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Punctured local holomorphic de Rham cohomology

By Xiaojun HUANG*, Hing Sun LUK† and Stephen S.-T. YAU*†

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Abstract. Let V be a complex analytic space and x be an isolated singular point of V. We define the q-th punctured local holomorphic de Rham cohomology $H_h^q(V,x)$ to be the direct limit of $H_h^q(U-\{x\})$ where U runs over strongly pseudoconvex neighborhoods of x in V, and $H_h^q(U-\{x\})$ is the holomorphic de Rahm cohomology of the complex manifold $U-\{x\}$. We prove that punctured local holomorphic de Rham cohomology is an important local invariant which can be used to tell when the singularity (V,x) is quasi-homogeneous. We also define and compute various Poincaré number $\tilde{p}_x^{(i)}$ and $\bar{p}_x^{(i)}$ of isolated hypersurface singularity (V,x).

1. Introduction.

Let M be a complex manifold. The q-th holomorphic de Rham cohomology of M, $H_h^q(M)$, is defined to be the d-closed holomorphic q-forms modulo the d-exact holomorphic q-forms. Holomorphic de Rham cohomology was studied by Hörmander [**Hö**]. It is well known that if M is a Stein manifold, then the holomorphic de Rham cohomology coincides with the ordinary de Rham cohomology [**Hö**]. The purpose of this paper is to introduce the notion of punctured local holomorphic de Rham cohomology. Let V be a complex analytic space and x be a point in V such that $V-\{x\}$ is nonsingular near x. We define the q-th punctured local holomorphic de Rham cohomology $H_h^q(V,x)$ to be the direct limit of $H_h^q(U-\{x\})$ where U runs over strongly pseudoconvex neighborhoods of x in V. If V is a complex manifold of dimension at least 2, then $H_h^q(V-\{x\})=H_h^q(V)$ by Hartog extension theorem. Therefore the punctured local holomorphic de Rham cohomology $H_h^q(V,x)$ vanishes if x is a smooth point of V. It turns out that punctured local holomorphic de Rham cohomology is an important local invariant which can be used to tell when the singularity (V,x) is quasi-homogeneous. The following is our Main Theorem.

MAIN THEOREM. *Let* $(V,0)=\{(z_0,\dots,z_n)\in \boldsymbol{C}^{n+1}:f(z_0,\dots,z_n)=0\}$ *be a hypersurface with origin as isolated singular point. Then*

(i) $\dim H_h^q(V,0)=0$ *for* $q\le n-2$

(ii) $\dim H_h^n(V,0)-\dim H_h^{n-1}(V,0)=\mu-\tau$ *where* $\mu=\dim \boldsymbol{C}\{z_0,\dots,z_n\}/(\partial f/\partial z_0,\dots,\partial f/\partial z_n)$ *is the Milnor number and* $\tau=\dim \boldsymbol{C}\{z_0,\dots,z_n\}/(f,\partial f/\partial z_0,\dots,\partial f/\partial z_n)$ *is the Tjurina number of the singularity* $(V,0)$ *respectively.*

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In particular, $\dim H_h^n(V,0) = \dim H_h^{n-1}(V,0)$ *if and only if* $(V,0)$ *is a quasi-homogeneous singularity, i.e. f is a weighted homogeneous polynomial after holomorphic change of coordinates.*

The above theorem says that we can determine the quasi-homogeneity of the singularity by the punctured neighborhood of the singularity. This follows from the celebrated theorem of Saito [**Sa**] which states that $\mu = \tau$ if and only if f is a weighted homogeneous polynomial after holomorphic change of coordinates.

In the course of the proof of our Main Theorem above, we also define and compute various Poincare numbers $\tilde{p}_x^{(i)}$ and $\bar{p}_x^{(i)}$ of isolated hypersurface singularity (V,x).

