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A construction of equivalence subrelations for intermediate subalgebras

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Abstract. If M is a (separable) von Neumann algebra and A is a Cartan subalgebra of M, then M is determined by an equivalence relation and a 2-cocycle. By constructing an equivalence subrelation, we show that for any intermediate von Neumann subalgebra N between M and A, there exists a faithful normal conditional expectation from M onto N.

1. Introduction.

Let M be a (separable) von Neumann algebra. A Cartan subalgebra A of M is a maximal abelian von Neumann subalgebra of M which is regular in M and the range of a faithful normal conditional expectation from M. By [**7**], for each such an inclusion $A \subseteq M$, there exists a discrete measured equivalence relation $\mathscr{R}$ on a standard Borel space $(X, \mathfrak{B}, \mu)$ such that $(A \subseteq M) \cong (W^*(X) \subseteq W^*(\mathscr{R}, \sigma))$, i.e., M is the "twisted matrix algebra" over $\mathscr{R}$ and A is the algebra of "diagonal matrices". Very roughly, M is a sort of a set of matrices $(a_{x,y})_{(x,y) \in \mathscr{R}}$. Under the above isomorphism, a faithful normal conditional expectation from M onto A is defined by the restriction of each matrix over $\mathscr{R}$ to the diagonal: $(a_{x,y})_{(x,y) \in \mathscr{R}} \mapsto (a_{x,x})_{x \in X}$. (A precise definition is in Section 2.)

In this situation, for each subrelation $\mathscr{S}$ of $\mathscr{R}$, we can construct an intermediate subalgebra $W^*(\mathscr{S}, \sigma|_{\mathscr{S}})$. A faithful normal conditional expectation from $W^*(\mathscr{R}, \sigma)$ onto $W^*(\mathscr{S}, \sigma|_{\mathscr{S}})$ can be defined by restricting of each matrix over $\mathscr{R}$ to $\mathscr{S}$: $(a_{x,y})_{(x,y) \in \mathscr{R}} \mapsto (a_{x,y})_{(x,y) \in \mathscr{S}}$. Since $W^*(\mathscr{S}, \sigma|_{\mathscr{S}})$ and $W^*(\mathscr{R}, \sigma)$ have a common maximal abelian subalgebra $W^*(X)$, by [**1**, Theorem 1.5.5], this is the unique conditional expectation from $W^*(\mathscr{R}, \sigma)$ onto $W^*(\mathscr{S}, \sigma|_{\mathscr{S}})$.

Conversely, if N is an intermediate subalgebra with a (unique) faithful normal conditional expectation from M onto N, then A is also a Cartan subalgebra of N ([**10**, Remark 2.4]). So N can be expressed by $W^*(\mathscr{S}, \sigma|_{\mathscr{S}})$ for some subrelation $\mathscr{S}$ of $\mathscr{R}$.

Therefore, a question arises whether every intermediate subalgebra comes from a subrelation, i.e., whether every intermediate subalgebra is the range of a faithful normal conditional expectation from M. In this paper, we will give an affirmative answer to this question. Our main theorem in this article is the following:

Theorem 1.1 (cf. [**12**, Theorem 1.1]). *Let M be a von Neumann algebra and A be a Cartan subalgebra of M. If N is a von Neumann subalgebra of M such that $A \subseteq N \subseteq M$, then there exists a unique faithful normal conditional expectation from M onto N, and A is also a Cartan subalgebra of N.*

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We note that Sutherland pointed out in [**12**, Theorem 1.1] that every such an intermediate von Neumann subalgebra comes from an equivalence subrelation. But his proof has a gap. So we will give a complete proof in this paper. The gap is in his proof of the claim $M\xi_0 \cap \overline{N\xi_0} = N\xi_0$, where ξ_0 is a cyclic and separating vector for M. To show the claim, he did not use the property that M contains a Cartan subalgebra. So, if his argument were valid, then the above equation would hold for any inclusions of von Neumann algebras $N \subseteq M$ with a cyclic and separating vector ξ_0 for M. This contradicts the existence of a common cyclic and separating vector for any inclusions of properly infinite von Neumann algebras (see [**3**, Corollaire 2] and [**4**, Proposition 1.2]).

The idea to prove our theorem is to construct a subrelation $\mathscr{S}$ of $\mathscr{R}$ from an intermediate subalgebra N. By making use of [**6**, Theorem 1], $\mathscr{R}$ is a disjoint countable union of graphs of partial Borel transformations $\{\rho_n\}_{n\in I}$. For each $n \in I$, we define an element of normalizing groupoid v_n which is defined by a graph of ρ_n. In this situation, for each intermediate subalgebra N, we define a subset A_n of A by

$$A_n := E_A(Nv_n^*)$$

for each $n \in I$. By Lemma 3.1, each A_n is two-sided ideal of A and determines a subset E_n of $\mathrm{Dom}(\rho_n)$. So we focus on the subrelation which is generated by a union of graphs of $\rho_n|_{E_n}$'s and denote it by $\mathscr{S}$. Since A_n is equal to $Av_nv_n^* \cap Nv_n^*$ for each $n \in I$ (Lemma 3.1), we obtain that $W^*(\mathscr{S}, \sigma|_{\mathscr{S}})$ contains N. The converse can be proved by the same argument as in [**9**]. Hence we conclude that N is equal to $W^*(\mathscr{S}, \sigma|_{\mathscr{S}})$, which ensure the existence of a unique conditional expectation from M onto N.

The organization of the paper is as follows. Section 2 contains some preliminaries. We recall a decomposition for a discrete measured equivalence relation by graphs of partial Borel transformations. We give a slight modification about the decomposition in Lemma 2.3. In Section 3, we prove our main theorem. For each intermediate subalgebra, we construct a subrelation which determines the subalgebra (Lemma 3.2, Proposition 3.4). By using our main theorem, we also show some corollaries which generalize results of Dye, Jones, Sutherland and Popa. Since each intermediate subalgebra is the range of a conditional expectation, we can use the same argument as the case of von Neumann algebras being finite. For example, Corollary 3.5 shows that for each (not necessary finite) von Neumann algebra M and Cartan subalgebra A, there exists a Galois correspondence between the set of intermediate subalgebras of $(A \subseteq M) \cong (W^*(X) \subseteq W^*(\mathscr{R}, \sigma))$ and the set of Borel equivalence subrelations of $\mathscr{R}$ on X.

