J. Math. Soc. Japan
Vol. 55, No. 2, 2003

Domination of unbounded operators and commutativity

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(Received Oct. 24, 2001)

Abstract. It is proved that pointwise commuting formally normal operators which are dominated by a single essentially normal operator are essentially normal and essentially spectrally commuting. The question when essential normality of a polynomial in an operator implies essential normality of that operator is solved in this way. Furthermore, domination by essentially normal powers of formally normal operators are studied and, as a consequence, extended versions of Nelson's criterion for essential spectral commutativity are proposed. Subsequent domination results ensuring joint subnormality of systems of operators are proved. Several applications to multidimensional moment problems are found.

Introduction.

The problem whether a pair of essentially selfadjoint operators which pointwise commute may spectrally commute has had an interesting history. In the 1959 paper [**19**] Nelson constructed his famous example (it has to be mentioned that a prototype of this can be found in [**14**]) and gave sufficient conditions for the problem to be answered affirmatively. Since then there is a long-lasting demand for finding any kind of competitive conditions (and, if possible, to clarify the circumstances); see [**2**] and papers quoted therein. One of the possibilities is to work under the assumption of inclusions of domains of involved operators like in [**22**]. Another is to maintain the *domination* idea originated in [**19**]. In this paper we develop the domination approach with an extensive use of the technique of bounded vectors (in fact, bounded vectors, which is the simplest class of so called $\mathscr{C}^{\infty}$-vectors, play a substantial role for most of our presentation) and this makes our considerations as much independent of the spectral theorem as possible. Besides its simplicity, which also means it to be easily tested, this technique allows the operators in question to be filled up with bounded ones (cf. [**18**] and [**36**]). The main results of this part of the paper, Theorems 10 and 12, are inspired by [**24**].

As a step further we consider more subtle domination involving powers or polynomials in operators, the case not studied so far. This results in producing diverse forms of Nelson's theorem of polynomial type. Applying this to the multidimensional moment problem we come in particular to a substantially simplified version of [**25**, Theorem 2.7] avoiding any use of Hilbert's *Nullstellensatz* by the way.

It turns out that much of the results can be carried over to essentially normal operators as those which generalize essentially selfadjoint ones. However, though it

2000 *Mathematics Subject Classification*. Primary 47B15, 47B25, 44A60; Secondary 47B20, 43A35.
Key Words and Phrases. Selfadjoint operator, normal operator, spectral commutativity, symmetric operator, formally normal operator, subnormal operator, domination relation.
The research resulting in this paper was supported by the KBN grant #2 P03A 004 17.

may look like it, this is *not* doubling the problem for essentially selfadjoint operators. This is because it may happen [**23**] that

$$\overline{A + \mathrm{i}B} \subsetneq \bar{A} + \mathrm{i}\bar{B},$$

for two essentially selfadjoint operators A and B which pointwise commute on a common domain and which closures $\bar{A}$ and $\bar{B}$ spectrally commute (this means that while $\bar{A} + \mathrm{i}\bar{B}$ is normal $\overline{A + \mathrm{i}B}$ is exclusively formally normal). The main reason for engaging normality comes from our interest in subnormal operators, which, in turn, goes back to mathematical physics, cf. [**46**]. As a consequence, we establish relevant domination results for subnormal operators and get more applications to the complex moment problem.

Preliminaries.

Let $\mathscr{H}$ be a Hilbert space (we consider complex spaces exclusively). Denote by $\boldsymbol{B}(\mathscr{H})$ the $\mathscr{C}^*$-algebra of all bounded linear operators *on* $\mathscr{H}$ (all the operators are linear here). For a linear operator A *in* $\mathscr{H}$ denote by $\mathscr{D}(A)$ and $\mathscr{R}(A)$ its domain and range, respectively; A^* and $\bar{A}$ stand for the adjoint and the closure of A, respectively. The graph norm of A is denoted by $\|\cdot\|_A$, i.e. $\|f\|_A^2 = \|f\|^2 + \|Af\|^2$ for $f \in \mathscr{D}(A)$. Set $\mathscr{D}^\infty(A) = \bigcap_{n=1}^\infty \mathscr{D}(A^n)$ and

$$\mathscr{B}_a(A) = \{f \in \mathscr{D}^\infty(A) : \exists c > 0 \ \forall n \geq 0, \|A^n f\| \leq c a^n\}, \quad a \geq 0.$$

A vector $f \in \mathscr{D}^\infty(A)$ is said to be a *bounded vector*[1] of A (in short $f \in \mathscr{B}(A)$) if there exists $a > 0$ such that $f \in \mathscr{B}_a(A)$. We denote by $\mathscr{Q}(A)$ the set of all $f \in \mathscr{D}^\infty(A)$ such that $\sum_{n=1}^\infty \|A^n f\|^{-1/n} = +\infty$. Members of $\mathscr{Q}(A)$ are called *quasianalytic vectors* of A. It is clear that $\mathscr{B}(A) \subset \mathscr{Q}(A)$.

Recall that a linear subspace $\mathscr{E}$ of $\mathscr{D}(A)$ is said to be a *core* of A if the graph of A is contained in the closure of the graph of the restriction $A|_{\mathscr{E}}$ of A to $\mathscr{E}$. Here we need the notion of core of a finite system $\boldsymbol{A} = (A_1, \ldots, A_\kappa)$ of operators in $\mathscr{H}$. First, following [**15**], define the *graph* $\mathscr{G}(\boldsymbol{A})$ of $\boldsymbol{A}$ as

$$\mathscr{G}(\boldsymbol{A}) \stackrel{\mathrm{df}}{=} \{(f, A_1 f, \ldots, A_\kappa f); f \in \mathscr{D}(A_1) \cap \cdots \cap \mathscr{D}(A_\kappa)\}.$$

A linear subspace $\mathscr{E}$ of $\mathscr{D}(A_1) \cap \cdots \cap \mathscr{D}(A_\kappa)$ is said to be a *core* of $\boldsymbol{A} = (A_1, \ldots, A_\kappa)$ if $\mathscr{G}(A_1, \ldots, A_\kappa) \subset \overline{\mathscr{G}(A_1|_{\mathscr{E}}, \ldots, A_\kappa|_{\mathscr{E}})}$. If $\mathscr{E}$ is a core of $\boldsymbol{A}$, then $\overline{\mathscr{G}(A_1, \ldots, A_\kappa)} = \overline{\mathscr{G}(A_1|_{\mathscr{E}}, \ldots, A_\kappa|_{\mathscr{E}})}$. The following facts can be easily verified.

PROPOSITION 1. *Let $A_1, \ldots, A_\kappa$ be operators in $\mathscr{H}$ $(\kappa \geq 1)$.*

If $A_1, \ldots, A_\kappa$ are closable, then $\overline{\mathscr{G}(A_1, \ldots, A_\kappa)} \subset \mathscr{G}(\bar{A}_1, \ldots, \bar{A}_\kappa)$. If $A_1, \ldots, A_\kappa$ are closed, then $\mathscr{G}(A_1, \ldots, A_\kappa)$ is closed.

If $\mathscr{E}$ is a core of $(A_1, \ldots, A_\kappa)$ and $\mathscr{D}(A_1) \cap \cdots \cap \mathscr{D}(A_\kappa)$ is a core of A_i for a fixed $i \in \{1, \ldots, \kappa\}$, then $\mathscr{E}$ is a core of A_i. If $\mathscr{E} \subset \mathscr{D}(A_1) \cap \cdots \cap \mathscr{D}(A_\kappa)$ is a core of $(\bar{A}_1, \ldots, \bar{A}_\kappa)$, then $\mathscr{E}$ is a core of $(A_1, \ldots, A_\kappa)$.

[1] the term can be found in [**12**]

We say that two operators A and B in $\mathscr{H}$ *pointwise commute on* a linear subspace $\mathscr{E}$ of $\mathscr{H}$ if $\mathscr{E} \subset \mathscr{D}(AB) \cap \mathscr{D}(BA)$ and $ABf = BAf$ for every $f \in \mathscr{E}$; saying "pointwise commute" without "on ..." refers to $\mathscr{E} = \mathscr{D}(AB) \cap \mathscr{D}(BA)$. We state now a criterion for pointwise commutativity of closures of operators.

Proposition 2. *If A and B are closable operators in $\mathscr{H}$ which pointwise commute on a dense linear subspace $\mathscr{E}$ of $\mathscr{H}$, and $\mathscr{D}(A^*B^*) \cap \mathscr{D}(B^*A^*)$ is dense in $\mathscr{H}$, then $\bar{A}$ and $\bar{B}$ pointwise commute.*

Proof. Set $C = \bar{A}\bar{B} - \bar{B}\bar{A}$. Then $B^*A^* - A^*B^* \subset C^*$, which implies that C^* is densely defined. By the von Neumann theorem, C is closable. Since $Cf = 0$ for $f \in \mathscr{E}$, it must be $Cf = 0$ for $f \in \mathscr{D}(C)$, which completes the proof. $\square$

An operator A in $\mathscr{H}$ is said to be *paranormal* if $\|Af\|^2 \le \|f\| \cdot \|A^2 f\|$ for all $f \in \mathscr{D}(A^2)$ (cf. [**16**]). Paranormal operators need not be closable and the closures of paranormal operators need not be paranormal (cf. [**11**]). A densely defined operator A in $\mathscr{H}$ is said to be *formally normal* (resp. *hyponormal*) if $\mathscr{D}(A) \subset \mathscr{D}(A^*)$ and $\|A^*f\| = \|Af\|$ (resp. $\|A^*f\| \le \|Af\|$) for $f \in \mathscr{D}(A)$. If A is formally normal and $\mathscr{D}(A) = \mathscr{D}(A^*)$, then A is called *normal*. We say that a closable operator is *essentially normal* if its closure is normal. Formally normal (resp. hyponormal) operators are closable and their closures are formally normal (resp. hyponormal) as well. A densely defined operator A in $\mathscr{H}$ is said to be *subnormal* if there exist a Hilbert superspace $\mathscr{K}$ of $\mathscr{H}$ and a normal operator N in $\mathscr{K}$ such that $A \subset N$, i.e. $\mathscr{D}(A) \subset \mathscr{D}(N)$ and $Af = Nf$ for $f \in \mathscr{D}(A)$. Subnormal operators are hyponormal and hyponormal operators are paranormal.

Let $\mathscr{D}$ be a dense linear subspace of $\mathscr{H}$. Denote by $\boldsymbol{L}(\mathscr{D})$ the algebra of all operators in $\mathscr{H}$ with invariant domain $\mathscr{D}$. The symbol $I_{\mathscr{D}}$ stands for the identity operator on $\mathscr{D}$. Denote by $\boldsymbol{L}^{\#}(\mathscr{D})$ the $*$-algebra of all $A \in \boldsymbol{L}(\mathscr{D})$ for which there exists $A^{\#} \in \boldsymbol{L}(\mathscr{D})$ such that $\langle Af, g\rangle = \langle f, A^{\#}g\rangle$ for all $f, g \in \mathscr{D}$; the operator $A^{\#}$ is unique and the mapping $A \mapsto A^{\#}$ is an involution in $\boldsymbol{L}^{\#}(\mathscr{D})$. An operator $N \in \boldsymbol{L}^{\#}(\mathscr{D})$ is formally normal (in $\mathscr{H}$) if and only if $N^{\#}N = NN^{\#}$.

We now formulate some properties of a formally normal operator which are forced by a related normal one.

Proposition 3. *If N is a normal operator in $\mathscr{H}$ and A is a formally normal operator in $\mathscr{H}$ such that $N \subset A$, then $N = A$.*

Proof. Indeed, we have $\mathscr{D}(A) \subset \mathscr{D}(A^*) \subset \mathscr{D}(N^*) = \mathscr{D}(N) \subset \mathscr{D}(A)$. $\square$

Proposition 4. *Let A be a formally normal operator in $\mathscr{H}$ and let $\mathscr{E}$ be a dense linear subspace of $\mathscr{D}(A)$ such that $A|_{\mathscr{E}}$ is essentially normal. Then $\mathscr{E}$ is a core of A, $\bar{A}$ is normal, $\mathscr{D}(\bar{A}^n) = \mathscr{D}(A^{*n})$ for $n \ge 0$, $\mathscr{D}^\infty(\bar{A}) = \mathscr{D}^\infty(A^*)$ and A pointwise commutes on $\mathscr{D}^\infty(\bar{A})$ with A^*. Moreover[2], if E is the spectral measure of $\bar{A}$, then $\mathscr{B}_a(\bar{A}) = \mathscr{B}_a(A^*) = \mathscr{R}(E(\{z \in \boldsymbol{C}; |z| \le a\}))$ for $a \ge 0$, and $\mathscr{B}(\bar{A})$ is a core of $\bar{A}$.*

[2] cf. [**17**] for a prototype of this formula.

Proof. Since $\bar{A}$ is formally normal and $\overline{A|_{\mathscr{E}}} \subset \bar{A}$, Proposition 3 gives us $\overline{A|_{\mathscr{E}}} = \bar{A}$, so $\bar{A}$ is normal. Without loss of generality we can assume that A is closed. Since

$$A^n = \int_{\boldsymbol{C}} z^n E(\mathrm{d}z) \quad \text{and} \quad A^{*n} = \int_{\boldsymbol{C}} \bar{z}^n E(\mathrm{d}z) \quad \text{for } n \geq 0, \tag{1}$$

we get $\mathscr{D}(A^n) = \mathscr{D}(A^{*n})$ for $n \geq 0$ (and consequently $\mathscr{D}^\infty(A) = \mathscr{D}^\infty(A^*)$) as well as[3] $A^*Af = AA^*f$ for $f \in \mathscr{D}^\infty(A)$. By (1), $\mathscr{B}_a(A^*) = \mathscr{B}_a(A)$. If $f \in \mathscr{B}_a(A)$, then

$$\lim_{n\to\infty} \left(\int_{\boldsymbol{C}} |z|^{2n} \langle E(\mathrm{d}z)f, f\rangle \right)^{1/(2n)} = \lim_{n\to\infty} \|A^n(f)\|^{1/n} \leq a,$$

so $\langle E(\{z \in \boldsymbol{C}; |z| > a\})f, f\rangle = 0$ (cf. [**26**, page 73]), which in turn is equivalent to $f \in \mathscr{R}(E(\{z \in \boldsymbol{C}; |z| \leq a\}))$. The converse implication is easily seen to be true. Basing on the equality $\mathscr{B}(A) = \bigcup_{n=1}^\infty \mathscr{R}(E(\{z \in \boldsymbol{C}; |z| \leq n\}))$ one can show that $\mathscr{B}(\bar{A})$ is a core of $\bar{A}$, which completes the proof. □

We conclude this section with a criterion for essential normality.

Theorem 5. *Assume that N is a densely defined operator in $\mathscr{H}$ and $\mathscr{E}$ is a dense subspace of $\mathscr{H}$ such that*
(i) $\mathscr{E} \subset \mathscr{D}(N^*N) \cap \mathscr{D}(NN^*)$,
(ii) $N^*N|_{\mathscr{E}} = NN^*|_{\mathscr{E}}$,
(iii) *the operator $N^*N|_{\mathscr{E}}$ is essentially selfadjoint.*
Then N is closable, $\bar{N}$ is a normal operator and $\mathscr{E}$ is a core of N.

Proof. Since $\mathscr{E} \subset \mathscr{D}(N^*)$ and $\bar{\mathscr{E}} = \mathscr{H}$, the operator N is closable. Set $C = N^*N|_{\mathscr{E}}$. On account of (i), (ii) and (iii), $\bar{C}$ is a selfadjoint operator such that $\bar{C} \subset \bar{N}^*\bar{N}$ and $\bar{C} \subset \bar{N}\bar{N}^*$. Since the operators $\bar{N}^*\bar{N}$ and $\bar{N}\bar{N}^*$ are selfadjoint (cf. [**47**, Theorem 5.39]), we conclude that $\bar{N}^*\bar{N} = \bar{C} = \bar{N}\bar{N}^*$, which in turn implies that $\bar{N}$ is normal (cf. [**47**, Proposition, page 125]) and

$$\mathscr{E} \text{ is a core of } \bar{N}^*\bar{N}. \tag{2}$$

To show that $\mathscr{E}$ is a core of N, take $f \in \mathscr{D}(\bar{N}^*\bar{N})$. Then, by (2), there exists $\{f_n\}_{n=1}^\infty \subset \mathscr{E}$ such that $f = \lim_{n\to\infty} f_n$ and $\bar{N}^*\bar{N}f = \lim_{n\to\infty} \bar{N}^*\bar{N}f_n$. Since

$$\|\bar{N}(f - f_n)\|^2 = \langle \bar{N}^*\bar{N}(f - f_n), f - f_n\rangle, \quad n \geq 1,$$

we get $\lim_{n\to\infty} \bar{N}f_n = \bar{N}f$. This shows that $(\bar{N}|_{\mathscr{D}(\bar{N}^*\bar{N})})^- \subset (\bar{N}|_{\mathscr{E}})^- \subset \bar{N}$. However $\mathscr{D}(\bar{N}^*\bar{N})$ is a core of $\bar{N}$ (cf. [**47**, Theorem 5.39]), so $\mathscr{E}$ is a core of N. □

Domination.

Domination and pointwise commutativity.

Given two operators A and B in a Hilbert space $\mathscr{H}$ and a linear subspace $\mathscr{E}$ of $\mathscr{D}(A) \cap \mathscr{D}(B)$, we say that A *dominates* B *on* $\mathscr{E}$ (this notion is widely used in the

[3] This fact does not depend on the spectral theorem, because if C is a formally normal operator and $\mathscr{D}(C^*C) \cap \mathscr{D}(CC^*) = \mathscr{H}$, then $C^*Cf = CC^*f$ for $f \in \mathscr{D}(C^*C) \cap \mathscr{D}(CC^*)$.

perturbation theory and known there as A-boundedness of B on $\mathscr{E}$) if there exists $c > 0$ such that $\|Bf\| \le c(\|f\| + \|Af\|)$ for every $f \in \mathscr{E}$. In this section we discuss the influence of the domination relation on pointwise commutativity of two operators on the set of $\mathscr{C}^\infty$-vectors of the dominator. By the way, we obtain the relationship between bounded vectors of such operators (see also Lemma 38). We begin with a fact of a general nature which connects the domination with the inclusion of domains of operators in question. Its proof (based on the closed graph theorem) is left to the reader.

Proposition 6. *Assume A and B are closable operators in a Hilbert space $\mathscr{H}$.*

(i) *If $\mathscr{D}(\bar{A}) \subset \mathscr{D}(\bar{B})$, then there exists $c > 0$ such that $\|\bar{B}f\| \le c(\|f\| + \|\bar{A}f\|)$ for every $f \in \mathscr{D}(\bar{A})$.*

(ii) *If $\mathscr{D}(A) \subset \mathscr{D}(B)$ and there exists $c > 0$ such that*

$$\|Bf\| \le c(\|f\| + \|Af\|), \quad f \in \mathscr{D}(A),$$

then $\mathscr{D}(\bar{A}) \subset \mathscr{D}(\bar{B})$ and $\|\bar{B}f\| \le c(\|f\| + \|\bar{A}f\|)$ for $f \in \mathscr{D}(\bar{A})$.

The following theorem opens a series of *domination* results. It corresponds somehow to the commutative part of Lemma 5.2 in [**19**].

Theorem 7. *Let A be a formally normal operator in $\mathscr{H}$, B be a densely defined operator in $\mathscr{H}$ and $\mathscr{E}$ be a dense linear subspace of $\mathscr{H}$. Assume that*

(a) *$\mathscr{E} \subset \mathscr{D}(A) \cap \mathscr{D}(B) \cap \mathscr{D}(B^*)$ and $\langle Af, Bg\rangle = \langle B^*f, A^*g\rangle$ for $f, g \in \mathscr{E}$,*

(b) *$A|_{\mathscr{E}}$ is essentially normal in $\mathscr{H}$,*

(c) *there exists $c > 0$ such that $\|Bf\| \le c(\|f\| + \|Af\|)$ for $f \in \mathscr{E}$.*

Then B is closable and

(i) *$\mathscr{D}(\bar{A}^{n+1}) \subset \mathscr{D}(A^{*n}\bar{B}) \cap \mathscr{D}(\bar{B}A^{*n})$, $\bar{B}\mathscr{D}(\bar{A}^{n+1}) \subset \mathscr{D}(\bar{A}^n)$ and $\bar{B}A^{*n}f = A^{*n}\bar{B}f$ for $n \ge 0$ and $f \in \mathscr{D}(\bar{A}^{n+1})$,*

(ii) *$\mathscr{D}^\infty(\bar{A}) \subset \mathscr{D}(A^*) \cap \mathscr{D}(\bar{B})$, $A^*\mathscr{D}^\infty(\bar{A}) \subset \mathscr{D}^\infty(\bar{A})$ and $\bar{B}\mathscr{D}^\infty(\bar{A}) \subset \mathscr{D}^\infty(\bar{A})$,*

(iii) *$A^*\bar{B}f = \bar{B}A^*f$ for $f \in \mathscr{D}^\infty(\bar{A})$,*

(iv) *$\|\bar{B}f\| \le c(\|f\| + \|\bar{A}f\|)$ for $f \in \mathscr{D}(\bar{A})$,*

(v) *$\mathscr{B}(\bar{A}) \subset \mathscr{B}(\bar{B})$,*

(vi) *the operators $\bar{A}$, A^* and $\bar{B}$ leave the spaces $\mathscr{B}_a(\bar{A})$, $a \ge 0$, and $\mathscr{B}(\bar{A})$ invariant; $\bar{A}|_{\mathscr{B}_a(\bar{A})}, \bar{B}|_{\mathscr{B}_a(\bar{A})} \in \boldsymbol{B}(\mathscr{B}_a(\bar{A}))$ and $\bar{A}\bar{B}f = \bar{B}\bar{A}f$ for $f \in \mathscr{B}(\bar{A})$.*

Proof. B is closable because $\overline{\mathscr{D}(B^*)} = \mathscr{H}$. By (b), $\bar{A}$ is normal and $\bar{A} = \overline{A|_{\mathscr{E}}}$ (cf. Proposition 4). This and (c) lead (via Proposition 6) to $\mathscr{D}(\bar{A}) \subset \mathscr{D}(\bar{B})$ and (iv). Since $\mathscr{D}(\bar{A}^k) = \mathscr{D}(A^{*k})$, we get $A^{*k}\mathscr{D}(\bar{A}^{k+1}) \subset \mathscr{D}(A^*) = \mathscr{D}(\bar{A}) \subset \mathscr{D}(\bar{B})$, so

$$\mathscr{D}(\bar{A}^{k+1}) \subset \mathscr{D}(\bar{B}A^{*k}), \quad k \ge 0. \tag{3}$$

We now show that

$$\langle Af, \bar{B}g\rangle = \langle B^*f, A^*g\rangle, \quad f \in \mathscr{E}, g \in \mathscr{D}(\bar{A}). \tag{4}$$

Indeed, since $\mathscr{E}$ is a core of A, there exists a sequence $\{g_n\}_{n=1}^\infty \subset \mathscr{E}$ such that $g_n \to g$ and $Ag_n \to \bar{A}g$ as $n \to \infty$. By (c) and formal normality of A we have $Bg_n \to \bar{B}g$ and $A^*g_n \to A^*g$ as $n \to \infty$. This and (a) imply the equality (4).