2. Poincaré numbers and punctured local holomorphic de Rham cohomology.

Let V be an n-dimensional complex analytic subvariety in $\boldsymbol{C}^N$ with only isolated singularities. There are four kinds of sheaves of germs of holomorphic p forms defined on V (cf. [**Ya 2**])

(i) $\Omega_V^p := \Omega_{\boldsymbol{C}^N}^p / \{f\alpha + dg \wedge \beta : \alpha \in \Omega_{\boldsymbol{C}^N}^p, \beta \in \Omega_{\boldsymbol{C}^N}^{p-1}, f, g \in I\}$, where I is the ideal sheaf of V in $\mathcal{O}_{\boldsymbol{C}^N}$

(ii) $\tilde{\Omega}_V^p := \Omega_V^p/\text{torsion subsheaf of } \Omega_V^p$

(iii) $\bar{\Omega}_V^p := R^0\pi_*\Omega_M^p$, where $\pi : M \to V$ is a resolution of singularities of V

(iv) $\bar{\bar{\Omega}}_V^p := \theta_*\Omega_{V-\operatorname{sing} V}^p$, where $\theta : V - \operatorname{sing} V \to V$ is the inclusion map and $\operatorname{sing} V$ is the singular set of V.

Remark 2.1. Clearly Ω_V^p, $\tilde{\Omega}_V^p$ and $\bar{\Omega}_V^p$ are coherent sheaf. $\bar{\bar{\Omega}}_V^p$ is also a coherent sheaf by a Theorem of Siu [**Si**]. In case V is a normal variety, then the dualizing sheaf ω_V of Grothendieck is actually the sheaf $\bar{\bar{\Omega}}_V^n$.

Definition 2.2. Let H^p, K^p and J^p be coherent sheaves, supported on the singular points of V, defined by the following exact sequences:

$$0 \to K^p \to \Omega_V^p \to \tilde{\Omega}_V^p \to 0$$

$$0 \to \tilde{\Omega}_V^p \to \bar{\Omega}_V^p \to H^p \to 0$$

$$0 \to \bar{\Omega}_V^p \to \bar{\bar{\Omega}}_V^p \to J^p \to 0$$

Then the invariants $g^{(p)}$, $m^{(p)}$ and $s^{(p)}$ at x are defined to be $\dim(H^p)_x$, $\dim(K^p)_x$, and $\dim(J^p)_x$ respectively.

Definition 2.3. Let x be a singular point of V. Let

$$0 \to \boldsymbol{C} \to \mathcal{O}_{V,x} \xrightarrow{d^0} \Omega_{V,x}^1 \xrightarrow{d^1} \Omega_{V,x}^2 \xrightarrow{d^2} \cdots$$

be the Poincaré complex at x. Then the Poincare numbers of V at x are defined as

$$p_x^{(i)} = \dim \operatorname{Ker} d^i / \operatorname{Im} d^{i-1}, \quad i \geq 0.$$

Remark 2.4. By Bloom and Herrera [**Bl-He**], all these Poincaré numbers are finite. If x is a smooth point of V, then all the Poincaré numbers of V at x vanish.

Remark 2.5. Now let (V,x) be an isolated hypersurface singularity, $\dim(V,x) = n = N-1$. In [**Br**], Brieskorn proved that $p_x^{(i)} = 0$ if $i \le n-2$. Later Sebastiani [**Se**] proved that $p_x^{(n-1)}$ is also equal to zero. In [**Sa**], Saito proved that $p_x^{(n)} = 0$ if and only if (V,x) is quasi-homogeneous. The proofs of Brieskorn, Sebastiani and Saito are purely local (the global argument of Brieskorn in his coherence theorem can be avoided by using the main theorem of Kiehl-Verdier).

In the following, we shall introduce generalized Poincaré numbers $\tilde{p}_x^{(i)}$, $\bar{p}_x^{(i)}$.

Definition 2.6. Let x be a singular point of V. Consider the complexes

$$0 \to \boldsymbol{C} \to \mathcal{O}_{V,x} \xrightarrow{\tilde{d}^0} \tilde{\Omega}^1_{V,x} \xrightarrow{\tilde{d}^1} \tilde{\Omega}^2_{V,x} \to \cdots,$$

$$0 \to \boldsymbol{C} \to \mathcal{O}_{V,x} \xrightarrow{\bar{d}^0} \bar{\Omega}^1_{V,x} \xrightarrow{\bar{d}^1} \bar{\Omega}^2_{V,x} \to \cdots.$$

Then the generalized Poincaré numbers $\tilde{p}_x^{(i)}$, $\bar{p}_x^{(i)}$ are defined by $\dim \ker \tilde{d}^i / \mathrm{Im}\, \tilde{d}^{i-1}$ and $\dim \ker \bar{d}^i / \mathrm{Im}\, \bar{d}^{i-1}$ respectively.

