-parametrizable if and only if f^n is equivalent to the degenerate mating of two polynomials for some $n > 0$. We use 'equivalence' in Thurston's sense. See [**3**] Definition 4.2.

6.1. Definitions.

A mating of two (topological) polynomials is a branched covering constructed in a certain way. First we give the definition of formal matings and degenerate matings for polynomials.

Definition 6.1. Let $f_1 : \boldsymbol{C}_1 \to \boldsymbol{C}_1$ and $f_2 : \boldsymbol{C}_2 \to \boldsymbol{C}_2$ be two monic polynomial maps of degree d, where $\boldsymbol{C}_i$ is a copy of the complex plane $\boldsymbol{C}$. Let $\tilde{R}_i(t)$ denote the external ray of angle t for f_i (see [**1**] for the definition). Adding a circle $C_i = \{\exp(2\pi\sqrt{-1}t) \cdot \infty_i \mid t \in \boldsymbol{T}\}$ at infinity such that $\exp(2\pi\sqrt{-1}t) \cdot \infty_i \in C_i$ is an endpoint of $\tilde{R}_i(t)$, we can consider f_i as a map of the closed disc $S_i = \boldsymbol{C}_i \cup C_i$ to itself, where $f_i(\exp(2\pi\sqrt{-1}t)\cdot\infty_i) = \exp(2\pi\sqrt{-1}\,dt)\cdot\infty_i$. Then

$$S = S_1 \sqcup S_2/(\exp(2\pi\sqrt{-1}t)\cdot\infty_1 \sim \exp(-2\pi\sqrt{-1}t)\cdot\infty_2 : t \in \boldsymbol{T})$$

is a 2-dimensional sphere. The branched covering $F : S \to S$ defined by $F|S_i = f_i$ is called the *formal mating* of f_1 and f_2.

If f_1 and f_2 are postcritically finite, then so is F. From now on, we suppose f_1 and f_2 are postcritically finite.

For $t \in \boldsymbol{T}$, we denote by $R_i(t)$ the closure of external ray of angle t for f_i. We consider that the endpoint of $R_i(t)$ on the infinity side is $\exp(2\pi\sqrt{-1}t)\cdot\infty_i$.

Definition 6.2. For $x, y \in S_i$, we say $x \sim_i y$ if x and y are contained in $R_i(t)$ for some t. The equivalence relation $\sim$ on S is defined to be the equivalence relation generated by $\sim_1$ and $\sim_2$. Note that $x \sim y$ implies $F(x) \sim F(y)$. The equivalence class of $x \in S$, which we denote by $[x]$, is a union of external rays. Each connected component of $F^{-1}([x])$ is also an equivalence class.

Let $[x_1], [x_2], \dots, [x_m]$ be the equivalence classes containing at least two postcritical points. Let $[y_1], [y_2], \dots, [y_n]$ be the equivalence classes such that $F^k([y_j]) = [x_i]$ for some i and for some $k \geq 0$ and that $[y_j]$ contains a point of $P_F \cup C_F$. Suppose that each $[y_j]$ is simply connected. Then $S' = S/\simeq$ is a 2-dimensional sphere, where $x \simeq y$ if $x, y \in [y_j]$ for some j. We define a branched covering $F' : S' \to S'$ as follows (see [**4**] §5). Let $U_1, U_2, \dots, U_n$ be disjoint topological open discs such that $[y_j] \subset U_j$. Then $U_j' = U_j/\simeq$ is also a topological open disc. Let $V_1, V_2, \dots, V_l$ be the connected components of $F^{-1}(\bigcup_{j=1}^n U_j)$ such that $V_i \cap \bigcup_{j=1}^n [y_j] = \emptyset$. Set $F'(x) = F(x)$ if $x \in S - (\bigcup_{j=1}^l [y_j] \cup \bigcup_{i=1}^l V_i)$, $F'([y_j]) = F([y_j])$ for $j = 1, 2, \dots, n$, and $F'|V_i$ to be homeomorphic. Since $F'|V_i$ is arbitrary, F' is not unique. However, it is uniquely determined up to the Thurston equivalence. We call F' the *degenerate mating* of f_1 and f_2.

Now we define matings for topological polynomials.

