

Cohomology groups for recurrence relations and contiguity relations of hypergeometric systems

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Abstract. We develop the theory of cohomology groups for recurrence relations, based upon the asymptotic analysis of finite difference equations carried out in a previous paper. We apply it to compute the Gevrey extension groups of the \mathcal{D} -modules associated to some confluent hypergeometric systems. In those applications, recurrence relations appear as contiguity relations of hypergeometric systems.

Introduction.

In this paper we shall develop the theory of cohomology groups for recurrence relations and apply it to compute the Gevrey extension groups of the \mathcal{D} -modules associated to some confluent hypergeometric systems.

At this stage, it is not so easy to explain in a few words what we mean by the terms “cohomology groups for recurrence relations”, and developing the theory without giving a motivation might appear somewhat too abstract even to the specialists of hypergeometric functions. Thus it would be better to begin with applications to confluent hypergeometric systems, with emphasis on particular examples, so as to motivate the reader to follow the general theory. In those applications, recurrence relations will appear as contiguity relations of hypergeometric systems.

The main theorem of this paper will be stated in §8 (Theorem 8.6). Roughly speaking, what we shall do in this paper is to associate to each contiguity relation a cochain complex, called the *harmonic complex*, which is expected to be useful in studying hypergeometric systems. Theorem 8.6 then presents a quite explicit formula for the harmonic complex.

1. Contiguity relation.

A characteristic feature of a *hypergeometric system* is that it admits a *contiguity relation* with respect to a parameter. Here a contiguity relation, or more precisely, a *contiguity operator* is a (differential) operator sending the solutions of a hypergeometric system with a given parameter into those with parameter shifted by one. In what follows we use the notation: $\partial_x = \partial/\partial x$, $\delta_x = x\partial_x$, etc.

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EXAMPLE 1.1. Let us consider the Kummer confluent hypergeometric equation:

$$L(c)f := \{x\partial_x^2 + (c-x)\partial_x - b\}f = 0. \quad (1.1)$$

If one sets $P(c) = 1 + (1/c)\partial_x$, then there exists a commutation relation:

$$L(c)P(c) = P(c)L(c+1).$$

So if f is a solution of $L(c+1)f = 0$, then $g = P(c)f$ is a solution of $L(c)g = 0$. Thus $P(c)$ is a contiguity operator of (1.1) with respect to the parameter c .

For our purpose, however, it is more convenient to transform (1.1) into a system $Q^0(c)u = 0$ for the vector-valued function $u = {}^t(f, \partial_x f)$, where

$$Q^0(c) = \begin{pmatrix} \partial_x & -1 \\ -b & \partial_x + c - x \end{pmatrix}.$$

Then the contiguity relation for (1.1) is expressed by a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(D)^2 & \xrightarrow{Q^0(c+1)} & \mathcal{O}(D)^2 & \longrightarrow & 0 \\ & & \downarrow P^0(c) & & \downarrow P^1(c) & & \\ 0 & \longrightarrow & \mathcal{O}(D)^2 & \xrightarrow{Q^0(c)} & \mathcal{O}(D)^2 & \longrightarrow & 0 \end{array} \quad (1.2)$$

where $\mathcal{O}(D)$ denotes the set of holomorphic functions on a domain $D \subset \mathbf{C}$, and

$$P^0(c) = \begin{pmatrix} 1 & x/c \\ b/c & x/c \end{pmatrix}, \quad P^1(c) = \begin{pmatrix} 1 & 1/c \\ bx/c & x/c \end{pmatrix}.$$

An advantage of this expression is that the operators $P^0(c)$ and $P^1(c)$ are matrices of *functions*, not of differential operators.

It may be interpreted that in diagram (1.2), the horizontal sequences give a solution complex of the \mathcal{D} -module associated to the differential equation (1.1), while the vertical arrows yield contiguity operators. With this interpretation, we give another example which is somewhat more involved.

EXAMPLE 1.2. Consider the Humbert confluent hypergeometric system ([4]):

$$\begin{cases} L_1(c)f := \{x\partial_x^2 + y\partial_x\partial_y + (c-x)\partial_x - b_1\}f = 0; \\ L_2(c)f := \{y\partial_y^2 + x\partial_x\partial_y + (c-y)\partial_y - b_2\}f = 0. \end{cases} \quad (1.3)$$

Then it turns out that (1.3) has a contiguity relation:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(D)^6 & \xrightarrow{Q^0(c+1)} & \mathcal{O}(D)^9 & \xrightarrow{Q^1(c+1)} & \mathcal{O}(D)^3 & \longrightarrow & 0 \\ & & \downarrow P^0(c) & & \downarrow P^1(c) & & \downarrow P^2(c) & & \\ 0 & \longrightarrow & \mathcal{O}(D)^6 & \xrightarrow{Q^0(c)} & \mathcal{O}(D)^9 & \xrightarrow{Q^1(c)} & \mathcal{O}(D)^3 & \longrightarrow & 0, \end{array} \quad (1.4)$$

where D is a domain in \mathbf{C}^2 , and the operators $P^i(c)$ and $Q^i(c)$ are given by

$$P^0(c) = \left(\begin{array}{c|cccccc} 1 & x/c & y/c & 0 & 0 & 0 \\ \hline b_1/c & x/c & 0 & 0 & 0 & 0 \\ b_2/c & 0 & y/c & 0 & 0 & 0 \\ 0 & (1+b_1)/c & 0 & x/c & 0 & 0 \\ 0 & b_2/(2c) & b_1/(2c) & 0 & (x+y)/(2c) & 0 \\ 0 & 0 & (1+b_2)/c & 0 & 0 & y/c \end{array} \right)$$

$$P^1(c) = \left(\begin{array}{cc|cccccc} 1 & 0 & x/c & 0 & y/c & 0 & 1/c & 0 & 0 \\ 0 & 1 & 0 & x/c & 0 & y/c & 0 & 1/c & 0 \\ \hline b_1/c & 0 & x/c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1/c & 0 & x/c & 0 & 0 & 0 & 0 & 1/(2c) \\ b_2/c & 0 & 0 & 0 & y/c & 0 & 0 & 0 & -1/(2c) \\ 0 & b_2/c & 0 & 0 & 0 & y/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x/c & 0 & -y/(2c) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y/c & x/(2c) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (x+y)/(2c) \end{array} \right)$$

$$P^2(c) = \left(\begin{array}{c|cc} 1 & 1/c & 0 \\ \hline (b_1x+b_2y)/c & (x+y)/c & -1/c \\ (b_1+b_2)xy/c & xy/c & 0 \end{array} \right)$$

$$Q^0(c) = \left(\begin{array}{c|ccccc} \partial_x & -1 & 0 & 0 & 0 & 0 \\ \partial_y & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & \partial_x & 0 & -1 & 0 & 0 \\ 0 & \partial_y & 0 & 0 & -1 & 0 \\ 0 & 0 & \partial_x & 0 & -1 & 0 \\ 0 & 0 & \partial_y & 0 & 0 & -1 \\ -b_1 & c-x & 0 & x & y & 0 \\ -b_2 & 0 & c-y & 0 & x & y \\ 0 & -b_2 & b_1 & 0 & x-y & 0 \end{array} \right)$$

$$Q^1(c) = \left(\begin{array}{cc|cccc} \partial_y & -\partial_x & 0 & 1 & -1 \\ \hline -b_2 & b_1 & x\partial_y & -(\delta_x - x + c) & \delta_y - y + c \\ -b_2x & b_1y & x(\delta_y + b_2) & -y(\delta_x - x + c - b_2) & x(\delta_y - y + c - b_1) \\ \hline 0 & 0 & 0 & 0 & 0 \\ -y\partial_x & \partial_y & -\partial_x & 1 & \\ -y(\delta_x + b_1) & \delta_y + b_2 & -(\delta_x + b_1) & \delta_x + \delta_y + c & \end{array} \right).$$

It can be seen that the horizontal sequences in (1.4) give a solution complex of the

\mathcal{D} -module associated to the differential system (1.3). We have obtained the contiguity relation (1.4) with the help of computer algebra system KAN ([11]).

Taking Examples 1.1 and 1.2 into account, we make the following definition.

DEFINITION 1.3. A contiguity relation of a \mathcal{D} -module $\mathcal{M} = \mathcal{M}(c)$ with a parameter c is a commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{O}(D)^{m_0} & \xrightarrow{Q^0(c+1)} & \mathcal{O}(D)^{m_1} & \xrightarrow{Q^1(c+1)} & \mathcal{O}(D)^{m_2} & \xrightarrow{Q^2(c+1)} & \dots \\
 & & \downarrow P^0(c) & & \downarrow P^1(c) & & \downarrow P^2(c) & & \\
 0 & \longrightarrow & \mathcal{O}(D)^{m_0} & \xrightarrow{Q^0(c)} & \mathcal{O}(D)^{m_1} & \xrightarrow{Q^1(c)} & \mathcal{O}(D)^{m_2} & \xrightarrow{Q^2(c)} & \dots
 \end{array} \tag{1.5}$$

such that the top (resp. bottom) sequence gives a solution complex of $\mathcal{M}(c + 1)$ (resp. $\mathcal{M}(c)$) with values in $\mathcal{O}(D)$, where $P^i(c)$ are matrices of functions, while $Q^i(c)$ are matrices of differential operators.

2. Lattice.

We continue to begin with examples. Assume that c is not an integer, and set

$$\begin{cases} P_n^i := P^i(c - n); \\ Q_n^i := Q^i(c - n), \end{cases} \quad (n \in \mathbf{Z}). \tag{2.1}$$

Then Examples 1.1 and 1.2 lead to the following:

EXAMPLE 2.1. The contiguity relation (1.2) induces a commutative diagram:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{O}(D)^2 & \xrightarrow{Q_{n-1}^0} & \mathcal{O}(D)^2 & \longrightarrow & 0 \\
 & & \downarrow P_n^0 & & \downarrow P_n^1 & & \\
 0 & \longrightarrow & \mathcal{O}(D)^2 & \xrightarrow{Q_n^0} & \mathcal{O}(D)^2 & \longrightarrow & 0 \\
 & & \downarrow P_{n+1}^0 & & \downarrow P_{n+1}^1 & & \\
 0 & \longrightarrow & \mathcal{O}(D)^2 & \xrightarrow{Q_{n+1}^0} & \mathcal{O}(D)^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
 \end{array} \tag{2.2}$$

EXAMPLE 2.2. The contiguity relation (1.4) induces a commutative diagram:

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \mathcal{O}(D)^6 & \xrightarrow{Q_{n-1}^0} & \mathcal{O}(D)^9 & \xrightarrow{Q_{n-1}^1} & \mathcal{O}(D)^3 \longrightarrow 0 \\
 & & \downarrow P_n^0 & & \downarrow P_n^1 & & \downarrow P_n^2 \\
 0 & \longrightarrow & \mathcal{O}(D)^6 & \xrightarrow{Q_n^0} & \mathcal{O}(D)^9 & \xrightarrow{Q_n^1} & \mathcal{O}(D)^3 \longrightarrow 0 \\
 & & \downarrow P_{n+1}^0 & & \downarrow P_{n+1}^1 & & \downarrow P_{n+1}^2 \\
 0 & \longrightarrow & \mathcal{O}(D)^6 & \xrightarrow{Q_{n+1}^0} & \mathcal{O}(D)^9 & \xrightarrow{Q_{n+1}^1} & \mathcal{O}(D)^3 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & \vdots & & \vdots & & \vdots &
 \end{array} \tag{2.3}$$

We would like to abstract some common features from these examples. First we notice that the spaces $\mathcal{O}(D)^2$, $\mathcal{O}(D)^6$, etc. appearing in (2.2) and (2.3) are complete, locally convex spaces, with the topology of uniform convergence on every compact subsets in D , with respect to which the (matrices of) differential operators Q_n^i are continuous, i.e., bounded. Next the fact that the contiguity operators P_n^i are matrices of *functions*, not of differential operators, can abstractly be stated as the condition that they are strongly bounded. Here a linear transformation P of a locally convex space U is said to be *strongly bounded* if for each semi-norm $|\cdot|$ on U , there exists a constant M such that $|Pu| \leq M|u|$ for any $u \in U$. (Compare this with the mere boundedness defined by the condition that for each semi-norm $|\cdot|$, there exist a constant M and another semi-norm $\|\cdot\|$ such that $|Pu| \leq M\|u\|$ for any $u \in U$.) Finally, we recall that the horizontal sequences in (2.3) are cochain complexes, i.e., $Q_n^1 Q_n^0 = 0$.

These observations lead us to consider a general commutative diagram:

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \dots & \longrightarrow & U^{i-1} & \xrightarrow{Q_{n-1}^{i-1}} & U^i & \xrightarrow{Q_{n-1}^i} & U^{i+1} \longrightarrow \dots \\
 & & \downarrow P_n^{i-1} & & \downarrow P_n^i & & \downarrow P_n^{i+1} \\
 \dots & \longrightarrow & U^{i-1} & \xrightarrow{Q_n^{i-1}} & U^i & \xrightarrow{Q_n^i} & U^{i+1} \longrightarrow \dots \\
 & & \downarrow P_{n+1}^{i-1} & & \downarrow P_{n+1}^i & & \downarrow P_{n+1}^{i+1} \\
 \dots & \longrightarrow & U^{i-1} & \xrightarrow{Q_{n+1}^{i-1}} & U^i & \xrightarrow{Q_{n+1}^i} & U^{i+1} \longrightarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & \vdots & & \vdots & & \vdots &
 \end{array} \tag{2.4}$$

where U^i is a complete, locally convex \mathbf{C} -linear space; P_n^i is a *strongly bounded* linear transformation of U^i ; $Q_n^i : U^i \rightarrow U^{i+1}$ is a bounded linear operator, and the indices i

and n range over all integers \mathbf{Z} . Assume that each horizontal sequence in (2.4) is a *cochain complex*, i.e., $Q_n^{i+1}Q_n^i = 0$. Thus (2.4) represents an infinite sequence of morphisms of cochain complexes, which will be referred to as a *lattice*. Note that the contiguity relation (1.5) induces a lattice (2.4) with $U^i = \mathcal{O}(D)^{m_i}$ ($i \geq 0$) and $U^i = 0$ ($i < 0$).