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2. Preparation.

In this section, we summarize the basic facts about discrete measured equivalence relations and von Neumann algebras which have Cartan subalgebras that we shall need for the subsequent arguments. Further details can be found for examples in [**2**], [**6**], [**7**].

We assume that all von Neumann algebras in this paper have separable preduals.

2.1. A characterization of an inclusion of a von Neumann algebra with a Cartan subalgebra.

Let $(X, \mathfrak{B}, \mu)$ be a standard Borel space and $\mathcal{R}$ be a discrete measured equivalence relation on X such that μ is quasi-invariant for $\mathcal{R}$. We write $x \sim y$ when $(x, y) \in \mathcal{R}$.

We denote the full group of $\mathcal{R}$ by $[\mathcal{R}]$ and the groupoid of $\mathcal{R}$ by $[\mathcal{R}]_*$, i.e.,

$[\mathcal{R}] := \{\varphi : \varphi$ is a bimeasurable nonsingular transformation on X such that $\varphi(x) \sim x$ up to a μ-null set.$\}$,

$[\mathcal{R}]_* := \{\varphi : \varphi$ is a bimeasurable nonsingular map from a measurable subset $\mathrm{Dom}(\varphi)$ of X onto a measurable subset $\mathrm{Im}(\varphi)$ of X such that $\varphi(x) \sim x$ up to a μ-null set.$\}$.

For each $\rho \in [\mathcal{R}]_*$, we write $\Gamma(\rho) \subseteq \mathcal{R}$ for the graph of ρ:

$$\Gamma(\rho) := \{(x, \rho(x)) \mid x \in \mathrm{Dom}(\rho)\}.$$

We denote by π_l the left-hand projection on $\mathcal{R}$, i.e., $\pi_l(x, y) = x$ for each $(x, y) \in \mathcal{R}$. The left counting measure $\nu := \mu_l$ is defined by

$$\mu_l(C) := \int_X |\pi_l^{-1}(x) \cap C| \, d\mu(x) \quad C \in \mathfrak{B} \times \mathfrak{B}|_{\mathcal{R}},$$

where $|\pi_l^{-1}(x) \cap C|$ is the cardinality of $\pi_l^{-1}(x) \cap C$. In the same way, we denote by π_r the right-hand projection $(x, y) \mapsto y$ on $\mathcal{R}$, and the right counting measure μ_r is defined by

$$\mu_r(C) := \int_X |\pi_r^{-1}(x) \cap C| \, d\mu(x) \quad C \in \mathfrak{B} \times \mathfrak{B}|_{\mathcal{R}}.$$

Since π_l and π_r are countable-to-one and μ is quasi-invariant for $\mathcal{R}$, they are equivalent σ-finite measures. We denote the Radon-Nikodym derivative $d\mu_l/d\mu_r$ by D_μ.

In general, for each $n \in \boldsymbol{N}$, we can define a σ-finite measure $\nu^{(n)}$ on

$$\mathcal{R}^{(n)} := \{(x_0, x_1, \ldots, x_n) : (x_0, x_i) \in \mathcal{R} \text{ for each } i\}$$

by the same manner as $\nu = \nu^{(1)}$ on $\mathcal{R} = \mathcal{R}^{(1)}$. A 2-cocycle σ on $\mathcal{R}$ is a Borel map from $\mathcal{R}^{(2)}$ to the one-dimensional torus $\boldsymbol{T}$ which satisfies

$$\sigma(x, y, z)\sigma(x, z, w) = \sigma(x, y, w)\sigma(y, z, w)$$

for almost all (x, y, z, w) in $\mathcal{R}^{(3)}$. If a 2-cocycle σ satisfies $\sigma(x, y, z) = 1$ whenever two of x, y, z are equal, then σ is said to be a normalized 2-cocycle. For each discrete measured equivalence relation $\mathcal{R}$ and a normalized 2-cocycle σ on $\mathcal{R}$, we define a von Neumann algebra $W^*(\mathcal{R}, \sigma)$ which acts on $L^2(\mathcal{R}, \nu)$ by the following:

Definition 2.1. (1) Let f be a Borel function on $\mathcal{R}$. We call f a left finite function if $D_\mu^{1/2} f$ is a finite function and f satisfies the following:

$$\sup_{(x,y) \in \mathcal{R}} \{|\{z : z \sim x \text{ and } f(x, z) \neq 0\}| + |\{z : z \sim y \text{ and } f(z, y) \neq 0\}|\} < \infty.$$

(2) For each left finite function f on $\mathscr{R}$, a bounded operator $L^\sigma(f)$ on $L^2(\mathscr{R}, \nu)$ is defined by

$$(L^\sigma(f)\xi)(x,z) := \sum_{y\sim x} f(x,y)\xi(y,z)\sigma(x,y,z)$$

for any $\xi \in L^2(\mathscr{R}, \nu)$. We denote by $W^*(\mathscr{R}, \sigma)$ the von Neumann algebra which is generated by $\{L^\sigma(f) : f \text{ is a left finite function}\}$.

By [7], for each element T in $W^*(\mathscr{R}, \sigma)$, there exists a square integrable function f_T on $\mathscr{R}$ such that

$$(T\xi)(x,z) = \sum_{y\sim x} f_T(x,y)\xi(y,z)\sigma(x,y,z)$$

for any $\xi \in L^2(\mathscr{R}, \nu)$. We denote T by $L^\sigma(f_T)$. For each $L^\sigma(f)$, $L^\sigma(g) \in W^*(\mathscr{R}, \sigma)$, we have $L^\sigma(f)^* = L^\sigma(f^*)$ and $L^\sigma(f)L^\sigma(g) = L^\sigma(f * g)$, where f^* and $f * g$ are square integrable functions on $\mathscr{R}$ which are defined by

$$f^*(x,z) := D_\mu^{-1}(x,z)\overline{f(z,x)},$$

$$(f * g)(x,z) := \sum_{y\sim x} f(x,y)g(y,z)\sigma(x,y,z).$$

For each $a \in L^\infty(X, \mu)$, we regard it as a function on the diagonal D of $\mathscr{R}$ and write $L(a)$ for $L^\sigma(a)$, i.e.,

$$L(a)\xi(x,y) := a(x)\xi(x,y).$$

The von Neumann algebra which is generated by $\{L(a) : a \in L^\infty(X, \mu)\}$ is denoted by $W^*(X)$. It is easy to see that the map $L^\sigma(f) \mapsto L(f|_D)$ is a faithful normal conditional expectation from $W^*(\mathscr{R}, \sigma)$ onto $W^*(X)$.