Definition 6.3. A branched covering f is called a *topological polynomial* if there exists a distinguished point $\infty \in S^2$ such that $f^{-1}(\infty) = \{\infty\}$.

Definition 6.4. Let f_1 and f_2 be two postcritically finite topological polynomials of degree d. Then there exist simple closed curves γ_i $(i = 1, 2)$ encircling ∞ such that $f_i^{-1}(\gamma_i)$ is connected and isotopic to γ_i. We modify f_i in the neighborhood of ∞ so that $f_i^{-1}(\gamma_i) = \gamma_i$ and $f_i : \gamma_i \to \gamma_i$ is conjugate to $q_d : \boldsymbol{T} \to \boldsymbol{T}$. Let U_i be the simply connected domain bounded by γ_i which does not contain ∞. Then

$$S = \overline{U_1} \sqcup \overline{U_2}/(\phi_1(t) \sim \phi_2(-t) : t \in \boldsymbol{T})$$

is a 2-dimensional sphere, where $\phi_i : \boldsymbol{T} \to \gamma_i$ is a conjugacy between q_d and $f|\gamma_i$. Define a branched covering $F : S \to S$ such that $F|U_i = f_i$. Then F is postcritically finite and $P_F \cap U_i = P_{f_i} - \{\infty\}$ and the circle $\gamma = [\gamma_i] \subset S$ is F-invariant. We say F is a *formal mating* of f and g. Note that F depends on the choice of the conjugacies ϕ_1 and ϕ_2.

Definition 6.5. Suppose $F : S \to S$ is a formal mating of postcritically finite topological polynomials f_1 and f_2. We define a *degeneration* of F as follows.

We say a topological tree T in $S - P_F^a$ is an *equivalence tree* if $\# T \cap P_F^r \geq 2$, $F^k(T) = T$ for some k, $F^k : T \to T$ is a homeomorphism, and $T \cap \bigcup_{i=1}^{k-1} F^i(T) = \emptyset$.

Let $T_1, T_2, \dots, T_m$ be a collection of disjoint equivalence trees. Let $S_1, S_2, \dots, S_n$ be the components of $\bigcup_{j=0}^{\infty} F^{-j}(\bigcup_{i=1}^{m} T_i)$ which contain a point of $P_F \cup C_F$. Suppose each S_j is simply connected. Then the quotient space $S' = S/\simeq$ is a 2-dimensional sphere, where $x \simeq y$ if $x, y \in S_j$ for some j. We define a branched covering $\tilde{F} : S' \to S'$ by the same construction as in Definition 6.2. Then we say $\tilde{F}$ is a degeneration of F with respect to $T_1, T_2, \dots, T_m$. For convenience, we consider F itself as a degeneration of F with respect to the empty tree.

Proposition 6.6. *Let $F : S \to S$ be a formal mating of postcritically finite topological polynomials, and $\tilde{F} : S' \to S'$ a degeneration of F. Then there exists an oriented closed curve which is fully $\tilde{F}$-invariant up to homotopy.*

Moreover, if an expanding postcritically finite branched covering f is equivalent to $\tilde{F} : S' \to S'$, then J_f is orientedly S^1-parametrizable.

Proof. Let γ be the closed curve defined in Definition 6.4, and let $\gamma' = \gamma/\simeq \subset S'$. Since $F^{-1}(\gamma) = \gamma$, we have γ' is fully $\tilde{F}$-invariant up to homotopy. It is easily seen that γ' is oriented. The second assertion is verified by Theorem 3.5. □

6.2. Statement and proof of the main theorem.

Let γ be an oriented closed curve, and $p \in \gamma$ a self-intersection point of γ. We construct an 'unlacing' of γ at p as follows.