3. Basic assumption.

To set up a basic assumption (Assumption 3.1) on the operators P_n^i and Q_n^i in (2.4), we introduce some notations. First, given a locally convex space U , let $B(U)$ denote the space of strongly bounded linear transformations of U . To each semi-norm $|\cdot|$ on U , one can associate a semi-norm of $B(U)$, denoted by the same symbol $|\cdot|$, in the following manner: For each $P \in B(U)$, let $|P|$ be the infimum of those constants M which satisfy $|Pu| \leq M|u|$ for any $u \in U$. These semi-norms provide $B(U)$ with a locally convex topology. Throughout this paper, $B(U)$ is understood to have this topology. Secondly, for locally convex spaces U and U' , let $L(U, U')$ denote the set of all bounded linear operators of U into U' (no topology on $L(U, U')$ will be necessary in what follows). Finally we introduce (inverse and direct) *factorial monomials*:

$$(x)_j := \frac{(-1)^j(j-1)!}{x(x+1)(x+2)\cdots(x+j-1)}, \quad \langle x \rangle_j := \frac{x(x-1)(x-2)\cdots(x-j+1)}{(-1)^j(j-1)!},$$

for $j \in \mathbf{N}$, with the convention $(x)_0 = \langle x \rangle_0 = 1$. Note that they satisfy

$$(x)_j - (x-1)_j = (x)_{j-1}, \quad \langle x \rangle_j - \langle x-1 \rangle_j = \langle x \rangle_{j-1}.$$

ASSUMPTION 3.1. Assume that the operators P_n^i and Q_n^i in (2.4) admit the following factorial expansions:

$$P_n^i \sim \sum_{j=0}^{\infty} P^{i,j}(n-c)_j \quad \text{in } B(U^i), \tag{3.1}$$

$$Q_n^i = \sum_{j=0}^{N^i} Q^{i,j}\langle n-c \rangle_j \quad \text{in } L(U^i, U^{i+1}), \tag{3.2}$$

with some $c \in \mathbf{C} \setminus \mathbf{Z}$, where N^i is a non-negative integer, $P^{i,j} \in B(U^i)$ and $Q^{i,j} \in L(U^i, U^{i+1})$. By (3.1), we mean that for every positive integer $k \in \mathbf{N}$,

$$P_n^i = \sum_{j=0}^k P^{i,j}(n-c)_j + O(1/n^{k+1}) \quad \text{as } n \rightarrow +\infty$$

in the topology of $B(U^i)$. Assume that each U^i admits a direct sum decomposition:

$$U^i = U_0^i \oplus U_1^i, \tag{3.3}$$

with some *closed* subspaces U_0^i, U_1^i of U^i . Let $X^i : U^i \rightarrow U_0^i$ and $Y^i : U^i \rightarrow U_1^i$ denote

the associated projections. Assume that the leading coefficients $P^{i,0}$ and $P^{i,1}$ in the asymptotic expansion (3.1) satisfy the conditions:

$$\begin{cases} X^i P^{i,0} X^i = X^i, & X^i P^{i,0} Y^i = 0, \\ Y^i P^{i,0} X^i = 0, & X^i P^{i,1} X^i = 0. \end{cases} \quad (3.4)$$

Finally let $Z^i \in B(U_1^i)$ be defined by

$$Z^i := Y^i P^{i,0} Y^i, \quad (3.5)$$

and assume that there exists a constant ρ^i with $0 \leq \rho^i < 1$ such that

$$|Z^i| \leq \rho^i \quad (3.6)$$

for every $|\cdot| \in \mathcal{N}^i$, where \mathcal{N}^i is a system of semi-norms defining the locally convex topology of U^i .

Note that condition (3.6) implies that the operator $I_1 - Z^i$ has the inverse $(I_1 - Z^i)^{-1}$ in $B(U_1^i)$, where I_1 denotes the identity operator on U_1^i . We shall check that Assumption 3.1 is satisfied for the operators in Examples 2.1 and 2.2.

EXAMPLE 3.2. Consider the lattice (2.2). Then we have $U^0 = U^1 = \mathcal{O}(D)^2$ and $U^i = 0$ otherwise. The operators P_n^i and Q_n^i in (2.2) have factorial expansions:

$$P_n^0 = P^{0,0} + P^{0,1}(n-c)_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & x \\ b & x \end{pmatrix}(n-c)_1, \quad (3.7)$$

$$P_n^1 = P^{1,0} + P^{1,1}(n-c)_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ bx & x \end{pmatrix}(n-c)_1, \quad (3.8)$$

$$Q_n^0 = Q^{0,0} + Q^{0,1}\langle n-c \rangle_1 = \begin{pmatrix} \partial_x & -1 \\ -b & \delta_x - x \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\langle n-c \rangle_1. \quad (3.9)$$

For (3.3), we take the standard decomposition $U^i = U_0^i \oplus U_1^i$ with $U_0^i = U_1^i = \mathcal{O}(D)$, the associated projections $X^i : U^i \rightarrow U_0^i$ and $Y^i : U^i \rightarrow U_1^i$ being given by

$$X^i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y^i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (i = 0, 1). \quad (3.10)$$

As for the operator Z^i in (3.5), we have $Z^i = 0$. With these data, all the conditions in Assumption 3.1 are satisfied.

EXAMPLE 3.3. Consider the lattice (2.3). We have $U^0 = \mathcal{O}(D)^6$, $U^1 = \mathcal{O}(D)^9$, $U^2 = \mathcal{O}(D)^3$ and $U^i = 0$ otherwise. The operators P_n^i and Q_n^i in (2.3) have factorial expansions:

$$P_n^i = P^{i,0} + P^{i,1}(n-c)_1, \quad Q_n^i = Q^{i,0} + Q^{i,1}\langle n-c \rangle_1,$$

where

$$P^{0,0} = \left(\begin{array}{c|cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad (3.11)$$

$$P^{0,1} = \left(\begin{array}{c|ccccc} 0 & x & y & 0 & 0 & 0 \\ \hline b_1 & x & 0 & 0 & 0 & 0 \\ b_2 & 0 & y & 0 & 0 & 0 \\ 0 & 1+b_1 & 0 & x & 0 & 0 \\ 0 & b_2/2 & b_1/2 & 0 & (x+y)/2 & 0 \\ 0 & 0 & 1+b_2 & 0 & 0 & y \end{array} \right) \quad (3.12)$$

$$P^{1,0} = \left(\begin{array}{cc|cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad (3.13)$$

$$P^{1,1} = \left(\begin{array}{cc|ccccccc} 0 & 0 & x & 0 & y & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & y & 0 & 1 & 0 \\ \hline b_1 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & x & 0 & 0 & 0 & 0 & 1/2 \\ b_2 & 0 & 0 & 0 & y & 0 & 0 & 0 & -1/2 \\ 0 & b_2 & 0 & 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & -y/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & x/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (x+y)/2 \end{array} \right) \quad (3.14)$$

$$P^{2,0} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad (3.15)$$

$$P^{2,1} = \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline b_1x + b_2y & x + y & -1 \\ (b_1 + b_2)xy & xy & 0 \end{array} \right) \quad (3.16)$$

$$Q^{0,0} = \left(\begin{array}{c|cccccc} \partial_x & -1 & 0 & 0 & 0 & 0 \\ \partial_y & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & \partial_x & 0 & -1 & 0 & 0 \\ 0 & \partial_y & 0 & 0 & -1 & 0 \\ 0 & 0 & \partial_x & 0 & -1 & 0 \\ 0 & 0 & \partial_y & 0 & 0 & -1 \\ -b_1 & -x & 0 & x & y & 0 \\ -b_2 & 0 & -y & 0 & x & y \\ 0 & -b_2 & b_1 & 0 & x-y & 0 \end{array} \right) \quad (3.17)$$

$$Q^{0,1} = \left(\begin{array}{c|cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad (3.18)$$

$$Q^{1,0} = \left(\begin{array}{cc|cccc} \partial_y & -\partial_x & 0 & 1 & -1 & \\ \hline -b_2 & b_1 & x\partial_y & -(\delta_x - x) & \delta_y - y & \\ -b_2x & b_1y & x(\delta_y + b_2) & -y(\delta_x - x - b_2) & x(\delta_y - y - b_1) & \\ \hline & & 0 & 0 & 0 & 0 \\ & & -y\partial_x & \partial_y & -\partial_x & 1 \\ & & -y(\delta_x + b_1) & \delta_y + b_2 & -(\delta_x + b_1) & \delta_x + \delta_y \end{array} \right) \quad (3.19)$$

$$Q^{1,1} = \left(\begin{array}{cc|cccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y & x & 0 & 0 & 1 \end{array} \right). \quad (3.20)$$

For (3,3), we take standard decompositions $U^i = U_0^i \oplus U_1^i$, with

$$\begin{cases} U_0^0 = \mathcal{O}(\mathcal{D}); & U_1^0 = \mathcal{O}(\mathcal{D})^5; \\ U_0^1 = \mathcal{O}(\mathcal{D})^2; & U_1^1 = \mathcal{O}(\mathcal{D})^7; \\ U_0^2 = \mathcal{O}(\mathcal{D}); & U_1^2 = \mathcal{O}(\mathcal{D})^2. \end{cases} \quad (3.21)$$

As for the operator Z^i in (3.5), we have $Z^i = 0$. With these data, all the conditions in Assumption 3.1 are satisfied.

In Examples 3.2 and 3.3, the operators Z^i defined by (3.5) were the zero operator. However, the contiguity relation in c of the Gauss hypergeometric equation:

$$L(c)f := \{x(1-x)\partial_x^2 + [c - (a+b+1)x]\partial_x - ab\}f = 0$$

would give an example with non-zero Z^i ([6]). In those examples (including the Gauss equation) the expansion (3.2) is of order $N^i = 1$. An example with $N^i = 2$ will be given in Example 9.4. There are an abundance of examples fulfilling the conditions of Assumption 3.1.

We shall eventually work with Assumption 3.1, though intermediate assumptions (Assumptions 5.1 and 6.1) will be introduced in the course of the discussion. In particular, our main theorem (Theorem 8.6) will be stated under Assumption 3.1.

4. Mapping cone.

Let us rewrite the lattice (2.4) more concisely by introducing some sequence spaces and relevant linear operators. Let

$$V^i = \{v^i = (u_n^i) : u_n^i \in U^i, u_n^i = 0 \ (n \ll 0)\} \tag{4.1}$$

denote the linear space of all infinite sequences $v^i = (u_n^i)_{n \in \mathbf{Z}}$ in U^i such that $u_n^i = 0$ for every sufficiently small $n \in \mathbf{Z}$. We refer to V^i as the *formal* sequence space associated to U^i , since the behavior of a sequence in V^i as $n \rightarrow +\infty$ can be arbitrary. (We will later encounter some linear subspaces of V^i whose elements are characterized in terms of certain asymptotic behaviors as $n \rightarrow +\infty$. See §7.)

Let $T : V^i \rightarrow V^i$ be the *translation* (or shift) operator defined by

$$(Tv^i)_n = u_{n-1}^i \quad \text{for } v^i = (u_n^i). \tag{4.2}$$

The collections of operators $(P_n^i)_{n \in \mathbf{Z}}$ and $(Q_n^i)_{n \in \mathbf{Z}}$ define “term-wise” linear operators $P^i : V^i \rightarrow V^i$ and $Q^i : V^i \rightarrow V^{i+1}$ by

$$(P^i v^i)_n = P_n^i u_n^i, \quad (Q^i v^i)_n = Q_n^i u_n^i \quad \text{for } v^i = (u_n^i). \tag{4.3}$$

Let $F^i : V^i \rightarrow V^i$ be a linear endomorphism defined by

$$F^i = I - P^i T, \tag{4.4}$$

where I is the identity operator on V^i . Then F^i becomes an *isomorphism*, since the equation $F^i \phi^i = v^i$, or more explicitly, the finite difference equation:

$$\begin{cases} \phi_n^i = P_n^i \phi_{n-1}^i + u_n^i & (n \in \mathbf{Z}); \\ \phi_n^i = 0 & (n \ll 0), \end{cases} \tag{4.5}$$

admits a unique (formal) solution $\phi^i = \phi^i(v^i) \in V^i$ for each $v^i \in V^i$.

The commutative diagram (2.4) now induces a commutative diagram:

$$\begin{array}{ccccccc} \dots & \longrightarrow & V^{i-1} & \xrightarrow{Q^{i-1}} & V^i & \xrightarrow{Q^i} & V^{i+1} & \longrightarrow & \dots \\ & & \downarrow F^{i-1} & & \downarrow F^i & & \downarrow F^{i+1} & & \\ \dots & \longrightarrow & V^{i-1} & \xrightarrow{Q^{i-1}} & V^i & \xrightarrow{Q^i} & V^{i+1} & \longrightarrow & \dots \end{array} \tag{4.6}$$

Let V denote the horizontal sequence in (4.6). Since each horizontal sequence in (2.4) is a cochain complex, V is also a cochain complex and the vertical maps F^i yield an

endomorphism $F : V \rightarrow V$ of cochain complex, which is in fact an isomorphism, since so is each $F^i : V^i \rightarrow V^i$.