We recall that a subalgebra A of a von Neumann algebra M is called a Cartan subalgebra of M if A satisfies the following:

(i) A is maximal abelian in M,

(ii) A is regular in M, i.e., the normalizer

$$\mathscr{N}_M(A) := \{u \in M : u \text{ is unitary and } uAu^* = A\}$$

generates M,

(iii) there exists a (unique) faithful normal conditional expectation E_A from M onto A.

It is known that $W^*(X)$ is a Cartan subalgebra of $W^*(\mathscr{R}, \sigma)$. Indeed, by the proof of [7, Proposition 2.9], for each u in $W^*(\mathscr{R}, \sigma)$, u is in the normalizer of $W^*(X)$ if and only if u is of the form $L^\sigma(a(g, \rho))$, with a measurable function g on X of absolute value one and $\rho \in [\mathscr{R}]$, where $a(g, \rho)$ is defined by the following:

$$a(g,\rho)(x,y) := D_\mu^{-1/2}(x,y)g(x)\chi_{\Gamma(\rho^{-1})}(x,y).$$

(In general, χ_E stands for the characteristic function of a subset E.) So the normalizer of $W^*(X)$ in $W^*(\mathscr{R}, \sigma)$ generates $W^*(\mathscr{R}, \sigma)$.

Conversely, Feldman and Moore also show that each inclusion of a von Neumann algebra and a Cartan subalgebra arises from an equivalence relation and a 2-cocycle on it.

Theorem 2.2 ([**7**, Theorem 1]). *For each inclusion of a von Neumann algebra M and a Cartan subalgebra A of M, there exists a standard Borel space $(X, \mathfrak{B}, \mu)$ and a discrete measured equivalence relation $\mathscr{R}$ on X with a normalized 2-cocycle σ such that $(A \subseteq M)$ is isomorphic to $(W^*(X) \subseteq W^*(\mathscr{R}, \sigma))$.*

2.2. A decomposition of an equivalence relation by graphs of partial transformations. In this subsection, we fix an inclusion of a von Neumann algebra M and a Cartan subalgebra A of M with the faithful normal conditional expectation E_A from M onto A. By Theorem 2.2, we assume that $(A \subseteq M)$ is isomorphic to $(W^*(X) \subseteq W^*(\mathscr{R}, \sigma))$ for some discrete measured equivalence relation $\mathscr{R}$ on $(X, \mathfrak{B}, \mu)$ and a normalized 2-cocycle σ.

The following lemma may be a folklore, but we contain its proof for readers' convenience.

Lemma 2.3. *Let $\mathscr{R}$ be a measured equivalence relation on $(X, \mathfrak{B}, \mu)$. There exists a subset $\{\rho_n\}_{n \in I}$ of $[\mathscr{R}]_*$, where $I := \{n \in \boldsymbol{Z} : |n| < m\}$ for some $m \in \boldsymbol{N} \cup \{\infty\}$ such that $\rho_0 = \mathrm{id}$, $\rho_n^{-1} = \rho_{-n}$ for each $n \in I$ and $\mathscr{R}$ is a disjoint union of $\{\Gamma(\rho_n)\}_{n \in I}$ up to null sets.*

Proof. By [**6**, Theorem 1], there exists a countable group G of Borel automorphisms of X such that

$$\mathscr{R} = \mathscr{R}_G := \{(x, gx) : x \in X, g \in G\}.$$

Since G is countable, there exists $l \in \boldsymbol{N} \cup \{\infty\}$ such that $J := \{n \in \boldsymbol{Z} : |n| < l\}$ and

$$G = \{g_n : n \in J\}, \quad g_0 = \mathrm{id}, \quad g_{-n} = g_n^{-1} \text{ for each } n \in J.$$

For each $n \in J$, we define a Borel subset E_n by the following:

$$E_n := \begin{cases} X, & n = 0, \\ \{x \in X : (x, g_n(x)) \notin \bigcup_{j=-n+1}^{n-1} \Gamma(g_j)\}, & n > 0, \\ \{x \in X : (x, g_n(x)) \notin \bigcup_{j=n+1}^{-n-1} \Gamma(g_j)\} = g_{-n}(E_{-n}), & n < 0. \end{cases}$$

Now, we may assume that X is a Borel subset of $[0, 1]$. Let us denote by "$<$" the usual order on $[0, 1]$. For each $n \in J$, we define a Borel subset F_n of $E_n \cap E_{-n}$ by the following:

$$F_n := \begin{cases} \{x \in E_n \cap E_{-n} : g_n(x) = g_{-n}(x) \text{ and } x < g_n(x)\}, & n \geqq 0, \\ \{x \in E_n \cap E_{-n} : g_n(x) = g_{-n}(x) \text{ and } x > g_n(x)\} = g_{-n}(F_{-n}), & n < 0. \end{cases}$$

By the definition of $\{F_n \subseteq E_n\}_{n \in J}$, we obtain that $\mathscr{R}$ is a disjoint union of $\{\Gamma(g_n|_{E_n \setminus F_n})\}_{n \in J}$ up to a ν-null set. We set $I := \{n \in J : \mu(E_n \setminus F_n) > 0\}$ and $\rho_n := g_n|_{E_n \setminus F_n}$ for each $n \in I$. Since $\rho_n(E_n \setminus F_n) = E_{-n} \setminus F_{-n}$ up to a μ-null set, we have $\rho_{-n} = \rho_n^{-1}$ for each $n \in I$ and $\mathscr{R} = \bigcup_{n \in I} \Gamma(\rho_n)$ up to null sets. By relabeling I, we get the conclusion. □

The set of partial isometries v of M which satisfy v^*v, $vv^* \in A$ and $vAv^* = Avv^*$ is denoted by $\mathscr{GN}_M(A)$ and called the normalizing groupoid of A in M.