Lemma 2.7. *Let (V,x) be an isolated singularity. Then the punctured local holomorphic de Rham cohomology $H_h^q(V,x)$ is isomorphic to cohomology $\ker \bar{\bar{d}}^q / \mathrm{Im}\, \bar{\bar{d}}^{q-1}$ of the following complex*

$$0 \to \boldsymbol{C} \to \mathcal{O}_{V,x} \xrightarrow{\bar{\bar{d}}^0} \bar{\bar{\Omega}}^1_{V,x} \xrightarrow{\bar{\bar{d}}^1} \bar{\bar{\Omega}}^2_{V,x} \to \cdots.$$

Proof. It is obvious. □

Proposition 2.8. *Let (V,x) be an isolated singularity. Then the generalized Poincaré numbers $\tilde{p}_x^{(i)}$, $\bar{p}_x^{(i)}$ and $\dim H_h^q(V,x)$ are finite numbers.*

Proof. Consider the short exact sequence of complexes

$$0 \to K_x^\bullet \to \Omega_{V,x}^\bullet \to \tilde{\Omega}_{V,x}^\bullet \to 0.$$

Since K^i's are coherent sheaves with support at $\{x\}$, $(K_x^\bullet, d)$ is a complex of finite dimensional vector spaces. Observe that $p^{(i)} = \dim H^i(\Omega_{V,x}^\bullet)$, and $\tilde{p}^{(i)} = \dim H^i(\tilde{\Omega}_{V,x}^\bullet)$. So the long cohomology sequence

$$\cdots \to H^i(K^\bullet) \to H^i(\Omega_{V,x}^\bullet) \to H^i(\tilde{\Omega}_{V,x}) \to H^{i+1}(K_x^\bullet) \to \cdots$$

shows that $\tilde{p}_x^{(i)}$ is finite for all i because $\dim H^i(K_x^\bullet)$ and $p_x^i = \dim H^i(\Omega_{V,x}^\bullet)$ are finite numbers.

Similarly by considering the short exact sequence of the complexes

$$0 \to \tilde{\Omega}_{V,x}^\bullet \to \bar{\Omega}_{V,x}^\bullet \to H_x^\bullet \to 0,$$

$$0 \to \bar{\Omega}_{V,x}^\bullet \to \bar{\bar{\Omega}}_{V,x}^\bullet \to J_x^\bullet \to 0,$$

one can show that $\bar{p}_x^{(i)}$ and $\dim H_h^i(V,x)$ are finite for all i. □

Theorem 2.9. *Let $V = \{(z_0, \dots, z_n) \in \boldsymbol{C}^{n+1} : f(z_0, \dots, z_n) = 0\}$ be a hypersurface with the origin as its only singular point. Let*

$$\tau = \dim \boldsymbol{C}\{z_0, \dots, z_n\} \Big/ \left(f, \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right)$$

$$\mu = \dim \boldsymbol{C}\{z_0, \dots, z_n\} \Big/ \left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right).$$

Then

(a) $\Omega_V^i = \tilde{\Omega}_V^i$ *for* $0 \le i \le n-1$, *i.e.* $K^i = 0$ *for* $0 \le i \le n-1$; $\tilde{\Omega}_V^{n+1} = 0$; *and* $K^{n+1} = \Omega_V^{n+1}$ *with dimension* τ,

(b) $\dim K^n = \dim K^{n+1} = \tau$,

(c) $p^{(i)} = 0$ *for* $i \le n-1$ *and* $i = n+1$; $p^{(n)} = \mu - \tau$,

(d) $\tilde{p}^{(i)} = 0$ *for* $i \le n-2$, *and* $i = n+1$; $\tilde{p}^{(n)} - \tilde{p}^{(n-1)} = \mu - \tau$.

PROOF.

(a) See Theorem 2.9 of [**Ya2**].

(b) $\dim K^n = \tau$ is due to Greuel ([**Gr**], Proposition 1.11 (iii)). $\dim K^{n+1} = \tau$ follows easily from the definition (cf. Theorem 2.9 of [**Ya2**]).