Consider the *mapping cone* W of the endomorphism $F : V \rightarrow V$. By definition,

$$W^i := V^i \oplus V^{i-1}, \tag{4.7}$$

and the coboundary operator $D^i : W^i \rightarrow W^{i+1}$ is represented by the matrix:

$$D^i = \begin{pmatrix} -Q^i & 0 \\ F^i & Q^{i-1} \end{pmatrix}, \tag{4.8}$$

where each element $w^i = v^i \oplus v^{i-1} \in W^i$ is identified with the column vector $w^i = {}^t(v^i, v^{i-1})$. Since F is an isomorphism, its mapping cone W is cohomologically trivial.

A more interesting situation occurs when one considers a collection $(\mathcal{V}^i)_{i \in \mathbb{Z}}$, where each \mathcal{V}^i is a linear subspace of V^i , such that

$$T \text{ and } P^i \text{ keep } \mathcal{V}^i \text{ invariant, and } Q^i \text{ maps } \mathcal{V}^i \text{ into } \mathcal{V}^{i+1}. \tag{4.9}$$

Then the commutative diagram (4.6) restricts to a subdiagram:

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathcal{V}^{i-1} & \xrightarrow{Q^{i-1}} & \mathcal{V}^i & \xrightarrow{Q^i} & \mathcal{V}^{i+1} & \longrightarrow & \dots \\ & & \downarrow F^{i-1} & & \downarrow F^i & & \downarrow F^{i+1} & & \\ \dots & \longrightarrow & \mathcal{V}^{i-1} & \xrightarrow{Q^{i-1}} & \mathcal{V}^i & \xrightarrow{Q^i} & \mathcal{V}^{i+1} & \longrightarrow & \dots \end{array} \tag{4.10}$$

Namely, one obtains a cochain subcomplex \mathcal{V} of V , along with an endomorphism $F : \mathcal{V} \rightarrow \mathcal{V}$. One can now speak of its mapping cone \mathcal{W} :

$$\mathcal{W}^i := \mathcal{V}^i \oplus \mathcal{V}^{i-1}, \tag{4.11}$$

with the coboundary operator $D^i : \mathcal{W}^i \rightarrow \mathcal{W}^{i+1}$ defined by (4.8). In general, \mathcal{W} is not necessarily cohomologically trivial, since the endomorphism $F : \mathcal{V} \rightarrow \mathcal{V}$ is not necessarily isomorphic; it remains to be injective, but may not be surjective. Thus computing the cohomology groups $H^i(\mathcal{W})$ could be an interesting problem. We shall see this by taking \mathcal{V}^i to be the space of rapidly decreasing sequences or a Gevrey sequence space (for the definition of these spaces, see §7). To distinguish W from \mathcal{W} , we refer to W as the *formal* mapping cone.

A few words here should be in order as to the reason why it is important to consider mapping cones. As before, we explain this with an example.

EXAMPLE 4.1. The Kummer confluent hypergeometric equation (1.1) and the Humbert confluent hypergeometric system (1.3) are closely related to each other through the contiguity relation (1.2), where the parameter b in (1.1) is assumed to be the same as the parameter b_1 in (1.3). More precisely, it may be said that *the Humbert system (1.3) is obtained as the mapping cone of the contiguity relation (1.2) of the Kummer equation (1.1)*. To explain what we mean by this, let us consider the formal mapping cone W :

$$0 \rightarrow W^0 \xrightarrow{D^0} W^1 \xrightarrow{D^1} W^2 \rightarrow 0, \tag{4.12}$$

arising from the lattice (2.2), which in turn arises from the contiguity relation (1.2). Let \mathcal{M} be the \mathcal{D} -module associated to the Humbert system (1.3). Then it turns out that

(4.12) represents the solution complex $\mathbf{RHom}_{\mathcal{O}}(\mathcal{M}, \mathcal{F})$, where $\mathcal{F} := \mathcal{O}(D)((1/y))$ is the ring of formal Laurent series in $1/y$ with coefficients in $\mathcal{O}(D)$. Indeed, there is an isomorphism of complexes:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & W^0 & \xrightarrow{D^0} & W^1 & \xrightarrow{D^1} & W^2 \longrightarrow 0 \\
 & & \downarrow J^0 & & \downarrow J^1 & & \downarrow J^2 \\
 0 & \longrightarrow & \mathcal{F}^{\oplus 2} & \xrightarrow{\nabla^0} & \mathcal{F}^{\oplus 4} & \xrightarrow{\nabla^1} & \mathcal{F}^{\oplus 2} \longrightarrow 0,
 \end{array} \tag{4.13}$$

where ∇^0 and ∇^1 are given by

$$\nabla^0 = \begin{pmatrix} \partial_x & -1 \\ -b_1 & \delta_x + \delta_y + c - x \\ (\delta_y + c)\partial_y - (\delta_y + b_2) & x\partial_y \\ b_1\partial_y & x\partial_y - \delta_y - b_2 \end{pmatrix},$$

$$\nabla^1 = \begin{pmatrix} (\delta_y + c)\partial_y - (\delta_y + b_2) & \partial_y & -\partial_x & 1 \\ b_1\partial_y & x\partial_y - (\delta_y + b_2) & b_1 & -(\delta_x + \delta_y + c - x) \end{pmatrix},$$

and the bottom sequence in (4.13) is easily seen to be a solution complex of \mathcal{M} with coefficients in \mathcal{F} . Moreover, the vertical maps in (4.13) are given by

$$\begin{aligned}
 J^0 : W^0 &\rightarrow \mathcal{F}^{\oplus 2}, & (f_n) &\mapsto f := \sum c_n f_n(x) y^{-n}; \\
 J^1 : W^1 &\rightarrow \mathcal{F}^{\oplus 4}, & (g_n) &\mapsto g := \sum c_n A_n f_n(x) y^{-n}; \\
 J^2 : W^2 &\rightarrow \mathcal{F}^{\oplus 2}, & (h_n) &\mapsto h := \sum c_n B_n f_n(x) y^{-n},
 \end{aligned}$$

with diagonal matrices $A_n := \text{diag}(1, 1, n - b_2, n - b_2)$, $B_n := \text{diag}(n - b_2, n - b_2)$ and constants:

$$c_n := \begin{cases} \frac{(-1)^n (n-1)! (1-c, n)}{(1-b_2, n)} & (n \geq 1); \\ 1 & (n = 0); \\ \frac{(b_2, -n)}{(-n)! (c, -n)} & (n \leq -1), \end{cases} \tag{4.14}$$

where (c, n) is the Pochhammer symbol defined by

$$(c, n) := \begin{cases} c(c+1)(c+2)\cdots(c+n-1) & (n \in \mathbf{N}); \\ 1 & (n = 0). \end{cases} \tag{4.15}$$

If formal sequence spaces (V^i) are replaced by a collection (\mathcal{V}^i) of subspaces satisfying (4.9), then (4.13) restricts to an isomorphism of complexes:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{W}^0 & \xrightarrow{D^0} & \mathcal{W}^1 & \xrightarrow{D^1} & \mathcal{W}^2 \longrightarrow 0 \\
 & & \downarrow J^0 & & \downarrow J^1 & & \downarrow J^2 \\
 0 & \longrightarrow & \mathcal{G}^{\oplus 2} & \xrightarrow{\nabla^0} & \mathcal{G}^{\oplus 4} & \xrightarrow{\nabla^1} & \mathcal{G}^{\oplus 2} \longrightarrow 0,
 \end{array} \tag{4.16}$$

with some subspace $\mathcal{G} \subset \mathcal{F}$, which in turn induces an isomorphism:

$$H^i(\mathcal{W}) \cong \text{Ext}_{\mathcal{D}}^i(\mathcal{M}, \mathcal{G}). \tag{4.17}$$

If (\mathcal{V}^i) is taken to be a collection of Gevrey sequence spaces, then \mathcal{G} will be a space of formal Gevrey functions in $1/y$. (See §9 for a rigorous statement.) Thus computing the Gevrey extension groups $\text{Ext}_{\mathcal{D}}^i(\mathcal{M}, \mathcal{G})$ of the \mathcal{D} -module associated to the Humbert system (1.3) is reduced to computing the cohomology groups $H^i(\mathcal{W})$ of the complex \mathcal{W} arising from the contiguity relation (1.2) of the Kummer equation (1.1). This fact illustrates the necessity of dealing with the mapping cone (4.11); it is of particular importance to find out how to compute its cohomology groups.

Similarly, one can check that the Humbert system in *three* variables ([1], [3]):

$$\begin{cases} L_1(c)f := \{x\partial_x^2 + y\partial_x\partial_y + z\partial_x\partial_z + (c-x)\partial_x - b_1\}f = 0; \\ L_2(c)f := \{y\partial_y^2 + z\partial_y\partial_z + x\partial_y\partial_x + (c-y)\partial_y - b_2\}f = 0; \\ L_3(c)f := \{z\partial_z^2 + x\partial_z\partial_x + y\partial_z\partial_y + (c-z)\partial_z - b_3\}f = 0, \end{cases} \tag{4.18}$$

arises as the mapping cone of the contiguity relation (1.4) of the Humbert system in *two* variables (1.3). It often occurs that a confluent hypergeometric system is obtained as the mapping cone of a contiguity relation of another hypergeometric system. This phenomenon will be discussed more systematically elsewhere.

5. Harmonic complex.

Let (\mathcal{V}^i) and \mathcal{W} be as in Section 4. We shall consider how to compute the cohomology groups $H^i(\mathcal{W})$ of the complex \mathcal{W} . In order for this problem to be tractable, some other conditions than (4.9) should be imposed upon the collection (\mathcal{V}^i) . To state them (Assumption 5.1), we introduce two linear subspaces V_c^i and V_0^i of the formal sequence space V^i . Let V_c^i denote the space of all convergent sequences $v^i = (u_n^i) \in V^i$ as $n \rightarrow +\infty$. Then the limit operator:

$$\lim : V_c^i \rightarrow U^i, \quad v^i \mapsto \lim_{n \rightarrow +\infty} u_n^i$$

is a well-defined linear operator. Let $V_0^i := \{v^i \in V_c^i : \lim v^i = 0\}$ be the space of all sequences converging to zero. Recall that the operator $F^i : V^i \rightarrow V^i$ in (4.4) is an isomorphism, whose inverse, or the resolvent operator, is denoted by

$$R^i := (F^i)^{-1} : V^i \rightarrow V^i. \tag{5.1}$$

ASSUMPTION 5.1. Assume that (\mathcal{V}^i) satisfies the following conditions:

- (A1) T and P^i keep \mathcal{V}^i invariant, and Q^i maps \mathcal{V}^i into \mathcal{V}^{i+1} ;
- (A2) \mathcal{V}^i is a linear subspace of V_0^i ;
- (A3) R^i maps \mathcal{V}^i into V_c^i ;
- (A4) for any $v^i \in \mathcal{V}^i$, if $R^i v^i \in V_0^i$, then $R^i v^i \in \mathcal{V}^i$.

We remark that (A1) is nothing other than (4.9). In view of (A2), condition (A4) asserts that for any $v^i \in \mathcal{V}^i$, one has $R^i v^i \in \mathcal{V}^i$ if and only if $R^i v^i \in V_0^i$. Our purpose in this section is to establish the following theorem.

THEOREM 5.2. *Under Assumption 5.1, there exist a cochain complex C and a quasi-isomorphism $\pi : \mathcal{W} \rightarrow C$ such that C^i is a linear subspace of U^{i-1} :*

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \mathcal{W}^{i-1} & \xrightarrow{D^{i-1}} & \mathcal{W}^i & \xrightarrow{D^i} & \mathcal{W}^{i+1} \longrightarrow \dots \\
 & & \downarrow \pi^{i-1} & & \downarrow \pi^i & & \downarrow \pi^{i+1} \\
 \dots & \longrightarrow & C^{i-1} & \xrightarrow{d^{i-1}} & C^i & \xrightarrow{d^i} & C^{i+1} \longrightarrow \dots
 \end{array} \tag{5.2}$$

By this theorem, we have an isomorphism $H^i(\pi) : H^i(\mathcal{W}) \cong H^i(C)$ of cohomology groups, and hence the computation of $H^i(\mathcal{W})$ is reduced to that of $H^i(C)$. We now notice that C^i is much “smaller” in size than \mathcal{W}^i , since C^i is a subspace of U^{i-1} , while \mathcal{W}^i is a space of infinite sequences of vectors in $U^i \oplus U^{i-1}$. This can most clearly be seen when each U^i is a finite-dimensional linear space. Indeed, in this case, C is a cochain complex of *finite*-dimensional linear spaces, while \mathcal{W} is a cochain complex of *infinite*-dimensional linear spaces. So it is expected that computing $H^i(C)$ is much more accessible than computing $H^i(\mathcal{W})$ directly. The complex C will be called the *harmonic complex*.

The rest of this section is devoted to the proof of Theorem 5.2, which will be completed only after Lemma 5.5. Let us introduce an operator:

$$p^i := \lim R^i : \mathcal{V}^i \rightarrow U^i, \tag{5.3}$$

which is well defined by virtue of (A3). We define C^i by

$$C^i := \text{Im } p^{i-1} = \{p^{i-1}v^{i-1} : v^{i-1} \in \mathcal{V}^{i-1}\}. \tag{5.4}$$

The coboundary operators of C are defined in the following lemma.