Let the notations be as in Lemma 2.3. For each $n \in I$, ρ_n determines a partial isometry v_n of M by $v_n := L^\sigma(a(1, \rho_n^{-1}))$. It is easy to see that v_n belongs to $\mathscr{GN}_M(A)$.

Suppose $L^\sigma(f)$ is in M. For each $n \in I$, we set $L^\sigma(f_n) := E_A(L^\sigma(f)v_n^*)v_n$. A direct computation shows that $f * a(1, \rho_n)$ satisfies

$$(2.1)\qquad (f * a(1,\rho_n))(x,x) = \sum_{y \sim x} f(x,y) D_\mu^{-1/2}(y,x) \chi_{\Gamma(\rho_n^{-1})}(y,x)$$

$$= \chi_{\mathrm{Dom}(\rho_n)}(x) f(x, \rho_n(x)) D_\mu^{1/2}(x, \rho_n(x))$$

for almost all $x \in X$. So f_n is determined by

$$(2.2)\qquad f_n(x,y) = (\chi_{\Gamma(\rho_n)} f)(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in \Gamma(\rho_n), \\ 0, & \text{otherwise} \end{cases}$$

for almost all $(x,y) \in \mathscr{R}$. Let ξ_0 be a characteristic function of the diagonal. It is well-known that ξ_0 is a cyclic and separating vector for M. In fact, $L^\sigma(f)\xi_0 = f$ for each $L^\sigma(f) \in M$. So we have the following results.

Lemma 2.4. *Let the notations be as above. Then, for each $L^\sigma(f) \in M$ and $\rho \in [\mathscr{R}]_*$, $E_A(L^\sigma(f)L^\sigma(a(1,\rho)))L^\sigma(a(1,\rho^{-1}))$ is equal to $L^\sigma(\chi_{\Gamma(\rho)}f)$. In particular, for each $T \in M$, the following equation holds up to a ν-null set*:

$$T\xi_0 = \sum_{n \in I} E_A(Tv_n^*)v_n\xi_0.$$

3. Proof of main theorem.

In the discussion that follows, we fix a von Neumann algebra M and a Cartan subalgebra A of M. By Theorem 2.2, we suppose that $(A \subseteq M) \cong (W^*(X) \subseteq W^*(\mathscr{R},\sigma))$ with a characteristic function of the diagonal ξ_0 on $\mathscr{R}$. By Lemma 2.3, there exists a subset $\{\rho_n\}_{n \in I}$ of $[\mathscr{R}]_*$ with $I = \{n \in \boldsymbol{Z} : |n| < m\}$ for some $m \in \boldsymbol{N} \cup \{\infty\}$ such that $\rho_0 = \mathrm{id}$, $\rho_{-n} = \rho_n^{-1}$ for each $n \in I$ and $\mathscr{R} = \bigcup_{n \in I} \Gamma(\rho_n)$ (disjoint union) up to null sets. For each $n \in I$, we set $v_n := L^\sigma(a(1,\rho_n^{-1})) \in \mathscr{GN}_M(A)$.

Suppose that N is a von Neumann subalgebra of M which contains A. We first show the next lemma which will be crucial in our argument.

Lemma 3.1. *For each $n \in I$, $E_A(Nv_n^*)$ is equal to $Av_nv_n^* \cap Nv_n^*$. Moreover, this is a two-sided ideal of A.*

Proof. Since v_n belongs to $\mathscr{GN}_M(A)$, $Av_nv_n^* \cap Nv_n^*$ is a two-sided ideal of A. So it suffices to prove that $E_A(Nv_n^*) \subseteq Nv_n^*$. For each $T \in N$, we have

$$\begin{aligned} E_A(Tv_n^*) &\in \mathrm{conv}\{uTv_n^*u^* : u \text{ is unitary in } A\}^{-\mathrm{stg}} \\ &= \mathrm{conv}\{uTv_n^*v_nv_n^*u^* : u \text{ is unitary in } A\}^{-\mathrm{stg}} \\ &= \mathrm{conv}\{uTv_n^*u^*v_nv_n^* : u \text{ is unitary in } A\}^{-\mathrm{stg}} \quad (\text{since } v_nv_n^* \in A) \\ &\subseteq \mathrm{conv}\{Sv_n^* : S \text{ is in } N\}^{-\mathrm{stg}} \quad (\text{since } v_n^*u^*v_n \in N) \\ &= Nv_n^* \quad (\text{since } v_n^*v_n \in N). \end{aligned}$$

So we get the conclusion. □

Now we construct a Borel subrelation of $\mathscr{R}$ associated to N as follows. For each $n \in I$, we set

$$A_n := E_A(Nv_n^*)(= Av_nv_n^* \cap Nv_n^*).$$

By Lemma 3.1, there exists a projection e_n in A such that $e_n \leqq v_nv_n^*$ and $Ae_n = A_n$. For each e_n, we obtain a Borel subset E_n of $\mathrm{Dom}(\rho_n)$ such that $e_n = L(\chi_{E_n})$ and $E_{-n} = \rho_n(E_n)$. We define a subset $\mathscr{S}_0$ of $\mathscr{R}$ by the following:

$$\mathscr{S}_0 := \bigcup_{n \in I} \Gamma(\rho_n|_{E_n}).$$

Moreover, we define $\mathscr{S}$ as a subset of $\mathscr{R}$ which is constructed by $\Gamma(\rho_n|_{E_n})$'s, i.e.,

$$\mathscr{S} := \langle \mathscr{S}_0 \rangle = \bigcup_{k \geqq 1} \bigcup_{l_1,\dots,l_k \in I} F_{l_1,\dots,l_k},$$

where

$$F_{l_1,\dots,l_k} := \Gamma(\rho_{l_k}\rho_{l_{k-1}} \cdots \rho_{l_1}|_{E_{l_1} \cap \rho_{l_1}^{-1}(E_{l_2}) \cap \cdots \cap \rho_{l_1}^{-1} \cdots \rho_{l_{k-1}}^{-1}(E_{l_k})}).$$

Lemma 3.2. *The subset $\mathscr{S}$ defined above is a Borel equivalence subrelation of $\mathscr{R}$.*

Proof. Since $\rho_l \in [\mathscr{R}]_*$ and E_l is a Borel subset of X for each $l \in I$, $\mathscr{S}$ is a Borel subset of $\mathscr{R}$. So it suffices to prove that $\mathscr{S}$ is an equivalence relation.