The condition (i) will be proved by induction. The case $n = 0$ is done. Assume (i) holds for a fixed $n \geq 0$. Let $f \in \mathscr{D}(\bar{A}^{n+2})$. Then clearly $f \in \mathscr{D}(\bar{A}^{n+1})$ and consequently, by virtue of the induction assumption, (4) and (3), we obtain

$$\langle A^{*n}\bar{B}f, Ah\rangle = \langle \bar{B}A^{*n}f, Ah\rangle = \langle A^{*(n+1)}f, B^*h\rangle = \langle \bar{B}A^{*(n+1)}f, h\rangle, \quad h \in \mathscr{E}.$$

Since $\mathscr{E}$ is a core of A, we get $A^{*n}\bar{B}f \in \mathscr{D}(A^*)$ (equivalently: $f \in \mathscr{D}(A^{*(n+1)}\bar{B})$) and $\bar{B}A^{*(n+1)}f = A^{*(n+1)}\bar{B}f$. In particular $\bar{B}f \in \mathscr{D}(A^{*(n+1)}) = \mathscr{D}(\bar{A}^{n+1})$.

(vi) According to (ii) and Proposition 4, $\mathscr{B}_a(\bar{A})$ is closed and $\mathscr{B}_a(\bar{A}) \subset \mathscr{D}(\bar{B})$. Hence, by the closed graph theorem, the operator $\bar{B}|_{\mathscr{B}_a(\bar{A})} : \mathscr{B}_a(\bar{A}) \to \mathscr{H}$ is bounded. Set $\varphi(a) = \|\bar{B}|_{\mathscr{B}_a(\bar{A})}\|$. If $f \in \mathscr{B}_a(\bar{A})$, then (ii) and (iii) lead to

$$\|\bar{A}^n\bar{B}f\| = \|A^{*n}\bar{B}f\| = \|\bar{B}A^{*n}f\| \leq \varphi(a)\|A^{*n}f\| = \varphi(a)\|\bar{A}^nf\|, \quad n \geq 0,$$

so $\bar{B}f \in \mathscr{B}_a(\bar{A})$. In consequence, $\bar{B}|_{\mathscr{B}_a(\bar{A})}$ belongs to $\boldsymbol{B}(\mathscr{B}_a(\bar{A}))$. From (iii) we get $(\bar{A}|_{\mathscr{B}_a(\bar{A})})^*\bar{B}|_{\mathscr{B}_a(\bar{A})} = \bar{B}|_{\mathscr{B}_a(\bar{A})}(\bar{A}|_{\mathscr{B}_a(\bar{A})})^*$. Since $\bar{A}|_{\mathscr{B}_a(\bar{A})}$ is normal, the Fuglede theorem yields $\bar{A}\bar{B}|_{\mathscr{B}_a(\bar{A})} = \bar{B}\bar{A}|_{\mathscr{B}_a(\bar{A})}$ $(a \geq 0)$, which completes the proof of (vi).

(v) comes from (vi) immediately. $\square$

It is worthwhile to point out that if A is a formally normal operator in $\mathscr{H}$, B is a densely defined operator in $\mathscr{H}$ and $\mathscr{E}$ is a dense linear subspace of $\mathscr{H}$ such that $\mathscr{E} \subset \mathscr{D}(B) \cap \mathscr{D}(AB^*) \cap \mathscr{D}(B^*A)$ and $AB^*f = B^*Af$ for $f \in \mathscr{E}$, then A and B satisfy the condition (a) of Theorem 7.

The following is a direct consequence of Proposition 6 and Theorem 7.

COROLLARY 8. *If A is a normal operator in $\mathscr{H}$, B is a closed densely defined operator in $\mathscr{H}$ and $\mathscr{E}$ is a dense linear subspace of $\mathscr{H}$ such that*

1° $\mathscr{D}(A) \subset \mathscr{D}(B)$,

2° $\mathscr{E} \subset \mathscr{D}(A) \cap \mathscr{D}(B^*)$ *and* $\langle Af, Bg\rangle = \langle B^*f, A^*g\rangle$ *for* $f, g \in \mathscr{E}$,

3° $\mathscr{E}$ *is a core of* A,

then the conditions (i), (ii), (iii), (iv), (v) *and* (vi) *of Theorem* 7 *hold.*

LEMMA 9. *Let A be formally normal and B be hyponormal, both in $\mathscr{H}$, and let $\mathscr{E}$ be a dense linear subspace of $\mathscr{H}$ such that $\mathscr{E} \subset \mathscr{D}(A) \cap \mathscr{D}(B)$. Suppose that the operators A and B satisfy conditions* (b) *and* (c) *of Theorem* 7. *If moreover one of the following two conditions holds*

(i) $\langle Af, B^*g\rangle = \langle Bf, A^*g\rangle$ *for* $f, g \in \mathscr{E}$,

(ii) $\langle Af, Bg\rangle = \langle B^*f, A^*g\rangle$ *for* $f, g \in \mathscr{E}$,

then $\mathscr{D}(\bar{A}) \subset \mathscr{D}(\bar{B})$, $\mathscr{B}(\bar{A}) \subset \mathscr{B}(\bar{B})$, the operators $\bar{A}$, A^*, $\bar{B}$ and B^* leave the spaces $\mathscr{D}^\infty(\bar{A})$ and $\mathscr{B}(\bar{A})$ invariant, $\mathscr{B}_a(\bar{A})$ reduces $\bar{B}$ to a bounded hyponormal operator $(a \geq 0)$, $\bar{A}\bar{B}f = \bar{B}\bar{A}f$ and $A^*\bar{B}f = \bar{B}A^*f$ for $f \in \mathscr{D}^\infty(\bar{A})$.

PROOF. It follows from (b) and (c), as in the proof of Theorem 7, that $\mathscr{D}(\bar{A}) \subset \mathscr{D}(\bar{B})$ and (because $\bar{B}$ is hyponormal)

$$\|B^*f\| \leq \|\bar{B}f\| \leq c(\|f\| + \|\bar{A}f\|), \quad f \in \mathscr{D}(\bar{A}). \tag{5}$$

Assume (i). By (5), we can apply Theorem 7 to the triplet $(A, B^*, \mathscr{E})$. Hence the operators $\bar{A}$, A^* and B^* leave the spaces $\mathscr{D}^\infty(\bar{A})$, $\mathscr{B}(\bar{A})$ and $\mathscr{B}_a(\bar{A})$ invariant,

(6) $$A^*B^*f = B^*A^*f, \quad f \in \mathscr{D}^\infty(\bar{A})$$

and $\bar{A}B^*f = B^*\bar{A}f$ for $f \in \mathscr{B}(\bar{A})$. The latter implies that $\langle \bar{A}f, \bar{B}g\rangle = \langle B^*f, A^*g\rangle$ for $f, g \in \mathscr{B}(\bar{A})$. This, Proposition 4 and (5) enable us to apply Theorem 7 to the triplet $(\bar{A}, \bar{B}, \mathscr{B}(\bar{A}))$. In consequence, $\mathscr{B}(\bar{A}) \subset \mathscr{B}(\bar{B})$, the operator $\bar{B}$ leaves $\mathscr{D}^\infty(\bar{A})$, $\mathscr{B}(\bar{A})$ and $\mathscr{B}_a(\bar{A})$ invariant, and $A^*\bar{B}f = \bar{B}A^*f$ for $f \in \mathscr{D}^\infty(\bar{A})$. Since $\bar{B}(\mathscr{D}^\infty(\bar{A})) \subset \mathscr{D}^\infty(\bar{A})$, (6) leads to $\bar{A}\bar{B}f = \bar{B}\bar{A}f$ for $f \in \mathscr{D}^\infty(\bar{A})$. As $\mathscr{B}_a(\bar{A})$ is invariant for $\bar{B}$ and B^*, $\mathscr{B}_a(\bar{A})$ reduces $\bar{B}$ to a bounded hyponormal operator.

The same arguments come into force in the case (ii) (first we apply Theorem 7 to $(A, B, \mathscr{E})$ and then to $(\bar{A}, B^*, \mathscr{B}(\bar{A}))$). The details are left to the reader. □

It is right time to relate our results to those of [**22**]. Notice that some parts of Lemma 9 and Theorems 10 and 12 can be derived from what is in [**22**]. However, we have made our proofs independent of [**22**] because we want them to be consistent with the main theme of the paper: *domination*. Let us mention that we can retrieve in this way information on cores of operators in question. Another favouring circumstance is that it is just domination what can be explicitly assumed in the case of moment problems.

Domination and spectral commutativity.

Recall that normal operators in a Hilbert space $\mathscr{H}$ are said to *spectrally commute* if their spectral measures commute. We now formulate the main criterion for the spectral commutativity of normal operators one of which dominates others. Its proof is based on Lemma 9.

Theorem 10. *Assume that $A_0, A_1, \ldots, A_\kappa$ $(\kappa \geq 1)$ are formally normal operators in $\mathscr{H}$ and $\mathscr{E}_{i,j}$, $0 \leq i < j \leq \kappa$, are dense linear subspaces of $\mathscr{H}$ such that*

(i) *$\mathscr{E}_{0,j} \subset \mathscr{D}(A_0) \cap \mathscr{D}(A_j)$ and either $\langle A_0 f, A_j^* g\rangle = \langle A_j f, A_0^* g\rangle$, $f, g \in \mathscr{E}_{0,j}$, or $\langle A_0 f, A_j g\rangle = \langle A_j^* f, A_0^* g\rangle$, $f, g \in \mathscr{E}_{0,j}$, for $j = 1, \ldots, \kappa$;*

(ii) *either $\mathscr{E}_{i,j} \subset \mathscr{D}(A_iA_j) \cap \mathscr{D}(A_jA_i)$ and $A_iA_jf = A_jA_if$, $f \in \mathscr{E}_{i,j}$, or $\mathscr{E}_{i,j} \subset \mathscr{D}(A_iA_j^*) \cap \mathscr{D}(A_j^*A_i)$ and $A_iA_j^*f = A_j^*A_if$, $f \in \mathscr{E}_{i,j}$, for $1 \leq i < j \leq \kappa$;*

(iii) *$A_0|_{\mathscr{E}_{0,j}}$ is essentially normal for $j = 1, \ldots, \kappa$;*

(iv) *there is $c > 0$ such that $\|A_j f\| \leq c(\|f\| + \|A_0 f\|)$ for $f \in \mathscr{E}_{0,j}$ and $j = 1, \ldots, \kappa$.*

Then $\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_\kappa$ are spectrally commuting normal operators. Moreover, if $\mathscr{E} = \mathscr{E}_{0,j}$ for all $j = 1, \ldots, \kappa$, then $\mathscr{E}$ is a core of any subsystem of $\{\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_\kappa\}$.

Proof. Set $\mathscr{H}_1 = \mathscr{B}_1(\bar{A}_0)$ and $\mathscr{H}_n = \mathscr{B}_n(\bar{A}_0) \ominus \mathscr{B}_{n-1}(\bar{A}_0)$ for $n \geq 2$. By Proposition 4 and Lemma 9, $\mathscr{H} = \bigoplus_{n=1}^\infty \mathscr{H}_n$, the closed linear space $\mathscr{H}_n$ reduces $\bar{A}_i$ to a bounded normal operator $N_{i,n} \in \boldsymbol{B}(\mathscr{H}_n)$ and $N_{0,n}N_{i,n} = N_{i,n}N_{0,n}$ for all $n \geq 1$ and $i = 0, \ldots, \kappa$. If $1 \leq i < j \leq \kappa$ and $A_iA_jf = A_jA_if$ for $f \in \mathscr{E}_{i,j}$, then $N_{i,n}N_{j,n} = N_{j,n}N_{i,n}$ for $n \geq 1$ (use Proposition 2). Likewise, if $A_iA_j^*f = A_j^*A_if$ for $f \in \mathscr{E}_{i,j}$, then $N_{i,n}N_{j,n}^* = N_{j,n}^*N_{i,n}$, and consequently, by the Fuglede theorem, $N_{i,n}N_{j,n} = N_{j,n}N_{i,n}$. Hence, the normal operators $\bigoplus_{n=1}^\infty N_{i,n}$, $i = 0, \ldots, \kappa$, spectrally commute (because spectral measures of appropriate summands commute). Since $\bigoplus_{n=1}^\infty N_{i,n} \subset \bar{A}_i$, Proposition 3 yields $\bar{A}_i = \bigoplus_{n=1}^\infty N_{i,n}$, so $\bar{A}_0, \ldots, \bar{A}_\kappa$ are spectrally commuting normal operators.

Suppose that $\mathscr{E}_{0,j} = \mathscr{E}$ for all $j = 1, \ldots, \kappa$. Take a finite sequence of integers $0 \leq \nu_1 < \cdots < \nu_m \leq \kappa$ $(1 \leq m \leq \kappa + 1)$. Let E be the spectral measure of the system

$(\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_\kappa)$, i.e. $\bar{A}_j = \int_{\boldsymbol{C}^{\kappa+1}} z_j E(\mathrm{d}z_0, \mathrm{d}z_1, \ldots, \mathrm{d}z_\kappa)$ for $j = 0, 1, \ldots, \kappa$ (cf. [**8**]). Set $\Delta_n = \bigcap_{j=0}^{\kappa} \{(z_0, z_1, \ldots, z_\kappa) \in \boldsymbol{C}^{\kappa+1} : |z_j| \le n\}$, $n \ge 1$, and define a dense linear subspace $\mathscr{X}$ of $\mathscr{H}$ via $\mathscr{X} = \bigcup_{n=1}^{\infty} \mathscr{R}(E(\Delta_n))$. By Lemma 9, $\mathscr{X} \subset \mathscr{D}(\bar{A}_0) \cap \cdots \cap \mathscr{D}(\bar{A}_\kappa) = \mathscr{D}(\bar{A}_0)$. If $f \in \mathscr{D}(\bar{A}_{\nu_1}) \cap \cdots \cap \mathscr{D}(\bar{A}_{\nu_m})$, then clearly $\mathscr{X} \ni E(\Delta_n)f \to f$ and $\bar{A}_{\nu_j} E(\Delta_n) f \to \bar{A}_{\nu_j} f$ as $n \to \infty$ for $j = 1, \ldots, m$. This implies

$$\mathscr{G}(\bar{A}_{\nu_1}, \ldots, \bar{A}_{\nu_m}) \subset \overline{\mathscr{G}(\bar{A}_{\nu_1}|_{\mathscr{X}}, \ldots, \bar{A}_{\nu_m}|_{\mathscr{X}})}. \tag{7}$$

Take $h \in \mathscr{X}$. Since, by (iii), $\mathscr{E}$ is a core of A_0, there exists a sequence $\{h_n\}_{n=1}^{\infty} \subset \mathscr{E}$ such that $h = \lim_{n\to\infty} h_n$ and $\bar{A}_0 h = \lim_{n\to\infty} A_0 h_n$. This and (iv) imply that $\bar{A}_{\nu_j} h = \lim_{n\to\infty} A_{\nu_j} h_n$ for $j = 1, \ldots, m$. In this way we have proved that

$$\mathscr{G}(\bar{A}_{\nu_1}|_{\mathscr{X}}, \ldots, \bar{A}_{\nu_m}|_{\mathscr{X}}) \subset \overline{\mathscr{G}(\bar{A}_{\nu_1}|_{\mathscr{E}}, \ldots, \bar{A}_{\nu_m}|_{\mathscr{E}})}. \tag{8}$$

Combining (7) and (8) we get $\mathscr{G}(\bar{A}_{\nu_1}, \ldots, \bar{A}_{\nu_m}) \subset \overline{\mathscr{G}(\bar{A}_{\nu_1}|_{\mathscr{E}}, \ldots, \bar{A}_{\nu_m}|_{\mathscr{E}})}$, which completes the proof. □

REMARK 11. In fact we have proved the following fact (see the second part of the proof of Theorem 10 and apply Proposition 6): *if $A_0, A_1, \ldots, A_\kappa$ are spectrally commuting normal operators such that $\mathscr{D}(A_0) \subset \mathscr{D}(A_i)$ for all $i = 1, \ldots, \kappa$, and $\mathscr{E}$ is a core of A_0, then $\mathscr{E}$ is a core of any subsystem of $\{A_0, A_1, \ldots, A_\kappa\}$.*

The next result is a direct consequence of Proposition 6 and Theorem 10.

THEOREM 12. *Let A_0 be a normal operator in $\mathscr{H}$, $A_1, \ldots, A_\kappa$ be closed formally normal operators in $\mathscr{H}$ and $\mathscr{E}_{i,j}$, $0 \le i < j \le \kappa$, be dense linear subspaces of $\mathscr{H}$. Assume that*

(i) *$\mathscr{E}_{0,j} \subset \mathscr{D}(A_0) \subset \mathscr{D}(A_j)$ and either $\langle A_0 f, A_j^* g\rangle = \langle A_j f, A_0^* g\rangle$, $f, g \in \mathscr{E}_{0,j}$, or $\langle A_0 f, A_j g\rangle = \langle A_j^* f, A_0^* g\rangle$, $f, g \in \mathscr{E}_{0,j}$, for $j = 1, \ldots, \kappa$;*

(ii) *either $\mathscr{E}_{i,j} \subset \mathscr{D}(A_i A_j) \cap \mathscr{D}(A_j A_i)$ and $A_i A_j f = A_j A_i f$, $f \in \mathscr{E}_{i,j}$, or $\mathscr{E}_{i,j} \subset \mathscr{D}(A_i A_j^*) \cap \mathscr{D}(A_j^* A_i)$ and $A_i A_j^* f = A_j^* A_i f$, $f \in \mathscr{E}_{i,j}$, for $1 \le i < j \le \kappa$;*

(iii) *$\mathscr{E}_{0,j}$ is a core of A_0 for $j = 1, \ldots, \kappa$.*

Then $A_0, A_1, \ldots, A_\kappa$ are spectrally commuting normal operators. Moreover, if $\mathscr{E} = \mathscr{E}_{0,j}$ for all $j = 1, \ldots, \kappa$, then $\mathscr{E}$ is a core of any subsystem of $\{A_0, A_1, \ldots, A_\kappa\}$.

The particular case of Theorem 12 for symmetric operators is stronger than Lemma 2 of [**24**] and Proposition 2 of [**31**] (which, in turn, generalizes the result of [**24**]). It differs from those results by having assumed here weak commutativity instead of pointwise one as well as by allowing varying subspaces $\mathscr{E}_{i,j}$ to constitute a core exclusively for $i = 0$ (as compared with [**31**]) and by having gained the core conclusion.

REMARK 13. In addition to the circumstances of Theorem 10 (and, in a sense, of Theorem 12) the fact that the closed support of the spectral measure E of the system $(\bar{A}_0, \ldots, \bar{A}_\kappa)$ of spectrally commuting normal operators is contained in the semianalytic set $\bigcap_{j=1}^{\kappa} \mathscr{Z}_j$, where $\mathscr{Z}_j = \{(z_0, \ldots, z_\kappa) \in \boldsymbol{C}^{\kappa+1}; |z_j|^2 \le b(1 + |z_0|^2)\}$, is equivalent to the domination condition (iv) appearing therein (the smallest nonnegative constants b and c are related to each other as follows: $c^2 \le b \le 2c^2$). Indeed, we see that: $\|\bar{A}_j f\|^2 \le b(\|f\|^2 + \|\bar{A}_0 f\|^2)$, $f \in \mathscr{D}(\bar{A}_0)$, if and only if $\int_{\boldsymbol{C}^{\kappa+1}} (b(1 + |z_0|^2) - |z_j|^2)\langle E(\mathrm{d}z) f, f\rangle \ge 0$,

$f \in \mathscr{D}(\bar{A}_0)$, if and only if (as $E\bar{A}_0 \subset \bar{A}_0 E$) $\int_\sigma (b(1+|z_0|^2) - |z_j|^2)\langle E(\mathrm{d}z)f, f\rangle \geq 0$, σ a Borel subset of $\boldsymbol{C}^{\kappa+1}$ and $f \in \mathscr{D}(\bar{A}_0)$, if and only if $\langle E(\boldsymbol{C}^{\kappa+1}\backslash\mathscr{Z}_j)f, f\rangle = 0$, $f \in \mathscr{D}(\bar{A}_0)$, if and only if $E(\boldsymbol{C}^{\kappa+1}\backslash\mathscr{Z}_j) = 0$. Roughly speaking, Theorem 10 guarantees essential spectral commutativity of a finite system of (formally) normal operators in case it is dominated by some (essentially) normal operator. Conversely, if $A_1, \ldots, A_\kappa$ are spectrally commuting normal operators, then the operator $A_0 \stackrel{\mathrm{df}}{=} A_1^* A_1 + \cdots + A_\kappa^* A_\kappa$ is selfadjoint (because $A_0 = \int_{\boldsymbol{C}^\kappa} (|z_1|^2 + \cdots + |z_\kappa|^2) E(\mathrm{d}z)$, where E is the spectral measure of the system $(A_1, \ldots, A_\kappa)$), the operators $A_0, A_1, \ldots, A_\kappa$ spectrally commute and finally A_0 dominates every A_i on $\mathscr{D}(A_0)$, $i = 1, \ldots, \kappa$ (because $\mathscr{D}(A_0) \subset \mathscr{D}(A_i)$).