(c) $p^{(i)} = 0$ for $i \le n-2$ is due to Brieskorn. $p^{(n-1)} = 0$ is due to Sebastiani [**Se**]. One checks easily that $\Omega_V^n \xrightarrow{d} \Omega_V^{n+1}$ is surjective. So $p^{(n+1)} = 0$ follows. Finally $p^{(n)} = \mu - \tau$ can be found, for example, in Corollary 5.6 of [**Ya2**].

(d) Consider the exact sequence of complexes

$$\begin{array}{ccccccccc}
 & & & & \Omega_V^{n-2} & \xrightarrow{\approx} & \tilde{\Omega}_V^{n-1} & & \\
 & & & & \downarrow{\scriptstyle d_V^{n-2}} & & \downarrow{\scriptstyle \tilde{d}^{n-2}} & & \\
0 & \longrightarrow & 0 & \longrightarrow & \Omega_V^{n-1} & \xrightarrow{\approx} & \tilde{\Omega}_V^{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle d^{n-1}} & & \downarrow{\scriptstyle \tilde{d}^{n-1}} & & \\
0 & \longrightarrow & K^n & \longrightarrow & \Omega_V^n & \longrightarrow & \tilde{\Omega}_V^n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^n} & & \downarrow{\scriptstyle d^n} & & \downarrow{\scriptstyle \tilde{d}^n} & & \\
0 & \longrightarrow & K^{n+1} & \longrightarrow & \Omega_V^{n+1} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}.$$

We have the following long cohomology exact sequence

$$0 = H^{n-1}(\Omega_V^\bullet) \to H^{n-1}(\tilde{\Omega}_V^{n-1}) \to H^n(K^\bullet) \to H^n(\Omega_V^\bullet)$$

$$\to H^n(\tilde{\Omega}_V^\bullet) \to H^{n+1}(K^\bullet) \to H^{n+1}(\Omega_V^\bullet) = 0.$$

Recall that the Euler characteristic of a complex is equal to the Euler characteristic of its cohomology. We have $\dim H^n(K^\bullet) - \dim H^{n+1}(K^1) = \dim K^n - \dim K^{n+1} = 0$. Therefore from the above exact sequence, we conclude that

$$\dim H^n(\tilde{\Omega}_V^\bullet) - \dim H^{n-1}(\tilde{\Omega}_V^{n-1}) = \dim H^n(\Omega_V^\bullet)$$

which is equivalent to

$$\tilde{p}^{(n)} - \tilde{p}^{(n-1)} = p^{(n)} = \mu - \tau$$

Clearly $p^{(i)} = \tilde{p}^{(i)} = 0$ for $i \le n-2$ from definitions and parts (a) and (c). □

THEOREM 2.10. *Let $V = \{(z_0, \dots, z_n) \in \boldsymbol{C}^{n+1} : f(z_0, \dots, z_n) = 0\}$ be a hypersurface with origin as its only singular point. Then*

(a) $\tilde{\Omega}_V^i = \overline{\Omega}_V^i$ *for* $0 \le i \le n-2$; $\tilde{\Omega}_V^{n+1} = \overline{\Omega}_V^{n+1} = 0$,

(b) $\bar{p}^{(i)} = \tilde{p}^{(i)} = 0$ *for* $i \le n-2$
$\bar{p}^{(n)} - \bar{p}^{(n-1)} = \mu - \tau + g^{(n)} - g^{(n-1)}$

where $g^{(n)} = \dim H^n$ *and* $g^{(n-1)} = \dim H^{n-1}$.

PROOF.
(a) See Theorem 2.9 of [**Ya2**].
(b) It is clear that $\bar{p}^{(i)} = 0$ for $i \le n-2$.
Consider the following exact sequence of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{\Omega}_V^{n-2} & \longrightarrow & \overline{\Omega}_V^{n-2} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \tilde{d}^{n-2}} & & \Big\downarrow{\scriptstyle \bar{d}^{n-2}} & & \Big\downarrow & & \\
0 & \longrightarrow & \tilde{\Omega}_V^{n-1} & \longrightarrow & \overline{\Omega}_V^{n-1} & \longrightarrow & H^{n-1} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \tilde{d}^{n-1}} & & \Big\downarrow{\scriptstyle \bar{d}^{n-1}} & & \Big\downarrow{\scriptstyle \bar{d}^{n-1}} & & \\
0 & \longrightarrow & \tilde{\Omega}_V^{n} & \longrightarrow & \overline{\Omega}_V^{n} & \longrightarrow & H^{n} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \tilde{d}^{n}} & & \Big\downarrow{\scriptstyle \bar{d}^{n}} & & \Big\downarrow{\scriptstyle \bar{d}^{n}} & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