Since $\rho_0 = \mathrm{id}$ and $E_0 = X$ up to a μ-null set, $\mathscr{S}$ contains the diagonal D. If $(x, y) \in \mathscr{S}$, then there exist $l_1, \dots, l_k \in I$ such that $(x, y) \in F_{l_1,\dots,l_k}$. So we conclude that (y, x) is in $F_{-l_k,\dots,-l_1} \subseteq \mathscr{S}$. Finally, if (y, z) is also in $\mathscr{S}$, then $(y, z) \in F_{m_1,\dots,m_j}$ for some $m_1, \dots, m_j \in I$ and we get $(x, z) \in F_{l_1,\dots,l_k,m_1,\dots,m_j} \subseteq \mathscr{S}$. Therefore we complete the proof. □

Lemma 3.3. *The above subrelation $\mathscr{S}$ coincides with $\mathscr{S}_0$ up to a ν-null set, i.e., $\nu(\mathscr{S} \setminus \mathscr{S}_0) = 0$.*

Proof. If $\nu(\mathscr{S} \setminus \mathscr{S}_0) > 0$, then there exist $l_1, \dots, l_k \in I$ such that

$$\nu(F_{l_1,\dots,l_k} \setminus \mathscr{S}_0) > 0.$$

We set $F := F_{l_1,\dots,l_k} \setminus \mathscr{S}_0$ and define measurable functions $\{f_i\}_{i=1}^k$ on $\mathscr{R}$ and $w \in \mathscr{GN}_M(A)$ by the following:

$$f_i := a(1, \rho_{l_i}^{-1}|_{\rho_{l_i} \cdots \rho_{l_1}(\pi_l(F))}) = D_\mu^{1/2} \chi_{\Gamma(\rho_{l_i}|_{\rho_{l_{i-1}} \cdots \rho_{l_1}(\pi_l(F))})},$$

$$w := L^\sigma(f_1 * \cdots * f_k).$$

It is easy to see that $\mathrm{supp}(f_1 * \cdots * f_k) = F$ and $E_A(wv_n^*e_n) = 0$ for each $n \in I$. Indeed, a direct computation and (2.1) show that $v_n^*e_n = L^\sigma(a(1, \rho_n|_{E_n}))$ and $(f_1 * \cdots * f_k * a(1, \rho_n|_{E_n}))(x, x) = 0$ for almost all $x \in X$.

On the other hand, since $L^\sigma(f_i) \in Ae_{l_i}v_{l_i} \subseteq N$ for each $i = 1, \dots, k$, we get $w \in N$. In particular, by Lemma 3.1, $E_A(wv_n^*)e_n = E_A(wv_n^*)$ for each $n \in I$. So $E_A(wv_n^*) = 0$ for each $n \in I$. By Lemma 2.4, we obtain $w = 0$, i.e., $\nu(F) = 0$, a contradiction. Thus $\nu(\mathscr{S} \setminus \mathscr{S}_0) = 0$. □

By this lemma, we obtain $\mu(\pi_l(\Gamma(\rho_n) \cap (\mathscr{S} \setminus \mathscr{S}_0))) = 0$ for each $n \in I$. So we can replace E_n by $E_n \cup \pi_l(\Gamma(\rho_n) \cap (\mathscr{S} \setminus \mathscr{S}_0))$ and get $\mathscr{S} \cap \bigcup_{n \in I} \Gamma(\rho_n) = \mathscr{S}_0$.

Proposition 3.4. *The von Neumann subalgebra* $W^*(\mathscr{S},\sigma|_{\mathscr{S}})$ *of* M *is equal to* N.

Proof. We set $L := W^*(\mathscr{S},\sigma|_{\mathscr{S}})$.

We first show that $L \subseteq N$. For this, it suffices to prove that $\mathscr{N}_L(A) \subseteq N$. If $u \in \mathscr{N}_L(A)$, then there exists $\rho \in [\mathscr{S}]$ such that $u = L^\sigma(a(g,\rho))$ for some $g \in L^\infty(X)$. For each $n \in I \cap \boldsymbol{N}$, we define $u_n \in M$ by the following:

$$u_n := \sum_{k=-n+1}^{n-1} E_A(uv_k^*)v_k.$$

Since $\{\Gamma(\rho_k)\}_{k\in I}$ are mutually disjoint up to ν-null sets, we have

$$\Gamma(\rho^{-1}) \cap \Gamma(\rho_k) \subseteq \Gamma(\rho_k|_{E_k})$$

up to a ν-null set for each $k \in N$. In particular, a Borel subset $F_k := \pi_l(\Gamma(\rho^{-1}) \cap \Gamma(\rho_k))$ is contained in E_k up to a μ-null set. By (2.2) and Lemma 3.1, we get

$$E_A(uv_k^*)v_k = L^\sigma(a(\chi_{F_k}g,\rho)) \in Ae_kv_k = Av_k \cap Nv_k^*v_k \subseteq N$$

for each $k \in I$. Hence u_n also belongs to N for each $n \in I$. Moreover, since $\{F_k\}_{k\in I}$ are mutually disjoint up to μ-null sets, we have

$$u_n = \sum_{k=-n+1}^{n-1} L^\sigma(a(\chi_{F_k}g,\rho)) = L(\chi_{\bigcup_{k=-n+1}^{n-1}F_k})u.$$