Remark 14. The conditions (i) and (ii) of Theorem 10 are apparently necessary for the operators $\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_\kappa$ to be spectrally commuting. Indeed, the spaces $\mathscr{E}_{0,j} = \mathscr{D}(A_0) \cap \mathscr{D}(A_j)$ and $\mathscr{E}_{i,j} = \mathscr{D}(A_i A_j) \cap \mathscr{D}(A_j A_i)$ (resp. $\mathscr{E}_{i,j} = \mathscr{D}(A_i A_j^*) \cap \mathscr{D}(A_j^* A_i)$), $i, j \geq 1$ being fixed, are the largest spaces satisfying conditions (i) and (ii) (use the spectral measure of the system $(\bar{A}_0, \ldots, \bar{A}_\kappa)$ mimicking a part of the proof of Theorem 10). It turns out that none of the remaining conditions (iii) and (iv) of Theorem 10 can be omitted without spoiling its conclusion. Let us discuss the case $\kappa = 2$ in detail.

In [**19**] two pointwise commuting symmetric operators $A_0, A_1 \in \boldsymbol{L}(\mathscr{D})$, acting in a Hilbert space $\mathscr{H}$, whose closures $\bar{A}_0, \bar{A}_1$ are selfadjoint but not spectrally commuting are constructed. Thus the operators A_0, A_1 satisfy all the assumptions of Theorem 10 (with $\mathscr{E}_{0,1} = \mathscr{D}$) except for (iv). As is shown in [**42**, Section 16], the pair (A_0, A_1) does not extend to any pair of spectrally commuting selfadjoint operators even in a larger Hilbert space.

Let $A \in \boldsymbol{L}^\#(\mathscr{D})$ be a formally normal operator, which is not subnormal (cf. [**10**], [**30**]; it is possible to choose A to be $*$-cyclic, see [**33**]). Set $A_i = A$ for $i = 0, 1$. Then the formally normal operators A_0 and A_1 satisfy all the assumptions of Theorem 10 except (iii). In this particular case there is no pair of spectrally commuting normal operators (even in a larger Hilbert space) which extends (A_0, A_1). Another possibility for this is to consider $A_0 = A$ and $A_1 = (1/2)(A + A^\#)$; the conclusion is as before, however now A_1 is symmetric. Below we present a more advanced example.

Example 15. By [**33**, Proposition 7.3] there exist pointwise commuting symmetric operators $A, B \in \boldsymbol{L}(\mathscr{D})$, acting in a Hilbert space $\mathscr{H}$, such that

(A1) the pair (A, B) does not extend to any pair of spectrally commuting selfadjoint operators in a larger Hilbert space,

(A2) $B(B - A^2) = 0$.

Set $A_0 = A^2$ and $A_1 = B$. Then $A_0 A_1 = A_1 A_0$ and, according to (A2), we have

$$\|A_1 f\|^2 = \langle A_0 f, A_1 f\rangle \leq \|A_0 f\| \, \|A_1 f\|, \quad f \in \mathscr{D},$$

which implies that $\|A_1 f\| \leq \|A_0 f\| \leq \|f\| + \|A_0 f\|$ for $f \in \mathscr{D}$. This means that the operators A_0, A_1 satisfy all the assumptions of Theorem 10 (with $\mathscr{E}_{0,1} = \mathscr{D}$) except for (iii). Indeed, suppose contrary to our claim that $A^2 = A_0$ is essentially selfadjoint. Then, by Theorem 24, $\bar{A}$ and $\bar{B}$ are spectrally commuting selfadjoint operators, which contradicts (A1). Notice however that the pair (A_0, A_1) extends to a pair of spectrally commuting selfadjoint operators in a larger Hilbert space; this is because $A_1(A_1 - A_0) = 0$ (cf. [**9**, Theorem 3]; see also [**33**, Proposition 5.3]).

Domination in higher powers.

Polynomially essentially normal operators.

In this section we try to answer the following question: when does (essential) normality of a complex polynomial in an operator imply essential normality of the operator in question? Notice that a very special case of Theorem 16 below (namely saying that if A is a closed symmetric operator and A^2 is selfadjoint, then A is itself selfadjoint) has been used in the proof of [**21**, Theorem 2].

The symbol $\boldsymbol{C}[X_1, \dots, X_\kappa]$ stands as usual for the ring of all polynomials in κ commuting formal variables $X_1, \dots, X_\kappa$ with complex coefficients; in the case $\kappa = 1$ we write simply $\boldsymbol{C}[X]$. For $p \in \boldsymbol{C}[X]$, we define $p^* \in \boldsymbol{C}[X]$ via $p^*(z) = \overline{p(\bar{z})}$, $z \in \boldsymbol{C}$.

Theorem 16. *If A is a formally normal operator in $\mathscr{H}$ such that $p(A)$ is normal for some polynomial $p \in \boldsymbol{C}[X]$ of degree $n \geq 1$, then for every $q \in \boldsymbol{C}[X]$ with $\deg q \leq n$, the operator $q(A)$ is essentially normal, $\mathscr{D}^\infty(A)$ is a core of $q(A)$, $\overline{q(A)} = q(\bar{A})$ and $q(A)^* = q^*(A^*)$. Moreover, the operator A^n is normal.*

Proof. First we prove that $\bar{A}$ is normal. Using an induction argument one can show that $\mathscr{D}^\infty(p(A)) \subset \mathscr{D}(A^{kn})$ for every $k \geq 1$, so $\mathscr{D}^\infty(p(A)) = \mathscr{D}^\infty(A)$ (this is true for any *linear* operator A). By Proposition 4, the linear space $\mathscr{E}_{0,1} \overset{\mathrm{df}}{=} \mathscr{D}^\infty(A)$ is a core of $A_0 \overset{\mathrm{df}}{=} p(A)$. Since $\mathscr{D}(p(A)) \subset \mathscr{D}(\bar{A})$, the triplet $(A_0, A_1, \mathscr{E}_{0,1})$, where $A_1 \overset{\mathrm{df}}{=} \bar{A}$, satisfies all the assumptions of Theorem 12. Hence $\bar{A}$ is normal.

Take a polynomial $q \in \boldsymbol{C}[X]$ with $\deg q \leq n$. Since $p(A) = p(\bar{A})$ (use Proposition 3), we see that $p(A)$ and $q(\bar{A})$ are spectrally commuting normal operators such that $\mathscr{D}(p(A)) \subset \mathscr{D}(q(\bar{A}))$. Applying Remark 11 to the triplet $(A_0, A_1', \mathscr{E}_{0,1})$ with $A_1' \overset{\mathrm{df}}{=} q(\bar{A})$, we conclude that $\mathscr{D}^\infty(A)$ is a core of $q(\bar{A})$. Thus $q(\bar{A}) = \overline{q(A)|_{\mathscr{D}^\infty(A)}} \subset \overline{q(A)} \subset q(\bar{A})$, which means that the operator $\overline{q(A)} = q(\bar{A})$ is normal and $\mathscr{D}^\infty(A)$ is a core of $q(A)$. Since $q^*(A^*) \subset q(A)^* = (\overline{q(A)})^* = q(\bar{A})^*$ and all these operators are normal, Proposition 3 gives us $q(A)^* = q^*(A^*)$.

In view of the previous part of the proof, it suffices to show that if B is a closable operator in $\mathscr{H}$ such that $\bar{B}$ is paranormal and $p(B)$ is closed, then B^n is closed (this is a more general result than we need). Suppose that sequences $\{f_k\}_{k=1}^\infty \subset \mathscr{D}(B^n)$ and $\{B^n f_k\}_{k=1}^\infty$ are convergent to vectors f and g, respectively. Since the graph norm of B^n is equivalent to the norm $(\sum_{i=0}^n \|B^i(\cdot)\|^2)^{1/2}$ (cf. [**41**, Proposition 6]), the sequence $\{p(B)f_k\}_{k=1}^\infty$ is convergent. Thus, by the closedness of $p(B)$, $f \in \mathscr{D}(p(B)) = \mathscr{D}(B^n)$. However the operator $\bar{B}^n$ is closed as the nth power of the paranormal operator $\bar{B}$ (cf. [**41**, Proposition 6]). Hence $g = \bar{B}^n f = B^n f$, which shows that B^n is closed. The proof is complete. $\square$

Even for bounded operators Theorem 16 is false if A itself is not (formally) normal: take the rank one operator $A = e \otimes f$ with $e, f \in \mathscr{H}$ such that $\langle e, f \rangle = 1$ and $\|e\| \, \|f\| \neq 1$; then A is not normal though $A^2 - A = 0$.

Lemma 17. *If A is a formally normal operator in $\mathscr{H}$ such that $p(A)$ is essentially normal, $p(\bar{A})$ is closed and $\mathscr{D}(\bar{A}^n) \subset \mathscr{D}(p(\bar{A})^*)$ for some polynomial $p \in \boldsymbol{C}[X]$ of degree $n \geq 1$, then for every $q \in \boldsymbol{C}[X]$ with $\deg q \leq n$, the operator $q(A)$ is essentially normal, $\mathscr{D}(A^n)$ is a core of $q(A)$, $\overline{q(A)} = q(\bar{A})$ and $q(A)^* = q^*(A^*)$.*

Proof. Notice first that according to $\overline{p(A)} \subset p(\bar{A})$, we have

$$\mathscr{D}(p(\bar{A})) = \mathscr{D}(\bar{A}^n) \subset \mathscr{D}(p(\bar{A})^*) \subset \mathscr{D}(p(A)^*) = \mathscr{D}(\overline{p(A)}),$$

so $p(\bar{A}) = \overline{p(A)}$ and consequently $p(\bar{A})$ is normal. By Theorem 16, the operator $\bar{A}$ is normal as well. Let $q \in \boldsymbol{C}[X]$ be a polynomial of $\deg q \le n$. Since $A_0 \stackrel{\mathrm{df}}{=} p(\bar{A})$ and $A_1 \stackrel{\mathrm{df}}{=} q(\bar{A})$ are spectrally commuting normal operators such that $\mathscr{D}(p(\bar{A})) \subset \mathscr{D}(q(\bar{A}))$, we can apply Remark 11 to the triplet $(A_0, A_1, \mathscr{E}_{0,1})$ with $\mathscr{E}_{0,1} \stackrel{\mathrm{df}}{=} \mathscr{D}(A^n)$. We conclude that $\mathscr{D}(A^n)$ is a core of $q(\bar{A})$. Now we can follow arguments used in the middle part of the proof of Theorem 16. □

Let us recall that formally normal operators need not be subnormal (cf. [**10**], [**30**], [**33**]). We show now that the property of being essentially normal is inherited by roots, at least within the class of subnormal and formally normal operators.

Proposition 18. *If A is a subnormal and formally normal operator in $\mathscr{H}$ such that $p(A)$ is essentially normal for some polynomial $p \in \boldsymbol{C}[X]$ of degree $n \ge 1$, then the conclusion of Lemma 17 holds true.*

Proof. Suppose that $p(X) = \sum_{j=0}^n p_j X^j$ with $p_n \ne 0$. By [**35**, Proposition 5.3] (see also [**29**] for the case of symmetric operators) $p(\bar{A})$ is closed. Let N be a normal extension of A. Then clearly $\bar{A}^j \subset N^j$. Moreover, if $f \in \mathscr{D}(\bar{A}^n)$, then

$$\langle f, p(\bar{A})h\rangle = \sum_{j=0}^n \bar{p}_j \langle f, \bar{A}^j h\rangle = \sum_{j=0}^n \bar{p}_j \langle f, N^j h\rangle = \left\langle \sum_{j=0}^n \bar{p}_j N^{*j} f, h\right\rangle, \quad h \in \mathscr{D}(\bar{A}^n),$$

so $f \in \mathscr{D}(p(\bar{A})^*)$. Applying Lemma 17 completes the proof. □

In the case of symmetric operators (which always have selfadjoint extensions possibly in larger Hilbert spaces; cf. [**1**, §111 Theorem 1] and [**40**, Proposition 1]) Proposition 18 simplifies to

Corollary 19. *If A is a symmetric operator in $\mathscr{H}$ such that $p(A)$ is essentially selfadjoint for some polynomial $p \in \boldsymbol{R}[X]$ of degree $n \ge 1$, then for every $q \in \boldsymbol{R}[X]$ with $\deg q \le n$, the operator $q(A)$ is essentially selfadjoint, $\mathscr{D}(A^n)$ is a core of $q(A)$ and $\overline{q(A)} = q(\bar{A})$.*

Notice that essential selfadjointness of A follows from that of $p(A)$ $(p \in \boldsymbol{R}[X])$ via the following simple arguments: since $\deg p \ge 1$, there exists $\xi \in \boldsymbol{C} \backslash \boldsymbol{R}$ such that $p(\xi) \in \boldsymbol{C} \backslash \boldsymbol{R}$ (because otherwise $p(\boldsymbol{C}) \subset \boldsymbol{R}$ which, by analyticity of p, implies $p \equiv$ constant, a contradiction) and, in consequence, $p(\bar{\xi}) \in \boldsymbol{C} \backslash \boldsymbol{R}$; hence $\mathscr{N}(A^* - \xi) \subset \mathscr{N}(p(A)^* - p(\xi)) = \{0\}$ and $\mathscr{N}(A^* - \bar{\xi}) \subset \mathscr{N}(p(A)^* - p(\bar{\xi})) = \{0\}$, which means that the defect indices of $\bar{A}$ are both equal to 0. However, this simple argument is not applicable in the context of Proposition 18.

Corollary 19 says in particular that if A^n is selfadjoint then so are all the powers $A, A^2, \dots, A^{n-1}$. For A cyclic this can be get directly from the $\mathscr{L}^2$-model of the symmetric operator A, the fact which has been overlooked in [**7**], Theorem 1, where the so called index of determinacy of a determinate positive measure on $\boldsymbol{R}$ is considered.

Corollary 20. *Let A be a formally normal operator in $\mathscr{H}$ such that A^n is essentially normal for some $n \geq 1$. If either $A(\mathscr{D}(A)) \subset \mathscr{D}(A)$ or $\bar{A}^n$ is formally normal, then the conclusion of Lemma 17 holds true.*

Proof. Suppose that $A(\mathscr{D}(A)) \subset \mathscr{D}(A)$. Then, by part (iii) of [**41**, Proposition 6], the operator $A_0 \stackrel{\mathrm{df}}{=} A^n$ dominates $A_1 \stackrel{\mathrm{df}}{=} A$ on $\mathscr{E}_{0,1} \stackrel{\mathrm{df}}{=} \mathscr{D}(A)$. Since $\langle A_0 f, A_1^* g\rangle = \langle A^{n+1} f, g\rangle = \langle A_1 f, A_0^* g\rangle$ for all $f, g \in \mathscr{E}_{0,1}$, Theorem 10 implies that $\bar{A}$ is normal.

Assume now that $\bar{A}^n$ is formally normal. By part (iv) of [**41**, Proposition 6], the operator $\bar{A}^n$ is closed, so $\overline{A^n} \subset \bar{A}^n$. This and Proposition 3 imply that $\overline{A^n} = \bar{A}^n$ and consequently that $\bar{A}^n$ is normal. It follows from Theorem 16 that $\bar{A}$ is normal.

In both cases we can apply Proposition 18, which completes the proof. □

Corollary 21. *If $\mathscr{D}$ is a dense linear subspace of $\mathscr{H}$ and $A \in \boldsymbol{L}(\mathscr{D})$ is formally normal, then the following conditions are equivalent for every $n \geq 1$*

(i) *A^n is essentially normal,*

(ii) *A^i is essentially normal for every $i = 1, \ldots, n$,*

(iii) *A is essentially normal and $\mathscr{D}$ is a core of every $\bar{A}^i$, $i = 1, \ldots, n$,*

(iv) *A is essentially normal and $\mathscr{D}$ is a core of $\bar{A}^n$.*

Proof. The implications (i) ⇒ (ii) and (i) ⇒ (iii) are true due to Corollary 20 while (ii) ⇒ (i) and (iii) ⇒ (iv) are trivial. If (iv) holds, then $\bar{A}^n$ is normal and, in consequence, $\overline{A^n} = (\bar{A}^n|_{\mathscr{D}})^- = \bar{A}^n$, so $\overline{A^n}$ is normal, which gives us (i). □

Domination in powers.

The two theorems which follow extend applicability of Theorem 10 to the case when either some power of the dominator is essentially normal or the dominator itself, still being essentially normal, is a power of some formally normal operator. First we prove

Lemma 22. *Let $\mathscr{D}$ be a dense linear subspace of $\mathscr{H}$, $A_0, A_1, \ldots, A_\kappa \in \boldsymbol{L}(\mathscr{D})$ be pointwise commuting operators and $c > 0$ be such that*

(a) *$\|A_i f\| \leq c(\|f\| + \|A_0 f\|)$ for $f \in \mathscr{D}$ and $i = 1, \ldots, \kappa$.*

Then for every $p \in \boldsymbol{C}[X_0, X_1, \ldots, X_\kappa]$, there exists $c_p > 0$ such that

(i) *$\|p(A_0, A_1, \ldots, A_\kappa) f\| \leq c_p \sum_{j=0}^{\deg p} \|A_0^j f\|$ for $f \in \mathscr{D}$.*

If A_0 is paranormal, then for every $n \geq 0$ and for every polynomial p with $\deg p \leq n$ *there exists $d_{n,p} > 0$ such that*

(ii) *$\|p(A_0, A_1, \ldots, A_\kappa) f\| \leq d_{n,p}(\|f\| + \|A_0^n f\|)$ for $f \in \mathscr{D}$.*

Proof. (i) Set $\boldsymbol{Z}_+ = \{0, 1, 2, \ldots\}$. It suffices to show that

$$\|\boldsymbol{A}^\alpha f\| \leq c_{|\alpha|} \sum_{j=0}^{|\alpha|} \|A_0^j f\|, \quad f \in \mathscr{D}, \alpha \in \boldsymbol{Z}_+^{\kappa+1}, \tag{9}$$

where $\boldsymbol{A}^\alpha = A_0^{\alpha_0} \cdots A_\kappa^{\alpha_\kappa}$ and $|\alpha| = \alpha_0 + \cdots + \alpha_\kappa$ for $\alpha = (\alpha_0, \ldots, \alpha_\kappa) \in \boldsymbol{Z}_+^{\kappa+1}$. The proof is by induction on $n = |\alpha|$. Assume (9) holds for a fixed $n \geq 0$. If $\alpha \in \boldsymbol{Z}_+^{\kappa+1}$ and $|\alpha| = n+1$, then there exists $i \in \{0, \ldots, \kappa\}$ such that $\alpha_i \geq 1$. Set $\alpha' = (\alpha_0, \ldots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \ldots, \alpha_\kappa) \in \boldsymbol{Z}_+^{\kappa+1}$. Since $|\alpha'| = n$, (a) yields

$$\|\boldsymbol{A}^{\alpha}f\| = \|A_i\boldsymbol{A}^{\alpha'}f\| \leq \tilde{c}(\|\boldsymbol{A}^{\alpha'}f\| + \|\boldsymbol{A}^{\alpha'}A_0f\|) \leq 2\tilde{c}c_n\sum_{j=0}^{n+1}\|A_0^jf\|, \quad f \in \mathscr{D},$$

where $\tilde{c} = \max\{1, c\}$. This completes the proof of (9) and, in consequence, of (i).