Take a small open disc U centered at p so that $U \cap \gamma$ is homeomorphic to a tree with only one branch point. Then each connected component C_i of $U \backslash \gamma$ is considered as a sector bounded by two radii $H_i^+, H_i^- \subset \gamma$ and an arc $I_i \subset \partial U$. Since γ is oriented, there exists at least one sector C_i such that $\gamma^{-1}(H_i^+ \cup H_i^-)$ is connected. Thus we can construct a homotopy $h' : S^2 \times [0,1] \to S^2$ such that $h'(x,t) = x$ for $x \in S^2 - C_i$, $h'(\cdot, 0)$ is the identity, $h'(\cdot, t) : S^2 \to S^2$ is a homeomorphism for $0 \leq t < 1$, and $h'(I_i', 1) = H_i^+ \cup H_i^-$, where I_i' is a simple curve in C_i homotopic to I_i keeping the endpoints fixed. Doing this operation finite times, we obtain a homotopy $h : S^2 \times [0,1] \to S^2$ such that $h(x,t) = x$ for $x \in S^2 - U$, $h(\cdot, 0)$ is the identity, and $h(\gamma', 1) = \gamma$, where γ' is an oriented closed curve which has no self-intersection in U. We can modify h so that $h(\cdot, t) : S^2 \to S^2$ is a homeomorphism for $0 \leq t < 1$ and $h(\cdot, 1) : S^2 - (\gamma' \cup T_p) \to S^2 - \gamma$ is a homeomorphism, where $T_p = h(\cdot, 1)^{-1}(p)$ is homeomorphic to the tree $\tilde{T}$ defined below. We say T_p is the *tree of self-intersection* at p. See Figures 3.1, 3.2 and 3.3.

Let $\mathscr{P}$ be the set of connected component of $U \cap \gamma'$ and $\mathscr{Q}$ be the set of connected component of $U - \gamma'$. If $a = (A, B) \in \mathscr{P} \times \mathscr{Q}$ satisfy $A \subset \partial B$, then we define I_a to be an arc with endpoints s_a and t_a. Then we set $\tilde{T} = (\bigsqcup_{a=(A,B): A \subset \partial B} I_a)/\sim$, where for $a = (A, B)$ and $a' = (A', B')$ we set $s_a \sim s_{a'}$ if $A = A'$, and $t_a \sim t_{a'}$ if $B = B'$.

Definition 6.7. Let γ be an oriented closed curve, and $P \subset \gamma$ a collection of self-intersection points of γ. Take a small neighborhood U of P. By the above method, there exist an oriented closed curve γ' which has no self-intersection in U and a homotopy $h : S^2 \times [0,1] \to S^2$ such that $h(x,t) = x$ for $x \in S^2 - U$, $h(\cdot,0)$ is the identity, and $h(\gamma',1) = \gamma$. Moreover, we can assume that $h(\cdot,t) : S^2 \to S^2$ is a homeomorphism for $0 \le t < 1$ and $h(\cdot,1) : S^2 - (\gamma' \cup \bigcup_{p \in P} T_p) \to S^2 - \gamma$ is a homeomorphism. We say γ' is an *unlacing* of γ with respect to P.

Theorem 6.8. *Let f be a postcritically finite branched covering of degree d. Suppose there exists an oriented curve γ in $S^2 - P_f^a$ which is fully f-invariant up to homotopy. Then there exist two topological polynomials f_1, f_2 and an integer n such that f^n is equivalent to a degeneration of a formal mating of f_1 and f_2.*

Proof. We can assume that $\gamma : S^1 \to S^2 - P_f^a$ has no self-intersection except in P_f^r. Let $\gamma_k : S^1 \to S^2 - P_f^a$, $k = 1,2,\dots$ be the oriented closed curve homotopic to γ with P_f^r fixed such that $f^k \circ \gamma_k = \gamma \circ q_{d^k}$. Then γ_k has no self-intersection except in $f^{-k}(P_f^r)$. We write $P = \gamma \cap P_f^r$. Note that $f(P) \subset P$. Let γ_k' be an unlacing of γ_k with respect to $f^{-k}(P) - P$. Then γ_k' are all homotopic to one another with P_f^r fixed. Hence there exist $1 \le t < t'$ such that γ_t' is carried to $\gamma_{t'}'$ by an ambient isotopy in $S^2 - P_f^a$ keeping P_f^r fixed. Indeed, it is easily checked that the set of oriented closed curves without self-intersections except in P_f^r that are homotopic to γ with P_f^r fixed are divided into finite classes up to ambient isotopy in $S^2 - P_f^a$ keeping P_f^r fixed. Now adopting γ_t' instead of γ as the starting curve (and renaming $\gamma = \gamma_t'$), we see that γ is carried to γ_n' by an ambient isotopy in $S^2 - P_f^a$ keeping P_f^r fixed, where $n = t' - t$. We can assume $\gamma_n' = \gamma$ by replacing f^n with an equivalent branched covering. For simplicity, we consider $n = 1$.