This shows that $\|u_n\| \leqq \|u\| = 1$. On the other hand, by Lemma 2.4, $u_n\xi_0$ converges to $u\xi_0$. Thus u_n strongly converges to u, i.e., for each $\xi \in L^2(\mathscr{R})$, $u_n\xi$ converges to $u\xi$. Indeed, for each $\varepsilon > 0$, since ξ_0 is a cyclic vector for M', there exists $T' \in M'$ such that $\|T'\xi_0 - \xi\| < \varepsilon/3$. By Lemma 2.4, there exists $n_0 \in \boldsymbol{N}$ such that $\|T'u_n\xi_0 - T'u\xi_0\| < \varepsilon/3$ for each $n > n_0$. So we have

$$\begin{aligned}\|u_n\xi - u\xi\| &\leqq \|u_n\xi - u_nT'\xi_0\| + \|u_nT'\xi_0 - uT'\xi_0\| + \|uT'\xi_0 - u\xi\| \\ &\leqq \|u_n\|\,\|\xi - T'\xi_0\| + \|T'u_n\xi_0 - T'u\xi_0\| + \|u\|\,\|T'\xi_0 - \xi\| \\ &\leqq \|\xi - T\xi_0\| + \|T'u_n\xi_0 - T'u\xi_0\| + \|T'\xi_0 - \xi\| \\ &< \varepsilon,\end{aligned}$$

for each $n > n_0$. This shows that $u_n\xi$ converges to $u\xi$, and u belongs to N.

Conversely, if $L^\sigma(f) \in N\backslash L$, then we get $\nu(\mathrm{supp}(f) \cap (\mathscr{R}\backslash\mathscr{S})) > 0$ and

$$\nu\left(\mathrm{supp}(f) \cap \bigcup_{n\in I}\Gamma(\rho_n|_{\mathrm{Dom}(\rho_n)\backslash E_n})\right) = \nu\left(\mathrm{supp}(f) \cap (\mathscr{R}\backslash\mathscr{S}) \cap \bigcup_{n\in I}\Gamma(\rho_n)\right) > 0.$$

So there exists $n \in I$ such that $\nu(\mathrm{supp}(f) \cap \Gamma(\rho_n|_{\mathrm{Dom}(\rho_n)\backslash E_n})) > 0$. On the other hand, $E_A(L^\sigma(f)v_n^*)$ is of the form $L(h)$ for some $h \in L^\infty(X)$. By (2.1), $\mathrm{supp}(h)$ is equal to $\pi_l(\mathrm{supp}(f) \cap \Gamma(\rho_n))$. Since $\mu(\pi_l(\mathrm{supp}(f) \cap \Gamma(\rho_n|_{\mathrm{Dom}(\rho_n)\backslash E_n}))) > 0$, we obtain $L(h)\cdot(1 - e_n) \neq 0$, i.e., $E_A(L^\sigma(f)v_n^*) \notin Ae_n = E_A(Nv_n^*)$. So we get $L^\sigma(f) \notin N$, a contradiction. Hence we have proved the proposition. $\square$

REMARK. By using the above argument, we can show the following: for each $T \in \mathcal{GN}_M(A)$,

$$T = \sum_{n \in I} E_A(Tv_n^*)v_n$$

in the sense of the strong operator topology.

We are now in a position to prove our main theorem.

PROOF OF THEOREM 1.1. By Theorem 2.2 and Proposition 3.4, we obtain a discrete measured equivalence relation $\mathscr{R}$ on a standard Borel space $(X, \mathfrak{B}, \mu)$, a Borel subrelation $\mathscr{S}$ of $\mathscr{R}$ and a normalized 2-cocycle σ on $\mathscr{R}$ such that

$$(A \subseteq N \subseteq M) \cong (W^*(X) \subseteq W^*(\mathscr{S}, \sigma|_{\mathscr{S}}) \subseteq W^*(\mathscr{R}, \sigma)).$$

So A is also a Cartan subalgebra of N and a conditional expectation E_N from M onto N is defined by the following:

$$E_N(L^\sigma(f)) := L^\sigma(f|_{\mathscr{S}}).$$

By [**1**, Theorem 1.5.5], this is the unique faithful normal conditional expectation from M onto N. Therefore we complete the proof. □

We conclude this paper with some results which follow from the main theorem.

First, we characterize intermediate von Neumann subalgebras between an inclusion of a von Neumann algebra and a Cartan subalgebra. Many properties concerning a measure preserving full groups are proved by Nakamura, Dye, Takeda, Choda and so on. Our corollary extends one of their results to general full groups.

COROLLARY 3.5 (cf. [**5**, Proposition 6.1]). *Suppose M is a von Neumann algebra with a Cartan subalgebra A of M such that $M = W^*(\mathscr{R}, \sigma)$ and $A = W^*(X)$, where $\mathscr{R}$ is an equivalence relation on $(X, \mathfrak{B}, \mu)$ with a 2-cocycle σ. Then there exists a bijective correspondence between the set of Borel subrelations $\mathscr{S}$ of $\mathscr{R}$ on $(X, \mathfrak{B}, \mu)$ and the set of von Neumann subalgebras N of M which contain A*:

$$N \mapsto \mathscr{S}_N := \left\langle \bigcup_{n \in I} \Gamma(\rho_n|_{E_n}) \right\rangle, \tag{3.1}$$

$$\mathscr{S} \mapsto W^*(\mathscr{S}, \sigma|_{\mathscr{S}}), \tag{3.2}$$

where $\{\rho_n\}_{n \in I} \subseteq [\mathscr{R}]_*$ and $\{E_n\}_{n \in I} \subseteq \mathfrak{B}$ are defined by the process described at the beginning of this section.

PROOF. By Theorem 1.1, for each such an N, we get a Borel subrelation $\mathscr{S}_N$ of $\mathscr{R}$ such that $N = W^*(\mathscr{S}_N, \sigma|_{\mathscr{S}_N})$. So the above correspondence is bijective. □

REMARK. (1) Suppose $\mathscr{R}$ is a discrete measured equivalence relation on $(X, \mathfrak{B}, \mu)$. There exists a countable group action G on X so that $\mathscr{R} = \mathscr{R}_G$. In this situation, the above Galois correspondence is between the set of full subgroups of $[G]$ and the set of intermediate subalgebras.