(ii) Since A_0 is paranormal, one can deduce from part (iii) of Proposition 6 in [**41**] that there exists $b_n > 0$ such that $\sum_{j=0}^{n}\|A_0^jf\| \leq b_n(\|f\| + \|A_0^nf\|)$ for every $f \in \mathscr{D}$. This and (i) complete the proof. $\square$

Theorem 23. *Let $\mathscr{D}$ be a dense linear subspace of $\mathscr{H}$ and $A_0, A_1, \dots, A_\kappa \in \boldsymbol{L}(\mathscr{D})$ be formally normal operators ($\kappa \geq 1$) satisfying conditions* (i), (ii) *and* (iv) *of Theorem* 10 *with $\mathscr{E}_{i,j} = \mathscr{D}$, $0 \leq i < j \leq \kappa$. If the operator $\overline{A_0^n}$ is normal for some $n \geq 1$, then the operators $A_0, A_1, \dots, A_\kappa$ pointwise commute, and for any choice $p_1, \dots, p_s$ of complex polynomials in $\kappa+1$ variables with $\deg p_i \leq n$, the operators $\overline{p_1(A_0, A_1, \dots, A_\kappa)}, \dots, \overline{p_s(A_0, A_1, \dots, A_\kappa)}$ are normal, they spectrally commute and $\mathscr{D}$ is a core of the system $(\overline{p_1(A_0, A_1, \dots, A_\kappa)}, \dots, \overline{p_s(A_0, A_1, \dots, A_\kappa)})$.*

Proof. It follows from Corollary 21 and Theorem 10 that $\bar{A}_0, \bar{A}_1, \dots, \bar{A}_\kappa$ are spectrally commuting normal operators. This in turn implies that the operators $A_0, A_1, \dots, A_\kappa$ pointwise commute. Let E be the spectral measure of the system $(\bar{A}_0, \bar{A}_1, \dots, \bar{A}_\kappa)$. Since $p_i(A_0, A_1, \dots, A_\kappa) \subset \int_{\boldsymbol{C}^{\kappa+1}} p_i \, \mathrm{d}E$, we infer from part (ii) of Lemma 22 that the operators A_0^n, $p_1(A_0, A_1, \dots, A_\kappa), \dots, p_s(A_0, A_1, \dots, A_\kappa)$ are formally normal and that they satisfy all the assumptions of Theorem 10 with $\mathscr{E}_{i,j} = \mathscr{D}$, $0 \leq i < j \leq s$. This completes the proof. $\square$

Theorem 24. *Let $\mathscr{D}$ be a dense linear subspace of $\mathscr{H}$, $A_0, A_1, \dots, A_\kappa \in \boldsymbol{L}(\mathscr{D})$ be formally normal operators and n be a positive integer ($\kappa \geq 1$). Assume that*

(i) *$A_iA_j = A_jA_i$ for all $0 \leq i < j \leq \kappa$,*

(ii) *A_0^n is essentially normal,*

(iii) *there is $c > 0$ such that $\|A_jf\| \leq c(\|f\| + \|A_0^nf\|)$ for $f \in \mathscr{D}$ and $j = 1, \dots, \kappa$.*

Then $\bar{A}_0, \bar{A}_1, \dots, \bar{A}_\kappa$ are spectrally commuting normal operators, and $\mathscr{D}$ is a core[4] *of any subsystem of $\{\bar{A}_0, \bar{A}_1, \dots, \bar{A}_\kappa\}$.*

Proof. We proceed as in the proof of Theorem 10 to conclude that $\bar{A}_1, \dots, \bar{A}_\kappa$ are spectrally commuting normal operators, $\overline{A_0^n}$ is normal and the space $\mathscr{B}_a(\overline{A_0^n})$ reduces every $\bar{A}_1, \dots, \bar{A}_\kappa$ to a bounded normal operator for all $a \geq 0$. By Corollary 21, the operator $\bar{A}_0$ is normal and $\overline{A_0^n} = \bar{A}_0^n$, so $\mathscr{D}^\infty(\overline{A_0^n}) = \mathscr{D}^\infty(\bar{A}_0)$. Using either [**41**, Lemma 8] or Proposition 4 one can prove that

$$\mathscr{B}_{a^{1/n}}(\bar{A}_0) = \mathscr{B}_a(\bar{A}_0^n) = \mathscr{B}_a(\overline{A_0^n}), \quad a \geq 0, \tag{10}$$

and consequently that $\mathscr{B}(\bar{A}_0) = \mathscr{B}(\overline{A_0^n})$. Therefore, by (i) and Propositions 2 and 4, we have $\bar{A}_0\bar{A}_if = \bar{A}_i\bar{A}_0f$ for $f \in \mathscr{B}(\overline{A_0^n})$ and $i = 1, \dots, \kappa$. This and (10) imply that for every $i = 1, \dots, \kappa$, the space $\mathscr{B}_a(\overline{A_0^n})$ reduces operators $\bar{A}_0$ and $\bar{A}_i$ to pointwise commuting bounded normal operators. Applying the argument (via orthogonal sums) from

[4] In fact, a small modification of the proof of Theorem 24 leads to more, in particular to: $\mathscr{D}$ is a core of any subsystem of $\{\bar{A}_0, \dots, \bar{A}_0^n, \bar{A}_1, \dots, \bar{A}_\kappa\}$.

the first part of the proof of Theorem 10, we conclude that $\bar{A}_0$ spectrally commutes with every $\bar{A}_i$.

Let E be the spectral measure of the system $(\bar{A}_0, \bar{A}_1, \dots, \bar{A}_\kappa)$. Let $\mathscr{X} = \bigcup_{i=1}^\infty \mathscr{R}(E(\Delta_i))$ and $0 \le v_1 < \dots < v_m \le \kappa$ be as in the proof of Theorem 10. Like there, we show that (7) holds true. Take $h \in \mathscr{X}$. Since $\mathscr{X} \subset \mathscr{D}(\bar{A}_0^n) = \mathscr{D}(\overline{A_0^n})$, there exists a sequence $\{h_i\}_{i=1}^\infty \subset \mathscr{D}$ such that $h = \lim_{i\to\infty} h_i$ and $\bar{A}_0^n h = \lim_{i\to\infty} A_0^n h_i$. It follows from part (iii) of [**41**, Proposition 6] and our assumption (iii) that $\bar{A}_0^j h = \lim_{i\to\infty} A_0^j h_i$ and $\bar{A}_{v_l} h = \lim_{i\to\infty} A_{v_l} h_i$ for all $j = 1, \dots, n$ and $l = 1, \dots, m$. In this way we have proved that (8) holds true with $\mathscr{E} = \mathscr{D}$. Combining inclusions (7) and (8) completes the proof. □

In connection with Theorem 24 we have to point out that in general two bounded normal operators A_1 and A_0 may not spectrally commute though A_1 spectrally commutes with some nth power of A_0, $n \ge 2$ (e.g. consider a normal nth root A_0 of the identity operator). This means that we can not replace in Theorem 24 the condition (i) by a weaker one: $A_0^n A_j = A_j A_0^n$ for every $j = 1, \dots, \kappa$ and $A_i A_j = A_j A_i$ for all $1 \le i < j \le \kappa$.

In view of Corollary 19, it is possible to formulate more general versions of Theorems 23 and 24 for symmetric operators weakening a little bit the assumption on domains of operators in question (mainly that about a common invariant domain). The details are left to the reader.

Nelson's type criterion for spectral commutativity.

Theorem 10 leads to a useful criterion for spectral commutativity of normal operators. It generalizes among other things the commutative part of Theorem 5 in [**19**] (see also Corollary 9.2 therein) to the context of formally normal operators. What has to be pointed out is the role played by bounded vectors in the background so as to make the arguments we have used in our proof to work as smoothly as possible. We want to take the opportunity here to indicate that [**22**] contains a result which is much like Nelson's (with a simpler proof). Though it is also somehow in flavour of our considerations the main difference is in the fact that existence and some behaviour of bounded vectors is in [**22**] made as an explicit assumption.

THEOREM 25. *Let $A_1, \dots, A_\kappa$ be formally normal operators in $\mathscr{H}$ and $\mathscr{E}$ be a dense linear subspace of $\mathscr{H}$. Suppose that*

(i) *$\mathscr{E} \subset \mathscr{D}(A_i A_j) \cap \mathscr{D}(A_j A_i)$ and $A_i A_j f = A_j A_i f$, $f \in \mathscr{E}$, for $1 \le i < j \le \kappa$,*

(ii) *$\mathscr{E} \subset \mathscr{D}(A_i^* A_i)$ and $\langle A_i^* A_i f, A_j g\rangle = \langle A_j^* f, A_i^* A_i g\rangle$, $f, g \in \mathscr{E}$, for $i, j = 1, \dots, \kappa$,*

(iii) *$(A_1^* A_1 + \dots + A_\kappa^* A_\kappa)|_{\mathscr{E}}$ is essentially selfadjoint.*

Then $\bar{A}_1, \dots, \bar{A}_\kappa$ are spectrally commuting normal operators and $\mathscr{E}$ is a core of any subsystem of $\{\bar{A}_1, \dots, \bar{A}_\kappa\}$.

PROOF. According to (ii) and (iii) the operator $A_0 \stackrel{\mathrm{df}}{=} A_1^* A_1 + \dots + A_\kappa^* A_\kappa$ is symmetric and $A_0|_{\mathscr{E}}$ is essentially selfadjoint. Notice that

$$\|A_j f\| = \langle A_j^* A_j f, f\rangle^{1/2} \le \langle A_0 f, f\rangle^{1/2} \le \|A_0 f\|^{1/2} \|f\|^{1/2} \le \frac{1}{2}(\|f\| + \|A_0 f\|)$$

for all $f \in \mathscr{E}$ and $1 \le j \le \kappa$. It follows from (ii) that for every $j = 1, \dots, \kappa$

$$\langle A_0 f, A_j g\rangle = \sum_{i=1}^{\kappa} \langle A_i^* A_i f, A_j g\rangle = \sum_{i=1}^{\kappa} \langle A_j^* f, A_i^* A_i g\rangle = \langle A_j^* f, A_0 g\rangle, \quad f, g \in \mathscr{E}.$$

Applying Theorem 10 completes the proof. □

REMARK 26. Theorem 25 is still true if condition (ii) is replaced by

(ii′) $\mathscr{E} \subset \mathscr{D}(A_i A_j^*) \cap \mathscr{D}(A_i^2)$ and $A_i A_j^* f = A_j^* A_i f$, $f \in \mathscr{E}$, for all $i, j \in \{1, \ldots, \kappa\}$ such that $i \neq j$.

Indeed, by formal normality of A_i we have $\langle A_i f, A_i g\rangle = \langle A_i^* f, A_i^* g\rangle$ for $f, g \in \mathscr{D}(A_i)$. This and $\mathscr{E} \subset \mathscr{D}(A_i^2)$ imply that $\mathscr{E} \subset \mathscr{D}(A_i^* A_i)$ and $\langle A_i^* A_i f, A_i g\rangle = \langle A_i f, A_i A_i g\rangle = \langle A_i^* f, A_i^* A_i g\rangle$ for $f, g \in \mathscr{E}$. On the other hand, conditions (i) and (ii′) lead to

$$\begin{aligned}\langle A_i^* A_i f, A_j g\rangle &= \langle A_i f, A_i A_j g\rangle = \langle A_i f, A_j A_i g\rangle = \langle A_j^* A_i f, A_i g\rangle = \langle A_i A_j^* f, A_i g\rangle \\ &= \langle A_j^* f, A_i^* A_i g\rangle, \quad f, g \in \mathscr{E}, i \neq j.\end{aligned}$$

This shows that condition (ii) of Theorem 25 is satisfied.

Conditions (i) and (ii) of Theorem 25 are also satisfied if the operators $A_i, A_j, A_i^* A_i$, $i, j = 1, \ldots, \kappa$, pointwise commute on a dense linear subspace $\mathscr{E}$ of $\mathscr{H}$.

The next result (except its part concerning cores) is often referred to as *Nelson's criterion* [**19**].

COROLLARY 27. *Let $A_1, \ldots, A_\kappa$ be symmetric operators in $\mathscr{H}$ and $\mathscr{E}$ be a dense linear subspace of $\mathscr{H}$. Suppose that*

(i) *$\mathscr{E} \subset \mathscr{D}(A_i A_j)$ and $A_i A_j f = A_j A_i f$, $f \in \mathscr{E}$, for all $i, j = 1, \ldots, \kappa$;*

(ii) *$(A_1^2 + \cdots + A_\kappa^2)|_{\mathscr{E}}$ is essentially selfadjoint in $\mathscr{H}$.*

Then $\bar{A}_1, \ldots, \bar{A}_\kappa$ are spectrally commuting selfadjoint operators and $\mathscr{E}$ is a core of any subsystem of $\{\bar{A}_1, \ldots, \bar{A}_\kappa\}$.

REMARK 28. It turns out that the two-operator version of Nelson's criterion is stronger (at least *a priori*) than its general form (assuming a little bit more about the domains of operators in question). Namely, if the symmetric operators $A_1, \ldots, A_\kappa$ ($\kappa \geq 2$) satisfy conditions (i) and (ii) of Corollary 27 and, moreover, $\langle A_i^2 f, A_j^2 g\rangle = \langle A_j^2 f, A_i^2 g\rangle$ for all $f, g \in \mathscr{E}$ and $i, j = 1, \ldots, \kappa$, then the operators $(A_p^2 + A_q^2)|_{\mathscr{E}}$, $1 \leq p < q \leq \kappa$, are essentially selfadjoint[5]. Indeed, we can apply Theorem 10 to symmetric operators $\tilde{A}_0 \stackrel{\mathrm{df}}{=} A_1^2 + \cdots + A_\kappa^2$ and $\tilde{A}_1 \stackrel{\mathrm{df}}{=} A_p^2 + A_q^2$ (with $\mathscr{E}_{0,1} = \mathscr{E}$) because

$$\begin{aligned}\|\tilde{A}_1 f\|^2 &= \int_{\boldsymbol{R}^\kappa} (x_p^2 + x_q^2)^2 \langle E(\mathrm{d}x_1, \ldots, \mathrm{d}x_\kappa) f, f\rangle \\ &\leq \int_{\boldsymbol{R}^\kappa} (x_1^2 + \cdots + x_\kappa^2)^2 \langle E(\mathrm{d}x_1, \ldots, \mathrm{d}x_\kappa) f, f\rangle = \|\tilde{A}_0 f\|^2, \quad f \in \mathscr{E},\end{aligned}$$

[5] The same conclusion is valid if symmetric operators $A_1, \ldots, A_\kappa$ satisfy condition (ii) of Corollary 27 with $\mathscr{E} \subset \bigcap_{i,j=1}^{\kappa} \mathscr{D}(A_i A_j)$ such that $\langle A_i^2 f, A_j^2 g\rangle = \langle A_j^2 f, A_i^2 g\rangle = \langle A_i A_j f, A_i A_j g\rangle$ for all $f, g \in \mathscr{E}$ and $i, j = 1, \ldots, \kappa$. Indeed, Theorem 10 is to be applied because

$$\|\tilde{A}_0 f\|^2 = \sum_{i,j=1}^{\kappa} \langle A_i^2 f, A_j^2 f\rangle = \sum_{i,j=1}^{\kappa} \|A_i A_j f\|^2 \geq \|\tilde{A}_1 f\|^2, \quad f \in \mathscr{E}.$$

where E is the spectral measure of the system $(\bar{A}_1, \dots, \bar{A}_\kappa)$ (such a measure exists due to Corollary 27). The same arguments can be used to show that any subsystem of $(A_1, \dots, A_\kappa)$ satisfies the condition (ii) of Corollary 27.

In view of Remark 28, the authors have decided to include a direct proof of Nelson's criterion for spectral commutativity of two symmetric operators based on the spectral theorem for a normal operator. Below we write $\operatorname{Re} A = (1/2)(A + A^*)$ and $\operatorname{Im} A = (1/(2\mathrm{i}))(A - A^*)$ for a densely defined operator A in $\mathscr{H}$ such that $\mathscr{D}(A) \subset \mathscr{D}(A^*)$.

Proof of Corollary 27 for $\kappa = 2$. By (i), $N \stackrel{\mathrm{df}}{=} A_1 + \mathrm{i}A_2$ is densely defined, $\mathscr{E} \subset \mathscr{D}((A_1 - \mathrm{i}A_2)(A_1 + \mathrm{i}A_2)) \subset \mathscr{D}(N^*N)$, $\mathscr{E} \subset \mathscr{D}((A_1 + \mathrm{i}A_2)(A_1 - \mathrm{i}A_2)) \subset \mathscr{D}(NN^*)$ and $N^*N|_{\mathscr{E}} = NN^*|_{\mathscr{E}} = (A_1^2 + A_2^2)|_{\mathscr{E}}$. Hence, according to (ii) and Theorem 5, N is closable, $\mathscr{E}$ is a core of $\bar{N}$ and $\bar{N}$ is normal. The latter implies that $S_1 \stackrel{\mathrm{df}}{=} \overline{\operatorname{Re} \bar{N}}$ and $S_2 \stackrel{\mathrm{df}}{=} \overline{\operatorname{Im} \bar{N}}$ are spectrally commuting selfadjoint operators and $\bar{N} = S_1 + \mathrm{i}S_2$ (cf. [**47**, Theorem 7.32]).

Take $f \in \mathscr{D}(\bar{N})$. Since $\mathscr{E}$ is a core of $\bar{N}$, there exists $\{f_n\}_{n=1}^\infty \subset \mathscr{E}$ such that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} Nf_n = \bar{N}f$. It follows from (i) that $\|N^*h\|^2 = \|Nh\|^2$ for $h \in \mathscr{E}$, so the sequence $\{N^*f_n\}_{n=1}^\infty$ is convergent. Thus $f \in \mathscr{D}(N^*)$ and $\lim_{n\to\infty} N^*f_n = N^*f$. Since $A_1 - \mathrm{i}A_2 \subset N^*$, we conclude that the sequence $A_1 f_n = (1/2)(Nf_n + N^*f_n)$, $n \geq 1$, converges to $\operatorname{Re} \bar{N}f$. Hence $f \in \mathscr{D}(\bar{A}_1)$ and $\bar{A}_1 f = \operatorname{Re} \bar{N}f$. This implies that $S_1 \subset \bar{A}_1$. Likewise $S_2 \subset \bar{A}_2$. By maximality of S_1 and S_2, we get $S_1 = \bar{A}_1$ and $S_2 = \bar{A}_2$, which means that $\bar{A}_1$ and $\bar{A}_2$ are spectrally commuting selfadjoint operators such that $\bar{N} = \bar{A}_1 + \mathrm{i}\bar{A}_2$.

If $f \in \mathscr{D}(\bar{A}_1) \cap \mathscr{D}(\bar{A}_2) = \mathscr{D}(\bar{N})$ and $\{f_n\}_{n=1}^\infty \subset \mathscr{E}$ are as in the previous paragraph, then $(f, \bar{A}_1 f, \bar{A}_2 f) = \lim_{n\to\infty}(f_n, \bar{A}_1 f_n, \bar{A}_2 f_n)$. Hence $\mathscr{G}(\bar{A}_1, \bar{A}_2) \subset \overline{\mathscr{G}(\bar{A}_1|_{\mathscr{E}}, \bar{A}_2|_{\mathscr{E}})}$, $\bar{A}_1 = (\bar{A}_1|_{\mathscr{D}(\bar{N})})^- \subset (\bar{A}_1|_{\mathscr{E}})^- \subset \bar{A}_1$ and $\bar{A}_2 = (\bar{A}_2|_{\mathscr{D}(\bar{N})})^- \subset (\bar{A}_2|_{\mathscr{E}})^- \subset \bar{A}_2$, which completes the proof. □

Nelson's criteria of polynomial type.

In the context of Theorem 25 and Corollary 27 (also referring to [**25**]) the following question seems to be natural (for simplicity we formulate it only for symmetric operators): does essential selfadjointness of $q(A_1, \dots, A_\kappa)$, where q is a polynomial with $\deg q \geq 1$ and $A_1, \dots, A_\kappa \in \boldsymbol{L}(\mathscr{D})$ are pointwise commuting symmetric operators, imply selfadjointness and spectral commutativity of $\bar{A}_1, \dots, \bar{A}_\kappa$? In general, the answer to our question can be easily made negative. Indeed, if $S \in \boldsymbol{L}(\mathscr{D})$ is a symmetric operator which is not essentially selfadjoint and $q(X_1, X_2) = X_1 - X_2$, then $q(S, S) = 0$. On the other hand, if q is a polynomial with $1 \leq \deg q \leq 2$ and (A_1, A_2) is a (cyclic) pair of pointwise commuting symmetric operators such that $q(A_1, A_2) = 0$, then (A_1, A_2) always extends to a pair of spectrally commuting selfadjoint operators possible in a larger Hilbert space (cf. [**33**], [**9**]). However, if $\deg q \geq 3$, even that property, which can be treated as a poor substitute for an affirmative answer to our question, does not hold (cf. [**33**]). However, the answer to our question is in the affirmative if $q(X_1, \dots, X_\kappa) = X_1^2 + \dots + X_\kappa^2$; this is due to Corollary 27. Below we propose some other possible solutions.

Proposition 29. *Let $n_1, \dots, n_\kappa$ be positive integers, $\mathscr{D}$ be a dense linear subspace of a Hilbert space $\mathscr{H}$ and $A_1, \dots, A_\kappa \in \boldsymbol{L}^\#(\mathscr{D})$ be such that*

(i) $A_iA_j = A_jA_i$ *and* $A_iA_j^\# = A_j^\# A_i$ *for all* $1 \le i \le j \le \kappa$,

(ii) $A_1^{*n_1}A_1^{n_1} + \cdots + A_\kappa^{*n_\kappa}A_\kappa^{n_\kappa}$ *is essentially selfadjoint.*

Then for all $p_1, \dots, p_s \in \boldsymbol{C}[X_1, \dots, X_{2\kappa}]$ *with* $\deg p_i \le \min_{j=1}^\kappa n_j$, $\mathscr{D}$ *is a core of the system* $(\overline{p_1(\boldsymbol{A}, \boldsymbol{A}^\#)}, \dots, \overline{p_s(\boldsymbol{A}, \boldsymbol{A}^\#)})$, *and* $\overline{p_1(\boldsymbol{A}, \boldsymbol{A}^\#)}, \dots, \overline{p_s(\boldsymbol{A}, \boldsymbol{A}^\#)}$ *are spectrally commuting normal operators*; *here* $\boldsymbol{A} = (A_1, \dots, A_\kappa)$ *and* $\boldsymbol{A}^\# = (A_1^\#, \dots, A_\kappa^\#)$.