Let $\tilde{\gamma}$ be an unlacing of γ with respect to P. Connecting the homotopy which carries $\tilde{\gamma}$ to γ and the homotopy which carries γ to γ_1, we have a homotopy $h : S^2 \times [0,1] \to S^2$ which satisfies the following: Let U_1 and U_2 be small neighborhoods of P and $f^{-1}(P) - P$ respectively. We denote by T_p the tree of self-intersection at p; $h(\cdot,0)$ is the identity, $h(\tilde{\gamma},1/2) = \gamma$, $h(\tilde{\gamma},1) = \gamma_1$, $h(x,t) = x$ for $x \in S^2 - (U_1 \cup U_2)$ and $0 \le t \le 1$, $h(x,t) = h(x,1/2)$ for $x \in S^2 - U_2$ and $1/2 \le t \le 1$, $h(\cdot,t) : S^2 \to S^2$ is a homeomorphism for $0 \le t < 1/2$, $h(\cdot,t) : S^2 - (\tilde{\gamma} \cup \bigcup_{p \in P} T_p) \to S^2 - h(\tilde{\gamma},t)$ is a homeomorphism for $1/2 \le t < 1$, and $h(\cdot,1) : S^2 - (\tilde{\gamma} \cup \bigcup_{p \in f^{-1}(P)} T_p) \to S^2 - \gamma_1$ is a homeomorphism.

Write $\phi_1 = h(\cdot,1/2) : S^2 \to S^2$ and $\phi_2 = h(\cdot,1) : S^2 \to S^2$. Let D be one of the connected components of $S^2 - \tilde{\gamma}$. We define a branched covering $f_1 : \bar{D} \to \bar{D}$ as follows. For $x \in \bar{D} - (\bigcup_{p \in f^{-1}(P)} T_p \cup f^{-2}(P))$, set $f_1(x) = \phi_1^{-1} \circ f \circ \phi_2(x)$. Then $f_1 : \bar{D} - (\bigcup_{p \in f^{-1}(P)} T_p \cup f^{-2}(P)) \to \bar{D} - \bigcup_{p \in f^{-1}(P)} T_p$ is a branched covering with critical points in $C_f - P$. By modifying f_1 near $f^{-2}(P) - f^{-1}(P)$, we have a branched covering $f_1 : \bar{D} - \bigcup_{p \in f^{-1}(P)} T_p \to \bar{D} - \bigcup_{p \in P} T_p$ with critical points in $C_f - P$. Varying f_1 continuously in a neighborhood of $\bigcup_{p \in f^{-1}(P)} T_p$, we can extend f_1 to a branched covering on $\bar{D}$ such that its restriction to $\bar{D} - \bigcup_{p \in f^{-1}(P)} T_p$ is still a branched covering with critical points in $C_f - P$. Note that each connected component of $T_p \cap D$ contains at most one point of P_{f_1}. We will use this fact in the proof of Theorem 6.9.

Similarly, we have a branched covering $f_2 : \bar{E} \to \bar{E}$, where E is the other connected component of $S^2 - \tilde{\gamma}$. Thus we get a postcritically finite branched covering $F : S^2 \to S^2$ such that $F|\bar{D} = f_1$, $F|\bar{E} = f_2$. From f_1 and f_2, we obtain two topological polynomials

$\tilde{f}_1$ and $\tilde{f}_2$ on 2-dimensional sphere by collapsing $\tilde{\gamma}$ to one point. It is easily seen that F is a formal mating of $\tilde{f}_1$ and $\tilde{f}_2$, and f is a degeneration of F. □

Theorem 6.9. *Let f be an expanding postcritically finite branched covering. If J_f is orientedly S^1-parameterized, then f^n is equivalent to the degenerate mating of polynomials for some $n > 0$.*