LEMMA 5.3. *There exist unique linear operators $d^i : C^i \rightarrow C^{i+1}$ that make C a cochain complex and the following diagram commutative:*

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \mathcal{V}^{i-2} & \xrightarrow{Q^{i-2}} & \mathcal{V}^{i-1} & \xrightarrow{Q^{i-1}} & \mathcal{V}^i \longrightarrow \dots \\
 & & \downarrow p^{i-2} & & \downarrow p^{i-1} & & \downarrow p^i \\
 \dots & \longrightarrow & C^{i-1} & \xrightarrow{d^{i-1}} & C^i & \xrightarrow{d^i} & C^{i+1} \longrightarrow \dots
 \end{array} \tag{5.5}$$

PROOF. By (5.4), $p^{i-1} : \mathcal{V}^{i-1} \rightarrow C^i$ is surjective, that is, for any $c^i \in C^i$, there exists an element $v^{i-1} \in \mathcal{V}^{i-1}$ such that $c^i = p^{i-1}v^{i-1}$. We define $d^i : C^i \rightarrow C^{i+1}$ by

$$d^i c^i := p^i Q^{i-1} v^{i-1}. \tag{5.6}$$

This is well defined, namely, $p^i Q^{i-1} v^{i-1}$ is independent of the choice of v^{i-1} . To see this it is sufficient to show that if $v^{i-1} \in \mathcal{V}^{i-1}$ satisfies $p^{i-1}v^{i-1} = 0$, then $p^i Q^{i-1} v^{i-1} = 0$. Since $\lim R^{i-1} v^{i-1} = p^{i-1} v^{i-1} = 0$, (A4) implies $R^{i-1} v^{i-1} \in \mathcal{V}^{i-1}$. So (A1) yields $Q^{i-1} R^{i-1} v^{i-1} \in \mathcal{V}^i$, and (A2) implies $\lim Q^{i-1} R^{i-1} v^{i-1} = 0$. Hence $p^i Q^{i-1} v^{i-1} = \lim R^i Q^{i-1} v^{i-1} = \lim Q^{i-1} R^{i-1} v^{i-1} = 0$, where we have used the commutation relation:

$$R^i Q^{i-1} = Q^{i-1} R^{i-1}, \quad (5.7)$$

which follows from (4.6) and (5.1). Therefore $d^i : C^i \rightarrow C^{i+1}$ is well defined. The commutativity of (5.5) immediately follows from the definition (5.6) of d^i . Since the top sequence in (5.5) is a cochain complex and the vertical arrows there are surjective, the bottom sequence is also a cochain complex. The uniqueness of d^i is clear. The proof is complete. \square

We proceed to define a morphism $\pi : \mathcal{W} \rightarrow C$ of cochain complexes. Let

$$q^i : \mathcal{W}^i = \mathcal{V}^i \oplus \mathcal{V}^{i-1} \rightarrow \mathcal{V}^{i-1} \quad (5.8)$$

be the projection down to the second factor. Consider the diagram:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathcal{W}^{i-1} & \xrightarrow{D^{i-1}} & \mathcal{W}^i & \xrightarrow{D^i} & \mathcal{W}^{i+1} & \longrightarrow & \dots \\ & & \downarrow q^{i-1} & & \downarrow q^i & & \downarrow q^{i+1} & & \\ \dots & \longrightarrow & \mathcal{V}^{i-2} & \xrightarrow{Q^{i-2}} & \mathcal{V}^{i-1} & \xrightarrow{Q^{i-1}} & \mathcal{V}^i & \longrightarrow & \dots \\ & & \downarrow p^{i-2} & & \downarrow p^{i-1} & & \downarrow p^i & & \\ \dots & \longrightarrow & C^{i-1} & \xrightarrow{d^{i-1}} & C^i & \xrightarrow{d^i} & C^{i+1} & \longrightarrow & \dots \end{array} \quad (5.9)$$

The lower half of (5.9) is just the commutative diagram (5.5), but the upper half is generally non-commutative, and so is the total diagram (5.9). Nonetheless, one can show the following weaker version of commutativity:

$$p^i q^{i+1} D^i = p^i Q^{i-1} q^i. \quad (5.10)$$

Indeed, it follows from (4.8), (5.1) and (5.8) that for any $w^i = v^i \oplus v^{i-1} \in \mathcal{W}^i$,

$$(R^i q^{i+1} D^i - R^i Q^{i-1} q^i) w^i = v^i \in \mathcal{V}^i.$$

Applying the limit operator on both sides and using (A2), we obtain $(p^i q^{i+1} D^i - p^i Q^{i-1} q^i) w^i = \lim v^i = 0$. This proves (5.10). We now define $\pi^i : \mathcal{W}^i \rightarrow C^i$ to be the composite of the two vertical homomorphisms in (5.9), i.e.,

$$\pi^i := p^{i-1} q^i. \quad (5.11)$$

Then (5.10), together with the commutativity of (5.5), shows that the diagram (5.2) is commutative, namely, $\pi : \mathcal{W} \rightarrow C$ is a morphism of cochain complexes.

To establish Theorem 5.2, it only remains to show that $\pi : \mathcal{W} \rightarrow C$ is a quasi-isomorphism. In what follows, $Z^i(A)$ and $B^i(A)$ denote the i -th cocycles and the i -th coboundaries of a cochain complex A , respectively. We now need two lemmas.

LEMMA 5.4. $\pi^i : Z^i(\mathcal{W}) \rightarrow Z^i(C)$ is surjective.

PROOF. Since $p^{i-1} : \mathcal{V}^{i-1} \rightarrow C^i$ is surjective, any $z^i \in Z^i(C)$ is written $z^i = p^{i-1} v^{i-1}$ with some $v^{i-1} \in \mathcal{V}^{i-1}$. Since $z^i \in Z^i(C)$, the commutative diagram (5.5) yields $p^i Q^{i-1} v^{i-1} = 0$. Hence (A4) implies $R^i Q^{i-1} v^{i-1} \in \mathcal{V}^i$. We claim that if we set

$$w^i := (-R^i Q^{i-1} v^{i-1}) \oplus v^{i-1} \in \mathcal{W}^i = \mathcal{V}^i \oplus \mathcal{V}^{i-1},$$

then $w^i \in Z^i(\mathcal{W})$ and $\pi^i w^i = z^i$. Indeed, using (4.8), we obtain

$$\begin{aligned} D^i w^i &= \begin{pmatrix} -Q^i & 0 \\ F^i & Q^{i-1} \end{pmatrix} \begin{pmatrix} -R^i Q^{i-1} v^{i-1} \\ v^{i-1} \end{pmatrix} \\ &= \begin{pmatrix} Q^i R^i Q^{i-1} v^{i-1} \\ -F^i R^i Q^{i-1} v^{i-1} + Q^{i-1} v^{i-1} \end{pmatrix} \\ &= \begin{pmatrix} R^{i+1} Q^i Q^{i-1} v^{i-1} \\ -Q^{i-1} v^{i-1} + Q^{i-1} v^{i-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned}$$

where we have used (5.1) and (5.7) in the third equality. This means that $w^i \in Z^i(\mathcal{W})$. Moreover, (5.11) leads to $\pi^i w^i = p^{i-1} q^i w^i = p^{i-1} v^{i-1} = z^i$. Hence $\pi^i : Z^i(\mathcal{W}) \rightarrow Z^i(C)$ is surjective. The proof is complete. \square

LEMMA 5.5. *For any $w^i \in Z^i(\mathcal{W})$, if $\pi^i w^i \in B^i(C)$, then $w^i \in B^i(\mathcal{W})$.*

PROOF. Since $p^{i-2} : \mathcal{V}^{i-2} \rightarrow C^{i-1}$ is surjective, it follows from $\pi^i w^i \in B^i(C)$ that there exists an element $v^{i-2} \in \mathcal{V}^{i-2}$ such that $\pi^i w^i = d^{i-1} p^{i-2} v^{i-2}$. The commutative diagram (5.5) yields $\pi^i w^i = p^{i-1} Q^{i-2} v^{i-2}$. If we write $w^i = v^i \oplus v^{i-1}$, then $\pi^i w^i = p^{i-1} v^{i-1}$, and hence $p^{i-1}(v^{i-1} - Q^{i-2} v^{i-2}) = 0$. It follows from (A4) that $R^{i-1}(v^{i-1} - Q^{i-2} v^{i-2}) \in \mathcal{V}^{i-1}$. In view of this, we claim that if we set

$$w^{i-1} := R^{i-1}(v^{i-1} - Q^{i-2} v^{i-2}) \oplus v^{i-2} \in \mathcal{W}^{i-1} = \mathcal{V}^{i-1} \oplus \mathcal{V}^{i-2},$$

then $D^{i-1} w^{i-1} = w^i$. Indeed, since $w^i = v^i \oplus v^{i-1} \in Z^i(\mathcal{W})$, (4.8) leads to

$$D^i w^i = \begin{pmatrix} -Q^i & 0 \\ F^i & Q^{i-1} \end{pmatrix} \begin{pmatrix} v^i \\ v^{i-1} \end{pmatrix} = \begin{pmatrix} -Q^i v^i \\ F^i v^i + Q^{i-1} v^{i-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

the second component of which yields $F^i v^i = -Q^{i-1} v^{i-1}$. Applying R^i , we have

$$v^i = -R^i Q^{i-1} v^{i-1}. \quad (5.12)$$

Using (4.8) again, we obtain

$$\begin{aligned} D^{i-1} w^{i-1} &= \begin{pmatrix} -Q^{i-1} & 0 \\ F^{i-1} & Q^{i-2} \end{pmatrix} \begin{pmatrix} R^{i-1}(v^{i-1} - Q^{i-2} v^{i-2}) \\ v^{i-2} \end{pmatrix} \\ &= \begin{pmatrix} -Q^{i-1} R^{i-1}(v^{i-1} - Q^{i-2} v^{i-2}) \\ F^{i-1} R^{i-1}(v^{i-1} - Q^{i-2} v^{i-2}) + Q^{i-2} v^{i-2} \end{pmatrix} \\ &= \begin{pmatrix} -R^i Q^{i-1}(v^{i-1} - Q^{i-2} v^{i-2}) \\ v^{i-1} - Q^{i-2} v^{i-2} + Q^{i-2} v^{i-2} \end{pmatrix} \\ &= \begin{pmatrix} -R^i Q^{i-1} v^{i-1} \\ v^{i-1} \end{pmatrix} = \begin{pmatrix} v^i \\ v^{i-1} \end{pmatrix} = w^i, \end{aligned}$$

where we have used (5.1) and (5.7) in the third equality, and (5.12) in the fifth equality. This means that $w^i \in B^i(\mathcal{W})$, and the proof is complete. \square

We are now in a position to complete the proof of Theorem 5.2.

PROOF OF THEOREM 5.2. As was mentioned previously, it only remains to show that $\pi : \mathcal{W} \rightarrow C$ is a quasi-isomorphism, that is, the induced homomorphism $H^i(\pi) : H^i(\mathcal{W}) \rightarrow H^i(C)$ is an isomorphism. But Lemmas 5.4 and 5.5 immediately imply its surjectivity and injectivity, respectively. The proof is complete. \square

6. Formula for the harmonic complex.

Under Assumption 5.1, we were able to reduce the computation of $H^i(\mathcal{W})$ to that of $H^i(C)$. Our next task is to try to describe the harmonic complex C as explicitly as possible. For this purpose, however, we require some conditions stronger than those in Assumption 5.1. To state them (Assumption 6.1), recall that R^i is the resolvent operator of the finite difference equation $F^i \phi^i = v^i$, namely, its (formal) solution is given by $\phi^i = R^i v^i$. Hereafter we write

$$\phi^i(v^i) := R^i v^i \quad \text{with} \quad \phi^i(v^i) = (\phi_n^i(v^i))_{n \in \mathbb{Z}}. \tag{6.1}$$

In addition to the factorial monomials $(x)_j$ and $\langle x \rangle_j$ introduced in §3, we set

$$[x]_j = \frac{(-1)^j (j-1)!}{x(x-1)(x-2) \cdots (x-j+1)},$$

for $j \in \mathbb{N}$, with the convention $[x]_0 = 1$.