(2) To construct an equivalence subrelation for a subalgebra, we use only the sub-

algebra and the original equivalence relation, i.e., our construction which is determined by (3.1) does not use the arguments given in [**7**, Section 3].

The second corollary is concerned with the regularity. Some interesting results on maximal abelian $*$-subalgebras of finite von Neumann algebras were shown by Dye, Dixmer, Jones and Popa. Dye and Jones-Popa proved that the regularity is hereditary in the setting of a finite von Neumann algebra with a maximal abelian $*$-subalgebra. We generalize this result to a general von Neumann algebra with a maximal abelian $*$-subalgebra which is the range of a conditional expectation.

Corollary 3.6 (cf. [**5**, Lemma 6.1] and [**10**, Corollary 2.3]). *If A is a maximal abelian $*$-subalgebra of M with the faithful normal conditional expectation from M onto A, then the regularity of A in M is hereditary, i.e., if A is regular in M, then, for each von Neumann subalgebra N between M and A, A is also regular in N.*

Proof. By Theorem 1.1, if A is a Cartan subalgebra of M, then A is also a Cartan subalgebra of N. $\square$

By making use of our main theorem, we shall prove two corollaries about an inclusion of factors with a common Cartan subalgebra. For this, we recall the basic facts about an inclusion of ergodic equivalence relations with choice functions. For the details about these matters, refer to [**7**] and [**8**].

Let $\mathscr{R}$ be a discrete measured equivalence relation on a standard measure space $(X, \mathfrak{B}, \mu)$ and σ be a normalized 2-cocycle on $\mathscr{R}$. The following results are well-known:

(1) A von Neumann algebra $W^*(\mathscr{R}, \sigma)$ is a factor if and only if $\mathscr{R}$ is ergodic, i.e., for any $[\mathscr{R}]$ invariant Borel subset E of X, E satisfies $\mu(E) = 0$ or $\mu(X \backslash E) = 0$ ([**7**, Proposition 2.9]).

(2) The Murray-von Neumann algebraic type of $W^*(\mathscr{R}, \sigma)$ coincides with that of $\mathscr{R}$.

For each inclusion of an ergodic discrete measured equivalence relations $\mathscr{S} \subseteq \mathscr{R}$ on $(X, \mathfrak{B}, \mu)$, by [**8**], we get choice functions $\{\psi_i\}_{i \in I}$ for $\mathscr{S} \subseteq \mathscr{R}$, i.e., $\{\psi_i\}_{i \in I}$ satisfy the following:

(i) Each ψ_i is a Borel function on X and $(x, \psi_i(x)) \in \mathscr{R}$ up to a μ-null set,

(ii) there exists $0 \in I$ such that ψ_0 is the identity,

(iii) if $(x, y) \in \mathscr{R}$, then there exists a unique $i \in I$ such that $(x, \psi_i(y)) \in \mathscr{S}$ up to a ν-null set.

The cardinality of I does not depend upon the choice of $\{\psi_i\}_{i \in I}$. This constant is said to be the index of $\mathscr{S}$ in $\mathscr{R}$. By [**8**, Lemma 1.3], if the types of $\mathscr{S}$ and $\mathscr{R}$ are equal, there exist choice functions $\{\psi_i\}_{i \in I}$ for $\mathscr{S} \subseteq \mathscr{R}$ such that each ψ_i is bijective (i.e., $\psi_i \in [\mathscr{R}]$).

Corollary 3.7 (cf. [**12**, Corollary 1.3]). *Let M be a (separable) factor and A be a Cartan subalgebra of M. If N is a subfactor of M which contains A, then the Jones index of N in M is in $\mathbf{N}$ or ∞.*

Proof. By Theorem 1.1, there exist an inclusion of ergodic equivalence relations $\mathscr{S} \subseteq \mathscr{R}$ on $(X, \mathfrak{B}, \mu)$ and a normalized 2-cocycle σ on $\mathscr{R}$ such that

$$(A \subseteq N \subseteq M) \cong (W^*(X) \subseteq W^*(\mathscr{S}, \sigma|_{\mathscr{S}}) \subseteq W^*(\mathscr{R}, \sigma)).$$

By the same argument as in the case where σ is trivial, we conclude that the Jones index of N in M coincides with the index of $\mathscr{S}$ in $\mathscr{R}$. Hence the index is in $\boldsymbol{N} \cup \{\infty\}$. □

Finally, we will prove the following corollary. We note that Popa proved it for type II_1 factors that do not necessarily have separable preduals. So, in the case where factors have separable preduals, our corollary is a generalization of his result to general factors.

Corollary 3.8 (cf. [**11**, Theorem 2.3]). *Let M be a (separable) factor and A be a Cartan subalgebra of M. If N is a subfactor of M such that N contains A and the Murray-von Neumann algebraic type of N coincides with that of M, then for each faithful normal state ω on A, there exists a subset $\{u_i\}_{i \in I}$ of $\mathscr{N}_M(A)$ containing 1 such that*

$$\sum_{i \in I}^{\oplus} N u_i \xi = M \xi \quad \text{in } L^2(M, \omega \circ E_A),$$

where ξ is the implementing vector of $\omega \circ E_A$.

Proof. By Theorem 1.1, $(N \subseteq M) \cong (W^*(\mathscr{S}, \sigma|_{\mathscr{S}}) \subseteq W^*(\mathscr{R}, \sigma))$ for a pair of ergodic discrete measured equivalence relations $\mathscr{S} \subseteq \mathscr{R}$ and a 2-cocycle σ. Since the type of $W^*(\mathscr{R}, \sigma)$ is equal to that of $\mathscr{R}$, we obtain that the types of $\mathscr{S}$ and $\mathscr{R}$ are equal. So we get bijective choice functions $\{\psi_i\}_{i \in I}$ for $\mathscr{S} \subseteq \mathscr{R}$. For each $i \in I$, we set $u_i := L^{\sigma}(a(1, \psi_i)) \in \mathscr{N}_M(A)$. Since $\{\Gamma(\psi_i^{-1})\}_{i \in I}$ are mutually disjoint and $\mathscr{R}$ is generated by $\mathscr{S}$ and $\{\Gamma(\psi_i^{-1})\}_{i \in I}$ up to null sets, we conclude $M\xi = \sum_{i \in I}^{\oplus} N u_i \xi$. Indeed, if $i \neq j$, then, for any $\rho \in [\mathscr{S}]_*$, we have