Proof. Since there exists an oriented closed curve γ without self-intersection except in P_f which is fully f-invariant up to homotopy, by Theorem 6.8 we see that f^n is equivalent to a degeneration of a formal mating of topological polynomials for some n. Let $f_1 : \bar{D} \to \bar{D}$ and $f_2 : \bar{E} \to \bar{E}$ be branched coverings constructed in Theorem 6.8. We denote by T the union of the trees of self-intersection. Let $\pi : S^2 \to S^2$ be the projection that collapses each component of T into one point. For simplicity, we assume $n = 1$. Note that we can assume the following: Let $p \in P_f^r \cap \gamma$, and let $\alpha : [s, t] \to S^2$ be a part of γ such that $\alpha(s) = \alpha(t) = p$. Then α is not homotopic to a trivial curve relative to $\{s, t\}$ in $S^2 - (P_f - \{p\})$.

It is sufficient to show that f_1 and f_2 have no Levy cycle ([**3**] Definition 6.2). Indeed, a topological polynomial is equivalent to a polynomial if and only if there is no Levy cycle ([**3**] Fact 6.5).

Suppose that f_1 has a Levy cycle $\{\alpha_1, \alpha_2, \dots, \alpha_n = \alpha_0\}$, that is, there exists a component α'_{i-1} of $f_1^{-1}(\alpha_i)$ which is homotopic to α_{i-1} in $D - P_{f_1}$ such that $f_1 : \alpha'_{i-1} \to \alpha_i$ is one-to-one, and the simply connected domain C_i bounded by α_i includes at least two points of P_{f_1}. It is easy to see that $C_i \cap P_{f_1}$ consists of periodic points in $P_{f_1}^r$.

Let β_1 be an arc in C_1 joining two points of $C_1 \cap P_{f_1}$. Then there exists a component β_0 of $f_1^{-1}(\beta_1)$ such that β_0 joins two points of $C_n \cap P_{f_1}$ and $f_1 : \beta_0 \to \beta_1$ is one-to-one. Thus for $j = 0, -1, -2, \dots$, there exists an arc β_j such that β_j joins two points of $C_k \cap P_{f_1}$, $f_1^{1-j} : \beta_j \to \beta_1$ is one-to-one, where $k = j \mod n$.

Write $\beta'_i = \pi(\beta_i)$. If for every i, either $\beta'_i \cap P_f$ consists of more than one point or β'_i is not homotopic to a trivial curve with P_f fixed, then we have a contradiction for f is expanding.

Suppose that $\beta'_i \cap P_f$ is one point and β'_i is homotopic to a trivial curve with P_f fixed for some i. Let a and b be the endpoints of β_i. Then $\pi(a) = \pi(b) = p$ is a periodic point in P_f and β'_i is a closed curve such that a domain bounded by β'_i contains no point of P_f. Let S_1 and S_2 be the components of $T \cap D$ containing a and b respectively. As mentioned in the proof of Theorem 6.8, we have $S_1 \neq S_2$. Therefore there exists $\alpha : [s, t] \to S^2$ which is a part of γ such that $\alpha(s) = \alpha(t) = p$ and α is homotopic to a trivial curve with P_f fixed. This is a contradiction to the assumption. Thus f_1 has no Levy cycle. □

Example 6.10. (1) The rational map $f(z) = (z^2 - 2)/z^2$ is equivalent to the degenerate mating of $P(z) = z^2 - 2$ and $Q(z) = z^2 + c$, and is also equivalent to that of $P(z)$ and $\bar{Q}(z) = z^2 + \bar{c}$, where c is the root of $c^3 + 2c^2 + 2c + 2 = 0$ in the upper half plane. There are two resolutions because $f^{-1}(\gamma)$ has two parametrizations, where γ is the closed curve in Example 3.8. (2) The rational map $f(z) = (z^3 - 16/27)/z$ is equivalent to the formal mating of $P(z) = z^3 + 3z$ and $Q(z) = z^3 + (3/2)z^2$. We need not take the degenerate mating because the closed curve constructed in Theorem 5.7 has no self-intersection.

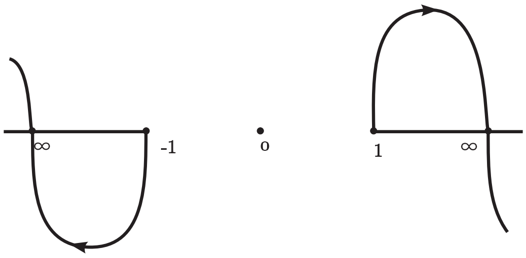

Figure 1.1. The oriented closed curve γ.