ASSUMPTION 6.1. In addition to (A1) and (A2) in Assumption 5.1, we assume:

- (B1) there exist linear operators $\Phi^i : \mathcal{V}^i \rightarrow U^i$ and $\Psi^{i,j} : U_0^i \rightarrow U^i$, where U_0^i is the image of Φ^i , and a constant $c \in \mathbb{C} \setminus \mathbb{Z}$ such that for any $v^i \in \mathcal{V}^i$, the following factorial asymptotic expansion holds as $n \rightarrow +\infty$:

$$\phi_n^i(v^i) \sim \left(\sum_{j=0}^{\infty} \Psi^{i,j} [n-c]_j \right) \Phi^i v^i \quad \text{in } U^i,$$

where $\Psi^{i,0} : U_0^i \hookrightarrow U^i$ is the inclusion operator;

- (B2) R^i maps \mathcal{V}_0^i into \mathcal{V}^i , where $\mathcal{V}_0^i := \{v^i \in \mathcal{V}^i : \Phi^i v^i = 0\}$;
- (B3) there exists a factorial expansion:

$$Q_n^i = \sum_{j=0}^{N^i} Q^{i,j} \langle n-c \rangle_j \quad \text{in } L(U^i, U^{i+1}).$$

Note that (B3) is nothing but (3.2) in Assumption 3.1. We observe that Assumption 6.1 implies Assumption 5.1. Indeed, if (B1) holds then for each $v^i \in \mathcal{V}^i$, $\phi_n(v^i) \rightarrow \Phi^i v^i$ as $n \rightarrow +\infty$. This shows that (B1) implies (A3), and moreover that the operator $p^i : \mathcal{V}^i \rightarrow U^i$ in (5.3) is given by

$$p^i = \Phi^i. \tag{6.2}$$

Under condition (B1), it is easy to see that (B2) implies (A4). Thus Assumption 5.1 implies Assumption 6.1. Our aim in this section is to establish the following:

THEOREM 6.2. Under Assumption 6.1, the complex C in Theorem 5.2 is given by

$$\begin{cases} C^i = U_0^{i-1} := \text{Im}[\Phi^i : \mathcal{V}^i \rightarrow U^i], \\ d^i = Q^{i-1,0} + \sum_{j=1}^{N^{i-1}} Q^{i-1,j} \sum_{k=1}^j \Psi^{i-1,k}. \end{cases} \quad (6.3)$$

The rest of this section is devoted to the proof of Theorem 6.2, which will be completed only after Lemma 6.4. The formula $C^i = U_0^{i-1}$ in (6.3) is an immediate consequence of (5.4), (6.2) and (B.1). So we have only to derive the formula for $d^i : C^i \rightarrow C^{i+1}$ in (6.3). We first notice that for any $u^{i-1} \in C^i = U_0^{i-1}$,

$$d^i u^{i-1} = \lim_{n \rightarrow +\infty} Q_n^{i-1} \phi_n^{i-1}(v^{i-1}), \quad (6.4)$$

where $v^{i-1} \in \mathcal{V}^{i-1}$ is any element such that $u^{i-1} = \Phi^{i-1} v^{i-1}$. Indeed, by (5.6) with c^i replaced by u^{i-1} , we obtain

$$\begin{aligned} d^i u^{i-1} &= p^i Q^{i-1} v^{i-1} = \lim R^i Q^{i-1} v^{i-1} \\ &= \lim Q^{i-1} R^{i-1} v^{i-1} = \lim_{n \rightarrow +\infty} Q_n^{i-1} \phi_n^{i-1}(v^{i-1}), \end{aligned}$$

where we have used (5.7) in the third equality. To evaluate the right-hand side of (6.4), we require two lemmas.

LEMMA 6.3. For each $v^{i-1} \in \mathcal{V}^{i-1}$, we have an asymptotic expansion:

$$Q_n^{i-1} \phi_n^{i-1}(v^{i-1}) = \sum_{j=0}^{N^{i-1}} \sum_{k=0}^j Q^{i-1,j} \Psi^{i-1,k} u^{i-1} \langle n-c \rangle_j [n-c]_k + o(1) \quad (6.5)$$

as $n \rightarrow +\infty$ in U^i .

PROOF. We set $u^{i-1} = \Phi^{i-1} v^{i-1}$. By (B1), there exists an infinite sequence $\varepsilon_n \in U^{i-1}$ such that $n^{N^{i-1}} \varepsilon_n \rightarrow 0$ in U^{i-1} as $n \rightarrow +\infty$, and that

$$\phi_n^{i-1}(v^{i-1}) = \left(\sum_{k=0}^{N^{i-1}} \Psi^{i-1,k} [n-c]_k \right) u^{i-1} + \varepsilon_n. \quad (6.6)$$

Applying the expansion $Q_n^{i-1} = \sum_{j=0}^{N^{i-1}} Q^{i-1,j} \langle n-c \rangle_j$ in (B3) to (6.6), we obtain

$$Q_n^{i-1} \phi_n^{i-1}(v^{i-1}) = \sum_{j=0}^{N^{i-1}} \sum_{k=0}^j Q^{i-1,j} \Psi^{i-1,k} u^{i-1} \langle n-c \rangle_j [n-c]_k + \alpha_n,$$

where $\alpha_n = \beta_n + \gamma_n$ is given by

$$\begin{cases} \beta_n = \sum_{k=1}^{N^{i-1}} \sum_{j=0}^{k-1} \langle n-c \rangle_j [n-c]_k Q^{i-1,j} \Psi^{i-1,k} u^{i-1}, \\ \gamma_n = \sum_{j=0}^{N^{i-1}} \frac{\langle n-c \rangle_j}{n^{N^{i-1}}} Q^{i-1,j} (n^{N^{i-1}} \varepsilon_n). \end{cases}$$

To prove the lemma it is sufficient to show $\alpha_n \rightarrow 0$. First, since $\langle n-c \rangle_j [n-c]_k \rightarrow 0$ for $j=0, 1, \dots, k-1$, we have $\beta_n \rightarrow 0$. Secondly, since $n^{N^{i-1}} \varepsilon_n \rightarrow 0$ and $Q^{i-1, j} \in L(U^i, U^{i+1})$, we have $Q^{i-1, j}(n^{N^{i-1}} \varepsilon_n) \rightarrow 0$. This, together with

$$\frac{\langle n-c \rangle_j}{n^{N^{i-1}}} \rightarrow \frac{(-1)^{N^{i-1}}}{(N^{i-1}-1)!} \delta_{j, N^{i-1}} \quad (j=0, 1, \dots, N^{i-1}),$$

yields $\gamma_n \rightarrow 0$, where δ_{ij} denotes the Kronecker delta. Therefore we obtain $\alpha_n \rightarrow 0$, which establishes the lemma. \square

In view of Lemma 6.3, let us consider an operator-valued polynomial:

$$G^i(x) = \sum_{j=0}^{N^{i-1}} \sum_{k=0}^j Q^{i-1, j} \Psi^{i-1, k} u^{i-1} \langle x-c \rangle_j [x-c]_k. \quad (6.7)$$

To simplify this expression, we introduce other factorial monomials:

$$\{x\}_j := \frac{x(x+1)(x+2) \cdots (x+j-1)}{(-1)^j j!},$$

for $j \in \mathbf{N}$, with the convention $\{x\}_0 = 1$. We need the following lemma.

LEMMA 6.4. *The factorial monomials $\langle x \rangle_j$, $[x]_j$ and $\{x\}_j$ satisfy the relations:*

$$\langle x \rangle_j [x]_k = \begin{cases} j \sum_{i=1}^j \binom{j-1}{j-i} \{x\}_i & (j \in \mathbf{N}, k=0); \\ \frac{(j-k)!(k-1)!}{(j-1)!} \sum_{i=0}^{j-k} \binom{j-1}{j-k-i} \{x\}_i & (j, k \in \mathbf{N}, j \geq k). \end{cases}$$

PROOF. It follows from the definitions of $\langle x \rangle_j$, $[x]_j$, $\{x\}_j$ that

$$\langle x \rangle_j [x]_k = \frac{(j-k)!(k-1)_+!}{(j-1)!} \{x-j+1\}_{j-k} \quad (j \geq k \geq 0, j \geq 1), \quad (6.8)$$

where $a_+ := \max\{a, 0\}$. For any non-negative integers m and k , we claim

$$\{x-m\}_k = \sum_{j=(k-m)_+}^k \binom{m}{k-j} \{x\}_j. \quad (6.9)$$

Indeed, applying the generating series $(1+t)^{-x} = \sum_{k=0}^{\infty} \{x\}_k t^k$ to the equality $(1+t)^{-x+m} = (1+t)^m (1+t)^{-x}$, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \{x-m\}_k t^k &= \sum_{p=0}^m \binom{m}{p} t^p \sum_{k=0}^{\infty} \{x\}_k t^k \\ &= \sum_{k=0}^{\infty} t^k \sum_{j=(k-m)_+}^k \binom{m}{k-j} \{x\}_j. \end{aligned}$$

Comparing the coefficients of t^k on both sides, we obtain (6.9). Finally, applying (6.9) to (6.8), we establish the lemma. \square

PROOF OF THEOREM 6.2. We are now in a position to complete the proof of Theorem 6.2. It follows from (6.4), (6.5) and (6.7) that for any $u^{i-1} \in C^i = U_0^{i-1}$,

$$d^i u^{i-1} = \lim_{n \rightarrow +\infty} G^i(n) u^{i-1}. \tag{6.10}$$

(The existence of the limit is already established.) Applying Lemma 6.4 to (6.7), we obtain $G^i(x) = \sum_{p=0}^{N^{i-1}} G^{i,p} \{x - c\}_p$, with

$$G^{i,p} = \begin{cases} Q^{i-1,0} \Psi^{i-1,0} + \sum_{j=1}^{N^{i-1}} Q^{i-1,j} \sum_{k=1}^j \Psi^{i-1,k} & (p = 0), \\ \sum_{j=p}^{N^{i-1}} Q^{i-1,j} \sum_{k=0}^{j-p} \frac{(j-k)!}{(j-k-p)!} \frac{(k-1)_+!}{(k-1+p)!} \Psi^{i-1,k} & (p = 1, 2, \dots, N^{i-1}). \end{cases}$$

Since $G^i(n)u^{i-1}$ converges as $n \rightarrow +\infty$, the polynomial $G^i(x)u^{i-1}$ must be a constant, i.e., one has $G^i(x)u^{i-1} = G^{i,0}u^{i-1}$ identically. Hence (6.10) shows that

$$d^i u^{i-1} = \lim_{n \rightarrow +\infty} G^i(n) u^{i-1} = G^{i,0} u^{i-1},$$

which completes the proof of Theorem 6.2. \square

7. Weighted sequence space.

As mentioned at the end of §3, we shall eventually work with Assumption 3.1. So we are naturally led to the following problem.

PROBLEM 7.1. How shall we choose a collection (\mathcal{V}^i) so that Assumption 3.1 implies Assumption 6.1? Moreover, with such a choice of (\mathcal{V}^i) , describe the formula (6.3) in Theorem 6.2 explicitly only in terms of the data given in Assumption 3.1.

A solution to this problem is taking \mathcal{V}^i to be the space of rapidly decreasing sequences or a Gevrey sequence space with a suitable Gevrey index. To explain this we begin with an (abstract) definition.

DEFINITION 7.2. A set A of infinite sequences of positive numbers indexed by $n \in \mathbb{N}$ is said to be a *weight* if the following conditions are satisfied:

(W1) for any $a = (a_n) \in A$, there is an element $b = (b_n) \in A$ such that

$$L(a, b) := \limsup_{n \rightarrow +\infty} \frac{b_n}{a_{n+1}} < \infty;$$

(W2) for any $k \in \mathbb{N}$ and $a = (a_n) \in A$, there is an element $b = (b_n) \in A$ such that

$$M_k(a, b) := \limsup_{n \rightarrow +\infty} \frac{n^k b_n}{a_n} < \infty;$$

(W3) there is an element $a = (a_n) \in A$ such that $a_n \rightarrow 0$ as $n \rightarrow +\infty$.

Let A be a weight. Given a locally convex space U , let $V = V(U)$ be the formal sequence space associated to U , as was defined in (4.1). We denote by $V_A = V_A(U)$ the linear space of all sequences $v = (u_n) \in V(U)$ such that $|v|_a < \infty$ for every semi-norms $|\cdot|$ of U and every $a \in A$, where

$$|v|_a := \limsup_{n \rightarrow +\infty} \frac{|u_n|}{a_n}.$$

The space $V_A = V_A(U)$ is referred to as the *weighted sequence space* with weight A associated to U . We set $V_A^i := V_A(U^i)$.

EXAMPLE 7.3 (The space of rapidly decreasing sequences). It is easy to see that

$$A = \{(n^{-k}) : k \in \mathbf{N}\}, \tag{7.1}$$

becomes a weight. The associated weighted sequence space V_A , which will be denoted by $\ell = \ell(U)$, is referred to as the space of *rapidly decreasing sequences* associated to U . We set $\ell^i := \ell(U^i)$.

EXAMPLE 7.4 (Gevrey sequence space). Let s be a non-negative number, and $A \subset (0, \infty)$ be an open interval. Assume that if $s = 0$ then $0 \leq \inf A < 1$ (this constraint is put to insure (W3)); otherwise, $\inf A \geq 0$ can be arbitrary. Then,

$$A = A_{s,A} := \{(a^n (n!)^{-s}) : a \in A\}, \tag{7.2}$$

becomes a weight. The associated weighted sequence space V_A , which will be denoted by $\mathcal{G}_{s,A} = \mathcal{G}_{s,A}(U)$, is referred to as the *Gevrey sequence space* with index (s, A) associated to U . In particular, the number s is called the *main Gevrey index*. We remark that $\mathcal{G}_{s,A}$ is a linear subspace of ℓ . We set $\mathcal{G}_{s,A}^i := \mathcal{G}_{s,A}(U^i)$.

The following proposition is a first step toward a solution to Problem 7.1.

PROPOSITION 7.5. Under Assumption 3.1, let $\mathcal{V}^i = V_A^i$ with A any weight. Then conditions (A1) and (A2) in Assumption 5.1 are satisfied.

To prove this proposition, we shall make use of the following lemma.

LEMMA 7.6. Let A be a weight. Then,

- (1) the translation operator T keeps V_A invariant;
- (2) let (P_n) be a sequence in $B(U)$ that is bounded as $n \rightarrow +\infty$. Then for any $v = (u_n) \in V_A$, the sequence $Pv := (P_n u_n)$ belongs to V_A ;
- (3) for any $v = (u_n) \in V_A$ and any non-negative integer k , the sequence $\tilde{v} := (n^k u_n)$ belongs to V_A ;
- (4) for any $v = (u_n) \in V_A(U)$ and $Q \in L(U, U')$, the sequence $Qv := (Q u_n)$ belongs to $V_A(U')$;
- (5) any sequence $v = (u_n) \in V_A$ converges to zero as $n \rightarrow +\infty$.