We have the following long cohomology exact sequence

$$0 \to H^{n-1}(\tilde{\Omega}_V^\bullet) \to H^{n-1}(\overline{\Omega}_V^\bullet) \to H^{n-1}(H^\bullet) \to H^n(\tilde{\Omega}_V^\bullet)$$
$$\to H^n(\overline{\Omega}_V) \to H^n(H^\bullet) \to 0.$$

So we have

$$\tilde{p}^{(n-1)} - \bar{p}^{(n-1)} + \dim H^{n-1}(H^\bullet) - \tilde{p}^{(n)} + \bar{p}^{(n)} - \dim H^n(H^\bullet) = 0$$

which is equivalent to

$$\begin{aligned}
\bar{p}^{(n)} - \bar{p}^{(n-1)} &= \tilde{p}^{(n)} - \tilde{p}^{(n-1)} + \dim H^n(H^\bullet) - \dim H^{n-1}(H^\bullet) \\
&= \mu - \tau + g^{(n)} - g^{(n-1)}.
\end{aligned}$$

Here we use the fact that Euler characteristic of a complex is equal to the Euler characteristic of its cohomology. □

THEOREM 2.11. *Let* $V = \{(z_0, \dots, z_n) \in \boldsymbol{C}^{n+1} : f(z_0, \dots, z_n) = 0\}$ *be a hypersurface with origin as its only singular point. Then*

(a) $\bar{\Omega}^i_V = \bar{\bar{\Omega}}^i_V$ *for* $0 \le i \le n-2$; $\bar{\Omega}^{n+1} = \bar{\bar{\Omega}}^{n+1} = 0$,

(b) $\dim H^i_h(V, x) = \bar{p}^{(i)} = 0$ *for* $i \le n-2$,
$\dim H^n_h(V, x) - \dim H^{n-1}_h(V, x) = \mu - g^{(n-1)} - s^{(n-1)}$.

PROOF.
(a) See Theorem 2.9 of [**Ya2**].
(b) It is clear that $\dim H^i_h(V, x) = \bar{p}^{(i)} = 0$ for $i \le n-2$.
Consider the following exact sequence of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{\Omega}^{n-2}_V & \longrightarrow & \bar{\bar{\Omega}}^{n-2}_V & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \bar{d}^{n-2}} & & \Big\downarrow{\scriptstyle \bar{\bar{d}}^{n-2}} & & \Big\downarrow & & \\
0 & \longrightarrow & \bar{\Omega}^{n-1}_V & \longrightarrow & \bar{\bar{\Omega}}^{n-1}_V & \longrightarrow & J^{n-1} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \bar{d}^{n-1}} & & \Big\downarrow{\scriptstyle \bar{\bar{d}}^{n-1}} & & \Big\downarrow{\scriptstyle \bar{\bar{d}}^{n-1}} & & \\
0 & \longrightarrow & \bar{\Omega}^{n}_V & \longrightarrow & \bar{\bar{\Omega}}^{n}_V & \longrightarrow & J^{n} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \bar{d}^{n}} & & \Big\downarrow{\scriptstyle \bar{\bar{d}}^{n}} & & \Big\downarrow{\scriptstyle \bar{\bar{d}}^{n}} & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

We have the following long cohomology exact sequence

$$0 \to H^{n-1}(\bar{\Omega}^\bullet_V) \to H^{n-1}(\bar{\bar{\Omega}}^\bullet_V) \to H^{n-1}(J^\bullet) \to H^n(\bar{\Omega}^\bullet_V)$$
$$\to H^n(\bar{\bar{\Omega}}^\bullet_V) \to H^n(J^\bullet) \to 0.$$