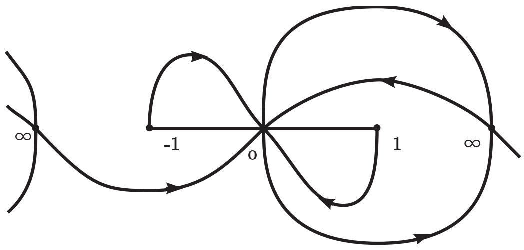

Figure 1.2. The inverse image of γ.

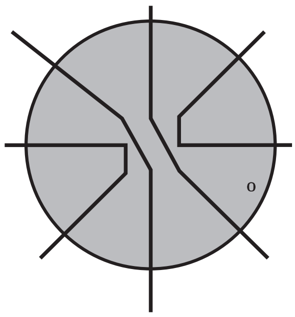

Figure 1.3. There are two ways of parametrization of $f^{-1}(\gamma)$ by which we obtain S^1-parametrization of the Julia sets. Here is the 'unlacing' of one parametrization near the origin.

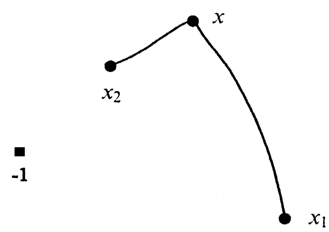

Figure 2.1. The radial r.

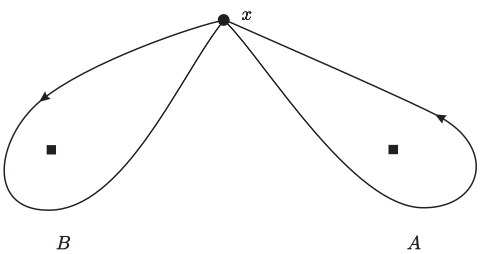

Figure 2.2. The generators A, B.

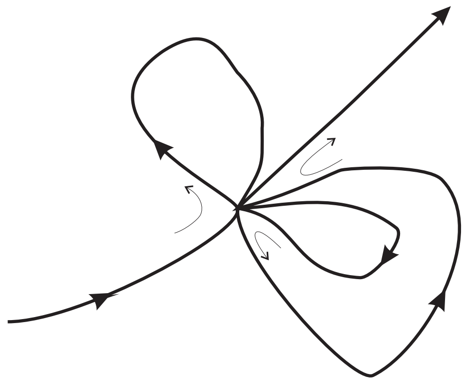

Figure 3.1. Here is a self-intersection point p.

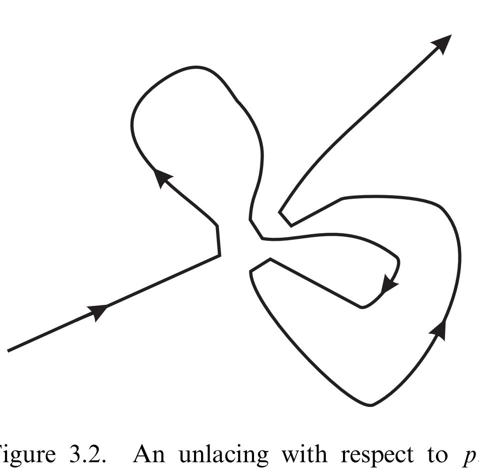

Figure 3.2. An unlacing with respect to p.

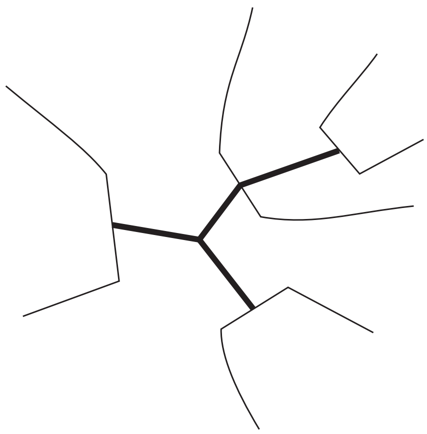

Figure 3.3. The thick tree is the tree of self-intersection at p.

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