PROOF. Let $v = (u_n)$ be any sequence in V_A , $a = (a_n)$ any element in A , and $|\cdot|$ any semi-norm on U . By (W1) there is an element $b = (b_n) \in A$ such that $L(a, b) < \infty$.

$$|Tv|_a = \limsup_{n \rightarrow +\infty} \frac{|u_{n-1}|}{a_n} = \limsup_{n \rightarrow +\infty} \frac{b_n}{a_{n+1}} \frac{|u_n|}{b_n} \leq L(a, b)|v|_b < \infty.$$

Thus $Tv \in V_A$, and hence assertion (1). Since P_n is bounded in $B(U)$ as $n \rightarrow +\infty$, there is a constant M such that $|P_n| \leq M$ for every $n \in \mathbf{N}$. We have

$$|Pv|_a = \limsup_{n \rightarrow +\infty} \frac{|P_n u_n|}{a_n} \leq \limsup_{n \rightarrow +\infty} |P_n| \frac{|u_n|}{a_n} \leq M|v|_a < \infty.$$

Thus $Pv \in V_A$, and hence assertion (2). By (W2) there is an element $b = (b_n) \in A$ such that $M_k(a, b) < \infty$. We have

$$|\tilde{v}|_a = \limsup_{n \rightarrow +\infty} \frac{|n^k u_n|}{a_n} = \limsup_{n \rightarrow +\infty} \frac{n^k b_n}{a_n} \frac{|u_n|}{b_n} \leq M_k(a, b)|v|_b < \infty.$$

Thus $\tilde{v} \in V_A$, and hence assertion (3). Let $|\cdot|'$ be any semi-norm on U' . Since $Q : U \rightarrow U'$ is bounded, there exist a semi-norm $\|\cdot\|$ on U and a constant M such that $|Qu|' \leq M\|u\|$ for every $u \in U$. Then we have

$$|Qv|'_a = \limsup_{n \rightarrow +\infty} \frac{|Qu_n|'}{a_n} \leq \limsup_{n \rightarrow +\infty} \frac{M\|u_n\|}{a_n} = M\|v\|_a < \infty.$$

Thus $Qv \in V_A(U')$, and hence assertion (4). Finally, assertion (5) immediately follows from (W3). The proof is complete. \square

PROOF OF PROPOSITION 7.5. In Lemma 7.6 we set $U = U^i$ and $P_n = P_n^i$. Then Lemma 7.6(1) implies that T keeps V_A^i invariant. By (3.1) in Assumption 3.1, P_n^i is bounded in $B(U^i)$ as $n \rightarrow +\infty$. It follows from Lemma 7.6(2) that P^i keeps V_A^i invariant. To show that Q^i maps V_A^i into V_A^{i+1} , we make use of Lemma 7.6(3) and (4) by setting $U = U^i$, $U' = U^{i+1}$ and $Q = Q^{i,j}$, where $Q^{i,j}$ ($j = 0, 1, \dots, N^i$) are the coefficients of the factorial expansion (3.2). It follows from Lemma 7.6(3) that for any $v = (u_n) \in V_A^i$, the sequence $(\langle n - c \rangle_j u_n)$ belongs to V_A^i . Then Lemma 7.6(4) implies that the sequence $(Q^{i,j} \langle n - c \rangle_j u_n)$ belongs to V_A^{i+1} . Since

$$(Q^i v^i)_n = \sum_{j=0}^{N^i} Q^{i,j} \langle n - c \rangle_j u_n,$$

we have $Q^i v^i \in V_A^{i+1}$, and hence Q^i maps \mathcal{V}^i into \mathcal{V}^{i+1} . Therefore (A1) is proved. (A2) is an immediate consequence of Lemma 7.6(3). The proof is complete. \square

8. Main theorem.

We are now in a position to give a complete solution to Problem 7.1. To state it (Theorem 8.6), we require some notations. We begin by introducing some operators which will be used to describe the harmonic complex C explicitly.

DEFINITION 8.1. With the notations in Assumption 3.1, we first set

$$P_{jk}^i := \sum_{m=1}^{j-k} \frac{(k-1)_+(m-1)! (m-1, j-k-m)}{(k+m-1)! (j-k-m)!} P^{i,m} \quad (0 \leq k < j), \quad (8.1)$$

where (c, j) is the Pochhammer symbol defined in (4.15). Using operators P_{jk}^i defined by (8.1), we next set

$$A_{jk}^i := X^i P_{j+1,k}^i + \left(I + \frac{1}{j} X^i P^{i,1} \right) (I_1 - Z^i)^{-1} (Y^i P_{jk}^i - \delta_{j,k+1} Z^i) \quad (0 \leq k < j), \quad (8.2)$$

where I and I_1 are the identity operators on U^i and on U_1^i , respectively, δ_{ij} being the Kronecker symbol. Recall that, under Assumption 3.1, the inverse operator $(I_1 - Z^i)^{-1}$ exists in $B(U^i)$, and so A_{jk}^i is well defined. For each finite set of positive integers $J = \{j_1, j_2, \dots, j_k\}$ with $j_1 < j_2 < \dots < j_k$, we finally set

$$A_J^i := A_{j_k j_{k-1}}^i A_{j_{k-1} j_{k-2}}^i \cdots A_{j_2 j_1}^i A_{j_1 0}^i. \quad (8.3)$$

EXAMPLE 8.2. We illustrate Definition 8.1 with the example arising from the contiguity relation (1.2) of the Kummer equation (recall the discussions in Examples 1.1, 2.1, 3.2 and 4.1). In this case, we are in a simple situation to the effect that (i) $P^{i,j} = 0$ for $j \geq 2$ in (3.1) and (ii) $Z^i = 0$ in (3.5). Then (8.1) reads

$$P_{jk}^i = \begin{cases} (1/k^+) P^{i,1} & (j = k + 1); \\ 0 & (\text{otherwise}), \end{cases} \quad (8.4)$$

where $k^+ := \max\{k, 1\}$, and so (8.2) reads

$$A_{jk}^i = \begin{cases} (1/k^+) (I + (1/(k + 1)) X^i P^{i,1}) (Y^i P^{i,1}) & (j = k + 1); \\ 0 & (\text{otherwise}). \end{cases} \quad (8.5)$$

From the fact that $A_{jk}^i = 0$ unless $j = k + 1$, (8.3) implies

$$A_J^i = \begin{cases} A_{[k]}^i = A_{k,k-1} A_{k-1,k-2} \cdots A_{21} A_{10} & (J = [k], k \in \mathbb{N}); \\ 0 & (\text{otherwise}), \end{cases} \quad (8.6)$$

where $[k] := \{1, 2, \dots, k\}$. Substituting (3.7), (3.8), (3.10) into (8.5), we have

$$A_{k+1,k}^0 = \frac{1}{k^+} \begin{pmatrix} bx/(k + 1) & x^2/(k + 1) \\ b & x \end{pmatrix}, \quad A_{k+1,k}^1 = \frac{x}{k^+} \begin{pmatrix} b/(k + 1) & 1/(k + 1) \\ b & 1 \end{pmatrix}. \quad (8.7)$$

By using these formulas, $A_{[k]}^i$ ($i = 0, 1, k \in \mathbb{N}$) can be computed as follows.

$$A_{[k]}^i = \begin{cases} \frac{(b + 1, k - 1)}{\{(k - 1)!\}^2} \begin{pmatrix} bx^k/k & x^{k+1}/k \\ bx^{k-1} & x^k \end{pmatrix} & (i = 0); \\ \frac{(b + 1, k - 1)}{\{(k - 1)!\}^2} x^k \begin{pmatrix} b/k & 1/k \\ b & 1 \end{pmatrix} & (i = 1). \end{cases} \quad (8.8)$$

When trying to solve Problem 7.1 in a Gevrey setting $\mathcal{V}^i = \mathcal{G}_{s,A}^i$, we have to impose some conditions on the Gevrey index (s, A) , so that this attempt should be successful. To state them (Condition 8.5), we make the following definition.

DEFINITION 8.3. With the notations in Assumption 3.1, let p^i be the *smallest* non-negative integer j such that $X^i P^{i,j} Y^i \neq 0$; if there is no such j , let $p^i = \infty$. We define q^i and r^i in the same manner by using $Y^i P^{i,j} X^i$ and $Y^i P^{i,j} Y^i$ in stead of $X^i P^{i,j} Y^i$,

respectively. (Here is an abuse of notation: p^i and q^i in this section are different from those in §5.) Note that condition (3.4) forces that $p^i, q^i \geq 1$ and $r^i \geq 0$. Let s_0 be a non-negative integer defined by

$$s_0 := \min_{i \in \mathbf{Z}} s^i, \quad \text{with } s^i := \min\{p^i + q^i - 1, r^i\}. \tag{8.9}$$

We remark that if $U^i = 0$ then $X^i = Y^i = 0$, so that $p^i = q^i = r^i = \infty$ and $s^i = \infty$. Thus the minimum above can be taken only over those $i \in \mathbf{Z}$ with $U^i \neq 0$. The integer s_0 will be used to define an *upper bound* of admissible main Gevrey indices (see Condition 8.5).

Let \mathcal{N}^i be a system of semi-norms defining the locally convex topology of U^i . For each semi-norm $v = |\cdot| \in \mathcal{N}^i$, we set

$$a_0^i(v) := \begin{cases} a_1^i(v)a_2^i(v) & \text{(if } p^i + q^i - 1 < r^i\text{);} \\ a_1^i(v)a_2^i(v) + a_3^i(v) & \text{(if } p^i + q^i - 1 = r^i\text{);} \\ a_3^i(v) & \text{(if } p^i + q^i - 1 > r^i\text{),} \end{cases} \tag{8.10}$$

where $a_1^i(v), a_2^i(v)$ and $a_3^i(v)$ are given by

$$a_k^i(v) := \begin{cases} (p^i - 1)! |X^i P^{i,p^i} Y^i| & (k = 1); \\ (q^i - 1)! |Y^i P^{i,q^i} X^i| & (k = 2); \\ (r^i - 1)_+! |Y^i P^{i,r^i} Y^i| & (k = 3). \end{cases} \tag{8.11}$$

Finally let a_0 be a non-negative constant, which may possibly be $+\infty$, defined by

$$a_0 := \sup_{i \in \mathbf{Z}} a_0^i, \quad \text{with } a_0^i := \sup_{v \in \mathcal{N}^i} a_0^i(v). \tag{8.12}$$

We remark that if $U^i = 0$ then $a_k^i(v) = 0$ for $k = 1, 2, 3$ and so $a_0^i = 0$. Thus the supremum above can be taken only over those $i \in \mathbf{Z}$ with $U^i \neq 0$. The constant a_0 will also play a role in formulating admissible Gevrey indices in Condition 8.5.

We shall explain in Appendix (§10) how to calculate the constants $a_k^i(v)$ in (8.11) when $U^i = \mathcal{O}(D)^{m_i}$ and $P^{i,j}$ are matrices of holomorphic functions on D , this case being relevant in applications to contiguity relations (1.5).

EXAMPLE 8.4. We illustrate Definition 8.3 with the same example as in Example 8.2. A simple check of (3.7), (3.8) and (3.10) shows that

$$p^i = r^i = 1, \quad q^i = \begin{cases} 1 & (b \neq 0); \\ \infty & (b = 0), \end{cases} \quad \text{and so } s^i = 1 \quad (i = 0, 1). \tag{8.13}$$

For $i \neq 0, 1$, we have $s^i = \infty$, since $U^i = 0$. Thus (8.9) yields

$$s_0 = 1. \tag{8.14}$$

We proceed to calculate the constant a_0 in (8.12). Recall that $U^0 = U^1 = \mathcal{O}(D)^2$ with a domain D in \mathbf{C} . Let \mathcal{K} be the set of all compact subsets in D . For each $K \in \mathcal{K}$ and $f = {}^t(f_1, f_2) \in \mathcal{O}(D)^2$, we set

$$|f|_K := \sup_{x \in K} \sqrt{|f_1(x)|^2 + |f_2(x)|^2}.$$

Then the semi-norms $|\cdot|_K$ ($K \in \mathcal{K}$) provide $\mathcal{O}(D)^2$ with a complete, locally convex topology. Identifying the semi-norm $\nu = |\cdot|_K$ with $K \in \mathcal{K}$, we set $\mathcal{N}^0 = \mathcal{N}^1 := \mathcal{K}$, and write $a_k(\nu) = a_k(K)$ in (8.10) and (8.11). Now a simple computation yields

$$\begin{aligned} X^0 P^{0,1} Y^0 &= \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, & Y^0 P^{0,1} X^0 &= \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, & Y^0 P^{0,1} Y^0 &= \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}, \\ X^1 P^{1,1} Y^1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & Y^1 P^{1,1} X^1 &= \begin{pmatrix} 0 & 0 \\ bx & 0 \end{pmatrix}, & Y^1 P^{1,1} Y^1 &= \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}. \end{aligned}$$

By using these data, if $b \neq 0$, the constants $a_k^i(K)$ in (8.11) can be computed as

$$\begin{cases} a_1^0(K) = r(K), & a_2^0(K) = |b|, & a_3^0(K) = r(K), \\ a_1^1(K) = 1, & a_2^1(K) = |b|r(K), & a_3^1(K) = r(K), \end{cases} \quad (8.15)$$

where $r(E)$ denotes the radius of a subset $E \subset D$ relative to the origin:

$$r(E) := \sup_{x \in E} |x|. \quad (8.16)$$

See Appendix (Lemma 10.1) for more detailed derivation of (8.15). From (8.13), if $b \neq 0$, our example falls into the second case in (8.10). Substituting (8.15) into (8.10), we have $a_i^i(K) = (1 + |b|)r(K)$ for $i = 0, 1$. This formula remains valid for $b = 0$, although we are then in the third case in (8.10). Taking the supremum over $K \in \mathcal{K}$, we obtain $a_i^i = (1 + |b|)r(D)$ for $i = 0, 1$, and hence

$$a_0 = (1 + |b|)r(D). \quad (8.17)$$

In particular, a_0 is finite if and only if D is a bounded domain in \mathbb{C} .