So we have

$$\bar{p}^{(n-1)} - \dim H^{n-1}_h(V, x) + \dim H^{n-1}(J^0) - \bar{p}^{(n)} + \dim H^n_h(V, x) - \dim H^n(J^\bullet) = 0$$

which is equivalent to

$$\begin{aligned}
\dim H^n_h(V, x) - \dim H^{n-1}_h(V, x) &= \bar{p}^{(n)} - \bar{p}^{(n-1)} + \dim H^n(J^\bullet) - \dim H^{n-1}(J^\bullet) \\
&= \mu - \tau + g^{(n)} - g^{(n-1)} + \dim J^n - \dim J^{n-1} \\
&= \mu - \tau + g^{(n)} - g^{(n-1)} + s^{(n)} - s^{(n-1)}.
\end{aligned}$$

Recall that $g^{(n)} = \tau - s^{(n)}$ by Theorem 2.9 and Theorem 4.2 of [**Ya2**]. Therefore we have $\dim H^n_h(V, x) - \dim H^{n-1}_h(V, x) = \mu - g^{(n-1)} - s^{(n-1)}$. □

THEOREM 2.12. *Let* $V = \{(z_0, \dots, z_n) \in \boldsymbol{C}^{n+1} : f(z_0, \dots, z_n) = 0\}$ *be a hypersurface with origin as its only singular point. Then* $g^{(n-1)} + s^{(n-1)} = \tau$. *In particular* $\dim H^n_h(V, x) - \dim H^{n-1}_h(V, x) = \mu - \tau$.

Proof. Let $\pi : M \to V$ be a resolution of V and $A = \pi^{-1}(0)$. In view of [**Ya2**], we know that

$$g^{(n-1)} + s^{(n-1)} = \dim \Gamma(M, \Omega_M^{n-1})/\pi^* \Gamma(V, \Omega_V^1) + \dim \Gamma(M - A, \Omega_M^{n-1})/\Gamma(M, \Omega_M^{n-1}).$$

By the exact sequence

$$0 \to \Gamma(M, \Omega_M^{n-1})/\pi^* \Gamma(V, \Omega_V^{n-1}) \to \Gamma(M \backslash A, \Omega_M^{n-1})/\pi^* \Gamma(V, \Omega_V^{n-1}) \to \Gamma(M - A, \Omega_M^{n-1})/\Gamma(M, \Omega_M^{n-1}) \to 0,$$

we have

$$\begin{aligned} g^{(n-1)} + s^{(n-1)} &= \dim \Gamma(M \backslash A, \Omega_M^{n-1})/\pi^* \Gamma(V, \Omega_V^{n-1}) \\ &= \dim \Gamma(V - \{0\}, \Omega_V^{n-1})/\Gamma(V, \Omega_V^{n-1}). \end{aligned}$$

Look at the local cohomology exact sequence

$$0 \to H_{\{0\}}^0(V, \Omega_V^{n-1}) \to H^0(V, \Omega_V^{n-1}) \to H^0(V - \{0\}, \Omega_V^{n-1}) \to H_{\{0\}}^1(V, \Omega_V^{n-1}) \to H^1(V, \Omega_V^{n-1}),$$

$H^1(V, \Omega_V^{n-1}) = 0$ because V is Stein. Hence

$$\begin{aligned} g^{(n-1)} + s^{(n-1)} &= \dim H_{\{0\}}^1(V, \Omega_V^{n-1}) \\ &= \tau \end{aligned}$$

in view of Theorem 4.1 of [**Ya1**]. Combining this result with Theorem 2.11 (b) the last statement of Theorem 2.12 follows. □

Now we are ready to prove the Main Theorem in section 1. (i) and (ii) of the Main Theorem follow directly from Theorem 2.11 and Theorem 2.12 respectively. The last statement of the Main Theorem follows from (ii) and a theorem of Saito [**Sa**].

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Xiaojun HUANG
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903
E-mail: huangx@math.rutgers.edu

Hing Sun LUK
Department of Mathematics
The Chinese University of Hong Kong
Shatin, N.T.
Hong Kong
E-mail: hsluk@math.cuhk.edu.hk

Stephen S.-T. YAU
Department of Mathematics, Statistics
and Computer Science (M/C 249)
University of Illinois at Chicago
851 South Morgan Street
Chicago, IL 60607-7045
U.S.A.
E-mail: yau@uic.edu