Proof. By (i), $A_1, A_1^\#, \dots, A_\kappa, A_\kappa^\#$ are pointwise commuting formally normal operators. Set $n = \min\{n_1, \dots, n_\kappa\}$. We show that $A_0 \stackrel{\mathrm{df}}{=} A_1^{*n_1}A_1^{n_1} + \cdots + A_\kappa^{*n_\kappa}A_\kappa^{n_\kappa}$ dominates every A_i on $\mathscr{D}$, $i = 1, \dots, \kappa$. Indeed, if $h \in \mathscr{D}$ is a normalized vector, then by [**45**, formula (4)] the sequence $\{\|A_i^j h\|^{1/j}\}_{j=1}^\infty$ is monotically increasing, so

$$\begin{aligned}\|A_i h\| &\le \|A_i^{n_i} h\|^{1/n_i} = \langle A_i^{*n_i}A_i^{n_i} h, h\rangle^{1/(2n_i)} \le \langle A_0 h, h\rangle^{1/(2n_i)} \\ &\le \|A_0 h\|^{1/(2n_i)} \le 1 + \|A_0 h\|, \quad i = 1, \dots, \kappa,\end{aligned}$$

because $t \le 1 + t^r$ for all real numbers $t \ge 0$ and $r \ge 1$. Thus $\|A_i f\| \le \|f\| + \|A_0 f\|$ for $f \in \mathscr{D}$ and $i = 1, \dots, \kappa$. By Theorem 10, $\bar{A}_1, \dots, \bar{A}_\kappa$ are spectrally commuting normal operators. Let E be the spectral measure of the system $(\bar{A}_1, \dots, \bar{A}_\kappa)$. Since there exists $d > 0$ such that $|p_i(z_1, \dots, z_{2\kappa})|^2 \le d(1 + |z_1|^{2n} + \cdots + |z_{2\kappa}|^{2n})$ for $(z_1, \dots, z_{2\kappa}) \in \boldsymbol{C}^{2\kappa}$ and $i = 1, \dots, s$, we obtain

$$\begin{aligned}\|p_i(\boldsymbol{A}, \boldsymbol{A}^\#) f\|^2 &= \int_{\boldsymbol{C}^\kappa} |p_i(z, \bar{z})|^2 \langle E(\mathrm{d}z) f, f\rangle \\ &\le c \int_{\boldsymbol{C}^\kappa} (1 + |z_1|^{2n_1} + \cdots + |z_\kappa|^{2n_\kappa}) \langle E(\mathrm{d}z) f, f\rangle \\ &= c(\|f\|^2 + \langle A_0 f, f\rangle) \le 2c(\|f\|^2 + \|A_0 f\|^2), \quad f \in \mathscr{D}, i = 1, \dots, s,\end{aligned}$$

where $c = 2(\kappa + 1)d$. This means that A_0 dominates every operator $p_i(\boldsymbol{A}, \boldsymbol{A}^\#)$ on $\mathscr{D}$. Applying Theorem 10 to $(A_0, p_1(\boldsymbol{A}, \boldsymbol{A}^\#), \dots, p_s(\boldsymbol{A}, \boldsymbol{A}^\#))$ completes the proof. □

Assuming subnormality of operators in question we can strengthen Proposition 29 as follows

Proposition 30. *Let* $q_1, \dots, q_\kappa \in \boldsymbol{C}[X]$ *be polynomials with* $\deg q_i \ge 1$, $\mathscr{D}$ *be a dense linear subspace of* $\mathscr{H}$ *and* $A_1, \dots, A_\kappa \in \boldsymbol{L}^\#(\mathscr{D})$ *be subnormal operators. If*

(i) $A_iA_j = A_jA_i$ *and* $A_iA_j^\# = A_j^\# A_i$ *for all* $1 \le i \le j \le \kappa$,

(ii) $q_1(A_1)^* q_1(A_1) + \cdots + q_\kappa(A_\kappa)^* q_\kappa(A_\kappa)$ *is essentially selfadjoint*,

then for all $p_1, \dots, p_s \in \boldsymbol{C}[X_1, \dots, X_{2\kappa}]$ *with* $\deg p_i \le \min_{j=1}^\kappa \deg q_j$, $\mathscr{D}$ *is a core of the system* $(\overline{p_1(\boldsymbol{A}, \boldsymbol{A}^\#)}, \dots, \overline{p_s(\boldsymbol{A}, \boldsymbol{A}^\#)})$, *and* $\overline{p_1(\boldsymbol{A}, \boldsymbol{A}^\#)}, \dots, \overline{p_s(\boldsymbol{A}, \boldsymbol{A}^\#)}$ *are spectrally commuting normal operators*; *here* $\boldsymbol{A} = (A_1, \dots, A_\kappa)$ *and* $\boldsymbol{A}^\# = (A_1^\#, \dots, A_\kappa^\#)$.

Proof. Set $A_0 = q_1(A_1)^* q_1(A_1) + \cdots + q_\kappa(A_\kappa)^* q_\kappa(A_\kappa)$ and $n_i = \deg q_i$. Since there exists $\alpha > 0$ such that $|z|^2 \le \alpha(1 + |q_i(z)|^2)$ for $z \in \boldsymbol{C}$ and $i = 1, \dots, \kappa$, one can show, using the spectral measure of a normal extension of A_i, that $\|A_i f\|^2 \le \alpha(\|f\|^2 + \|q_i(A_i) f\|^2)$ for $f \in \mathscr{D}$ and $i = 1, \dots, \kappa$. This implies that

$$\|A_i f\|^2 \le \alpha(\|f\|^2 + \langle A_0 f, f\rangle) \le 2\alpha(\|f\|^2 + \|A_0 f\|^2), \quad f \in \mathscr{D}, i = 1, \dots, \kappa.$$

By Theorem 10, $\bar{A}_1, \dots, \bar{A}_\kappa$ are spectrally commuting normal operators. Let E be the spectral measure of the system $(\bar{A}_1, \dots, \bar{A}_\kappa)$. Since there exists $\beta > 0$ such that $|z^{n_i}|^2 \le \beta(1 + |q_i(z)|^2)$ for $z \in \boldsymbol{C}$ and $i = 1, \dots, \kappa$, we see, as in the proof of Proposition 29, that for some $\gamma > 0$ the following inequalities hold

$$\begin{aligned}\|p_i(\boldsymbol{A}, \boldsymbol{A}^{\#})f\|^2 &= \int_{\boldsymbol{C}^\kappa} |p_i(z, \bar{z})|^2 \langle E(\mathrm{d}z)f, f\rangle \\ &\le \gamma \int_{\boldsymbol{C}^\kappa} (1 + |q_1(z_1)|^2 + \cdots + |q_\kappa(z_\kappa)|^2)\langle E(\mathrm{d}z)f, f\rangle \\ &= \gamma(\|f\|^2 + \langle A_0 f, f\rangle) \le 2\gamma(\|f\|^2 + \|A_0 f\|^2), \quad f \in \mathscr{D}, i = 1, \dots, s.\end{aligned}$$

This and Theorem 10 complete the proof. □

Since symmetric operators are subnormal, Proposition 30, when formulated for symmetric operators, simplifies to:

COROLLARY 31. *If $q_1, \dots, q_\kappa \in \boldsymbol{R}[X]$ are polynomials with $\deg q_i \ge 1$ and $A_1, \dots, A_\kappa \in \boldsymbol{L}(\mathscr{D})$ are pointwise commuting symmetric operators such that the operator $q_1(A_1)^2 + \cdots + q_\kappa(A_\kappa)^2$ is essentially selfadjoint, then for all $p_1, \dots, p_s \in \boldsymbol{R}[X_1, \dots, X_\kappa]$ with $\deg p_i \le \min_{j=1}^{\kappa} \deg q_j$, $\mathscr{D}$ is a core of $(\overline{p_1(\boldsymbol{A})}, \dots, \overline{p_s(\boldsymbol{A})})$, and $\overline{p_1(\boldsymbol{A})}, \dots, \overline{p_s(\boldsymbol{A})}$ are spectrally commuting selfadjoint operators.*

The idea used in the proofs of Propositions 29 and 30 enables us to build up more Nelson's criteria of polynomial type. Here we formulate a sample of what can be done in this matter for symmetric operators. Notice that Proposition 32 below includes essentially new criteria.

PROPOSITION 32. *Let $A_1^{(1)}, \dots, A_{n_1}^{(1)}, \dots, A_1^{(\kappa)}, \dots, A_{n_\kappa}^{(\kappa)} \in \boldsymbol{L}(\mathscr{D})$ be pointwise commuting symmetric operators such that the operator*

$$r_1(q_1^{(1)}(A_1^{(1)})^2 + \cdots + q_{n_1}^{(1)}(A_{n_1}^{(1)})^2)^2 + \cdots + r_\kappa(q_1^{(\kappa)}(A_1^{(\kappa)})^2 + \cdots + q_{n_\kappa}^{(\kappa)}(A_{n_\kappa}^{(\kappa)})^2)^2$$

is essentially selfadjoint, where $r_1, \dots, r_\kappa, q_1^{(1)}, \dots, q_{n_1}^{(1)}, \dots, q_1^{(\kappa)}, \dots, q_{n_\kappa}^{(\kappa)} \in \boldsymbol{R}[X]$ are polynomials of degree at least 1. Then $\overline{A_1^{(1)}}, \dots, \overline{A_{n_1}^{(1)}}, \dots, \overline{A_1^{(\kappa)}}, \dots, \overline{A_{n_\kappa}^{(\kappa)}}$ are spectrally commuting selfadjoint operators, and $\mathscr{D}$ is a core of any subsystem of $(\overline{A_1^{(1)}}, \dots, \overline{A_{n_1}^{(1)}}, \dots, \overline{A_1^{(\kappa)}}, \dots, \overline{A_{n_\kappa}^{(\kappa)}})$.

All this for subnormal operators.

Subnormality from domination.

Given a finite system $\boldsymbol{A} = (A_1, \dots, A_\kappa)$ of operators in $\mathscr{H}$, we define $\mathscr{D}(\boldsymbol{A}) = \mathscr{D}(A_1) \cap \cdots \cap \mathscr{D}(A_\kappa)$ and

$$\mathscr{D}^\infty(\boldsymbol{A}) = \bigcap\{\mathscr{D}(B_1 \cdots B_n); n \ge 1, B_i = A_1, \dots, A_\kappa \text{ for } i = 1, \dots, n\}.$$

A linear subspace $\mathscr{E}$ of $\mathscr{D}(\boldsymbol{A})$ is said to be *invariant* for $\boldsymbol{A}$ if $A_i\mathscr{E} \subset \mathscr{E}$ for every $i = 1, \dots, \kappa$. The set $\mathscr{D}^\infty(\boldsymbol{A})$ is the largest linear subspace of $\mathscr{D}(\boldsymbol{A})$ which is invariant for $\boldsymbol{A}$. We write $\boldsymbol{A}^* = (A_1^*, \dots, A_\kappa^*)$, provided all A_i are densely defined.

Let $\mathscr{E}$ be a linear subspace of $\mathscr{D}(\boldsymbol{A})$. $\boldsymbol{A}$ is said to satisfy the *Halmos-Bram-Ito condition with respect to* $\mathscr{E}$ (in short: $\mathsf{HBI}(\mathscr{E})$) if $\mathscr{E}$ is invariant for $\boldsymbol{A}$, $A_iA_j|_{\mathscr{E}} = A_jA_i|_{\mathscr{E}}$ for all $i, j = 1, \dots, \kappa$ and[6]

$$(11) \qquad \sum_{\alpha,\beta \in \boldsymbol{Z}_+^\kappa} \langle \boldsymbol{A}^\alpha f_\beta, \boldsymbol{A}^\beta f_\alpha \rangle \geq 0 \quad \text{for every finite multisequence } \{f_\alpha\}_{\alpha \in \boldsymbol{Z}_+^\kappa} \subset \mathscr{E},$$

where $\boldsymbol{A}^\alpha = A_1^{\alpha_1} \cdots A_\kappa^{\alpha_\kappa}$ for $\alpha \in \boldsymbol{Z}_+^\kappa$.

We say that a system $\boldsymbol{A} = (A_1, \dots, A_\kappa)$ of operators in $\mathscr{H}$ is *subnormal* if $\overline{\mathscr{D}(\boldsymbol{A})} = \mathscr{H}$ and there exists a Hilbert space $\mathscr{K} \supset \mathscr{H}$ and a system $\boldsymbol{N} = (N_1, \dots, N_\kappa)$ of spectrally commuting normal operators in $\mathscr{K}$ such that $A_i \subset N_i$ for $i = 1, \dots, \kappa$ (in short: $\boldsymbol{N}$ *is a normal extension of* $\boldsymbol{A}$). It is clear that if $\boldsymbol{A}$ is subnormal, then $(\bar{A}_1, \dots, \bar{A}_\kappa)$ is subnormal as well. A normal extension $\boldsymbol{N}$ of $\boldsymbol{A}$ is said to be *minimal of cyclic type* if there exists a linear subspace $\mathscr{E}$ of $\mathscr{H}$ which is an invariant core for every $\bar{A}_i$, $i = 1, \dots, \kappa$, and such that the linear span[7] $\mathscr{F}_{\boldsymbol{N}}[\mathscr{E}]$ of $\bigcup_{\alpha \in \boldsymbol{Z}_+^\kappa} \boldsymbol{N}^{*\alpha}(\mathscr{E})$ is a core for every N_i, $i = 1, \dots, \kappa$ (one may take $\mathscr{E}$ as $\mathscr{D}^\infty(\bar{A}_1, \dots, \bar{A}_\kappa)$). Like in the single operator case (cf. [**38**, Section 7]), any two normal extensions $\boldsymbol{N}'$ and $\boldsymbol{N}''$ of $\boldsymbol{A}$ acting in Hilbert spaces $\mathscr{K}'$ and $\mathscr{K}''$, respectively, which are minimal of cyclic type are $\mathscr{H}$-unitarily equivalent, i.e. there exists a unitary operator $U : \mathscr{K}' \to \mathscr{K}''$ such that $U|_{\mathscr{H}} = I_{\mathscr{H}}$ and $UN_i' = N_i''U$ for $i = 1, \dots, \kappa$. It is worth while to note that there exist subnormal operators having no minimal normal extension of cyclic type (cf. Example 1 and Theorem 3 in [**38**], and [**44**]).

The only relation between systems of operators satisfying the Halmos-Bram-Ito condition and those which are subnormal available at this stage is the following

Proposition 33. *If $\boldsymbol{A} = (A_1, \dots, A_\kappa)$ is a subnormal system of operators in $\mathscr{H}$, then $\boldsymbol{A}$ satisfies $\mathsf{HBI}(\mathscr{D}^\infty(\boldsymbol{A}))$.*

It follows from Proposition 33 that if $\boldsymbol{N} = (N_1, \dots, N_\kappa)$ is a normal extension of a subnormal system $\boldsymbol{A} = (A_1, \dots, A_\kappa)$, then $\mathscr{D}^\infty(\boldsymbol{A})$ is invariant for $\boldsymbol{N}$, $N_iN_jf = N_jN_if$, $\boldsymbol{A}^\alpha f = \boldsymbol{N}^\alpha f$ for $i, j = 1, \dots, \kappa$, $\alpha \in \boldsymbol{Z}_+^\kappa$ and $f \in \mathscr{D}^\infty(\boldsymbol{A})$.

Even in the case of a single operator, as is pretty well known (cf. [**10**], [**30**], [**33**]), the converse to Proposition 33 does not hold. However, we have

Proposition 34. *Suppose $\mathscr{E}$ is a dense linear subspace of $\mathscr{H}$. A system $\boldsymbol{A} = (A_1, \dots, A_\kappa) \in \boldsymbol{L}(\mathscr{E})^\kappa$ satisfies $\mathsf{HBI}(\mathscr{E})$ if and only if there exists a Hilbert space $\mathscr{K}$, a dense linear subspace $\mathscr{F}$ of $\mathscr{K}$ and a system $\boldsymbol{N} = (N_1, \dots, N_\kappa) \in \boldsymbol{L}^\#(\mathscr{F})^\kappa$ such that $\mathscr{H} \subset \mathscr{K}$, $A_i \subset N_i$, $N_iN_j = N_jN_i$ and $N_iN_j^\# = N_j^\# N_i$ for all $i, j = 1, \dots, \kappa$. If this happens, then $\mathscr{F}$ can be chosen so that it coincides with the linear span of the set $\bigcup\{\boldsymbol{N}^{\#\alpha}(\mathscr{E}); \alpha \in \boldsymbol{Z}_+^\kappa\}$, where $\boldsymbol{N}^\# = (N_1^\#, \dots, N_\kappa^\#)$.*

Theorem 35. *Let $\mathscr{E}$ be a dense linear subspace of $\mathscr{H}$. Suppose $A_0, A_1, \dots, A_\kappa \in \boldsymbol{L}(\mathscr{E})$ $(\kappa \geq 1)$ fulfil the following three conditions*

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[6] Under some circumstances (e.g. for bounded operators) the commutativity of $\boldsymbol{A}$ can be inferred from the positivity condition (11), cf. [**34**].

[7] Using arguments based on the spectral measure of $\boldsymbol{N}$ one can easily show (like in the proof of Proposition 4) that $\mathscr{D}^\infty(\boldsymbol{N}) = \mathscr{D}^\infty(N_1^*, \dots, N_\kappa^*)$; this justifies the definition of $\mathscr{F}_{\boldsymbol{N}}[\mathscr{E}]$.

(i) *A_i satisfies* $\mathsf{HBI}(\mathscr{E})$ *and* $A_0A_i = A_iA_0$ *for* $i = 1, \dots, \kappa$,

(ii) $\|A_i f\| \le c(\|f\| + \|A_0 f\|)$ *for* $f \in \mathscr{E}$ *and* $i = 1, \dots, \kappa$, *with some* $c > 0$,

(iii) *A_0 is essentially normal.*

Then spaces $\mathscr{D}^\infty(\bar{A}_0)$ *and* $\mathscr{B}(\bar{A}_0)$ *are invariant for* $(\bar{A}_1, \dots, \bar{A}_\kappa)$ *and* $(A_1^*, \dots, A_\kappa^*)$. *If, moreover, the system* $(\bar{A}_1, \dots, \bar{A}_\kappa)$ *satisfies* $\mathsf{HBI}(\mathscr{B}(\bar{A}_0))$, *then* $(A_0, A_1, \dots, A_\kappa)$ *is subnormal, it has a minimal normal extension of cyclic type and* $\mathscr{E}$ *(as well as* $\mathscr{B}(\bar{A}_0)$*) is a core of the system* $(\bar{A}_0, \bar{A}_1, \dots, \bar{A}_\kappa)$.

PROOF. By Proposition 34, for every $i = 1, \dots, \kappa$, there exists a Hilbert space $\mathscr{K}_i$, a dense linear subspace $\mathscr{F}_i$ of $\mathscr{K}_i$ and $N_i \in \boldsymbol{L}^\#(\mathscr{F}_i)$ such that $\mathscr{H} \subset \mathscr{K}_i$, $A_i \subset N_i$ and $N_iN_i^\# = N_i^\# N_i$. It follows from [**46**, Fact D] that

$$\mathscr{D}(A_i) \subset \mathscr{D}(A_i^*) \quad \text{and} \quad P_iN_i^\#|_{\mathscr{E}} \subset A_i^*,$$

where P_i is the orthogonal projection of $\mathscr{K}_i$ onto $\mathscr{H}$. This in turn implies that

$$\|A_i^* f\| = \|P_iN_i^\# f\| \le \|N_i^\# f\| = \|N_i f\| = \|A_i f\|, \quad f \in \mathscr{E},$$

so the operator A_i is hyponormal. Since $A_0A_i = A_iA_0$, we get $\langle A_0 f, A_i^* g\rangle = \langle A_i f, A_0^* g\rangle$ for $f, g \in \mathscr{E}$. By Lemma 9, the spaces $\mathscr{D}^\infty(\bar{A}_0)$ and $\mathscr{B}(\bar{A}_0)$ are invariant for $\bar{A}_0, \dots, \bar{A}_\kappa$, $A_0^*, \dots, A_\kappa^*$, and

$$A_0^*\bar{A}_i f = \bar{A}_i A_0^* f, \quad f \in \mathscr{D}^\infty(\bar{A}_0), \ i = 0, \dots, \kappa, \tag{12}$$

$$\mathscr{B}(\bar{A}_0) \subset \mathscr{B}(\bar{A}_i), \quad i = 1, \dots, \kappa. \tag{13}$$

Assume now that $(\bar{A}_1, \dots, \bar{A}_\kappa)$ satisfies $\mathsf{HBI}(\mathscr{B}(\bar{A}_0))$. Let us define a new system $\tilde{\boldsymbol{A}} = (\bar{A}_1|_{\mathscr{B}(\bar{A}_0)}, \dots, \bar{A}_\kappa|_{\mathscr{B}(\bar{A}_0)}) \in \boldsymbol{L}(\mathscr{B}(\bar{A}_0))^\kappa$. Then, by (12), we have

$$\sum_{\substack{m,n\ge 0\\ \alpha,\beta \in \boldsymbol{Z}_+^\kappa}} \langle \bar{A}_0^m \bar{A}_1^{\alpha_1} \cdots \bar{A}_\kappa^{\alpha_\kappa} f_{n,\beta}, \bar{A}_0^n \bar{A}_1^{\beta_1} \cdots \bar{A}_\kappa^{\beta_\kappa} f_{m,\alpha}\rangle$$

$$= \sum_{\substack{m,n\ge 0\\ \alpha,\beta \in \boldsymbol{Z}_+^\kappa}} \langle \tilde{\boldsymbol{A}}^\alpha A_0^{*n} f_{n,\beta}, \tilde{\boldsymbol{A}}^\beta A_0^{*m} f_{m,\alpha}\rangle = \sum_{\alpha,\beta \in \boldsymbol{Z}_+^\kappa} \langle \tilde{\boldsymbol{A}}^\alpha g_\beta, \tilde{\boldsymbol{A}}^\beta g_\alpha\rangle \ge 0,$$

for any finite sequence $\{f_{m,\alpha}\}_{m\ge 0, \alpha \in \boldsymbol{Z}_+^\kappa} \subset \mathscr{B}(\bar{A}_0)$ with $g_\beta = \sum_{n\ge 0} A_0^{*n} f_{n,\beta} \in \mathscr{B}(\bar{A}_0)$. Thus the system $(\bar{A}_0, \dots, \bar{A}_\kappa)$ satisfies $\mathsf{HBI}(\mathscr{B}(\bar{A}_0))$. By Proposition 34 there exists a Hilbert space $\mathscr{M}$, a dense linear subspace $\mathscr{D}$ of $\mathscr{M}$ and a system $\boldsymbol{M} = (M_0, \dots, M_\kappa) \in \boldsymbol{L}^\#(\mathscr{D})^{\kappa+1}$ such that $\mathscr{H} \subset \mathscr{M}$, $\bar{A}_i|_{\mathscr{B}(\bar{A}_0)} \subset M_i$, $M_iM_j = M_jM_i$ and $M_iM_j^\# = M_j^\# M_i$ for all $i, j = 0, \dots, \kappa$, and $\mathscr{D}$ is the linear span of the set $\bigcup\{\boldsymbol{M}^{\#\alpha}(\mathscr{B}(\bar{A}_0)); \alpha \in \boldsymbol{Z}_+^{\kappa+1}\}$. Since, due to (13), $\mathscr{B}(\bar{A}_0) \subset \mathscr{B}(M_i)$ for $i = 0, \dots, \kappa$, we infer from [**37**, Proposition 2] that $\mathscr{D} = \mathscr{B}(M_i)$ for $i = 0, \dots, \kappa$ and consequently, by [**37**, Theorem 2], the system $(\overline{M}_0, \dots, \overline{M}_\kappa)$ is a normal extension of $(\bar{A}_0|_{\mathscr{B}(\bar{A}_0)}, \dots, \bar{A}_\kappa|_{\mathscr{B}(\bar{A}_0)})$ which is minimal of cyclic type[8]. Below we show that $\mathscr{B}(\bar{A}_0)$ is a core for every $\bar{A}_i$, $i = 0, \dots, \kappa$, so we will get that $(\overline{M}_0, \dots, \overline{M}_\kappa)$ is a normal extension of $(\bar{A}_0, \dots, \bar{A}_\kappa)$, which is minimal of cyclic type.