CONDITION 8.5 (Admissible Gevrey indices). Let (s_0, a_0) be as in Definition 8.3, and $A \subset (0, \infty)$ be an open interval. Assume that (s, A) and (s_0, a_0) satisfy one of the following conditions:

- (C1) $s = s_0 = 0$ and $a_0 \leq \inf A < 1$;
- (C2) $s = 0 < s_0$ and $0 \leq \inf A < 1$;
- (C3) $0 < s < s_0$ and $0 \leq \inf A < \infty$;
- (C4) $s = s_0 > 0$ and $a_0 \leq \inf A < \infty$.

The main theorem of this paper is now stated as follows.

THEOREM 8.6. Under Assumption 3.1, let (\mathcal{V}^i) be either

- (1) $\mathcal{V}^i = \ell^i$, the space of rapidly decreasing sequences, or
- (2) $\mathcal{V}^i = \mathcal{G}_{s,A}^i$ with Gevrey index (s, A) satisfying Condition 8.5.

Then the harmonic complex C , which is quasi-isomorphic to \mathcal{W} , is expressed as

$$\begin{cases} C^i = U_0^{i-1}, \\ d^i = Q^{i-1,0} + \sum_{j=1}^{N^{i-1}} Q^{i-1,j} \sum_{J \in S_j} A_J^{i-1}, \end{cases} \quad (8.18)$$

where S_j is the set of all nonempty subsets of $\{1, 2, \dots, j\}$ and A_J^i are defined by (8.3) in Definition 8.1.

The formula (8.18) describes the harmonic complex C explicitly only in terms of the data in Assumption 3.1. So (8.18) is a final formula we have been seeking for. The proof of Theorem 8.6 is based upon the following very deep result which, together with Proposition 7.5, shows that if the collection (\mathcal{V}^i) is taken as in Theorem 8.6, then Assumption 3.1 implies Assumption 6.1.

LEMMA 8.7 (Key Lemma). *Under Assumption 3.1, let (\mathcal{V}^i) be as in Theorem 8.6. Then (B1) and (B2) in Assumption 6.1 are satisfied, with the operators $\Psi^{i,j} : U_0^i \rightarrow U^i$ given by*

$$\Psi^{i,j} = \sum_{J \in S'_j} A_J^i \quad (8.19)$$

where S'_j is the set of all subsets of $\{1, 2, \dots, j\}$ containing j .

The contents of this lemma, or more precisely, statements equivalent to this lemma in somewhat different notations, have been established in our previous paper [7] as its main results (Theorems I and II, Corollaries 3.4.2 and 3.6.1). Once we admit Lemma 8.7, leaving its verification (very hard analysis!) to [7], the proof of Theorem 8.6 becomes quite easy. So the analysis carried out there should be emphasized as the most essential ingredient of the proof.

PROOF OF THEOREM 8.6. It follows from Proposition 7.5 and Lemma 8.7 that if we take (\mathcal{V}^i) as in Theorem 8.6, then Assumption 3.1 leads to Assumption 6.1, with the operators $\Psi^{i,j}$ given by (8.19). So we can apply Theorem 6.2. Since S_j is the set of all nonempty subsets of $\{1, 2, \dots, j\}$ and S'_j is the set of all subsets of $\{1, 2, \dots, j\}$ containing j , S_j is the disjoint union of S'_k over $k = 1, 2, \dots, j$. Hence (8.19) yields

$$\sum_{k=1}^j \Psi^{i,k} = \sum_{k=1}^j \sum_{J \in S'_k} A_J^i = \sum_{J \in S_j} A_J^i. \quad (8.20)$$

Substituting (8.20) with i replaced by $i - 1$ into (6.3), we obtain (8.18). This completes the proof of Theorem 8.6. \square

EXAMPLE 8.8. We illustrate Theorem 8.6 with the example discussed in Examples 8.2 and 8.4. We begin by considering what Condition 8.5 designates in this example. Since $(s_0, a_0) = (1, (1 + |b|)r(D))$, as was observed in (8.14) and (8.17), the case (C1) does not occur and the remaining cases (C2), (C3), (C4) take the form:

- (D2) $s = 0$ and $0 \leq \inf I < 1$;
- (D3) $0 < s < 1$ and $0 \leq \inf I < \infty$;
- (D4) $s = 1$ and $(1 + |b|)r(D) \leq \inf I < \infty$,

respectively, with the case (D4) occurring only when D is a *bounded* domain in \mathcal{C} .

We proceed to consider what the formula (8.18) tells us in our situation. First, recall that for $i = 0, 1$, the decomposition $U^i = U_0^i \oplus U_1^i$ is given by $U_0^i = \mathcal{O}(D)e_0$ and $U_1^i = \mathcal{O}(D)e_1$, where $e_0 = {}^t(1, 0)$ and $e_1 = {}^t(0, 1)$ are standard unit vectors. Since $U^i = 0$ for $i \neq 0, 1$, we have

$$C^i = \begin{cases} \mathcal{O}(D)e_0 & (i = 1, 2); \\ 0 & (\text{otherwise}). \end{cases} \quad (8.21)$$

Next we shall describe $d^1 : C^1 \rightarrow C^2$ explicitly. In general, if $N^{i-1} = 1$ then the formula (8.18) takes a very simple form:

$$d^i = Q^{i-1,0} + Q^{i-1,1} A_{\{1\}}^{i-1} = Q^{i-1,0} + Q^{i-1,1} A_{10}^{i-1}. \tag{8.22}$$

Currently we are just in this situation with $i = 1$. Substituting (3.9) and (8.8) into (8.22), we find that d^1 is represented by the matrix:

$$d^1 = \begin{pmatrix} \partial_x & -1 \\ 0 & \delta_x \end{pmatrix}. \tag{8.23}$$

Upon identifying C^1 and C^2 with $\mathcal{O}(D)$ in an obvious manner, (8.21) and (8.23) lead to the following expression for the harmonic complex:

$$C : 0 \rightarrow \overset{0}{\mathcal{O}} \rightarrow \mathcal{O}(D) \xrightarrow{\partial_x} \overset{2}{\mathcal{O}}(D) \rightarrow 0, \tag{8.24}$$

where the operator ∂_x in (8.24) comes from the (1,1)-entry of the matrix (8.23). Indeed, only the (1,1)-entry is relevant when (8.23) is thought of as an operator from U_0^0 into U_0^1 . It follows from (8.24) that

$$H^i(C) \cong \begin{cases} H_{\text{DR}}^{i-1}(D) & (i = 1, 2); \\ 0 & (\text{otherwise}), \end{cases} \tag{8.25}$$

where $H_{\text{DR}}^i(D)$ denotes the i -th holomorphic de Rham cohomology group of the domain D , (which is isomorphic to the C^∞ de Rham group, since any domain in \mathbf{C} is a domain of holomorphy). Theorem 8.6 now tells us that if (\mathcal{V}^i) is either (ℓ^i) or $(\mathcal{G}_{s,A}^i)$ with any Gevrey index (s, A) satisfying one of the conditions (D2), (D3), (D4), where $U^i = \mathcal{O}(D)^2$ for $i = 0, 1$ and $U^i = 0$ otherwise, then \mathcal{W} and C are quasi-isomorphic and therefore, by (8.25),

$$H^i(\mathcal{W}) \cong \begin{cases} H_{\text{DR}}^{i-1}(D) & (i = 1, 2); \\ 0 & (\text{otherwise}). \end{cases} \tag{8.26}$$

9. Some applications.

We present some applications of Theorem 8.6. An application has already been given in Example 8.8, that is, the formula (8.26). First we shall combine it with the discussion in Example 4.1 to compute the Gevrey extension groups of the Humbert system (1.3). To state the result we make the following definition.

DEFINITION 9.1. Let t be a real number, and $A \subset (0, \infty)$ be an open interval. Denote by $\mathcal{O}(D)((1/y))_{t,A}$ the set of all formal Laurent series $f = \sum f_n y^{-n}$ in $1/y$ with coefficients $f_n \in \mathcal{O}(D)$ such that for any compact subset K in D and any $a \in A$, there exists a constant $C(K, a)$ satisfying

$$|f_n|_K := \max_{x \in K} |f_n(x)| \leq C(K, a) a^n (n!)^{t-1} \quad (n \in \mathbf{N}). \tag{9.1}$$

The space $\mathcal{O}(D)((1/y))_{t,A}$ will be referred to as the space of *formal Gevrey functions* with index (t, A) . This definition of the main Gevrey index t agrees with the standard one employed in the literature on Gevrey analysis (e.g., [9], [10]).

Let us return to Example 4.1. In view of (4.13) and (4.14), one observes that if \mathcal{V}^i are taken to be the Gevrey sequence space $\mathcal{G}_{s,A}^i$, then the space \mathcal{G} in (4.16) and (4.17) becomes $\mathcal{O}(D)((1/y))_{1-s,A}$, namely, the two main indices s and t are related by $s + t = 1$. Therefore, putting (4.17) and (8.26) together, we arrive at the following result on the Gevrey extension groups of the Humbert system in two variables.

THEOREM 9.2. *Let \mathcal{M} be the \mathcal{D} -module associated to the Humbert system (1.3), and D be a domain in \mathbb{C} . Assume that the Gevrey index (t, A) satisfies one of the following conditions:*

- (1) $t = 1$ and $0 \leq \inf A < 1$;
- (2) $0 < t < 1$ and $0 \leq \inf A < \infty$;
- (3) $t = 0$ and $(1 + |b_1|)r(D) \leq \inf A < \infty$ (in this case, D must be bounded),

where $r(D)$ is defined by (8.16), then there is an isomorphism of cohomology groups:

$$\text{Ext}_{\mathcal{D}}^i(\mathcal{M}, \mathcal{O}(D)((1/y))_{t,A}) \cong \begin{cases} H_{\text{DR}}^{i-1}(D) & (i = 1, 2); \\ 0 & (\text{otherwise}). \end{cases} \tag{9.2}$$

We remark that conditions (1), (2), (3) in Theorem 9.2 correspond to (D2), (D3), (D4) in Example 8.8, respectively. See also [8] for the computation of Gevrey cohomology groups of the Humbert system. Next we take up another example.

EXAMPLE 9.3. Let us consider the example arising from the contiguity relation (1.4) of the Humbert system (1.3). Recall here the discussions in Examples 1.2, 2.2 and 3.3. Only the main points will be sketched, since this case resembles to a large extent the one arising from the Kummer equation discussed previously. In particular, formulas (8.4), (8.5), (8.6) and (8.22) are available, which then simplify the computation of the harmonic complex C . By substituting the explicit data (3.11)–(3.20) into these formulas, the coboundary operators d^1, d^2 of C are computed as

$$d^1 = \left(\begin{array}{c|cccccc} \partial_x & -1 & 0 & 0 & 0 & 0 \\ \partial_y & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & \partial_x & 0 & -1 & 0 & 0 \\ 0 & \partial_y & 0 & 0 & -1 & 0 \\ 0 & 0 & \partial_x & 0 & -1 & 0 \\ 0 & 0 & \partial_y & 0 & 0 & -1 \\ 0 & 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & 0 & x & y \\ 0 & -b_2 & b_1 & 0 & x - y & 0 \end{array} \right) \tag{9.3}$$

$$d^2 = \left(\begin{array}{cc|ccc} \partial_y & -\partial_x & 0 & 1 & -1 \\ \hline 0 & 0 & x\partial_y & -\delta_x & \delta_y \\ 0 & 0 & x(\delta_y + b_2) & -y(\delta_x - b_2) & x(\delta_y - b_1) \\ \hline & & 0 & 0 & 0 & 0 \\ & & -y\partial_x & \partial_y & -\partial_x & 0 \\ & & -y(\delta_x + b_1) & \delta_y + b_2 & -(\delta_x + b_1) & \delta_x + \delta_y \end{array} \right). \tag{9.4}$$

Now the decompositions (3.21) allow us to identify C^1, C^2 and C^3 with $\mathcal{O}(D), \mathcal{O}(D)^2$ and $\mathcal{O}(D)$, respectively, in an obvious manner. With these identifications, the harmonic complex is given by

$$C : 0 \longrightarrow \overset{\circ}{0} \longrightarrow \mathcal{O}(D) \xrightarrow{\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}} \mathcal{O}(D)^2 \xrightarrow{(\partial_y, -\partial_x)} \mathcal{O}(D) \longrightarrow 0,$$

where the operators $\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$ and $(\partial_y, -\partial_x)$ above come from the (1.1)-blocks of the matrices (9.3) and (9.4), respectively. Thus the harmonic complex is quasi-isomorphic to the holomorphic de Rham complex, shifted by one, on the domain D . In particular we have

$$H^i(C) \cong \begin{cases} H_{\text{DR}}^{i-1}(D) & (i = 1, 2, 3); \\ 0 & (\text{otherwise}). \end{cases} \quad (9.5)$$