$$\begin{aligned}(a(1, \psi_j)^* * a(1, \rho) * a(1, \psi_i))(x, x) &= \sum_{y, z \sim x} \chi_{\Gamma(\psi_j)}(x, y) \chi_{\Gamma(\rho^{-1})}(y, z) \chi_{\Gamma(\psi_i^{-1})}(z, x) \\ &= \chi_{\Gamma(\rho^{-1})}(\psi_j(x), \psi_i(x)) \\ &= 0 \quad (\text{since } (\psi_j(x), \psi_i(x)) \notin \mathscr{S})\end{aligned}$$

for almost all $x \in X$. Thus $N v_i \xi \perp N v_j \xi$. Therefore we complete the proof. □

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Added in Proof. After this paper was accepted for publication professor K. Saito informed the author that P. S. Muhly, K. Saito and B. Solel had treated the same problem in [**MSS**], and also informed that their paper has a gap in the proof of [**MSS**, Theorem 2.5]. In the theorem, they claimed that for each σ-weakly closed linear subspace N of M which is a bimodule over a Cartan subalgebra A of M (the subspace N is not necessarily a subalgebra), N comes from a Borel subset of an equivalence relation. The author is grateful to them for informing the above and also for calling his attention to [**F**], [**MSS**] and [**MS**].

To prove the theorem, they defined a bimodule $\mathscr{A}(F(\xi), G(\xi))$ over A for each $\xi \in L^2(\mathscr{R})$, and proved that the bimodule is equal to $\{T \in M : T\xi \in \overline{N\xi}\}$, and N coincides with $\bigcap_{\xi \in L^2(\mathscr{R})} \mathscr{A}(F(\xi), G(\xi))$. In this situation, they claimed that the intersection coincides with that taken over any countable dense subset $\{\xi_n\}_{n\geqq 1}$ of $L^2(\mathscr{R})$, i.e.,

$$(*) \qquad \bigcap_{\xi \in L^2(\mathscr{R})} \mathscr{A}(F(\xi), G(\xi)) = \bigcap_{n\geqq 1} \mathscr{A}(F(\xi_n), G(\xi_n)).$$

But this claim was mentioned without proof in [**MSS**]. Moreover, the following two arguments tell us that the verification of the claim seems nontrivial. This is the gap which we mentioned above.

We first note that the property that the subset $\{\xi_n\}_{n\geq 1}$ is dense in $L^2(\mathscr{R})$ does not reflect that of the family of bimodules $\{\mathscr{A}(F(\xi_n), G(\xi_n))\}_{n\geq 1}$ over A. Very roughly, the map $\xi \to \mathscr{A}(F(\xi), G(\xi))$ is "not continuous". For example, for each $\xi \in L^2(\mathscr{R})$ which satisfies $\mathscr{A}(F(\xi), G(\xi)) \neq M$, set $\eta_n := (1/n)\xi$ for each $n \geqq 1$. Then it is trivial that each $\mathscr{A}(F(\eta_n), G(\eta_n))$ is equal to $\mathscr{A}(F(\xi), G(\xi))$. But we have

$$\mathscr{A}\Big(F\Big(\lim_{n\to\infty} \eta_n\Big), G\Big(\lim_{n\to\infty} \eta_n\Big)\Big) = \mathscr{A}(F(0), G(0)) = M,$$

which is not equal to $\mathscr{A}(F(\xi), G(\xi))$.

Secondly, since $\mathscr{A}(F(\xi), G(\xi))$ is equal to $\{T \in M : T\xi \in \overline{N\xi}\}$ as mentioned above, $(*)$ is equivalent to the following:

$$\bigcap_{\xi \in L^2(\mathscr{R})} \{T \in M : T\xi \in \overline{N\xi}\} = \bigcap_{n\geqq 1} \{T \in M : T\xi_n \in \overline{N\xi_n}\}.$$

But they did not use the property that N is a bimodule over A. So if their arguments were valid, then the following equation would hold for any von Neumann algebra N acting on a separable Hilbert space H and any countable dense subset $\{\xi_n\}_{n\geqq 1}$ of H:

$$(\dagger) \qquad \bigcap_{\xi \in H} \{T \in B(H) : T\xi \in \overline{N\xi}\} = \bigcap_{n\geqq 1} \{T \in B(H) : T\xi_n \in \overline{N\xi_n}\}.$$

But the above equation does not hold in general. Indeed, suppose that N is not equal to $B(H)$, and that N has a cyclic and separating vector η_0. Then, by [**3**, Corollaire 2], there exists a subset of cyclic vectors $\{\eta_\lambda\}_{\lambda \in \Lambda}$ for N which is dense in H. Since H is separable, we may choose a sequence $\{\lambda_n\}_{n \geqq 1} \subseteq \Lambda$ such that $\{\eta_{\lambda_n}\}_{n \geqq 1}$ is dense in H. So we have

$$\bigcap_{n \geqq 1} \{T \in B(H) : T\eta_{\lambda_n} \in \overline{N\eta_{\lambda_n}}\} = \bigcap_{n \geqq 1} \{T \in B(H) : T\eta_{\lambda_n} \in H\} = B(H).$$

On the other hand, since N has a separating vector, by [**LS**, Theorem 3.7], the left-hand side of (†) is equal to N. So this provides a counterexample to (†).

We last note that if the equivalence relation $\mathscr{R}$ is hyperfinite, i.e., $M = W^*(\mathscr{R}, \sigma)$ is a hyperfinite von Neumann algebra, then [**MSS**, Theorem 2.5] is true. Indeed, P. S. Muhly and B. Solel proved it in [**MS**, Theorem 3.10], and I. Fulman gave a proof to a more general setting ([**F**, Theorem 15.18]). Although our arguments are valid for any intermediate subalgebra, it seems difficult to apply them to bimodules of a (not necessary hyperfinite) von Neumann algebra.

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