[8] We can also apply a particular case of [**37**, Theorem 10] (see also [**36**, Theorem 2] for the case of a single operator) to conclude that the system $(\bar{A}_0|_{\mathscr{B}(\bar{A}_0)}, \dots, \bar{A}_\kappa|_{\mathscr{B}(\bar{A}_0)})$ is subnormal. It was mentioned in [**37**, Remark 8] that a system like this has a minimal normal extension of cyclic type (under a different name).

It follows from Proposition 6 that $\mathscr{D}(\bar{A}_0) = \bigcap_{j=0}^{\kappa} \mathscr{D}(\bar{A}_j)$. Take $f \in \mathscr{D}(\bar{A}_0)$. Since $\mathscr{B}(\bar{A}_0)$ (resp. $\mathscr{E}$) is a core of $\bar{A}_0$, there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathscr{B}(\bar{A}_0)$ (resp. $\{f_n\}_{n=1}^{\infty} \subset \mathscr{E}$) such that $f_n \to f$ and $\bar{A}_0(f_n) \to \bar{A}_0(f)$ as $n \to \infty$. By Proposition 6, $\bar{A}_j(f_n) \to \bar{A}_j(f)$ as $n \to \infty$ for every $j = 0, \ldots, \kappa$. Thus we have proved that $\mathscr{G}(\bar{A}_0, \ldots, \bar{A}_\kappa) \subset \overline{\mathscr{G}(\bar{A}_0|_{\mathscr{B}(\bar{A}_0)}, \ldots, \bar{A}_\kappa|_{\mathscr{B}(\bar{A}_0)})}$ and $\mathscr{G}(\bar{A}_0, \ldots, \bar{A}_\kappa) \subset \overline{\mathscr{G}(A_0, \ldots, A_\kappa)}$ which means that $\mathscr{B}(\bar{A}_0)$ and $\mathscr{E}$ are cores of the system $(\bar{A}_0, \ldots, \bar{A}_\kappa)$. We have also proved that $A_i \subset \overline{\bar{A}_i|_{\mathscr{B}(\bar{A}_0)}}$ for $i = 0, \ldots, \kappa$ (because evidently $\mathscr{E} \subset \mathscr{D}(\bar{A}_0)$). This completes the proof. □

Notice that under the assumptions of Theorem 35 the space $\mathscr{H}$ reduces the operator $\overline{M}_0$ to $\bar{A}_0$, where $(\overline{M}_0, \ldots, \overline{M}_\kappa)$ is the normal extension of the system $(\bar{A}_0, \ldots, \bar{A}_\kappa)$ appearing in the proof of Theorem 35. This follows from [**39**, Corollary 1].

Corollary 36. *Suppose the operators* $A_0, A_1, \ldots, A_\kappa \in \boldsymbol{L}(\mathscr{E})$ $(\kappa \geq 1)$ *fulfil conditions* (i), (ii) *and* (iii) *of Theorem* 35. *If the system* $(A_1, \ldots, A_\kappa)$ *is subnormal, then the system* $(A_0, A_1, \ldots, A_\kappa)$ *is subnormal, it has a minimal normal extension of cyclic type and* $\mathscr{E}$ *(as well as* $\mathscr{B}(\bar{A}_0)$*) is a core of the system* $(\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_\kappa)$.

Proof. It follows from Theorem 35 that $\mathscr{B}(\bar{A}_0)$ is invariant for $(\bar{A}_1, \ldots, \bar{A}_\kappa)$. Since the system $(\bar{A}_1, \ldots, \bar{A}_\kappa)$ is subnormal, so is $(\bar{A}_1|_{\mathscr{B}(\bar{A}_0)}, \ldots, \bar{A}_\kappa|_{\mathscr{B}(\bar{A}_0)})$. By Proposition 33, the system $(\bar{A}_1, \ldots, \bar{A}_\kappa)$ satisfies $\mathsf{HBI}(\mathscr{B}(\bar{A}_0))$. Applying Theorem 35 completes the proof. □

Corollary 37. *Suppose the operators* $A_0, A_1, \ldots, A_\kappa \in \boldsymbol{L}(\mathscr{E})$ $(\kappa \geq 1)$ *fulfil conditions* (i) *and* (ii) *of Theorem* 35. *If the system* $(A_1, \ldots, A_\kappa)$ *satisfies* $\mathsf{HBI}(\mathscr{E})$ *and* A_0^n *is essentially normal for every* $n \geq 1$, *then the system* $(A_0, A_1, \ldots, A_\kappa)$ *is subnormal, it has a minimal normal extension of cyclic type and* $\mathscr{E}$ *(as well as* $\mathscr{B}(\bar{A}_0)$*) is a core of the system* $(\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_\kappa)$.

Proof. Set $\boldsymbol{A} = (A_1, \ldots, A_\kappa)$, $\bar{\boldsymbol{A}} = (\bar{A}_1, \ldots, \bar{A}_\kappa)$. By Corollary 21 we have

$$\mathscr{E} \text{ is a core of every } \bar{A}_0^m, \quad m \geq 1. \tag{14}$$

It follows from Theorem 35 that the space $\mathscr{B}(\bar{A}_0)$ is invariant for $\bar{A}_j$ and A_j^*, $1 \leq j \leq \kappa$. This and the fact that $A_i A_j = A_j A_i$ for all $i, j \geq 1$ enables us to apply Proposition 2; what we get is $\bar{A}_i \bar{A}_j f = \bar{A}_j \bar{A}_i f$ for $f \in \mathscr{B}(\bar{A}_0)$ and $1 \leq i, j \leq \kappa$.

Fix $m \geq 1$ and take a finite sequence $\{f_\alpha; \alpha \in \boldsymbol{Z}_+^\kappa, |\alpha| \leq m\} \subset \mathscr{B}(\bar{A}_0)$. By (14), for every such α there exists a sequence $\{f_{\alpha,n}\}_{n=1}^{\infty} \subset \mathscr{E}$ such that $f_{\alpha,n} \to f_\alpha$ and $A_0^m f_{\alpha,n} \to \bar{A}_0^m f_\alpha$ as $n \to \infty$. Since A_0 is paranormal, one can deduce from part (ii) of Lemma 22 that $\boldsymbol{A}^\beta f_{\alpha,n} \to \bar{\boldsymbol{A}}^\beta f_\alpha$ as $n \to \infty$ for $\beta \in \boldsymbol{Z}_+^\kappa$ with $|\beta| \leq m$. This and the fact that $\boldsymbol{A}$ satisfies $\mathsf{HBI}(\mathscr{E})$ lead to

$$\sum_{\substack{\alpha, \beta \in \boldsymbol{Z}_+^\kappa \\ |\alpha|, |\beta| \leq m}} \langle \bar{\boldsymbol{A}}^\alpha f_\beta, \bar{\boldsymbol{A}}^\beta f_\alpha \rangle = \lim_{n \to \infty} \sum_{\substack{\alpha, \beta \in \boldsymbol{Z}_+^\kappa \\ |\alpha|, |\beta| \leq m}} \langle \boldsymbol{A}^\alpha f_{\beta,n}, \boldsymbol{A}^\beta f_{\alpha,n} \rangle \geq 0,$$

so $\bar{\boldsymbol{A}}$ satisfies $\mathsf{HBI}(\mathscr{B}(\bar{A}_0))$. Applying Theorem 35 completes the proof. □

Domination and quasianalyticity.

The result which follows is simple in essence and relates quasianalytic vectors of two pointwise commuting operators one of which dominates the other. It has its origin in Section 2 of [**19**].

LEMMA 38. *Suppose $\mathscr{D}$ is a dense linear subspace of a Hilbert space $\mathscr{H}$ and $A, B \in \boldsymbol{L}(\mathscr{D})$ are pointwise commuting operators such that*

$$\|Bf\| \le c(\|f\| + \|Af\|), \quad f \in \mathscr{D}, \tag{15}$$

for some $c > 0$. Then for every $n \ge 0$

$$\|B^n f\| \le c^n \sum_{j=0}^{n} \binom{n}{j} \|A^j f\|, \quad f \in \mathscr{D}. \tag{16}$$

Consequently, if A is paranormal, then $\mathscr{B}_a(A) \subset \mathscr{B}_{c(1+a)}(B)$ for every $a \ge 0$, $\mathscr{B}(A) \subset \mathscr{B}(B)$ and $\mathscr{Q}(A) \subset \mathscr{Q}(B)$.

PROOF. We prove (16) by induction on n. The case $n = 1$ is just (15). If (16) holds for a fixed $n \ge 1$, then (15) implies that

$$\begin{aligned}
\|B^{n+1} f\| &\le c^n \sum_{j=0}^{n} \binom{n}{j} \|BA^j f\| \le c^{n+1} \left(\sum_{j=0}^{n} \binom{n}{j} \|A^j f\| + \sum_{j=0}^{n} \binom{n}{j} \|A^{j+1} f\| \right) \\
&= c^{n+1} \left(\|f\| + \sum_{j=1}^{n} \left[\binom{n}{j} + \binom{n}{j-1} \right] \|A^j f\| + \|A^{n+1} f\| \right) \\
&= c^{n+1} \sum_{j=0}^{n+1} \binom{n+1}{j} \|A^j f\|, \quad f \in \mathscr{D},
\end{aligned}$$

which completes the proof of (16).

Suppose A is paranormal. Take $f \in \mathscr{D}$ with $\|f\| = 1$. Then, by formula (4) of [**45**], the sequence $\{\|A^n f\|^{1/n}\}_{n=1}^{\infty}$ is monotonically increasing. This and (16) give us

$$\begin{aligned}
\|B^n f\| &\le c^n \left(1 + \sum_{j=1}^{n} \binom{n}{j} (\|A^j f\|^{1/j})^j \right) \le c^n \left(1 + \sum_{j=1}^{n} \binom{n}{j} (\|A^n f\|^{1/n})^j \right) \\
&= c^n (1 + \|A^n f\|^{1/n})^n, \quad n \ge 1, f \in \mathscr{D}.
\end{aligned} \tag{17}$$

If $a \ge 0$ and $f \in \mathscr{B}_a(A)$, then part (b) of Lemma 8 in [**41**] and the monotonicity of the sequence $\{\|A^n f\|^{1/n}\}_{n=1}^{\infty}$ lead to $\|A^n f\|^{1/n} \le a$ for $n \ge 1$. This and (17) imply that $\|B^n f\| \le c^n (1+a)^n$ for $n \ge 0$. Thus $f \in \mathscr{B}_{c(1+a)}(B)$. In case $f \in \mathscr{Q}(A)$, we can assume that $\|A^k f\|^{1/k} > 1$ for some $k \ge 1$ (because otherwise $f \in \mathscr{B}_1(A)$). By the monotonicity of $\{\|A^n f\|^{1/n}\}_{n=1}^{\infty}$, we have $\|A^n f\|^{1/n} > 1$ for $n \ge k$, so (17) yields $\|B^n f\|^{1/n} \le 2c\|A^n f\|^{1/n}$ for $n \ge k$. Hence $f \in \mathscr{Q}(B)$. Removing the normalization condition $\|f\| = 1$ does not spoil the conclusion. $\square$

Presence of quasianalytic vectors allows us to make an immediate use of Lemma 38 so as to get a result which can be considered as a kind of complement to the main subject of the paper.

Proposition 39. *Let* $\boldsymbol{A} = (A_0, A_1, \dots, A_\kappa) \in \boldsymbol{L}(\mathscr{E})^{\kappa+1}$ $(\kappa \geq 1)$. *Suppose that*

(i) $\boldsymbol{A}$ *satisfies* $\mathsf{HBI}(\mathscr{E})$,

(ii) $\|A_i f\| \leq c(\|f\| + \|A_0 f\|)$ *for* $f \in \mathscr{E}$ *and* $i = 1, \dots, \kappa$, *with some* $c > 0$,

(iii) *the linear span of the set of all quasianalytic vectors of* A_0 *is equal to* $\mathscr{E}$.

Then $\boldsymbol{A}$ *is a subnormal system which has a minimal normal extension of cyclic type, and* $\mathscr{E}$ *is a core of the system* $(\bar{A}_0, \bar{A}_1, \dots, \bar{A}_\kappa)$.

Proof. It follows from (i) that A_0 is hyponormal and hence A_0 is paranormal (cf. the proof of Theorem 35). By (ii) and Lemma 38, $\mathscr{Q}(A_0) \subset \mathscr{Q}(A_i)$ for every $i = 1, \dots, \kappa$. Applying [**37**, Theorem 10], we get subnormality of $\boldsymbol{A}$. That $\boldsymbol{A}$ has a minimal normal extension of cyclic type can be proved in the same way as in Theorem 35. Repeating the last part of the proof of Theorem 35 shows that $\mathscr{E}$ is a core of the system $(\bar{A}_0, \bar{A}_1, \dots, \bar{A}_\kappa)$. $\square$

The assumption (iii) of Proposition 39 can not be omitted, cf. Example 15. Also (ii) can not be omitted when $\kappa \geq 1$. Indeed, consider $A_0 = I_{\mathscr{E}}$ and any pair (S, T) of pointwise commuting symmetric operators which does not extends to a pair of spectrally commuting selfadjoint operators (Nelson's example). Set $A_1 = S + \mathrm{i}T$. Then the pair (A_0, A_1) of double pointwise commuting formally normal operators is not subnormal, though it satisfies the assumptions (i) and (iii) of Proposition 39. The case $\kappa = 1$ is more delicate, because according to [**20**, Theorem 10] any cyclic pair of pointwise commuting symmetric operators which satisfies the assumption (iii) of Proposition 39 is subnormal.

Subnormality of algebraic operators.

Let $\mathscr{D}$ be a dense linear subspace of a Hilbert space $\mathscr{H}$. According to the Nelson criterion, if $A_1, A_2 \in \boldsymbol{L}(\mathscr{D})$ are pointwise commuting symmetric operators such that $A_1^2 + A_2^2$ is essentially selfadjoint, then $\bar{A}_1$ and $\bar{A}_2$ are spectrally commuting selfadjoint operators; the essential selfadjointness of $A_1^2 + A_2^2$ is equivalent to[9] $\overline{\mathscr{R}(\varepsilon + A_1^2 + A_2^2)} = \mathscr{H}$ for some $\varepsilon > 0$ (or equivalently: for every $\varepsilon > 0$). In particular this *occurs* when $\mathscr{R}(\varepsilon + A_1^2 + A_2^2) = \mathscr{D}$ for some $\varepsilon > 0$. The question is what happens in the limit case $\varepsilon = 0$. The answer has been given in [**42**, Corollary 42]: we lose essential spectral commutativity still preserving subnormality, i.e. the possibility of extending the pair (A_1, A_2) to other one composed of spectrally commuting selfadjoint operators acting in a larger Hilbert space. Let us discuss this question under an additional assumption that the pair (A_1, A_2) is cyclic with a cyclic vector $e \in \mathscr{D}$, i.e. $\mathscr{D}$ is equal to the linear span of the set $\{A_1^m A_2^n e; m, n \geq 0\}$. Then, as is easily seen (see the proof of Lemma 46 in [**42**]), $\mathscr{R}(\varepsilon + A_1^2 + A_2^2) = \mathscr{D}$ if and only if there exists a polynomial $r \in \boldsymbol{R}[X_1, X_2]$ such that $(\varepsilon + A_1^2 + A_2^2) r(A_1, A_2) = I_{\mathscr{D}}$. On the other hand $\mathscr{R}(A_1^2 + A_2^2) = \mathscr{D}$ if and only if there exists $r \in \boldsymbol{R}[X_1, X_2]$ such that $(A_1^2 + A_2^2) r(A_1, A_2) = I_{\mathscr{D}}$. Summarizing, if the polynomial

[9] because the operator $A_1^2 + A_2^2$ is positive.

$(\varepsilon + X_1^2 + X_2^2)r(X_1, X_2) - 1$ annihilates the pair (A_1, A_2) for some $\varepsilon \geq 0$ (no cyclicity is required now), then (A_1, A_2) is at least subnormal.

The essence of [**25**], for the 2-dimensional Hamburger moment problem say, is in considering a polynomial $p = (1 + X_1^2 + X_2^2)X_3 - 1$ (see also Theorem 45). The advantage is taken from the fact that the algebraic set induced by p is unbounded in variables X_1 and X_2 but it is bounded with respect to the additional variable X_3. Illustrating our "ε tends to 0" programme of the preceding paragraph let us replace in p the first 1 by 0 so as to get $(X_1^2 + X_2^2)X_3 - 1$. Then the algebraic set induced by the latter polynomial is unbounded in each variable which makes things much more complicated. Therefore allowing unbounded algebraic sets extends applicability of the "partially bounded" approach of [**25**]. However, unlike to [**25**], this focuses on solving the moment problem itself leaving its determinacy apart. In what follows we work out this in detail. Though we deal here with systems of symmetric operators, the reader can easily formulate versions for systems of doubly pointwise commuting formally normal operators.

Proposition 40. *Let $\mathscr{D}$ be a dense linear subspace of $\mathscr{H}$ and $A_1, A_2, B \in \boldsymbol{L}(\mathscr{D})$ be pointwise commuting symmetric operators. If $q \in \boldsymbol{C}[X_1, X_2]$ is such that*

$$(A_1 + \mathrm{i}A_2)q(A_1, A_2)B = I_{\mathscr{D}}, \tag{18}$$

then the triplet (A_1, A_2, B) extends to a triplet (S_1, S_2, T) of spectrally commuting selfadjoint operators in a Hilbert space $\mathscr{K} \supset \mathscr{H}$. In particular, this is the case when $(A_1^2 + A_2^2)r(A_1, A_2)B = I_{\mathscr{D}}$ for some $r \in \boldsymbol{R}[X_1, X_2]$.

Proof. It follows from (18) that $\mathscr{R}(A_1 + \mathrm{i}A_2) = \mathscr{D}$, so by [**42**, Corollary 42] there exists a Hilbert space $\mathscr{K} \supset \mathscr{H}$ and a pair (S_1, S_2) of spectrally commuting selfadjoint operators in $\mathscr{K}$ such that $A_i \subset S_i$ for $i = 1, 2$. Denote by E the spectral measure of the pair (S_1, S_2), i.e. $S_i = \int_{\boldsymbol{R}^2} x_i E(\mathrm{d}x_1, \mathrm{d}x_2)$ for $i = 1, 2$. The closed linear span $\mathscr{M}$ of the set $\{E(\sigma)(h); h \in \mathscr{H}, \sigma$ is a Borel subset of $\boldsymbol{R}^2\}$ reduces E, $\mathscr{H} \subset \mathscr{M}$ and the pair $(S_1|_{\mathscr{M}}, S_2|_{\mathscr{M}})$ is composed of spectrally commuting selfadjoint operators in $\mathscr{M}$ such that $A_i \subset S_i|_{\mathscr{M}}$ for $i = 1, 2$. One can show that $\mathscr{M}$ is the smallest closed linear subspace of $\mathscr{K}$ which reduces the operators S_1 and S_2, and which contains $\mathscr{H}$. Thus we can assume, without loss of generality, that $\mathscr{K}$ is minimal in a sense that $\mathscr{K} = \mathscr{M}$ (this is so-called *minimality of spectral type*, see [**38**]).

Set $N = \int_{\boldsymbol{R}^2} (x_1 + \mathrm{i}x_2) E(\mathrm{d}x_1, \mathrm{d}x_2)$ and $C = q(A_1, A_2)B$. By (18) we have

$$C^{-1} = A_1 + \mathrm{i}A_2 \subset S_1 + \mathrm{i}S_2 = N. \tag{19}$$

Since $\mathscr{N}(N^*) = \mathscr{N}(N) = \mathscr{R}(E(\{(0,0)\}))$, we deduce from (19) that

$$\mathscr{D} = C^{-1}(\mathscr{D}) = N(\mathscr{D}) \subset \mathscr{R}(N) \subset \mathscr{K} \ominus \mathscr{N}(N) = \mathscr{R}(E(\boldsymbol{R}^2 \backslash \{(0,0)\})).$$

This implies that the closed linear space $\mathscr{K} \ominus \mathscr{N}(N)$ contains $\mathscr{H}$ and reduces the operators S_1 and S_2. As $\mathscr{K}$ is minimal, it must be $E(\{(0,0)\}) = 0$. By (18) we have

$$\mathscr{D} = q(A_1, A_2)(\mathscr{D}) \subset \mathscr{R}\left(\int_{\boldsymbol{R}^2} q \, \mathrm{d}E\right) \subset \mathscr{K} \ominus \mathscr{N}\left(\int_{\boldsymbol{R}^2} \bar{q} \, \mathrm{d}E\right) = \mathscr{K} \ominus \mathscr{R}(E(q^{-1}(\{0\}))),$$

which, in turn, implies that the closed linear space $\mathscr{K} \ominus \mathscr{N}(\int_{\boldsymbol{R}^2} \bar{q}\,\mathrm{d}E)$ contains $\mathscr{H}$ and reduces the operators S_1 and S_2. Hence, by minimality of $\mathscr{K}$, $E(q^{-1}(\{0\})) = 0$. We can now infer from (18), (19) and equalities $E(\{(0,0)\}) = E(q^{-1}(\{0\})) = 0$ that

$$B = (A_1 + \mathrm{i}A_2)^{-1} q(A_1, A_2)^{-1} \subset N^{-1}\left(\int_{\boldsymbol{R}^2} q\,\mathrm{d}E\right)^{-1} \subset \int_{\boldsymbol{R}^2} \varphi\,\mathrm{d}E,$$

where $\varphi(x_1, x_2) = ((x_1 + \mathrm{i}x_2)q(x_1, x_2))^{-1}$. One can check that the selfadjoint operator $T = \int_{\boldsymbol{R}^2} \operatorname{Re}\varphi\,\mathrm{d}E$ extends B, which completes the proof of the first part of the conclusion. To get the other one set $q(X_1, X_2) = (X_1 - \mathrm{i}X_2)r(X_1, X_2)$. $\square$

Proposition 41. *Suppose that $A_1, \ldots, A_\kappa, B \in \boldsymbol{L}(\mathscr{D})$ are pointwise commuting symmetric operators such that*

$$(I_{\mathscr{D}} + q_1(A_1)^2 + \cdots + q_\kappa(A_\kappa)^2) r(A_1, \ldots, A_\kappa) B = I_{\mathscr{D}} \tag{20}$$

for some $r \in \boldsymbol{R}[X_1, \ldots, X_\kappa]$ and $q_1, \ldots, q_\kappa \in \boldsymbol{R}[X]$ with $\deg q_i \geq 1$. Then there exists a selfadjoint extension T of B acting in $\mathscr{H}$ such that $\bar{A}_1, \ldots, \bar{A}_\kappa, T$ are spectrally commuting selfadjoint operators. If $r \equiv 1$, then $T = \bar{B}$ is bounded, and $\mathscr{D}$ is a core of the system $(\bar{A}_1, \ldots, \bar{A}_\kappa, \bar{B})$.