We proceed to the calculation of the constants s_0 and a_0 defined by (8.9) and (8.12). We can determine s_0 just as in the Kummer case (8.14):

$$s_0 = 1. \quad (9.6)$$

However, due to the complexity of the data (3.12), (3.14), (3.16), it is not so easy to write down the constant a_0 in a simple manner as in the Kummer case. Indeed, it turns out that (10.1) of Appendix yields

$$\begin{aligned} a_1^0 &= \sup_{(x,y) \in D} \sqrt{|x|^2 + |y|^2}, & a_2^0 &= \sqrt{|b_1|^2 + |b_2|^2}; \\ a_1^1 &= \sup_{(x,y) \in D} \sqrt{1 + |x|^2 + |y|^2}, & a_2^1 &= \sqrt{|b_1|^2 + |b_2|^2}; \\ a_1^2 &= 1, & a_2^2 &= \sup_{(x,y) \in D} \sqrt{|b_1x + b_2y|^2 + |b_1 + b_2|^2|x|^2|y|^2}, \end{aligned}$$

but there is no simple way to write down a_3^i ($i = 0, 1, 2$). For them, nonetheless, we make use of (10.2) to obtain upper bounds:

$$\begin{aligned} a_3^0 &\leq \tilde{a}_3^0 \\ &:= \sup_{(x,y) \in D} \sqrt{|1 + b_1|^2 + |1 + b_2|^2 + (1/4)|b_1|^2 + (1/4)|b_2|^2 + 2|x|^2 + 2|y|^2 + (1/4)|x + y|^2}; \\ a_3^1 &\leq \tilde{a}_3^1 := \sup_{(x,y) \in D} \sqrt{1/2 + (13/4)|x|^2 + (13/4)|y|^2 + (1/4)|x + y|^2}; \\ a_3^2 &\leq \tilde{a}_3^2 := \sup_{(x,y) \in D} \sqrt{1 + |x + y|^2 + |x|^2|y|^2}. \end{aligned}$$

Using these informations and in view of (8.10), (8.12), we set

$$\tilde{a}_0 := \max\{\tilde{a}_0^0, \tilde{a}_0^1, \tilde{a}_0^2\}, \quad \text{with } \tilde{a}_0^i := a_1^i a_2^i + \tilde{a}_3^i \quad (i = 0, 1, 2). \quad (9.7)$$

Let \mathcal{M} be the \mathcal{D} -module associated to the Humbert system (4.18) in *three* variables, and D be a domain in \mathbf{C}^2 . In a similar manner as in the two variable case, one can show that if the Gevrey index (t, A) satisfies one of the following conditions:

- (1) $t = 1$ and $0 \leq \inf A < 1$;
- (2) $0 < t < 1$ and $0 \leq \inf A < \infty$;
- (3) $t = 0$ and $\tilde{a}_0 \leq \inf A < \infty$ (in this case, D must be bounded),

where \tilde{a}_0 is defined by (9.7), then there is an isomorphism of cohomology groups:

$$\text{Ext}_{\mathcal{D}}^i(\mathcal{M}, \mathcal{O}(D)((1/z))_{t,A}) \cong \begin{cases} H_{\text{DR}}^{i-1}(D) & (i = 1, 2, 3); \\ 0 & (\text{otherwise}). \end{cases} \tag{9.8}$$

We present still one more example.

EXAMPLE 9.4. An inspection shows that there is a contiguity relation of the form (1.2), if operators $P^0(c), P^1(c)$ and $Q^0(c)$ are taken so that

$$\begin{aligned} P^0(c-n) &= P^{0,0} + P^{0,1}(n-c)_1 + P^{0,2}(n-2)_2 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2x^2 \\ b^2 & 4bx \end{pmatrix} (n-c)_1 + \begin{pmatrix} b^2x^2 & 2x^2(bx-1) \\ -b^3x & 2bx(1-bx) \end{pmatrix} (n-c)_2; \\ P^1(c-n) &= P^{1,0} + P^{1,1}(n-c)_1 + P^{1,2}(n-2)_2 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ b^2x^2 & 4bx \end{pmatrix} (n-c)_1 + \begin{pmatrix} b^2x^2 & 2(bx-1) \\ -b^3x^3 & 2bx(1-bx) \end{pmatrix} (n-c)_2; \\ Q^0(c-n) &= Q^{0,0} + Q^{0,1}\langle n-c \rangle_1 + Q^{0,2}\langle n-c \rangle_2 \\ &= \begin{pmatrix} \partial_x^2 & 4x\partial_x + 2(1-3bx) \\ -2b^2x\partial_x + b^2(1+3bx) & x^2\partial_x^2 - 6bx^2\partial_x + 9b^2x^2 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 2 \\ -b^2 & 2x\partial_x - 2(1+3bx) \end{pmatrix} \langle n-c \rangle_1 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \langle n-c \rangle_2, \end{aligned}$$

where c is thought of as the contiguity parameter, b being a constant. These are obtained as a special solution of the ‘‘contiguity equation’’ $Q(c)P(c) = R(c)P(c+1)$, though a detailed account of this issue will be left for a separate paper. In this example the operator $Q^0(c)$ has an expansion of order two, i.e., $N^0 = 2$ in (3.2). We have $U^0 = U^1 = \mathcal{O}(D)^2$ and $U^i = 0$ otherwise. For $i = 0, 1$, the decomposition $U^i = U_0^i \oplus U_1^i = \mathcal{O}(D) \oplus \mathcal{O}(D)$ is standard. Finally, $Z^0 = Z^1 = 0$ in (3.5).

In general, if $N^{i-1} = 2$ then the formula (8.18) reads

$$\begin{aligned} d^i &= Q^{i-1,0} + Q^{i-1,1}A_{\{1\}}^{i-1} + Q^{i-1,2}(A_{\{1\}}^{i-1} + A_{\{2\}}^{i-1} + A_{\{1,2\}}^{i-1}) \\ &= Q^{i-1,0} + Q^{i-1,1}A_{10}^{i-1} + Q^{i-1,2}(A_{10}^{i-1} + A_{20}^{i-1} + A_{21}^{i-1}A_{10}^{i-1}), \end{aligned}$$

where if $Z^{i-1} = 0$ then, according to (8.1) and (8.2),

$$\begin{aligned}
 A_{10}^{i-1} &= X^{i-1}P^{i-1,2} + (I + X^{i-1}P^{i-1,1})(Y^{i-1}P^{i-1,1}); \\
 A_{20}^{i-1} &= X^{i-1}P^{i-1,2} + \left(I + \frac{1}{2}X^{i-1}P^{i-1,1}\right)(Y^{i-1}P^{i-1,2}); \\
 A_{21}^{i-1} &= \frac{1}{2}X^{i-1}P^{i-1,2} + \left(I + \frac{1}{2}X^{i-1}P^{i-1,1}\right)(Y^{i-1}P^{i-1,1}).
 \end{aligned}$$

Applying this formula in the present situation, we obtain

$$d^1 = \begin{pmatrix} \partial_x^2 + 2b^2 & 4x\partial_x + 2bx + 2 \\ 0 & x^2\partial_x^2 + 2bx^2\partial_x - b^2x^2 + 6bx \end{pmatrix}, \tag{9.9}$$

together with $d^i = 0$ for the remaining i 's. With an obvious identification $C^i = U_0^{i-1} = \mathcal{O}(D)$ ($i = 1, 2$), only the $(1, 1)$ -entry of (9.9) is relevant to describing $d^1 : C^1 \rightarrow C^2$, and the harmonic complex is given by

$$C : 0 \longrightarrow \overset{\circ}{0} \longrightarrow \mathcal{O}(D) \xrightarrow{\partial_x^2 + 2b^2} \mathcal{O}(D) \longrightarrow 0.$$

As for (s_0, a_0) in (8.9) and (8.12), we easily observe that

$$s_0 = \begin{cases} 1 & (b \neq 0); \\ \infty & (b = 0), \end{cases} \tag{9.10}$$

$$a_0 = 2|b|r(D)\{|b|r(D) + 2\}, \tag{9.11}$$

where $r(D)$ is the radius of D relative to the origin (see (8.16)). We remark that (9.11) holds whether $b \neq 0$ or not. Theorem 8.6 now tells us that if \mathcal{V}^i is taken to be the Gevrey sequence space $\mathcal{G}_{s,A}^i$, with $U^0 = U^1 = \mathcal{O}(D)^2$ and $U^i = 0$ otherwise, then there is an isomorphism of cohomology groups:

$$H^i(\mathcal{V}) \cong H^i(C) \cong \begin{cases} \text{Ker}[\partial_x^2 + 2b^2 : \mathcal{O}(D) \rightarrow \mathcal{O}(D)] & (i = 1); \\ \text{Coker}[\partial_x^2 + 2b^2 : \mathcal{O}(D) \rightarrow \mathcal{O}(D)] & (i = 2); \\ 0 & (\text{otherwise}), \end{cases}$$

provided that the Gevrey index (s, A) satisfies one of the following conditions:

- (i) case $b \neq 0$,
 - (1) $s = 0$ and $0 \leq \inf A < 1$;
 - (2) $0 < s < 1$ and $0 \leq \inf A < \infty$;
 - (3) $s = 1$ and $2|b|r(D)\{|b|r(D) + 2\} \leq \inf A < \infty$,
- (ii) case $b = 0$,
 - (1) $s = 0$ and $0 \leq \inf A < 1$;
 - (2) $0 < s < \infty$ and $0 \leq \inf A < \infty$.

In this paper applications have been presented quite fragmentarily, since they are only intended to illustrate the use of Theorem 8.6. More substantial applications will be given in forthcoming papers.

10. Appendix.

This appendix describes the semi-norm structure of the space $\mathcal{O}(D)^m$, where D is a domain in a complex manifold. Let \mathcal{K} denote the set of all compact subsets in D . For each $K \in \mathcal{K}$ and $f = {}^t(f_1, \dots, f_m) \in \mathcal{O}(D)^m$, set

$$|f|_K := \max_{x \in K} \sqrt{|f_1(x)|^2 + \cdots + |f_m(x)|^2}.$$

Then the semi-norms $|\cdot|_K$ ($K \in \mathcal{K}$) provide $\mathcal{O}(D)^m$ with a complete, locally convex topology. One observes that each $m \times m$ matrix of holomorphic functions $P = P(x) \in M(m, \mathcal{O}(D))$ defines a strongly bounded transformation of $\mathcal{O}(D)^m$, that is, $P \in B(\mathcal{O}(D)^m)$. As was explained in §3, to each $K \in \mathcal{K}$ one can associate the semi-norm $|P|_K$ of P , the value of which is computed as follows.

LEMMA 10.1. *For each $K \in \mathcal{K}$ and $P \in M(m, \mathcal{O}(D))$,*

$$|P|_K = \max_{x \in K} \sqrt{\lambda(x)}, \quad (10.1)$$

where $\lambda(x)$ is the maximal eigenvalue of the Hermitian matrix $P(x)^*P(x)$.

PROOF. For each $f \in \mathcal{O}(D)^m$ and $x \in K$, we have

$$\begin{aligned} |P(x)f(x)|^2 &= (P(x)f(x), P(x)f(x)) \\ &= (P(x)^*P(x)f(x), f(x)) \leq \lambda(x)|f(x)|^2, \end{aligned}$$

where (\cdot, \cdot) is the standard Hermitian inner product on \mathbf{C}^m . Thus $|P(x)f(x)| \leq \max_{x \in K} \lambda(x)^{1/2} |f|_K$, and so $|Pf|_K \leq \max_{x \in K} \lambda(x)^{1/2} |f|_K$. In view of the definition of the semi-norms on the strongly bounded operators (see §3), we have

$$|P|_K \leq \max_{x \in K} \sqrt{\lambda(x)}.$$

To show the reverse inequality, take a point $x_0 \in K$ so that $\lambda(x_0) = \max_{x \in K} \lambda(x)$ and let $v_0 \in \mathbf{C}^m$ be a unit eigenvector of $P(x_0)^*P(x_0)$ corresponding to the eigenvalue $\lambda(x_0)$. Since v_0 belongs to $\mathcal{O}(D)^m$ as a constant function, we have

$$\begin{aligned} \sqrt{\lambda(x_0)} &= \sqrt{(P(x_0)^*P(x_0)v_0, v_0)} \\ &= |P(x_0)v_0| \leq |Pv_0|_K \leq |P|_K |v_0|_K = |P|_K. \end{aligned}$$

Therefore $|P|_K \geq \lambda(x_0)^{1/2} = \max_{x \in K} \lambda(x)^{1/2}$, which establishes the lemma. \square

As a simple illustration we explain how the formula for $a_2^1(K)$ in (8.15) has been obtained. We apply Lemma 10.1 to the matrix:

$$P(x) := Y^1 P^{1,1} X^1 = \begin{pmatrix} 0 & 0 \\ bx & 0 \end{pmatrix}.$$

Since $P(x)^*P(x)$ is the diagonal matrix with diagonal entries $|b|^2|x|^2$ and 0, we have $\lambda(x) = |b|^2|x|^2$. Hence (10.1) yields

$$a_2^1(K) = |Y^1 P^{1,1} X^1|_K = \max_{x \in K} |b| |x| = |b| r(K).$$

In general, formula (10.1) may be used to calculate the constants $a_k^i(v) = a_k^i(K)$ in (8.11), when $U^i = \mathcal{O}(D)^{m_i}$ and $P^{i,j} \in M(m_i, \mathcal{O}(D))$ (as in Examples 8.4, 9.3 and 9.4). Finally we remark that applying the inequality $\lambda(x) \leq \text{Trace}\{P(x)^* P(x)\}$ to (10.1) yields an upper bound:

$$|P|_K \leq \max_{x \in K} \sqrt{\text{Trace}\{P(x)^* P(x)\}} = \max_{x \in K} \sqrt{\sum_{i,j=1}^m |p_{ij}(x)|^2}, \quad (10.2)$$

where $p_{ij}(x)$ denotes the (i, j) -entry of the matrix $P(x)$. It is not always easy to calculate $\lambda(x)$ explicitly for a given $P(x)$, and the estimate (10.2) will be of some help in that case (see Example 9.3).

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