Proof. By (20), $\mathscr{R}(I_{\mathscr{D}} + q_1(A_1)^2 + \cdots + q_\kappa(A_\kappa)^2) = \mathscr{D}$, so the positive operator $q_1(A_1)^2 + \cdots + q_\kappa(A_\kappa)^2$ is essentially selfadjoint. It follows from Corollary 31 that $\bar{A}_1, \ldots, \bar{A}_\kappa$ are spectrally commuting selfadjoint operators, and $\mathscr{D}$ is a core of the system $(\bar{A}_1, \ldots, \bar{A}_\kappa)$. If E is the spectral measure of the system $(\bar{A}_1, \ldots, \bar{A}_\kappa)$, then $E(r^{-1}(\{0\})) = 0$ (see the proof of Proposition 40). Thus the operator

$$T = \int_{\boldsymbol{R}^\kappa} \frac{1}{(1 + q_1(x_1)^2 + \cdots + q_\kappa(x_\kappa)^2) r(x_1, \ldots, x_\kappa)} E(\mathrm{d}x_1, \ldots, \mathrm{d}x_\kappa)$$

is a required selfadjoint extension of B. In case $r \equiv 1$, $T = \bar{B} \in \boldsymbol{B}(\mathscr{H})$ and consequently $\mathscr{D}$ is a core of $(\bar{A}_1, \ldots, \bar{A}_\kappa, \bar{B})$. $\square$

Below we present some other sufficient conditions for subnormality; this of course does not exhaust all the possibilities in this matter (see Proposition 32). We recall, by the way, that a surjective operator $A \in \boldsymbol{L}(\mathscr{D})$ which satisfies $\mathsf{HBI}(\mathscr{D})$ is automatically subnormal (cf. [**42**, Theorem 39]).

Remark 42. Modifying proofs of Propositions 40 and 41 one can show that:

1° If a system $\boldsymbol{A} = (A_1, \ldots, A_\kappa) \in \boldsymbol{L}(\mathscr{D})^\kappa$ is subnormal and every A_i is bijective, then the system $(A_1^{-1}, \ldots, A_\kappa^{-1})$ is subnormal as well; moreover, if $(N_1, \ldots, N_\kappa)$ is a normal extension of $\boldsymbol{A}$ which is minimal of spectral type, then $\mathscr{N}(N_j) = \{0\}$ for every $j = 1, \ldots, \kappa$, and $(N_1^{-1}, \ldots, N_\kappa^{-1})$ is a normal extension of $(A_1^{-1}, \ldots, A_\kappa^{-1})$ which is minimal of spectral type (minimality of this kind is implicitly defined in the first paragraph of the proof of Proposition 40).

2° If a system $(A_1, \ldots, A_\kappa) \in \boldsymbol{L}(\mathscr{D})^\kappa$ is subnormal, $B \in \boldsymbol{L}(\mathscr{D})$ is an operator which pointwise commutes with every A_i, $i = 1, \ldots, \kappa$, and $p(A_1, \ldots, A_\kappa)B = I_{\mathscr{D}}$ for some $p \in \boldsymbol{C}[X_1, \ldots, X_\kappa]$, then the system $(A_1, \ldots, A_\kappa, B)$ is subnormal.

3° If $A_1, A_2, B_0, \ldots, B_n \in \boldsymbol{L}(\mathscr{D})$ are pointwise commuting symmetric operators such that $(A_1^2 + A_2^2) r_0(A_1, A_2) B_0 = I_{\mathscr{D}}$ and $r_i(A_1, A_2, B_0, \ldots, B_{i-1}) B_i = I_{\mathscr{D}}$ for $i = 1, \ldots, n$, where $r_i \in \boldsymbol{R}[X_1, \ldots, X_{2+i}]$, then $(A_1, A_2, B_0, \ldots, B_n)$ extends to a system of spectrally commuting selfadjoint operators in a larger Hilbert space.

4° If $A_1, \ldots, A_\kappa, B_0, \ldots, B_n \in \boldsymbol{L}(\mathscr{D})$ are pointwise commuting symmetric operators satisfying conditions $(I_{\mathscr{D}} + q_1(A_1)^2 + \cdots + q_\kappa(A_\kappa)^2) r_0(A_1, \ldots, A_\kappa) B_0 = I_{\mathscr{D}}$ and $r_i(A_1, \ldots, A_\kappa, B_0, \ldots, B_{i-1}) B_i = I_{\mathscr{D}}$ for $i = 1, \ldots, n$, where $r_i \in \boldsymbol{R}[X_1, \ldots, X_{\kappa+i}]$ and $q_1, \ldots, q_\kappa \in \boldsymbol{R}[X]$ are polynomials of $\deg q_i \geq 1$, then $(A_1, \ldots, A_\kappa, B_0, \ldots, B_n)$ extends to a system of spectrally commuting selfadjoint operators in $\mathscr{K}$.

5° If $q_1, \ldots, q_\kappa, r_1, \ldots, r_n \in \boldsymbol{R}[X]$ are polynomials with $\deg q_i, \deg r_j \geq 1$, and $A_1, \ldots, A_\kappa, B_1, \ldots, B_n \in \boldsymbol{L}(\mathscr{D})$ are pointwise commuting symmetric operators such that $(I_{\mathscr{D}} + q_1(A_1)^2 + \cdots + q_\kappa(A_\kappa)^2)(r_1(B_1)^2 + \cdots + r_n(B_n)^2) = I_{\mathscr{D}}$, then $\bar{A}_1, \ldots, \bar{A}_\kappa, \bar{B}_1, \ldots, \bar{B}_n$ are spectrally commuting selfadjoint operators, $\bar{B}_1, \ldots, \bar{B}_n$ are bounded and $\mathscr{D}$ is a core of $(\bar{A}_1, \ldots, \bar{A}_\kappa, \bar{B}_1, \ldots, \bar{B}_n)$ (use also Corollary 19). And so on.

Moment problems.

Semialgebraic sets.

Let us begin with recalling some indispensable definitions. A κ-sequence $\gamma = \{\gamma_\alpha\}_{\alpha \in \boldsymbol{Z}_+^\kappa} \subset \boldsymbol{R}$ is said to be *positive definite* if

$$\sum_{\alpha, \beta \in \boldsymbol{Z}_+^\kappa} \gamma_{\alpha+\beta} \lambda_\alpha \bar{\lambda}_\beta \geq 0$$

for every κ-sequence $\{\lambda_\alpha\}_{\alpha \in \boldsymbol{Z}_+^\kappa} \subset \boldsymbol{C}$ which has a finite number of nonzero entries; γ is said to be a *Hamburger moment κ-sequence* if there exists a positive Borel measure μ on $\boldsymbol{R}^\kappa$ (called a *representing measure of* γ) such that

$$\gamma_\alpha = \int_{\boldsymbol{R}^\kappa} x^\alpha \, \mathrm{d}\mu(x), \quad \alpha \in \boldsymbol{Z}_+^\kappa, \tag{21}$$

with the usual multi-index notation: $x^\alpha = x_1^{\alpha_1} \cdots x_\kappa^{\alpha_\kappa}$. It is well known that a Hamburger moment multi-sequence is automatically positive definite, but not conversely (cf. [**5**], [**6**], [**28**], [**13**], [**33**]).

We say that a Hamburger moment κ-sequence γ of the form (21) is *ultradeterminate* if the set $\boldsymbol{C}[X_1, \ldots, X_\kappa]$ is dense in $\mathscr{L}^2(\boldsymbol{R}^\kappa, (1 + \|x\|^2) \, \mathrm{d}\mu(x))$, where $\|x\|^2 = x_1^2 + \cdots + x_\kappa^2$. An ultradeterminate Hamburger moment multi-sequence is *determinate*, i.e. it has a unique representing measure (cf. [**15**]), but not conversely (cf. [**32**]). Recall (cf. [**43**], [**15**], [**42**]) that a κ-sequence γ is positive definite if and only if there exists a Hilbert space $\mathscr{H}$, a dense linear subspace $\mathscr{D}$ of $\mathscr{H}$, a vector $f_0 \in \mathscr{D}$ and a system $\boldsymbol{S} = (S_1, \ldots, S_\kappa) \in \boldsymbol{L}(\mathscr{D})^\kappa$ such that

$$S_1, \ldots, S_\kappa \text{ are pointwise commuting symmetric operators,} \tag{22}$$

$$\boldsymbol{S} \text{ is } f_0\text{-cyclic, i.e. } \mathscr{D} \text{ is the linear span of the set } \{\boldsymbol{S}^\alpha f_0; \alpha \in \boldsymbol{Z}_+^\kappa\}, \tag{23}$$

$$\gamma_\alpha = \langle \boldsymbol{S}^\alpha f_0, f_0 \rangle \quad \text{for every } \alpha \in \boldsymbol{Z}_+^\kappa. \tag{24}$$

If $(\mathscr{H}', \mathscr{D}', f_0', \boldsymbol{S}')$ is another choice, then there exists a unique unitary isomorphism $U : \mathscr{H} \to \mathscr{H}'$ such that $U(\mathscr{D}) = \mathscr{D}'$, $Uf_0 = f_0'$ and $UA_ih = A_i'Uh$ for all $h \in \mathscr{D}$ and $i = 1, \ldots, \kappa$. The κ-sequence γ is a Hamburger moment κ-sequence if and only if $\boldsymbol{S}$ extends to a system of spectrally commuting selfadjoint operators in a larger Hilbert space; moreover, γ is an ultradeterminate Hamburger moment κ-sequence if and only if $\mathscr{D}$ is a core of $(\bar{A}_1, \ldots, \bar{A}_\kappa)$, and $\bar{A}_1, \ldots, \bar{A}_\kappa$ are spectrally commuting selfadjoint operators (cf. [**15**]).

The main question in the multidimensional moment problem is to find additional conditions under which positive definiteness ensure the solution of the Hamburger moment problem. The results which follow offer such conditions. The first of them is related to [**20**, Theorem 10] where is shown that every positive definite 2-sequence, which satisfies the Carleman condition with respect to the first "variable", is always a Hamburger moment 2-sequence (hence the part of the conclusion of Proposition 43 which deal with the solution of 2-dimensional Hamburger moment problem holds true without assuming (ii)). This is no longer true for positive definite 3-sequences (take any positive definite 2-sequence $\{\gamma_{i,j}\}_{i,j=0}^\infty$ which is not a Hamburger moment one and consider the 3-sequence $\{\gamma_{j,k}\}_{i,j,k=0}^\infty$).

Here and subsequently, e_j stands for the sequence $(0, \ldots, 0, 1, 0, \ldots, 0)$ of length κ with the digit 1 on the jth position, $j = 1, \ldots, \kappa$.

Proposition 43. *Let $\gamma = \{\gamma_\alpha\}_{\alpha \in \boldsymbol{Z}_+^\kappa} \subset \boldsymbol{R}$ be a κ-sequence $(\kappa \geq 2)$ such that*

(i) *γ is positive definite,*

(ii) *there exists a number $a > 0$ such that for every $i = 2, \ldots, \kappa$, the κ-sequence $\{a(\gamma_\alpha + \gamma_{\alpha+2e_1}) - \gamma_{\alpha+2e_i}\}_{\alpha \in \boldsymbol{Z}_+^\kappa}$ is positive definite,*

(iii) $\sum_{n=1}^\infty \gamma_{2ne_1}^{-1/(2n)} = +\infty$.

Then γ is an ultradeterminate Hamburger moment κ-sequence having a representing measure with closed support in the set $\bigcap_{i=2}^\kappa \{x \in \boldsymbol{R}^\kappa; x_i^2 \leq a(1 + x_1^2)\}$.

Proof. It follows from (i) that there exists a quadruplet $(\mathscr{H}, \mathscr{D}, f_0, \boldsymbol{S})$ which satisfies conditions (22), (23) and (24). It is a simple matter to verify that operators $A_0, \ldots, A_{\kappa-1}$, where $A_i = S_{i+1}$, satisfy conditions (i), (ii) and (iv) of Theorem 10. By our assumption (iii), $f_0 \in \mathscr{Q}(A_0)$, so (cf. [**37**, Proposition 2]) $\mathscr{D}$ is equal to the linear span of $\mathscr{Q}(A_0)$. Hence A_0 is essentially selfadjoint (cf. [**20**]). By Theorem 10, $\bar{S}_1, \ldots, \bar{S}_\kappa$ are spectrally commuting selfadjoint operators, and $\mathscr{D}$ is a core of the system $(\bar{S}_1, \ldots, \bar{S}_\kappa)$. This implies that γ is an ultradeterminate Hamburger moment κ-sequence with a representing measure $\mu(\cdot) = \langle E(\cdot)f_0, f_0\rangle$, where E is the spectral measure of $(\bar{S}_1, \ldots, \bar{S}_\kappa)$. The localization of the closed support of μ follows from that of E (see Remark 13). □

According to the discussion preceding Proposition 43, we know that condition (ii) can not be omitted in Proposition 43 for $\kappa \geq 3$. Likewise, condition (iii) can not be removed therein as is shown in the following

Example 44. Let $A, B \in \boldsymbol{L}(\mathscr{D})$ be pointwise commuting symmetric operators as in Example 15. We can assume, without loss of generality, that the pair (A, B) is f_0-cyclic (cf. [**33**, Proposition 7.3]). Define a triplet $\boldsymbol{S} = (S_1, S_2, S_3)$ by $S_1 = A^2$, $S_2 = A$ and

$S_3 = B$. Then $\boldsymbol{S}$ is f_0-cyclic and S_1 dominates both S_2 and S_3 on $\mathscr{D}$, however $\boldsymbol{S}$ is not subnormal (cf. Example 15). The 3-sequence γ defined by (24) with $\kappa = 3$ is positive definite, it satisfies condition (ii) of Proposition 43, however it is not a Hamburger moment 3-sequence.

The result which follows is a substantially simplified version of [**25**, Theorem 2.7] while Proposition 30 provides an alternative tool for proving it. Due to our approach it becomes independent of Hilbert's Nullstellensatz.

Theorem 45. *Let $\gamma = \{\gamma_\alpha\}_{\alpha \in \boldsymbol{Z}_+^\kappa}$ $(\gamma_0 > 0)$ be a κ-sequence of real numbers and let $p_i(x) = \sum_{\xi \in J_i} a_{i\xi} x^\xi$ with $a_{i\xi} \in \boldsymbol{R}$ and $J_i \subset \boldsymbol{Z}_+^\kappa$ finite for all $i = 1, \ldots, m$. Set $n = \max\{1, \deg p_1, \ldots, \deg p_m\}$. Then γ is a Hamburger moment κ-sequence with a representing measure on the set $\Sigma \overset{\mathrm{df}}{=} \bigcap_{i=1}^m p_i^{-1}([0, +\infty))$ if and only if there exists a positive definite $(\kappa+1)$-sequence $\boldsymbol{\delta} = \{\delta_{(\alpha,\beta)}\}_{(\alpha,\beta) \in \boldsymbol{Z}_+^\kappa \times \boldsymbol{Z}_+}$ such that*

(i) *$\gamma_\alpha = \delta_{(\alpha,0)}$ for all $\alpha \in \boldsymbol{Z}_+^\kappa$,*

(ii) *$\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+1)} + \sum_{j=1}^\kappa \delta_{(\alpha+2ne_j,\beta+1)}$ for all $(\alpha,\beta) \in \boldsymbol{Z}_+^\kappa \times \boldsymbol{Z}_+$,*

(iii) *the $(\kappa+1)$-sequence $\{\sum_{\xi \in J_i} a_{i\xi} \delta_{(\alpha+\xi,\beta)}\}_{(\alpha,\beta) \in \boldsymbol{Z}_+^\kappa \times \boldsymbol{Z}_+}$ is positive definite for every $i = 1, \ldots, m$.*

The κ-sequence γ has a uniquely determined representing measure on the set Σ if and only if the $(\kappa+1)$-sequence $\boldsymbol{\delta}$ is unique.

A sketch of the proof. If γ has a representing measure μ on Σ, then the $(\kappa+1)$-sequence $\boldsymbol{\delta}$ defined by

$$\delta_{(\alpha,\beta)} = \int_\Sigma \frac{x^\alpha}{(1 + x_1^{2n} + \cdots + x_\kappa^{2n})^\beta} \,\mathrm{d}\mu(x), \quad (\alpha,\beta) \in \boldsymbol{Z}_+^\kappa \times \boldsymbol{Z}_+, \tag{25}$$

is positive definite and it satisfies conditions (i), (ii) and (iii).

Conversely, if $\boldsymbol{\delta}$ is a positive definite $(\kappa+1)$-sequence, then there exists a Hilbert space $\mathscr{H}$, a dense linear subspace $\mathscr{D}$ of $\mathscr{H}$, a vector $f_0 \in \mathscr{D}$ and pointwise commuting symmetric operators $S_1, \ldots, S_\kappa, T \in \boldsymbol{L}(\mathscr{D})$ such that (with $\boldsymbol{S} = (S_1, \ldots, S_\kappa)$)

$$\mathscr{D} \text{ is the linear span of the set } \{\boldsymbol{S}^\alpha T^\beta f_0; (\alpha,\beta) \in \boldsymbol{Z}_+^\kappa \times \boldsymbol{Z}_+\}, \tag{26}$$

$$\delta_{(\alpha,\beta)} = \langle \boldsymbol{S}^\alpha T^\beta f_0, f_0 \rangle, \quad (\alpha,\beta) \in \boldsymbol{Z}_+^\kappa \times \boldsymbol{Z}_+. \tag{27}$$

Using (26) and (27), one can deduce from (ii) that

$$I_{\mathscr{D}} = (I_{\mathscr{D}} + S_1^{2n} + \cdots + S_\kappa^{2n})T. \tag{28}$$

This implies that $\mathscr{R}(I_{\mathscr{D}} + S_1^{2n} + \cdots + S_\kappa^{2n}) = \mathscr{D}$ is dense in $\mathscr{H}$, so the operator $S_1^{2n} + \cdots + S_\kappa^{2n}$ is essentially selfadjoint. By Corollary 31 (with $q_i(X) = X^n$), the operators $\bar{S}_1, \ldots, \bar{S}_\kappa$, $\overline{p_1(\boldsymbol{S})}, \ldots, \overline{p_m(\boldsymbol{S})}$ are selfadjoint and spectrally commuting. Condition (iii) implies that every $p_i(\boldsymbol{S})$ is positive. This in turn implies that the closed support of the spectral measure E of the system $(\bar{S}_1, \ldots, \bar{S}_\kappa)$ is contained in Σ (because $\int p_i \,\mathrm{d}E = \overline{p_i(\boldsymbol{S})} \geq 0$). One can infer from (27) and (28) that $\boldsymbol{\delta}$ satisfies (25) with $\mu(\cdot) = \langle E(\cdot) f_0, f_0 \rangle$. Hence, by (i), μ is a representing measure of γ.

The uniqueness assertion goes in the standard way (cf. [**15**]) provided one defines $\boldsymbol{S}$ as the system of the multiplications by coordinates and T as the multiplication by

$(1+x_1^{2n}+\cdots+x_\kappa^{2n})^{-1}$ considered as operators in $\mathscr{H}=\mathscr{L}^2(\mu)$ with the common dense domain $\mathscr{D}=$ the linear span of rational functions $x^\alpha(1+x_1^{2n}+\cdots+x_\kappa^{2n})^{-\beta}$, $\alpha\in\boldsymbol{Z}_+^\kappa$, $\beta\in\boldsymbol{Z}_+$; $f_0(x)\equiv 1$ is the cyclic vector of $(\boldsymbol{S},T)$ (however still no Nullstellensatz needed). □

Algebraic sets of type A.

Let $p\in\boldsymbol{R}[X_1,\dots,X_\kappa]$ be a polynomial with coefficients $\{a_\alpha\}_{\alpha\in\boldsymbol{Z}_+^\kappa}$. We say that the real algebraic set $p^{-1}(0)$ is of *type* A (cf. [**33**], [**42**]), if each positive definite κ-sequence $\gamma=\{\gamma_\alpha\}_{\alpha\in\boldsymbol{Z}_+^\kappa}$ of real numbers which satisfies the following equality

$$\sum_{\alpha,\beta\in\boldsymbol{Z}_+^\kappa} a_\alpha a_\beta\gamma_{\alpha+\beta}=0 \tag{A}$$

is a Hamburger moment κ-sequence. Notice that condition (A) is equivalent to

$$\sum_{\alpha\in\boldsymbol{Z}_+^\kappa} a_\alpha\gamma_{\alpha+\beta}=0,\quad \beta\in\boldsymbol{Z}_+^\kappa. \tag{A$'$}$$

A Hamburger moment κ-sequence γ satisfies condition (A) if and only if the closed support of every (equivalently: at least one) representing measure of γ is contained in $p^{-1}(0)$ (cf. [**33**, Proposition 2.1]). According to [**33**] there are real algebraic sets which are not of type A. Recall that a κ-sequence γ is positive definite and it satisfies condition (A) if and only if there exists a quadruplet $(\mathscr{H},\mathscr{D},f_0,\boldsymbol{S})$ fulfilling conditions (22), (23), (24) and the equality $p(\boldsymbol{S})=0$ (cf. [**42**]). In consequence, the algebraic set $p^{-1}(0)$ is of type A if and only if for every quadruplet $(\mathscr{H},\mathscr{D},f_0,\boldsymbol{S})$ fulfilling (22), (23) and $p(\boldsymbol{S})=0$, the system $\boldsymbol{S}$ extends to a system $(T_1,\dots,T_\kappa)$ of spectrally commuting selfadjoint operators acting in a larger Hilbert space. The above discussion enables us to present some new algebraic sets of type A (other examples can be produced with help of Remark 42).

Proposition 46. *The set* $p^{-1}(0)$ *is of type* A *provided* p *is of the form*:

1° $p=(X_1^2+X_2^2)r(X_1,X_2)X_3-1$, *where* $r\in\boldsymbol{R}[X_1,X_2]$,

2° $p=(1+q_1(X_1)^2+\cdots+q_\kappa(X_\kappa)^2)r(X_1,\dots,X_\kappa)X_{\kappa+1}-1$, *where* $q_1,\dots,q_\kappa\in\boldsymbol{R}[X]$ *are polynomials with* $\deg q_i\geq 1$ *and* $r\in\boldsymbol{R}[X_1,\dots,X_\kappa]$.

If $r\equiv 1$ *in* 2°, *then every positive definite* $(\kappa+1)$*-sequence* γ *satisfying condition* (A) *is an ultradeterminate Hamburger moment* $(\kappa+1)$*-sequence*.

Proof. Part 1° can be deduced from Proposition 40 while part 2° from Proposition 41 (see also the proof of Theorem 45). □

The complex moment problem.

A multi-sequence $\boldsymbol{c}=\{c_{\alpha,\beta}\}_{\alpha,\beta\in\boldsymbol{Z}_+^\kappa}\subset\boldsymbol{C}$ is said to be $*$*-positive definite* if

$$\sum_{\alpha,\beta\in\boldsymbol{Z}_+^\kappa}\sum_{\alpha',\beta'\in\boldsymbol{Z}_+^\kappa} c_{\alpha+\beta',\beta+\alpha'}\lambda_{\alpha,\beta}\bar{\lambda}_{\alpha',\beta'}\geq 0$$

for every multi-sequence $\{\lambda_{\alpha,\beta}\}_{\alpha,\beta\in\boldsymbol{Z}_+^\kappa}\subset\boldsymbol{C}$ which has a finite number of nonzero entries; $\boldsymbol{c}$ is said to be a *complex moment multi-sequence* if there exists a positive Borel measure μ on $\boldsymbol{C}^\kappa$ (called a *representing measure of* $\boldsymbol{c}$) such that

$$c_{\alpha,\beta} = \int_{\boldsymbol{C}^\kappa} z^\alpha \bar{z}^\beta \,\mathrm{d}\mu(z), \quad \alpha, \beta \in \boldsymbol{Z}_+^\kappa, \tag{29}$$

where $\bar{z} = (\bar{z}_1, \dots, \bar{z}_\kappa)$. Complex moment multi-sequences are $*$-positive definite, but not conversely (cf. [**6**, Theorem 6.3.5], [**33**]). We say that a complex moment multi-sequence $\boldsymbol{c}$ of the form (29) is *ultradeterminate* if the set $\mathscr{P}_\kappa$ of all functions $\varphi : \boldsymbol{C}^\kappa \to \boldsymbol{C}$ of the form $\varphi(z) = p(z, \bar{z})$ with $p \in \boldsymbol{C}[X_1, \dots, X_{2\kappa}]$ is dense in the space $\mathscr{L}^2(\boldsymbol{C}^\kappa, (1 + \|z\|^2)\,\mathrm{d}\mu(z))$, where $\|z\|^2 = |z_1|^2 + \cdots + |z_\kappa|^2$ (cf. [**15**]). An ultradeterminate complex moment multi-sequence is determinate, but not conversely. Let us mention here that there is a natural correspondence between κ-dimensional complex moment problem and 2κ-dimensional Hamburger moment problem (the case $\kappa = 1$ has been made explicit in [**42**, Section 20]; see also [**37**]).

PROPOSITION 47. *Let $\boldsymbol{c} = \{c_{\alpha,\beta}\}_{\alpha,\beta \in \boldsymbol{Z}_+^\kappa} \subset \boldsymbol{C}$ $(\kappa \geq 2)$ be such that*
(i) *$\boldsymbol{c}$ is $*$-positive definite,*
(ii) *there is $a > 0$ such that for all $i = 2, \dots, \kappa$ and for all finite sequences $\{\lambda_\alpha\}_{\alpha \in \boldsymbol{Z}_+^\kappa} \subset \boldsymbol{C}$, $\sum_{\alpha,\beta \in \boldsymbol{Z}_+^\kappa} [a(c_{\alpha,\beta} + c_{\alpha+e_1,\beta+e_1}) - c_{\alpha+e_i,\beta+e_i}]\lambda_\alpha \bar{\lambda}_\beta \geq 0$,*
(iii) *$\sum_{n=1}^\infty c_{ne_1,ne_1}^{-1/(2n)} = +\infty$.*
Then $\boldsymbol{c}$ is an ultradeterminate complex moment multi-sequence.

PROOF. By (i), there exists a Hilbert space $\mathscr{H}$, a dense linear subspace $\mathscr{D}$ of $\mathscr{H}$, a vector $f_0 \in \mathscr{D}$ and a system $\boldsymbol{N} = (N_1, \dots, N_\kappa) \in \boldsymbol{L}^\#(\mathscr{D})^\kappa$ such that $N_i N_j = N_j N_i$ and $N_i N_j^\# = N_j^\# N_i$ for all $i, j \in \{1, \dots, \kappa\}$, $c_{\alpha,\beta} = \langle \boldsymbol{N}^\alpha f_0, \boldsymbol{N}^\beta f_0 \rangle$ for all $\alpha, \beta \in \boldsymbol{Z}_+^\kappa$ and $\mathscr{D}$ is linearly spanned by the set $\{\boldsymbol{N}^{\#\beta} \boldsymbol{N}^\alpha f_0; \alpha, \beta \in \boldsymbol{Z}_+^\kappa\}$ (cf. [**43**]). Denote by $\mathscr{E}$ the linear span of the set $\{\boldsymbol{N}^\alpha f_0; \alpha \in \boldsymbol{Z}_+^\kappa\}$ and by A_i the restriction of N_i to $\mathscr{E}$, $i = 1, \dots, \kappa$. One can deduce from (i) (or from the inclusion $\boldsymbol{A} \subset \boldsymbol{N}$) that the system $\boldsymbol{A} = (A_1, \dots, A_\kappa)$ satisfies $\mathsf{HBI}(\mathscr{E})$. Condition (ii) implies that A_1 dominates every A_i on $\mathscr{E}$. Since $f_0 \in \mathscr{Q}(A_1)$, we see that $\mathscr{E}$ is equal to the linear span of $\mathscr{Q}(A_1)$ (cf. [**37**, Proposition 2]). Hence, by Proposition 39, $\boldsymbol{c}$ is a complex moment multi-sequence with a representing measure μ. To prove its ultradeterminacy we proceed as follows. One can infer from Lemma 38 that $f_0 \in \bigcap_{i=1}^\kappa \mathscr{Q}(A_i)$, so

$$\sum_{n=1}^\infty c_{ne_i,ne_i}^{-1/(2n)} = +\infty, \quad i = 1, \dots, \kappa. \tag{30}$$

We show that the Carleman condition (30) implies the ultradeterminacy of $\boldsymbol{c}$ (the assumption (ii) is needless for this).

First, we prove that $\mathscr{P}_\kappa$ is dense in $\mathscr{L}^2(\rho\,\mathrm{d}\mu)$ for every $\rho \in \mathscr{L}^2(\mu)$ such that $\rho \geq 0$ a.e. $[\mu]$. Indeed, applying the Schwarz inequality to the complex moment multi-sequence $\hat{\rho}_{\alpha,\beta} = \int_{\boldsymbol{C}^\kappa} z^\alpha \bar{z}^\beta \rho(z)\,\mathrm{d}\mu(z)$ $(\alpha, \beta \in \boldsymbol{Z}_+^\kappa)$ we obtain

$$\hat{\rho}_{ne_i,ne_i} = \int_{\boldsymbol{C}^\kappa} |z_i|^{2n} \rho(z)\,\mathrm{d}\mu(z) \leq c_{2ne_i,2ne_i}^{1/2} \left(\int_{\boldsymbol{C}^\kappa} \rho^2 \,\mathrm{d}\mu \right)^{1/2}, \quad n \geq 1.$$

Hence there exists $M > 0$ such that $\sum_{n=1}^\infty \hat{\rho}_{ne_i,ne_i}^{-1/(2n)} \geq M \sum_{n=1}^\infty c_{2ne_i,2ne_i}^{-1/(4n)} = +\infty$ for all $i = 1, \dots, \kappa$ (the last series is divergent due to (c), page 32 in [**37**]). Now the density of $\mathscr{P}_\kappa$ in $\mathscr{L}^2(\rho\,\mathrm{d}\mu)$ follows from [**37**, Corollary 6].

Applying the above to $\rho(z) = 1 + \|z\|^2$ completes the proof. □

REMARK 48. Under the assumption (30), we can prove more, namely that $\mathscr{P}_\kappa$ is dense in $\mathscr{L}^p(\mu)$ for every $p \geq 1$. Indeed, one can deduce from (30) that the Hamburger moment 2κ-sequence $\{\gamma_\alpha\}_{\alpha \in \boldsymbol{Z}_+^{2\kappa}}$ defined by

$$\gamma_\alpha = \int_{\boldsymbol{C}^\kappa} (\operatorname{Re} z_1)^{\alpha_1} \cdots (\operatorname{Re} z_\kappa)^{\alpha_\kappa} (\operatorname{Im} z_1)^{\alpha_{\kappa+1}} \cdots (\operatorname{Im} z_\kappa)^{\alpha_{2\kappa}} \, \mathrm{d}\mu(z), \quad \alpha \in \boldsymbol{Z}_+^{2\kappa},$$

satisfies the Carleman condition $\sum_{n=1}^\infty \gamma_{2ne_i}^{-1/(2n)} = +\infty$ for all $i = 1, \ldots, 2\kappa$. By Theorem 3 in [**4**] and part b) of Theorem, §11 in [**15**], the set $\boldsymbol{C}[X_1, \ldots, X_{2\kappa}]$ is dense in $\mathscr{L}^p(\boldsymbol{R}^{2\kappa}, \mu)$ for every $p \geq 1$, which completes the argument. We get in this way a stronger result than that in section 2 of [**3**] labeled as 5°.

Comments.

The content of last three sections can be adapted to the operator Hamburger and operator complex multi-dimensional moment problems as well as to algebraic sets of operator type A, where multi-sequences of numbers have to be replaced by multi-sequences of sesquilinear forms over an arbitrary linear space (the only exception is the ultradeterminacy in the conclusion of Proposition 47, which needs a separate interest). For more details we refer the reader to [**42**].

There is a simple way of producing new algebraic sets of (operator-) type A from a given one. Namely, by [**42**, Proposition 60], the set of all polynomials $p \in \boldsymbol{R}[X_1, \ldots, X_\kappa]$ for which $p^{-1}(0)$ is of (operator-) type A is invariant under the action $\Phi \mapsto p \circ \Phi$ of the group of all polynomial automorphisms Φ of $\boldsymbol{R}^\kappa$. For instance for $\kappa = 3$ we can consider products of polynomial automorphisms of $\boldsymbol{R}^3$ of the form

$$\begin{aligned}
&(x_1, x_2, x_3) \mapsto (x_1 - f_1(x_2, x_3), x_2, x_3), \\
&(x_1, x_2, x_3) \mapsto (x_1, x_2 - f_2(x_1, x_3), x_3), \\
&(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 - f_3(x_1, x_2)), \\
&(x_1, x_2, x_3) \mapsto (x_1 - g(x_3), x_2 - h(x_3), x_3), \\
&\qquad\qquad\vdots
\end{aligned}$$

where $f_1, f_2, f_3 \in \boldsymbol{R}[X_1, X_2]$ and $g, h, \ldots \in \boldsymbol{R}[X]$. For more information on generators of the group of all polynomial automorphisms of $\boldsymbol{R}^2$, see [**27**].

Similar phenomenon occurs for subnormal systems. We exemplify this in the context of symmetric operators. Let $\Phi = (\varphi_1, \ldots, \varphi_\kappa)$ be a polynomial automorphism of $\boldsymbol{R}^\kappa$, $p \in \boldsymbol{R}[X_1, \ldots, X_\kappa]$ and $\boldsymbol{S} = (S_1, \ldots, S_\kappa) \in \boldsymbol{L}(\mathscr{D})^\kappa$ be a system of pointwise commuting operators. If $\boldsymbol{S}$ is annihilated by p, i.e. $p(\boldsymbol{S}) = 0$, and $\boldsymbol{S}$ extends to a system $\boldsymbol{T} = (T_1, \ldots, T_\kappa)$ of spectrally commuting selfadjoint operators acting in a Hilbert space $\mathscr{K} \supset \mathscr{H}$, then the system $\Phi(\boldsymbol{S}) \stackrel{\mathrm{df}}{=} (\varphi_1(\boldsymbol{S}), \ldots, \varphi_\kappa(\boldsymbol{S}))$ is annihilated by the polynomial $p \circ \Phi^{-1}$, and $\Phi(\boldsymbol{S})$ extends to the selfadjoint system $\int_{\boldsymbol{R}^\kappa} \Phi \, \mathrm{d}E \stackrel{\mathrm{df}}{=} (\int_{\boldsymbol{R}^\kappa} \varphi_1 \, \mathrm{d}E, \ldots, \int_{\boldsymbol{R}^\kappa} \varphi_\kappa \, \mathrm{d}E)$, where E is the spectral measure of $\boldsymbol{T}$ (the reverse implication, written in an appropriate way, is true as well). Moreover, $\boldsymbol{T}$ is minimal of spectral type if and only if so is $\int_{\boldsymbol{R}^\kappa} \Phi \, \mathrm{d}E$ (minimality of spectral type is implicitly defined in the first paragraph of the proof of Proposition 40).

References

[1] N. I. Akhiezer and I. M. Glazman, Theory of linear operators in Hilbert space, vol. II, Pitman, Boston-London-Melbourne, 1981.

[2] Yu M. Berezanskiĭ, Spectral theory of commutative Jacobi fields: direct and inverse problems, in Operator theory and its applications (Winnipeg, MB, 1998), 211–224, Fields Inst. Commun., **25**, Amer. Math. Soc., Providence, RI, 2000.

[3] C. Berg, Recent results about moment problems, in Probability Measures on Groups and Related Structures, Proc., (ed. H. Heyer), World Scientific, Singapore, 1995.

[4] C. Berg and J. P. R. Christensen, Exposants critiques dans le problème de moments, C. R. Acad. Sci. Paris, **296** (1983), Sér. I, 661–662.

[5] C. Berg, J. P. R. Christensen and C. U. Jensen, A remark on the multidimensional moment problem, Math. Ann., **243** (1979), 163–169.

[6] C. Berg, J. P. R. Christensen and P. Ressel, Harmonic Analysis on Semigroups, Springer, Berlin, 1984.

[7] C. Berg and A. J. Duran, Measures with finite index of determinacy or a mathematical model for Dr. Jekyll and Mr. Hyde, Proc. Amer. Math. Soc., **125** (1997), 523–530.

[8] M. S. Birman and M. Z. Solomiak, Spectral theory of selfadjoint operators in Hilbert space (in Russian), Leningrad University Publishers, Leningrad, 1980.

[9] T. M. Bisgaard, Extensions of Hamburger's theorem, Semigroup Forum, **57** (1998), 397–429.

[10] E. A. Coddington, Formally normal operators having no normal extension, Canad. J. Math., **17** (1965), 1030–1040.

[11] A. Daniluk, private communication, 2000.

[12] W. G. Faris, Selfadjoint operators, Lecture Notes in Math., vol. 433, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[13] J. Friedrich, A note on the two-dimensional moment problem, Math. Nachr., **121** (1985), 285–286.

[14] B. Fuglede, Problem 3, Math. Scand., **1** (1953), 172 and **2** (1954), 346–347.

[15] B. Fuglede, The multidimensional moment problem, Expo. Math., **1** (1983), 47–65.

[16] T. Furuta, On the class of paranormal operators, Proc. Japan. Acad., **43** (1967), 594–598.

[17] P. R. Halmos, Commutativity and spectral properties of normal operators, Acta Sci. Math. (Szeged), **12** (1950), 153–156.

[18] H. Leinfelder, A geometric proof of the spectral theorem for unbounded self-adjoint operators, Math. Ann., **242** (1979), 85–96.

[19] E. Nelson, Analytic vectors, Ann. of Math., **70** (1959), 572–615.

[20] A. E. Nussbaum, Quasi-analytic vectors, Ark. Mat., **6** (1965), 179–191.

[21] A. E. Nussbaum, A note on quasi-analytic vectors, Studia Math., **33** (1969), 305–309.

[22] A. E. Nussbaum, A commutativity theorem for unbounded operators in Hilbert space, Trans. Amer. Math. Soc., **140** (1969), 485–491.

[23] S. Ôta, On strongly normal extensions of unbounded operators, Bull. Polish Acad. Sci. Math., **46** (1998), 291–301.

[24] N. S. Poulsen, On the canonical commutation relations, Math. Scand., **32** (1973), 112–122.

[25] M. Putinar and F.-H. Vasilescu, Solving moment problems by dimensional extension, Ann. of Math., **149** (1999), 1087–1107.

[26] W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1974.

[27] K. Rusek, Polynomial automorphisms, Institute of Mathematics, Polish Academy of Sciences, preprint #456, Warszawa, 1989.

[28] K. Schmüdgen, An example of a positive polynomial which is not a sum of squares of polynomials, A positive, but not strongly positive functional, Math. Nachr., **88** (1979), 385–390.

[29] K. Schmüdgen, On domains of powers of closed symmetric operators, J. Operator Theory, **9** (1983), 53–75.

[30] K. Schmüdgen, A formally normal operator having no normal extension, Proc. Amer. Math. Soc., **95** (1985), 503–504.

[31] K. Schmüdgen, Strongly commuting selfadjoint operators and commutants of unbounded operator algebras, Proc. Amer. Math. Soc., **102** (1988), 365–372.

[32] K. Schmüdgen, On determinacy notion for the two-dimensional moment problem, Ark. Mat., **29** (1991), 277–284.

[33] J. Stochel, Moment functions on real algebraic sets, Ark. Mat., **30** (1992), 133–148.

[34] J. Stochel, Positive definiteness and commutativity of operators, Proc. Amer. Math. Soc., **126** (1998), 431–440.
[35] J. Stochel, Lifting strong commutants of unbounded subnormal operators, Integral Equations Operator Theory, **43** (2002), 189–214.
[36] J. Stochel and F. H. Szafraniec, Bounded vectors and formally normal operators, Operator Theory: Adv. Appl., **11**, Birkhauser, Basel, 1983, 363–370.
[37] J. Stochel and F. H. Szafraniec, On normal extensions of unbounded operators, I, J. Operator Theory, **14** (1985), 31–55.
[38] J. Stochel and F. H. Szafraniec, On normal extensions of unbounded operators, III, Spectral properties, Publ. Res. Inst. Math. Sci. Kyoto Univ., **25** (1989), 105–139.
[39] J. Stochel and F. H. Szafraniec, The normal part of an unbounded operator, Nederl. Akad. Wetensch. Proc. Ser. A, **92** (1989), 495–503.
[40] J. Stochel and F. H. Szafraniec, A few assorted questions about unbounded subnormal operators, Univ. Iagel. Acta Math., **28** (1991), 163–170.
[41] J. Stochel and F. H. Szafraniec, $\mathscr{C}^\infty$-vectors and boundedness, Ann. Polon. Math., **66** (1997), 223–238.
[42] J. Stochel and F. H. Szafraniec, The complex moment problem and subnormality: a polar decomposition approach, J. Funct. Anal., **159** (1998), 432–491.
[43] F. H. Szafraniec, Boundedness of the shift operator related to positive definite forms: an application to moment problems, Ark. Mat., **19** (1981), 251–259.
[44] F. H. Szafraniec, A RKHS of entire functions and its multiplication operator, An explicit example, Operator Theory: Advances and Applications, vol. 43, Birkhäuser, Basel, 1990, 309–312.
[45] F. H. Szafraniec, Kato–Protter type inequalities, bounded vectors and the exponential function, Ann. Polon. Math., **51** (1990), 303–312.
[46] F. H. Szafraniec, Subnormality in the quantum harmonic oscillator, Commun. Math. Phys., **210** (2000), 323–334.
[47] J. Weidmann, Linear operators in Hilbert spaces, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

Added in proof.

Recently two papers related somehow to some parts of ours have appeared:

J. Eschmeier, F.-H. Vasilescu, On jointly essentially self-adjoint tuples of operators, Acta Sci. Math. (Szeged), **67** (2001), 373–386.

T. M. Bisgaard, Characterization of moment multisequences by a variation of positive definiteness, Collect. Math., **52** (2001), 205–218.

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